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CLASS NUMBER PROBLEMS FOR REAL QUADRATIC FIELDS WITH SMALL FUNDAMENTAL UNIT

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Abstract. Chowla conjectured that the instances of $\mathbf{Q}(\sqrt{4u^2+1})$ of class number 1 (with $4u^2+1$ squarefree and $u>0$) are $u\in\{1,2,3,5,7,13\}$, and this was shown by Biró in 2003 by generalizing a method of Beck involving Dedekind ζ-functions of ideal classes.

We provide a different method of showing this result, and indeed proceed to explicitly classify all cases with class number 5 or less. We similarly handle Yokoi's family $\mathbf{Q}(\sqrt{u^2+4})$ and Euler's $\mathbf{Q}(\sqrt{25u^2+14u+2})$.

We also show there are exactly 22 fundamental discriminants $D>0$ with fundamental unit $(A+B\sqrt{D})/2$ that have $B\le D^{1/4}$ and class number 1 (the largest is 1253), as another concrete example of our methods.

Our main technical tool improves Goldfeld's lower bound for $L_\chi(1)$. This uses six explicit elliptic curves E, each of rank 5; each has an L-function with an analytic rank of at least 3 plus exactly two more zeros nearby (which are conjecturally also at the central point). By symmetry these two zeros are on either the central line or the real axis. On the other hand, we find that a "Deuring decomposition" gives a local approximation (with $i\xi$ roughly $(s-1)\log D$) to the completed product L-function as $\Lambda_E(s)\Lambda_{E\chi}(s)\sim c\xi(\sin\xi-\xi)$ whose noncentral zeros are off the axes, with an error bound depending on $L_\chi(1)$. We only improve Goldfeld's bound by a constant (based on the precision that $L_E'''(1)\approx 0.0000\ldots$ is known), but this suffices for our results.

1. History and introduction

A conjecture [13, §4] of Chowla posits that the largest u such that $D=4u^2+1$ is squarefree[1] and $\mathbf{Q}(\sqrt{D})$ has class number one is $u=26$. Both this and the companion problem of Yokoi's conjecture for $\mathbf{Q}(\sqrt{u^2+4})$ were solved in 2003 by Biró [5] using a method involving ideal class ζ-functions.

Of course, the "point" with these conjectures is that we have an explicit fundamental unit of small height (asymptotically proportional to $\log D$), so class number 1 is equivalent (by Dirichlet's class number formula) to a small value of $L_\chi(1)$ for the associated L-function, namely of size $(\log D)/\sqrt{D}$. Via Siegel's ineffective theorem [65] one can thus show there are at most finitely many exceptions to either conjecture (or indeed, at most one to both together [39]). However, the best known effective lower bound for $L_\chi(1)$ is somewhat worse than $(\log D)/\sqrt{D}$, precluding resolving the conjectures by such means.

Let us recall the best known effective lower bound for $L_\chi(1)$, in the generality of χ as the quadratic character of conductor $D=|\Delta|$ associated to the field $\mathbf{Q}(\sqrt{\Delta})$ of fundamental discriminant Δ. A combination of the works of Goldfeld [28] (for the general framework), Gross and Zagier [31] (showing that a suitable elliptic curve of analytic rank 3 exists), and Oesterlé [56] (for various refinements) imply the result[2]

$$\sqrt{D}L_\chi(1)=\pi h_K\ge\frac{\pi}{7000}\cdot\log D\cdot\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr)\tag{1}$$

for $\Delta<0$ (and $D>4$ to avoid units); one can adapt this for real quadratic fields,[3] but the obtained constant will be (significantly) less than 1, so the result is inferior to the

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[1] If we omitted this squarefree condition, we would be led to consider solutions to $4u^2+1=dv^2$, which is the same as $4u^2-dv^2=-1$, so is thus a Pellian family for each d. For instance, with $d=13$ we can take $(u,v)=(9,5)$. Indeed, the family of Euler that we consider has $D=((25\tilde{u}+7)^2+1)/5^2$ (or 4 times this), and is thus the 5-imprimitive version (with $-2u=25\tilde{u}+7$) of the $4u^2+1$ family.

[2] Oesterlé's result is often incorrectly given with 55 replacing 7000, but this is only valid for D coprime to 5077. We discuss this more in §10.

[3] Byeon and Kim [11] have recently produced a version of this, essentially giving a real quadratic version of Oesterlé's refinements of Goldfeld's framework – however, the utility of their result is dependent on the existence of a suitable elliptic curve of analytic rank exceeding 3, which is not currently known.

"trivial" lower bound given by Dirichlet's class number formula, which is

$$\sqrt{D}L_\chi(1) = 2h_k \log \epsilon_0 \geq 2\log\Big(\frac{\sqrt{D-4}+\sqrt{D}}{2}\Big) \sim \log D. \qquad (2)$$

1.1. In the current work, we improve the above lower bound (1) on $L_\chi(1)$, albeit only by a constant, allowing us to solve Chowla's and Yokoi's conjectures (and other such class number problems) in an effective manner. We shall then be concerned with the computations of making a couple of cases completely explicit.

In §2 we further explain our method to improve the above lower bound. For now we give the barest sketch: rather than start with an elliptic curve (over $\mathbf{Q}$) of rank 3, we begin with an elliptic curve E of rank 5. This is known (by Kolyvagin's theorem) to have an analytic rank of at least 3. By computing the third central derivative of the L-series of E to high precision (and finding as expected that $L_E'''(1) \approx 0.0000\ldots$ to the computed precision), an application of Rouché's theorem shows that there are five zeros close to central point; by symmetry the two remaining zeros must be either on the central line or the real axis (indeed, likely both). This is then juxtaposed with an expansion (which we call a Deuring decomposition) for $\Lambda_E(s)\Lambda_{E\chi}(s)$ about $s=1$, which (when $E\chi$ has odd parity) we find is proportional to $\xi(\sin\xi-\xi)$ where $i\xi$ is roughly $(s-1)\log D$, with an error essentially bounded in size by $\sqrt{D}L_\chi(1)|s-1|^4$. Assuming this error is dominated, we again use Rouché's theorem; upon noting that the noncentral zeros of $\xi(\sin\xi-\xi)$ are off the axes, this contradicts the assumption of small $L_\chi(1)$.

Combined with some minor technical work to control root numbers in quadratic twist families, an improved lower bound for $L_\chi(1)$ then follows, which we now state.

Theorem 10.3.2. *For $D \geq 4\pi^2\exp(10^7)$ we have*

$$\sqrt{D}L_\chi(1) \geq \min\big(10^{1000}\log D, (\log D)^3/10^{13}\big)\cdot \prod_{p|D}\Big(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Big). \qquad (3)$$

The constant 10^{1000} here depends on computing the central triple derivative of $L_E(s)$ to 1025 digits for 6 explicit curves E, each of conductor no more than 10^{16}. Reaching such high precision is a nontrivial computation (indeed, an interesting side question on its own), and in [74] we give an analysis and implementation details (using the arbitrary-precision package Arb [37] of Fredrik Johansson) that sufficed to complete said computations in 2-3 weeks for each curve (on a single core). On the other hand, for self-containment of our main applications, we need only show the above bound with 10^{1000} replaced by 10^3, which readily follows upon computing each $L_E'''(1)$ to (say) 30 digits, taking only a few hours using off-the-shelf software. See §7 for a brief discussion.

1.1.1. Note that (3) and (2) are not necessarily comparable in strength, even for large D, as in particular the product over primes $p|D$ can be arbitrarily small. Relatedly, in cases where D has sufficiently many prime factors, we can use the theory of genera to obtain a beneficial lower bound on the (2-divisibility of the) class number.

1.1.2. The above stated Theorem has a codicil of $D \geq 4\pi^2\exp(10^7)$, which is mostly for convenience. For smaller D we can adapt a technique that already appeared in early class number calculations of Montgomery and Weinberger [54, §4], which we generalized in [71]. This uses a battery of auxiliary quadratic Dirichlet L-functions ψ, each with a zero of abnormally low height. A type of Deuring decomposition for $\Lambda_\psi(s)\Lambda_{\psi\chi}(s)$ shows it is proportional to $(\sin\xi)/\xi$ with $i\xi$ roughly $(s-1/2)\log\sqrt{D}$, with the error here bounded in size by $D^{1/4}L_\chi(1)$. This then implies that various periodic ranges of $\log D$ do not have small values of $L_\chi(1)$. By taking enough such auxiliary L-functions, we can ensure that such periodic ranges cover $10^3 \leq \log D \leq 10^8$, obtaining the following result.

Proposition 12.3.1. *We have $\sqrt{D}L_\chi(1) \geq 100\log D$ when $10^3 \leq \log D \leq 10^8$.*

Via the auxiliary modulus 12461947 we can similarly obtain a result for smaller D.

Lemma 15.1.1. *We have* $\sqrt{D}L_\chi(1) > 1.51\log D$ *when* $200 \le \log D \le 1000$.

For D even smaller we can use computational classifications to obtain similar lower bounds, which we state in a quotable manner. In the real quadratic case we use Theorem 1.2.6 below, while for the imaginary quadratic case we use our previous work [71] in conjunction with $101\pi > 1.5\log(e^{200})$, and obtain the following.

Lemma 15.5.2. *When* $1253 < D \le e^{200}$ *and* $\Delta > 0$ *we have* $\sqrt{D}L_\chi(1) \ge 1.5\log D$. *When* $907 < D \le e^{200}$ *and* $\Delta < 0$ *we have* $\sqrt{D}L_\chi(1) \ge 1.5\log D$.

1.2. Returning to our main application of this new lower bound (3) for $L_\chi(1)$, let us review some history of the conjectures of Chowla and Yokoi.

Chowla's conjecture is first stated in a paper of Chowla and Friedlander [13, §4], written in the guise of primes $p = m^2 + 1$ such that $\mathbf{Q}(\sqrt{p})$ has class number 1 (see our Footnote 32 below for the equivalence of this with other phrasings). The main genesis for this conjecture is undoubtedly the implied smallness of $L_\chi(1)$, in conjunction with either the Generalized Riemann Hypothesis or Siegel's ineffective theorem to propose its unlikelihood. Some additional analysis is given in terms of small splitting primes, but it is unclear to me whether this is merely an equivalent formulation of the smallness of $L_\chi(1)$, as opposed to truly independent evidence.

Somewhat later, Yokoi [79] made a similar conjecture for $p = m^2 + 4$. (As Lachaud notes [42, Theorem 5, Footnote 23], this case was considered by Frobenius [24, §5 (8)]).

These were solved by Biró [5] in 2003, adapting a method of Beck.

1.2.1. The literature often puts these conjectures in the context of a somewhat larger class of real quadratic fields, namely those of "Richaud-Degert" type,[4] which are essentially given by $\mathbf{Q}(\sqrt{(au)^2 + ka})$ where $k \in \{\pm1, \pm2, \pm4\}$, subject to various conditions to avoid trivialities and ensure the discriminant is positive. The particular cases where $|ka| \in \{1, 4\}$ are the "narrow" cases, where the fundamental unit is respectively $au + \sqrt{D}$ or this divided by 2, and its norm has the opposite sign as k. More generally, the fundamental unit is $\big((2au^2 + k) + 2u\sqrt{D}\big)/|k|$ and has norm 1 (see Degert's Satz 1 in [20], as quoted in Lemma 2.3 of [45]).

Many results for Richaud-Degert fields are phrased in terms of necessary and sufficient conditions for small class number, for instance being related to prime-producing polynomials or to special values of partial Dedekind ζ-functions (which are at the heart of Beck's method as employed by Biró). However, there are some unconditional results: for instance Biró's student Lapkova developed his method in her thesis [45], and together they explicitly solved [44, 7] the class number 1 problem for Richaud-Degert fields with $k = 4$. Another notable result is that of Byeon and Lee [12], who show that when $u^2 + 1$ squarefree with $u > 0$ odd, then $\mathbf{Q}(\sqrt{u^2 + 1})$ has class number 2 exactly for $u \in \{3, 5, 11, 19\}$. This generalizes Biró's method, and was further expounded by Biró and Granville [6].

1.2.2. Our method is more general in that it can (at least in theory) show a class number 1 result for any real quadratic field whose fundamental unit is of polynomial size up to large degree (roughly 10^{500}) in D. For instance, Euler [23, Ex. 1] already considered (as an illumination of an instance of solving Pell's equation) examples where we have $D = (3u+1)^2 + (4u+1)^2 = 25u^2 + 14u + 2 = (5u + 7/5)^2 + 1/25$, which has its fundamental unit as $(25u + 7) + 5\sqrt{D}$ when $u > 0$ is odd and D is squarefree (see [47, §2] for more about Euler's work). Similarly, at least for discriminants where primes $p|D$ do not intervene heavily (for instance, prime D), when the fundamental unit is of size $\sqrt{D}$ our method can (in theory) show a class number h result for h up to 10^{1000} or so.

[4] The term comes from Hasse (1965), denoting a combination of Richaud's work (1866) with that of Degert (1958), though as Lemmermeyer [47] notes, the history contains many more names (including Euler of course) that could be appended.

1.2.3. Let us record the results we obtain. We could be more general, but choose these as exemplary cases with a small fundamental unit.

Lemmata 13.2.2 & 13.2.3. *Suppose we are in one of the following cases, with D fundamental and $10^{26} \le D \le \exp(10^3)$:*

(1) $D = 4u^2 + 1$ *with* $u > 0$,
(2) $D = u^2 + 4$ *with u odd and* $u > 0$,
(3) $D = 4(u^2 + 1)$ *with u odd and* $u > 0$,
(4&5) $D = 25u^2 + 14u + 2$ *with u odd; and* $D = 4(25u^2 + 14u + 2)$ *with u even.*

Then $h_K \ge 6$ for $K = \mathbf{Q}(\sqrt{D})$.

Note that for $\log D \ge 10^3$ the above Theorem 10.3.2 and Proposition 12.3.1 already suffice to show that $h_K \ge 6$. The results here use more care in the mid-sized range, employing the auxiliary moduli 12461947 and 17923.

We then complete the classifications of cases with $h_K \le 5$ in these families by a routine computational sieve – in fact (due to various parts of this paper being in flux) we did so up to 10^{28} rather than merely 10^{26}, thus needing to consider about 10^{14} values of u, which took a couple of days. Explicitly, we have the following result.

Theorem 1.2.4. *Suppose that D is a fundamental discriminant and is in one of the above five families. Then the cases with $h_K \le 5$ are as in Tables 1 and 2.*

type	$h_K = 1$	$h_K = 2$	$h_K = 3$
(1)	$1, 2, 3, 5, 7, 13$	$4, 11, 17, 23, 29$	$8, 27, 37, 47$
(2)	$1, 3, 5, 7, 13, 17$	$9, 19, 23, 25, 31, 41, 43, 53$	$15, 27, 35, 37, 47, 67, 73, 97$
(3)	1	$3, 5, 11, 19$	
(4&5)	$-5, -1, 0, 1, 3$	$-3, -2, 4, 15$	$-21, -9, 19$

Table 1. Values of u that give $h_K \in \{1, 2, 3\}$ for the families

type	$h_K = 4$	$h_K = 5$
(1)	$6, 15, 25, 31, 43, 49, 53, 61, 71$	$10, 33, 55, 73, 103$
(2)	$21, 49, 55, 59, 71, 77, 79,$ $83, 101, 107, 113, 127, 157$	$33, 57, 85, 103, 115, 137, 167, 193$
(3)	$9, 13, 17, 23, 25, 31, 37$	
(4&5)	$-33, -17, -13, -8, -6, 2, 6, 10, 39$	$-41, 9, 11$

Table 2. Values of u that give $h_K \in \{4, 5\}$ for the families

The bound of $h_K \le 5$ is chosen to go further than any preceding result, but not be too large so as to lead to unwieldy complications.

1.2.5. Finally, as an example of a more general result, chosen so as to be exemplary and not too computationally arduous, we will show the following Theorem.

Theorem 1.2.6. *Suppose that $D > 0$ is a fundamental discriminant with $(A + B\sqrt{D})/2$ its fundamental unit, with $B \le D^{1/4}$ and $h_K = 1$. Then D is in the 22-element set*

$$\{5, 8, 12, 13, 17, 21, 24, 29, 37, 53, 77, 93, 101, 173, 197, 293, 413, 437, 677, 773, 1133, 1253\}.$$

Again the proof technique uses additional care with auxiliary moduli in a mid-sized range, and then a computational sieve for small discriminants.

2. Outline

Let us next turn to a fuller outline of our method, then give some related general comments, acknowledgments, and a list of contents and notation.

Our technique is to utilize a Deuring decomposition for modular form L-functions that is suited to discerning the behavior near the central point. We refer the reader to our recent work [73] for more of the context of Deuring decompositions.

We again use the notation that $K = \mathbf{Q}(\sqrt{\Delta})$ is a quadratic field of fundamental discriminant Δ, with χ the associated quadratic character of conductor $D = |\Delta|$.

2.1. We start with an elliptic curve $E/\mathbf{Q}$ of rank 5. One expects (by the conjecture of Birch and Swinnerton-Dyer [4]) that the L-series of E has central vanishing of order 5, but currently we can only show that the analytic rank is at least 3 (this lower bound is due to work of Kolyvagin [40] following upon that of Gross and Zagier [31]). Moreover, we want $E\chi$, the quadratic twist of E by Δ, to have odd parity (and thus nonzero analytic rank). In the context of small $L_\chi(1)$, we can ensure (see §10) such a root number condition by using a specific fixed selection of three elliptic curves E of rank 5 satisfying certain technical properties – either one of these E's has $E\chi$ with odd parity, or there are three small split primes in K (whence $L_\chi(1)$ cannot be small).

Furthermore, for each given E we can show by computational means that there are exactly two additional zeros of $L_E(s)$ with $|s-1|\le 10^{-510}$ (say) and by symmetry these zeros $1\pm i\kappa$ must be either on the central line or the real axis. To show this bound on zeros of $L_E(s)$ near $s=1$ we compute its third central derivative to high precision and use Rouché's theorem (see §7).

2.1.1. For notational reasons we write $L_f(s)$ instead of $L_E(s)$, with f the weight 2 modular newform associated to E.

We then use Lavrik's general method [46] for approximate functional equations to derive a "Deuring decomposition" for $\Lambda_E^K(s)=\Lambda_f^K(s)=\Lambda_f(s)\Lambda_{f\chi}(s)$ that accentuates the behavior around $s=1$. Taking $\tilde r$ as a lower bound (of the correct parity) for the order of central vanishing of $L_f(s)L_{f\chi}(s)$, we consider

$$\frac{\Lambda_f^K(z)}{(z-1)^{\tilde r}}=\left(\int_{(2)}-\int_{(0)}\right)\frac{\Lambda_f^K(s)}{(s-1)^{\tilde r}(s-z)}\frac{\partial s}{2\pi i}, \tag{4}$$

where the extra weighting by $1/(s-1)^{\tilde r}$ seems to be novel, and indeed our main contribution. Following Lavrik, we then use the functional equation to reflect the integral on the 0-line to the 2-line, replace $L_f(s)L_{f\chi}(s)$ by an approximation using the smallness of $L_\chi(1)$, move the integrals to the left to get the main terms from residues, and estimate the errors involved.

To state the result, let us write N_f for the level of f, then assume $\gcd(N_f,D)=1$ for notational simplicity, and similarly only consider K imaginary quadratic for now. The fact the product L-function has analytic rank at least 4 allows us to show the rescaled completed product L-function satisfies

$$\tilde\Lambda_f^K(s)=L_f^K(s)\Gamma(s)^2\Big(\frac{N_fD}{4\pi^2}\Big)^{s-1}=T_f(s)+T_f(2-s)-T_f''(1)(s-1)^2+O_f(\tilde h_K|s-1|^4),$$

where $\tilde h_K=h_K\sum_a 1/a$ includes a reciprocal sum over minima of reduced binary quadratic forms of discriminant Δ and

$$T_f(s)=\frac{L_{S^2f}(2s)}{\zeta(2s-1)}\Gamma(s)^2\Big(\frac{N_fD}{4\pi^2}\Big)^{s-1}E_f(s),$$

with $L_{S^2f}(s)$ being the symmetric-square L-function of f, and E_f is an adjustment for bad primes of f and small noninert primes of K.

In particular, the dominating effect in $T_f(s)$ near $s=1$ is from D^s, and so we have an approximation of $\tilde\Lambda_f^K(s)$ by $c\xi(\sin\xi-\xi)$ where $i\xi$ is essentially $(s-1)(\log D)$. The noncentral zeros of this approximation are off the axes, and by Rouché's theorem this can be transferred to $\tilde\Lambda_f^K(s)$ when the error is small, contradicting our previous observation concerning the known zeros $1\pm i\kappa$ of $L_f(s)$. A comparison then gives us that

$$\tilde h_K|\kappa|^4 \underset{f}{\gg} E_f(1)\cdot\min\Big((\log D)|\kappa|^2,(\log D)^3|\kappa|^4\Big),$$

from which we obtain a lower bound for h_K that is quadratic in $1/|\kappa|$.

2.1.2. A similar schema works for real quadratic fields, with a different selection of three curves of rank 5 to start. Here Goldfeld already exhibited some counting arguments that relate small values of $L_\chi(1)$ to the splitting of small primes in K, though we still need some effort for the adaptation to our setting. We again mention the recent work of Byeon and Kim [11] as giving a different perspective on this.[5]

2.1.3. It does not seem that our method gives an analogous result (in particular, a new proof of class number 1) when starting from elliptic curves of rank 4 (where we would know $\tilde{r} \geq 3$). The problem is that the local approximation would then be of the form $c\xi(\cos\xi - 1)$; this has double zeros at twice the expected (average) zero-spacing of $\tilde{\Lambda}_f^K(s)$, but they do indeed occur on the real line.

We can also note that the device we employ in §10.2 to ensure $E\chi$ has odd parity might not work so well with higher ranks. Indeed, for the main Theorem 10.3.2 we need to exhibit an elliptic curve of rank 5 in various quadratic twist families; a heuristic of Granville [75, §11] suspects 5 is generically the maximal rank in a quadratic twist family, and [58, §8.3] concurs in this guess. On the other hand, more recently the twist of 11a by -203145767 was found by Daniels and Goodwillie [18, Table 4] to have rank 6, and similarly they noted the twist of 550k by 4817182 has rank 6.

2.2. The Deuring decomposition around $s = 1$ given in §6 grew out of the work [73] on the Deuring-Heilbronn phenomenon (or Deuring's zero-spacing phenomenon) that was done in 2016 or earlier. The idea of using it to obtain the current result(s) was conceived at Bill Duke's 60th birthday conference at ETH Zürich in June 2019. I thank H. Iwaniec for his interest in this topic.

2.3. Let us outline the contents of what appears below.

In §3 we derive some results about the sparsity of noninert primes when $L_\chi(1)$ is small. This is easy in the imaginary quadratic case (essentially lattice point counting in ellipses), but requires a lengthy analysis (following Goldfeld [28], who notes its origins from Hecke) in the real quadratic case. Unfortunately it seems difficult to reduce the quantity of material here, as [28] has a couple of mis-statements and discrepancies with Goldfeld's precursor work [30] with Schinzel, so we thought it best to re-derive everything.

We then in §4 recall the theory of modular form L-functions, and merge the previous result concerning noninert primes into this setting, showing that the complementary error term to our choice of $E_f(z)$ with the Deuring approximant is suitably bounded. In §5 we then give sufficient bounds on $E_f(z)$ itself, to allow us to show the main Deuring decomposition in §6.

We then turn to a different direction in §7: given a modular form L-function coming from an elliptic curve of rank 5, it has at least a triple central zero by work of Gross and Zagier combined with that of Kolyvagin, and we show that by computing its third central derivative to high precision (and indeed getting it is numerically $0.0000\ldots$ to the computed precision, as predicted by the conjecture of Birch and Swinnerton-Dyer) it has exactly two other zeros close to the central point. We then specialize to 6 specific elliptic curves, and describe the calculations regarding $L_f'''(1)$ for each (described more in [74]).

We wish to juxtapose this computational result with the central behavior in our Deuring decomposition, and in §8 we prepare for this by deriving more results about $E_f(z)$ and its derivatives (again using the smallness of $L_\chi(1)$), and also for the symmetric-square L-functions of our 6 curves. In §9 we then obtain an explicit lower bound on $L_\chi(1)$ in terms of the precision to which the third central derivative is known to be zero. This requires $f\chi$ to have odd parity, and in §10 we describe how to ensure that either this parity condition holds, or there are already sufficiently many small split primes, in which

[5] I must admit to not seeing how the first line of their Proposition 18 follows (it is stated as an inequality, but with complex numbers on each side), but perhaps a suitable variation of the positivity argument we give in Lemma 3.7.4 below would remedy this.

case the lower bound on $L_\chi(1)$ follows readily in any event. This then completes the proof of Theorem 10.3.2.

We then switch to applying this result to some class number problems for real quadratic fields with small fundamental unit.

We first give (§11) a suitable variant of §3 (with a different Mellin transform weighting in the error term) for the case of a Deuring decomposition for Dirichlet L-functions. In §12 we use a battery of 60 auxiliary Dirichlet L-functions with zeros of low height to handle the range $10^3 \le \log D \le 10^8$, doing this in some generality. In §13 we specialize to our specific families, handling $10^{26} \le D \le \exp(1000)$, and then in §14 we describe the computational sieves for $D \le 10^{28}$ in these families, showing Theorem 1.2.4. In §15 we then similarly prove Theorem 1.2.6.

Finally, in the appendix we give some proofs that didn't fit well in the main text, including a derivation of some results with continued fractions and real quadratic fields.

2.4. **Notation.** We use the "∂"-symbol rather than "d" for measures in integrals, simply because the letter "d" can be used in other contexts. We write L-functions with subscripts and consider the objects involved always to be primitive. We use the $\ll$- and O-notation, and also with the latter employ the Θ-notation for a constant bounded by 1.

We let $K = \mathbf{Q}(\sqrt{\Delta})$ be our quadratic field of interest, with Δ a fundamental discriminant and $D = |\Delta|$. The associated character is χ, whose L-function is $L_\chi(s)$. We write h_K for the class number, and ϵ_0 for the fundamental unit when K is real. The Dirichlet series coefficients of $\zeta_K(s)/\zeta(2s)$ are denoted by $R_K^\star(n)$, and this is split at $\sqrt{D}/2$ according to $R_K^\star = R_K^{\le} + R_K^{>}$. The notation $\langle a, b, c\rangle$ denotes a binary quadratic form of discriminant Δ, and in sums this will range over canonical reduced forms, one per class. When the summand only depends (at most) on a, this notation will often be shortened to $\langle a\rangle$.

We will be particularly interested in L-functions of weight 2 newforms f of level N_f, writing $L_f(s)$. The completed version is $\tilde\Lambda_f(s) = \Gamma(s)L_f(s)(\sqrt{N_f}/2\pi)^{s-1}$ where we have chosen to scale the conductor factor to be 1 at the central point $s = 1$. Moreover the product L-function $L_f(s)L_{f\chi}(s)$ will be notated as an "induced" L-function as $L_f^K(s)$, and similarly with $\tilde\Lambda_f^K(s)$. The sign of the functional equation for $\tilde\Lambda_f^K(s)$ is ϵ_f^K, and the symmetric-square L-function of f is denoted by $L_{S^2f}(s)$. We frequently use the standard notation $s = \sigma + it$ for a complex variable, and write $t_\star = |t| + 5$. The complex line integral from $a - i\infty$ to $a + i\infty$ is denoted $\int_{(a)}$.

The Deuring approximant $T_f(z)$ is $\Gamma(z)^2(MD)^{z-1}E_f(z)L_{S^2f}(2z)/\zeta(2z-1)$, with here $4\pi^2MD = \sqrt{N_fN_{f\chi}}$, while $E_f(z)$ is defined in §4.3.1, along with the related coefficients $r_f^K(n)$. The product $\mathcal{P}_s(D)$ is defined in §5.2 and the convenient $\mathcal{R}(\chi)$ in §4.3.3.

We write ψ for a real primitive auxiliary Dirichlet character whose conductor is k, and $L_\psi^K(s) = L_\psi(s)L_{\psi\chi}(s)$. The approximant $E_\psi^{\mathbf{P}}(s)$ (for a given set of primes $\mathbf{P}$) is defined in §11.2, while $\zeta_u(s) = \zeta(s)P_u(s)$ with $P_u(s) = \prod_{p|u}(1 - 1/p^s)$.

The Mellin transform I_j of $\Gamma(s)^2/(s-1)^j$ is described in §3.3.5. We also have the Mellin transform $\tilde I$ of $\Gamma(s)/(s-1/2)$ in §11.1.

Finally, we write $\mathcal{F}(u, z) = \sum_j 2^j\binom{z}{j}\binom{u}{j}$ as appears with Lemma 5.1.1.

2.4.1. We shall frequently estimate integrals numerically; a prototypical example (similar to Lemma 13.3.1) is the vertical line integral $\int |\Gamma(s)\zeta(2s)|\,\partial s$ on the line $\sigma = -1/4$. Here the decay of the Γ-function in the vertical direction is sufficiently fast that we need not worry about a sharp bound on ζ, and rather than giving a ponderous rigourous method, we simply use a computer algebra system such as GP/PARI [57].

3. Background on quadratic fields and binary quadratic forms

We recall some assorted results in the theories of L-functions, quadratic fields, and binary quadratic forms, with Davenport [19] as a standard reference. The most classical material for binary quadratic forms is already due to Gauss [26] (or even Lagrange). Also, I found Lachaud's overview [42] to be useful for some aspects for the real quadratic case.

We then derive various summatory bounds for the coefficients of $\zeta(s)L_\chi(s)/\zeta(2s)$ in terms of $L_\chi(1)$. We make some attempt to achieve decent constants, particularly in the real quadratic case (which has been comparatively neglected in the literature).

As notation, we let $K = \mathbf{Q}(\sqrt{\Delta})$ be a quadratic field of fundamental discriminant Δ, with χ its quadratic character. We write $D = |\Delta|$ for the absolute value of the discriminant. So as to avoid units in the imaginary quadratic case, we assume that $D > 4$.

The character χ is odd when $\chi(-1) = -1$ and even when $\chi(-1) = +1$, the former corresponding to the imaginary quadratic case and the latter the real quadratic.

3.1. The Riemann ζ-function $\zeta(s) = \sum_m 1/m^s = \prod_p (1 - 1/p^s)^{-1}$ has its completed version as $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ which analytically continues to the complex plane away from poles at 0 and 1, and satisfies the functional equation $\Lambda(s) = \Lambda(1-s)$.

Similarly, for a primitive real nontrivial Dirichlet character ψ of conductor $k > 1$, the Dirichlet L-function $L_\psi(s) = \sum_m \psi(m)/m^s = \prod_p (1-\psi(p)/p^s)^{-1}$ in the same manner has an entire completion $\Lambda_\psi(s) = (k/\pi)^{s/2}\Gamma((s+a)/2)L_\psi(s)$ that satisfies $\Lambda_\psi(s) = \Lambda_\psi(1-s)$, where $a = 0$ when $\psi(-1) = +1$ and $a = 1$ when $\psi(-1) = -1$. See [19, §9] for this.

3.1.1. The primitivized Dedekind ζ-function for K is defined as

$$\frac{\zeta_K(s)}{\zeta(2s)} = \frac{\zeta(s)L_\chi(s)}{\zeta(2s)} = \prod_p \frac{1 + 1/p^s}{1 - \chi(p)/p^s} = \sum_{n=1}^{\infty} \frac{R_K^\star(n)}{n^s},$$

where in the imaginary quadratic case $R_K^\star(n)$ counts half the number of primitive (that is, coprime) representations of n by reduced binary quadratic forms of discriminant Δ. An analogous accounting for $R_K^\star(n)$ holds true in the real quadratic case when one also requires the representations to be primary (see [19, §6 (12)]).

3.2. We write $\langle a, b, c\rangle$ for an integral binary quadratic form $ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = \Delta$, and the operation of composition on the set of such forms yields a group ([26, §234ff]). The class group then identifies forms under G-equivalence where G is $\mathbf{SL}_2(\mathbf{Z})$ for K imaginary and $\mathbf{GL}_2(\mathbf{Z})$ for K real, and this class group is isomorphic to the ideal class group of K (see Cox [16, page 128ff]).[6] We write h_K for its size.

3.2.1. A positive definite binary quadratic form has $a > 0$ and $\Delta < 0$, and is reduced[7] when $-a < b \le a < c$ or $0 \le b \le a = c$. Each equivalence class of forms has exactly one reduced form, and $a \le \sqrt{D/3}$ is its minimum.

For $\Delta > 0$ we say $\langle a, b, c\rangle$ is reduced when $0 < \sqrt{D} - b < 2|a| < \sqrt{D} + b$. In this indefinite case, writing $\omega = (-b + \sqrt{D})/2|a|$ and $\bar\omega$ for its conjugate, we have $0 < \omega < 1$ and $\bar\omega < -1$. We can note that $b > 0$ and $|a| < \sqrt{D}$. There can be more than one reduced form in an equivalence class, and indeed by iteratively sending ω to the fractional part of $1/\omega$ (also known as taking the continued fraction expansion) we obtain a sequence of $\mathbf{SL}_2(\mathbf{Z})$-equivalent reduced forms, at least if we take some care with the sign of a when passing from ω to $\langle a, b, c\rangle$, alternating it at each iteration. As the continued fraction expansion of $1/\omega$ is purely periodic, there are finitely many such equivalent reduced forms. When the fundamental unit of K has norm -1 the period length will be odd, and $\langle a, b, c\rangle$ is $\mathbf{SL}_2(\mathbf{Z})$-equivalent to $\langle -a, b, -c\rangle$. Otherwise these are only equivalent under $\mathbf{GL}_2(\mathbf{Z})$, and in particular it is the case that $\langle -a, b, -c\rangle$ will be in the reduction orbit of $\langle a, -b, c\rangle$.

For each class we choose a canonical representative by considering forms with $a > 0$ and then minimizing a then b (if necessary), referring to this a as the "minimum" of the class. We then write sums over $\langle a, b, c\rangle$, or $\langle a\rangle$ as a shorthand, as being over such canonical reduced forms, one per equivalence class.

[6] Gauss worked with "proper" equivalence under $\mathbf{SL}_2(\mathbf{Z})$ (in contrast with his predecessors), which indeed expedites the group law in the imaginary case, but is more debatable in the real quadratic case.

[7] This definite case is essentially due to Lagrange, and while he had a similar criterion for the indefinite case, there we instead use the criterion due to Gauss [26, §183] (recalling he required (§153) the middle coefficient of a form to be even, so his formulæ must be conventionally interpreted).

3.2.2. In modern language (compare [26, §228, §231]), two forms are said to be in the same genus if they are locally equivalent at all primes (including ∞). The number of genera is 2^t, where in the imaginary quadratic case t is one less than the number of prime divisors of D, while in the real quadratic case it is also one less, except when the fundamental unit of K has norm $+1$ when it is two less.[8] The number of genera divides the class number, and indeed the genus class group is coarser than the class group.

3.2.3. In the imaginary quadratic case the class number formula of Dirichlet states that $L_\chi(1) = \pi h_K/\sqrt{D}$ (recall we are assuming $D > 4$ so only ± 1 are units).

In the real quadratic case the fundamental unit $\epsilon_0 = (u + v\sqrt{D})/2$ comes from a minimal solution of $u^2 - Dv^2 = \pm 4$, with $u, v > 0$. This fundamental unit thus has norm ± 1, and (since $v \geq 1$) we have the bound $\epsilon_0 \geq (\sqrt{D-4} + \sqrt{D})/2$. Here the class number formula is $L_\chi(1) = 2(h_K \log \epsilon_0)/\sqrt{D}$, and we note $\sqrt{D}L_\chi(1) \geq \pi h_K$ for $D > 21$. This is often given [19, §6 (16)] in an alternative form in terms of a totally positive fundamental unit and the class number of the quadratic forms under $\mathbf{SL}_2(\mathbf{Z})$ (corresponding to the narrow class number), but the given formulation with h_K and ϵ_0 is more useful for us.

3.3. Our main goal is to derive summation bounds for $R_K^\star(n)$, in the spirit of Goldfeld's work [28, Lemmata 5 & 6]. However, it is technically more convenient to include a weighting by a Mellin transform (some of this is inspired by Oesterlé's presentation [56]). This task will take us some time, with notable complications in the real quadratic case.

The final result combines Propositions 3.4.6 and 3.7.6, with I_j defined in the next paragraph as the Mellin transform of $\Gamma(s)^2/(s-1)^j$, and $R_K^>$ restricting $R_K^\star$ to $n > \sqrt{D}/2$.

Proposition 3.3.1. *For $X > 0$ and $j \geq 3$, when $D \geq 5$ we have*

$$\sum_{n=1}^{\infty} R_K^>(n)\sqrt{n} I_j(n/X) \leq 3.814 \cdot 2^j X^{3/2} L_\chi(1).$$

3.3.2. We define our main weighting functions, which for $j \geq 0$ are

$$I_j(w) = \int_{(2)} w^{-s} \frac{\Gamma(s)^2}{(s-1)^j} \frac{\partial s}{2\pi i} = \int_{(2)} \int_0^\infty \int_0^\infty (u_1 u_2/w)^s e^{-u_1} \frac{\partial u_1}{u_1} e^{-u_2} \frac{\partial u_2}{u_2} \frac{\partial s/2\pi i}{(s-1)^j},$$

where we expanded the Γ-factors as $\Gamma(s) = \int_0^\infty u_1^s e^{-u_1} \frac{\partial u_1}{u_1} = \int_0^\infty u_2^s e^{-u_2} \frac{\partial u_2}{u_2}$.

Our main interest is in $j \geq 2$, for which we switch the order of integration to get

$$I_j(w) = \int_0^\infty \int_0^\infty \int_{(2)} (u_1 u_2/w)^s \frac{\partial s/2\pi i}{(s-1)^j} e^{-u_1} \frac{\partial u_1}{u_1} e^{-u_2} \frac{\partial u_1}{u_2},$$

where the inner integral is nonnegative since (with $j \geq 2$ for absolute convergence)

$$\int_{(2)} y^{-s} \frac{\partial s/2\pi i}{(s-1)^j} = \begin{cases} (\log 1/y)^{j-1}/y(j-1)! \text{ for } y \leq 1, \\ 0 \text{ for } y \geq 1, \end{cases}$$

the proof following respectively by moving the contour to the left or the right. Thus the original integral $I_j(w)$ is nonnegative too for $j \geq 2$. Also, the derivatives are nonpositive (including at $y = 1$), so that these I_j are nonincreasing.

Although it is not necessary, we can similarly note that $\Gamma(s)/(s-1) = \Gamma(s-1)$ implies

$$I_1(w) = \int_0^\infty \int_{(2)} (u_1/w)^s e^{-u_1} \Gamma(s-1) \frac{\partial s}{2\pi i} \frac{\partial u_1}{u_1} = \int_0^\infty (u_1/w) e^{-w/u_1} e^{-u_1} \frac{\partial u_1}{u_1},$$

so that I_1 is nonnegative. As for $j = 0$, we have $I_0(w) = 2K_0(2\sqrt{w})$ in terms of K-Bessel functions (see (5) below); this again is nonnegative, being the self-convolution of e^{-w}

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[8] It is also this case of norm $+1$ where equivalence under $\mathbf{SL}_2(\mathbf{Z})$ differs from $\mathbf{GL}_2(\mathbf{Z})$; the former leads to isomorphism with the narrow class group of K, which in this case differs from the class group of K. This is again well-explained by Cox [16]. Gauss (§257) calculates of the number of genera under $\mathbf{SL}_2(\mathbf{Z})$.

(the Mellin transform for $\Gamma(s)$). Thus one can alternatively show positivity of I_j via

$$\int_w^\infty I_j(t)\,\partial t = \int_{(2)} w^{-s+1}\frac{\Gamma(s)^2}{(s-1)^{j+1}}\frac{\partial s}{2\pi i} = wI_{j+1}(w),$$

In any case, a similar argument shows the functions $\int_{(2)} w^{-s}\frac{\Gamma(s)}{(s-1)^j}\frac{\partial s}{2\pi i}$ are nonnegative.

3.3.3. Having noted the I_j are nonnegative, let us also give an upper bound for $I_j(w)$. For our later usage, it will be convenient to do so by integrating on the $\sigma = 3/2$ line.

Lemma 3.3.4. *We have* $0 \le I_j(w) \le 2^j/4w^{3/2}$.

Proof. The upper bound follows by moving the integral defining $I_j(w)$ to the $\sigma = 3/2$ line, and applying the standard integral $\int_{-\infty}^{\infty} |\Gamma(3/2+it)|^2 \frac{\partial t}{2\pi} = 1/4$ (see Lemma A.2.1). □

3.3.5. We also derive a bound for a Mellin transform that appears below.

Lemma 3.3.6. *For* $u > 0$ *and* $j \ge 2$ *we have*

$$0 \le \int_{(2)} u^s \frac{s-1/2}{s-3/2}\frac{\partial s/2\pi i}{(s-1)^j} \le 2^j u^{3/2}.$$

Proof. For $u \le 1$ we move the contour to the right, and the integral is thus seen to be 0. Otherwise we have $u > 1$, and for $k \ge 1$ we move the contour to the left with

$$W_k(u) = \int_{(2)} \frac{u^s}{s-3/2}\frac{\partial s/2\pi i}{(s-1)^k} = 2^k u^{3/2} - 2u \sum\sum_{l+m=k-1} 2^l \frac{(\log u)^m}{m!}$$
$$= 2^k u^{3/2} - 2u\sum_{m=0}^{k-1} 2^{k-1}\frac{(\log\sqrt{u})^m}{m!} = 2^k\Bigl(u^{3/2} - u\sum_{m=0}^{k-1}\frac{(\log\sqrt{u})^m}{m!}\Bigr),$$

where for the residue at $s = 1$ we used the expansions $u^s = u\sum_m (s-1)^m(\log u)^m/m!$ and $1/(s-3/2) = -2/[1-2(s-1)] = -2\sum_l 2^l(s-1)^l$; we see that $W_k(u)$ is positive for $u > 1$ by comparison to the Taylor series of $e^{\log\sqrt{u}}$, while also $W_k(u) \le 2^k u^{3/2}$. Thus the desired integral

$$W_{j-1}(u) + W_j(u)/2 = \int_{(2)} u^s \frac{(s-1)+1/2}{s-3/2}\frac{\partial s/2\pi i}{(s-1)^j}$$

satisfies the bounds given in the Lemma. □

3.4. Next we recall Goldfeld's decomposition [28] of $\zeta_K(s)$ in the imaginary quadratic case. We start with a lemma of Iseki [36], which is an elementary consequence of lattice point counting in an ellipse (we omit the proof), and then decompose $\zeta_K(s) = \zeta(s)L_\chi(s)$.

Lemma 3.4.1. [36, Lemma 6], [28, Lemma 3]. *Let* α, β, γ *be real numbers with* $\alpha > 0$ *and* $\delta = 4\alpha\gamma - \beta^2 > 0$. *Then for any* $T > 0$ *we have*

$$S_{\alpha,\beta,\gamma}(T) = \sum\sum_{\substack{\alpha m^2+\beta mn+\gamma n^2 \le T\\ n\ne 0}} 1 = \frac{2\pi T}{\sqrt{\delta}} + \Theta\Bigl(4\sqrt{\frac{\alpha T}{\delta}}\Bigr) + \Theta\Bigl(4\sqrt{\frac{T}{\alpha}}\Bigr)$$

where the Θ*-notation indicates a constant bounded by 1.*

Lemma 3.4.2. [28, Theorem 3]. *Suppose* $D > 4$ *and* $\Delta < 0$. *For* $\sigma > 1/2$ *we have*

$$\zeta_K(s) = \zeta(s)L_\chi(s) = \zeta(2s)\sum_{\langle a\rangle}\frac{1}{a^s} + \frac{\pi}{\sqrt{D}}\frac{s}{s-1}\sum_{\langle a\rangle}\Bigl(\frac{D}{4a}\Bigr)^{1-s} + Z_{\mathrm{i}}(s)$$

where the $\langle a\rangle$*-sums are over minima of reduced forms of discriminant* Δ *and*

$$|Z_{\mathrm{i}}(s)| \le \frac{|s|}{\sigma-1/2}\sum_{\langle a\rangle}\Bigl(1+\frac{\sqrt{D}}{a}\Bigr)\Bigl(\frac{D}{4a}\Bigr)^{-\sigma}.$$

Note that both sides of the equation have a pole at $s = 1$.

Proof. We have

$$\zeta(s)L_\chi(s) = \zeta(2s)\sum_{\langle a\rangle}\frac{1}{a^s} + \sum_{\langle a,b,c\rangle}\sum_{m=-\infty}^{\infty}\sum_{n\neq 0}\frac{1/2}{(am^2+bmn+cn^2)^s}$$

and upon writing $V_{a,b,c}(l)$ for half the number of solutions to $am^2+bmn+cn^2 = l$ with $n\neq 0$, by partial summation the latter inner double sum is

$$\sum_{l=1}^{\infty}\frac{V_{a,b,c}(l)}{l^s} = \frac{s}{2}\int_1^{\infty}\frac{S_{a,b,c}(u)}{u^{s+1}}\partial u$$

in terms of the counting function S of Lemma 3.4.1. We have $S_{a,b,c}(u) = 0$ for $u \le D/4a$, as follows by completing the square $am^2+bmn+cn^2 = a(m+nb/2a)^2+(D/4a)n^2 \ge D/4a$ for $n\neq 0$. Upon substituting Iseki's bound from Lemma 3.4.1, for each $\langle a,b,c\rangle$ we have

$$\begin{aligned}\frac{s}{2}\int_{D/4a}^{\infty}\frac{S_{a,b,c}(u)}{u^{s+1}}\partial u &= \frac{s}{2}\int_{D/4a}^{\infty}\left[\frac{2\pi u}{\sqrt{D}} + \Theta\left(4\sqrt{\frac{au}{D}}\right) + \Theta\left(4\sqrt{\frac{u}{a}}\right)\right]\frac{\partial u}{u^{s+1}}\\ &= \frac{\pi}{\sqrt{D}}\frac{s}{s-1}\left(\frac{D}{4a}\right)^{1-s} + 2|s|\Theta\left(\sqrt{\frac{a}{D}}+\sqrt{\frac{1}{a}}\right)\int_{D/4a}^{\infty}\frac{\partial u}{u^{\sigma+1/2}}.\end{aligned}$$

Thus we obtain

$$\zeta(s)L_\chi(s) = \zeta(2s)\sum_{\langle a\rangle}\frac{1}{a^s} + \frac{\pi}{\sqrt{D}}\frac{s}{s-1}\sum_{\langle a\rangle}\left(\frac{D}{4a}\right)^{1-s} + \frac{2|s|}{\sigma-1/2}\Theta\left(\sqrt{\frac{a}{D}}+\sqrt{\frac{1}{a}}\right)\left(\frac{D}{4a}\right)^{1/2-\sigma}$$

and upon distributing $\sqrt{D/4a}$ from the final parentheses the result follows. □

Corollary 3.4.3. *Suppose $D > 4$ and $\Delta < 0$. For $\sigma > 1/2$ we have*

$$\frac{\zeta_K(s)}{\zeta(2s)} = \sum_{\langle a\rangle}\frac{1}{a^s} + \frac{1}{\zeta(2s)}\frac{\pi}{\sqrt{D}}\frac{s}{s-1}\sum_{\langle a\rangle}\left(\frac{D}{4a}\right)^{1-s} + \frac{Z_{\mathrm{i}}(s)}{\zeta(2s)}$$

where

$$|Z_{\mathrm{i}}(s)| \le \frac{|s|}{\sigma-1/2}\sum_{\langle a,b,c\rangle}\left(1+\frac{\sqrt{D}}{a}\right)\left(\frac{D}{4a}\right)^{-\sigma}.$$

3.4.4. We split $R_K^\star(n) = R_K^{\le}(n) + R_K^{>}(n)$, the former supported on $n \le \sqrt{D}/2$ and the latter on $n > \sqrt{D}/2$. This splitting is a bit arbitrary, though one motivation is that the $n \le \sqrt{D}/2$ with $R_K^\star(n) \neq 0$ correspond to leading coefficients of reduced forms.[9]

We also introduce $R_K^{\mathrm{m}}(n)$, which is the number of times that n appears in the multiset of minima of reduced forms, and write $R_K^{\tilde{\mathrm{m}}}(n) = R_K^\star(n) - R_K^{\mathrm{m}}(n)$. We can note that $R_K^{>}(n)$ is 0 for $n \le \sqrt{D}/2$, so is trivially bounded above by $R_K^{\tilde{\mathrm{m}}}(n)$ for such n, while for $n > \sqrt{D}/2$ we see that $R_K^{>}(n)$ is equal to (and thus bounded by) $R_K^\star(n) = R_K^{\tilde{\mathrm{m}}}(n) + R_K^{\mathrm{m}}(n)$.

3.4.5. We show a summation bound for $R_K^{>}(n)$ when weighted by $\sqrt{n}I_j(n/X)$.

Proposition 3.4.6. *For $X > 0$ and $j \ge 2$, when $D > 4$ and $\Delta < 0$ we have*

$$\sum_{n=1}^{\infty} R_K^{>}(n)\sqrt{n}I_j(n/X) \le (1.194 + 1.192 + 0.160)\cdot 2^j X^{3/2} L_\chi(1) = 2.546\cdot 2^j X^{3/2} L_\chi(1).$$

Proof. Rather than work with $R_K^{>}(n)$, it is easier to use the decomposition above, considering $\sum_n \sqrt{n}R_K^{\tilde{\mathrm{m}}}(n)/n^s = \zeta_K(s-1/2)/\zeta(2s-1) - \sum_{\langle a\rangle}\sqrt{a}/a^s$.

[9] This is well-known (by completing the square) for positive definite forms, while for the indefinite case we note an improvement in the derivation of Goldfeld and Schinzel [30, §3] (who only obtained this for $n \le \sqrt{D}/4$) in Lemma A.1.3.

We thus start by replacing $R_K^{>}(n)$ by $R_K^{\tilde{\mathrm{m}}}(n)$, and bounding the error therein. By Lemma 3.3.4 we have $0 \le I_j(w) \le 2^j/4w^{3/2}$, so that

$$\begin{aligned}\sum_{n=1}^{\infty} R_K^{>}(n)\sqrt{n}I_j(n/X) &\le \sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\sqrt{n}I_j(n/X) + \sum_{\substack{\langle a,b,c\rangle \\ a\ge\sqrt{D}/2}} \sqrt{a}I_j(a/X)\\ &\le \sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\sqrt{n}I_j(n/X) + h_K\frac{2^jX^{3/2}}{4\sqrt{D}/2},\end{aligned}$$

and by Dirichlet's class number formula $h_K/\sqrt{D} = L_\chi(1)/\pi$, so the second term here gives the 0.160 contribution in the statement of the Proposition.

We then expand out $I_j(n/X)$ to get

$$\sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\sqrt{n}I_j(n/X) = \int_{(2)} X^s\frac{\Gamma(s)^2}{(s-1)^j}\left[\frac{\zeta_K(s-1/2)}{\zeta(2s-1)} - \sum_{\langle a,b,c\rangle}\frac{\sqrt{a}}{a^s}\right]\frac{\partial s}{2\pi i},$$

and proceed to replace the bracketed term using Corollary 3.4.3.

3.4.7. The main term is

$$T_j^{\mathrm{i}} = \frac{\pi}{\sqrt{D}}\int_{(2)}\frac{X^s}{\zeta(2s-1)}\frac{s-1/2}{s-3/2}\sum_{\langle a,b,c\rangle}\Bigl(\frac{D}{4a}\Bigr)^{3/2-s}\frac{\Gamma(s)^2}{(s-1)^j}\frac{\partial s}{2\pi i}.$$

We expand both Γ-factors as integrals via $\Gamma(s) = \int_0^\infty u_1^s e^{-u_1}\frac{\partial u_1}{u_1} = \int_0^\infty u_2^s e^{-u_2}\frac{\partial u_2}{u_2}$, and similarly $1/\zeta(2s-1) = \sum_m m\mu(m)/m^{2s}$ as a sum, to get

$$\begin{aligned}T_j^{\mathrm{i}} = \frac{\pi}{\sqrt{D}}\sum_{\langle a,b,c\rangle}\Bigl(\frac{D}{4a}\Bigr)^{3/2}\sum_{m=1}^{\infty} m\mu(m)\times&\\ \times\int_0^\infty\int_0^\infty\int_{(2)}\Bigl(\frac{Xu_1u_2}{m^2}\frac{4a}{D}\Bigr)^s\frac{s-1/2}{s-3/2}\frac{\partial s/2\pi i}{(s-1)^j}\,e^{-u_1}\frac{\partial u_1}{u_1}e^{-u_2}\frac{\partial u_2}{u_2}.&\end{aligned}$$

The inner s-integral is nonnegative by Lemma 3.3.6, and moreover the same Lemma gives an upper bound for it. We then use $\sum_m |\mu(m)|/m^2 = \prod_p(1+1/p^2) = \zeta(2)/\zeta(4)$ so that

$$\begin{aligned}|T_j^{\mathrm{i}}| &\le \frac{\pi}{\sqrt{D}}\sum_{\langle a,b,c\rangle}\Bigl(\frac{D}{4a}\Bigr)^{3/2}\sum_{m=1}^{\infty} m|\mu(m)|\int_0^\infty\int_0^\infty 2^j\Bigl(\frac{Xu_1u_2}{m^2}\frac{4a}{D}\Bigr)^{3/2}e^{-u_1}\frac{\partial u_1}{u_1}e^{-u_2}\frac{\partial u_2}{u_2}\\ &= 2^jX^{3/2}\cdot\frac{\pi}{\sqrt{D}}h_K\frac{\zeta(2)}{\zeta(4)}\Gamma(3/2)^2 = 2^jX^{3/2}\cdot L_\chi(1)\frac{\pi^2/6}{\pi^4/90}\frac{\pi}{4} = \frac{15}{4\pi}\cdot 2^jX^{3/2}L_\chi(1),\end{aligned}$$

and as $15/4\pi \approx 1.19366$ this gives the first contribution in the Proposition.

3.4.8. For the secondary term U_j^{i} with $Z_{\mathrm{i}}(s)$, we move the contour to $\sigma = 3/2$ and bound it there, getting

$$|U_j^{\mathrm{i}}| \le X^{3/2}\int_{(3/2)}\left|\frac{\Gamma(s)^2}{\zeta(2s-1)}\right|\cdot\frac{|s-1/2|}{\sigma-1}\frac{|\partial s|/2\pi}{|s-1|^j}\times\sum_{\langle a,b,c\rangle}\Bigl(1+\frac{\sqrt{D}}{a}\Bigr)\Bigl(\frac{D}{4a}\Bigr)^{1/2-3/2},$$

where the integral is bounded numerically as $\le 0.593\cdot 2^j$, and since we have $a \le \sqrt{D/3}$ and $L_\chi(1) = \pi h_K/\sqrt{D}$ the sum is $\le 4h_K(1/\sqrt{3}+1)/\sqrt{D} \le 2.009L_\chi(1)$. Upon noting that $0.593\cdot 2.009 \le 1.192$ the statement of the Proposition follows. □

3.5. The situation is more complicated for real quadratic fields, though as Goldfeld indicates, the ideas go back to Hecke [33].

However, a significant amount of notation is needed even to state Goldfeld's result, and as we additionally want to make a small modification to it, I've chosen to re-present his work here, with various corrections.[10] Moreover, some of this depends upon work of Goldfeld and Schinzel [30], which in turn borrows from standard texts on continued fractions, so having everything in one place should be beneficial. First we review some theory about periodic continued fractions, then derive a version of Goldfeld's decomposition of $\zeta_K(s)$, and finally obtain a summation bound for $R_K^{>}$.

3.5.1. We begin with basic facts about continued fraction expansions of quadratic surds. In particular, given a reduced form $\langle a,b,c\rangle$ we consider $\omega=(-b+\sqrt{D})/2|a|$, which has $0<\omega<1$ and $\bar\omega<-1$ so that $1/\omega$ is a reduced quadratic surd.[11]

We consider the continued fraction expansion of ω, noting Lang [43, Chapter IV] and Perron [59, §19ff] as references for this, with our notation largely following the latter. We write $\omega=[0,\overline{e_1,\dots,e_k}]$ with this a primitive period of length k. We have successive convergents $A_v/B_v=[0,e_1,\dots,e_v]$, with the conventional $(A_{-1},B_{-1},A_0,B_0)=(1,0,0,1)$, while $A_1=1$ and $B_1\ge 1$, with $A_v=e_vA_{v-1}+A_{v-2}$ and $B_v=e_vB_{v-1}+B_{v-2}$ in general. We have complete quotients $\omega_v=[\overline{e_v,\dots,e_{v+k-1}}]=(P_v+\sqrt{D})/Q_v$, where $\omega_v=\omega_{v+k}$ and the Q_v are even and positive. Here $P_v>0$ since $2P_v/Q_v=\omega_v+\bar\omega_v>0$, and $P_v<\sqrt{D}$ since $\bar\omega_v<0$, which in turn implies that $e_v=\lfloor\omega_v\rfloor\le 2\sqrt{D}/2=\sqrt{D}$, and moreover from $P_v<\sqrt{D}$ and $\omega_v>1$ we have $Q_v\le 2\sqrt{D}$.

The standard bound on approximants is $1/(B_v+B_{v+1})<|B_v\omega-A_v|<1/B_{v+1}$, which implies $|B_{v+1}\omega-A_{v+1}|<1/B_{v+2}\le 1/(B_v+B_{v+1})<|B_v\omega-A_v|$. In our case this further implies the quotients $|B_v\omega-A_v|/|B_v\bar\omega-A_v|$ are decreasing in v. We recall that the even approximants A_{2v}/B_{2v} increase to ω and the odd ones decrease to it. Also (see [30, (17)]), we can write the fundamental unit as $\epsilon_0=A_{k-1}-\bar\omega B_{k-1}$ (where again k is the primitive period), and more generally we have $\epsilon_0^l=A_{lk-1}-\bar\omega B_{lk-1}$ for $l\ge 1$. Note that k can depend on the class of $\langle a,b,c\rangle$ (see the $D=40$ example, end of [42, §2.3]).

3.5.2. We then recall some details[12] of the $\mathbf{SL}_2(\mathbf{Z})$-orbit of $\langle a,b,c\rangle$. For $v\ge 0$ we have

$$\langle a_v,b_v,c_v\rangle=\langle a,b,c\rangle\begin{pmatrix}A_{v-1}&A_v\\B_{v-1}&B_v\end{pmatrix}\begin{pmatrix}1&0\\0&(-1)^v\end{pmatrix}=\Big\langle(-1)^v\frac{Q_v}{2},P_{v+1},-(-1)^v\frac{Q_{v+1}}{2}\Big\rangle,$$

where $A_{v-1}B_v-A_vB_{v-1}=(-1)^v$ by the determinant for continued fraction convergents.

We see that $a_v=(-1)^vQ_v/2$ and also $a_v=aA_{v-1}^2+bA_{v-1}B_{v-1}+cB_{v-1}^2$, while

$$a(A_{v-1}-B_{v-1}\omega)(A_{v-1}-B_{v-1}\bar\omega)=aA_{v-1}^2+b\frac{a}{|a|}A_{v-1}B_{v-1}+cB_{v-1}^2,$$

so that when $a>0$ we have $|a_v|=|a|\cdot|B_{v-1}\omega-A_{v-1}|\cdot|B_{v-1}\bar\omega-A_{v-1}|$.

---

[10] One notable confusion in his work is with H_n' and H_n at various points, including with the stated φ-range of integration in his Theorem 4. Addtionally, the usage of ϵ_0 as the fundamental unit does not always match up with [30], where ϵ_0 is the minimal totally positive unit.

[11] Goldfeld [28, middle page 634] erroneously says ω is reduced, while as [30, page 578] notes, it is actually $1/\omega$ that is. Moreover, the convention of $\omega_0=\omega$ (extant in both papers) also seems dubious to me (note in fact that $\omega_1=1/\omega$), though I don't think it matters in the rendition we give here.

[12] There seems to be some discrepancy between page 636 of [28] and its citation to [30], already with a translation $v\to v+1$. Moreover, the formula near the bottom of page 579 of the latter seems wrong, and the t-shift to reduce the form appears to incorrectly use the bounds for a definite form.

To ensure notational matters, we give the example of $\langle a,b,c\rangle=\langle 4,9,-11\rangle$ with $D=257$. In this case we have $\omega=[0,\overline{1,3,7}]$, while $(A_0/B_0,A_1/B_1,A_2/B_2,A_3/B_3)=(0/1,1/1,7/8,22/25)$, with the complete quotients as $\omega_1=(9+\sqrt{D})/22$, $\omega_2=(13+\sqrt{D})/4$, and $\omega_3=(15+\sqrt{D})/8$, so that $(P_1,P_2,P_3)=(9,13,15)$ and $(Q_1,Q_2,Q_3)=(22,4,8)$. We write $F(x,y)=ax^2+bxy+cy^2$ and noting $\left(\begin{smallmatrix}A_0&A_1\\B_0&B_1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}x\\-y\end{smallmatrix}\right)=\left(\begin{smallmatrix}-y\\x-y\end{smallmatrix}\right)$ we have $F(-y,x-y)=-11x^2+13xy+2y^2$ as $\langle a_1,b_1,c_1\rangle$, and similarly we have $F(x+7y,x+8y)=2x^2+15xy-4y^2$ for $\langle a_2,b_2,c_2\rangle$ and finally $F(7x-22y,8x-25y)=-4x^2+9xy+11y^2$ for $\langle a_3,b_3,c_3\rangle$.

3.5.3. In the appendix (§A.1) we give some more basic results concerning real quadratic fields and continued fractions. In particular, we correct an error in [28, Lemma 4], and improve a statement appearing in [30, page 578].

3.6. We next derive Goldfeld's decomposition for $\zeta_K(s)$ in the real case.

3.6.1. First we recall the Mellin transform for $\Gamma(s)^2$, as derived from that of $\Gamma(s)$ by convolution. Indeed, with substitutions $e^u = \tilde{y} = y/\sqrt{t}$ we have (for $\sigma > 0$)

$$\begin{aligned}\Gamma(s)^2 &= \int_0^\infty t^s \int_0^\infty e^{-t/y}e^{-y}\frac{\partial y}{y}\frac{\partial t}{t} = \int_0^\infty t^s \int_0^\infty \exp\bigl(-\sqrt{t}/\tilde{y} - \sqrt{t}\tilde{y}\bigr)\frac{\partial \tilde{y}}{\tilde{y}}\frac{\partial t}{t}\\ &= \int_0^\infty t^s \int_{-\infty}^\infty \exp\bigl(-\sqrt{t}(e^u + e^{-u})\bigr)\,\partial u\frac{\partial t}{t}\\ &= \int_0^\infty t^s \int_{-\infty}^\infty \exp(-2\sqrt{t}\cosh u)\,\partial u\frac{\partial t}{t} = 2\int_0^\infty t^s K_0(2\sqrt{t})\frac{\partial t}{t} \qquad (5)\end{aligned}$$

where the last is in terms of the K-Bessel function $K_0(t) = \int_0^\infty \exp(-t\cosh u)\,\partial u$. From this, substituting $v = \sqrt{t}$ we have $\Gamma(s/2)^2 = 4\int_0^\infty v^s K_0(2v)\frac{\partial v}{v}$.

We then have a calculation Goldfeld notes, namely for real nonzero U, V we have

$$\begin{aligned}\Gamma(s)\int_{-\infty}^\infty \frac{\partial\varphi}{(U^2e^\varphi + V^2/e^\varphi)^s} &= \Gamma(s)\int_{-\infty}^\infty \frac{\partial\varphi}{|UV|^s((U/V)e^\varphi + (V/U)/e^\varphi)^s}\\ &= \frac{\Gamma(s)}{|UV|^s}\int_{-\infty}^\infty \frac{\partial\varphi}{(e^\varphi + 1/e^\varphi)^s} = \frac{1}{|UV|^s}\int_0^\infty t^s\int_{-\infty}^\infty \exp\bigl(-t(e^\varphi + 1/e^\varphi)\bigr)\partial\varphi\frac{\partial t}{t}\\ &= \frac{2}{|UV|^s}\int_0^\infty t^s\int_0^\infty \exp(-2t\cosh\varphi)\partial\varphi\frac{\partial t}{t} = \frac{2}{|UV|^s}\int_0^\infty t^s K_0(2t)\frac{\partial t}{t} = \frac{\Gamma(s/2)^2}{2|UV|^s}.\end{aligned}$$

We rewrite this (in the way we will most often use it) as

$$\frac{1}{|UV|^s} = 2\frac{\Gamma(s)}{\Gamma(s/2)^2}\int_{-\infty}^\infty \frac{\partial\varphi}{(U^2e^\varphi + V^2/e^\varphi)^s}.$$

Goldfeld applies this with $(U, V) = (\lambda, \bar{\lambda})$ where $\lambda \in \mathbf{Q}(\sqrt{D})$ and $\bar{\lambda}$ is its conjugate. We can also note another consequence: taking $U = V = 1$ and $\tilde{\varphi} = e^\varphi$ we have (for $\sigma > 0$)

$$1 = 2\frac{\Gamma(s)}{\Gamma(s/2)^2}\int_0^\infty \frac{1}{(\tilde{\varphi} + 1/\tilde{\varphi})^s}\frac{\partial\tilde{\varphi}}{\tilde{\varphi}}. \qquad (6)$$

.

3.6.2. Given a reduced form $\langle a, b, c\rangle$ we let $\mathfrak{a}$ be the ideal of norm $|a|$ generated by a and $(-b + \sqrt{D})/2$. From the calculation[13] in §3.6.1 the partial zeta-function of the class of $\mathfrak{a}$ can be written as

$$\sum_{0\neq\lambda\in\mathfrak{a}/\langle\pm\epsilon_0\rangle} \frac{|a|^s}{|\lambda\bar{\lambda}|^s} = 2|a|^s\frac{\Gamma(s)}{\Gamma(s/2)^2}\sum_{0\neq\lambda\in\mathfrak{a}/\langle\pm\epsilon_0\rangle}\int_{-\infty}^\infty \frac{\partial\varphi}{(\lambda^2e^\varphi + \bar{\lambda}^2/e^\varphi)^s}$$

We then sum over ideal classes (or equivalently canonical reduced forms), eliminate $\pm$ from the units for a factor of 2, replace $\tilde{\varphi} = e^\varphi$ in the integral, and split it up into multiplicative segments of size ϵ_0^2 to get

$$\begin{aligned}\zeta_K(s) &= \sum_{\langle a,b,c\rangle}|a|^s\frac{\Gamma(s)}{\Gamma(s/2)^2}\sum_{0\neq\lambda\in\mathfrak{a}/\langle\epsilon_0\rangle}\int_0^\infty \frac{1}{(\lambda^2\tilde{\varphi} + \bar{\lambda}^2/\tilde{\varphi})^s}\frac{\partial\tilde{\varphi}}{\tilde{\varphi}}\\ &= \sum_{\langle a,b,c\rangle}|a|^s\frac{\Gamma(s)}{\Gamma(s/2)^2}\sum_{0\neq\lambda\in\mathfrak{a}/\langle\epsilon_0\rangle}\sum_{u=-\infty}^\infty\int_{\eta\epsilon_0^{2u}}^{\eta\epsilon_0^{2u+2}} \frac{1}{(\lambda^2\tilde{\varphi} + \bar{\lambda}^2/\tilde{\varphi})^s}\frac{\partial\tilde{\varphi}}{\tilde{\varphi}}\end{aligned}$$

[13] Although I'm not sure he ever states it explicitly, here Goldfeld is using equivalence under $\mathbf{GL}_2(\mathbf{Z})$; for instance on page 633 he describes the correspondence to ideal classes.

where $\eta > 0$ is arbitrary. We then substitute $\tilde{\varphi} = \epsilon_0^{2u}\varphi$, and accumulate $\lambda\epsilon_0^u$ over u to thus remove the $\langle\epsilon_0\rangle$-quotient, and get (using $1/\epsilon_0^2 = \bar{\epsilon}_0^2$)

$$\begin{aligned}\zeta_K(s) &= \sum_{\langle a,b,c\rangle} |a|^s \frac{\Gamma(s)}{\Gamma(s/2)^2} \sum_{0\neq\lambda\in\mathfrak{a}/\langle\epsilon_0\rangle} \sum_{u=-\infty}^{\infty} \int_\eta^{\eta\epsilon_0^2} \frac{1}{(\lambda^2\epsilon_0^{2u}\varphi + \bar{\lambda}^2/\epsilon_0^{2u}\varphi)^s} \frac{\partial\varphi}{\varphi} \\ &= \sum_{\langle a,b,c\rangle} |a|^s \frac{\Gamma(s)}{\Gamma(s/2)^2} \sum_{0\neq\lambda\in\mathfrak{a}} \int_\eta^{\eta\epsilon_0^2} \frac{1}{(\lambda^2\varphi + \bar{\lambda}^2/\varphi)^s} \frac{\partial\varphi}{\varphi}.\end{aligned}$$

We then let $\omega = (-b + \sqrt{D})/2|a|$, with $0 < \omega < 1$ since $\langle a, b, c\rangle$ is reduced. The $\lambda \in \mathfrak{a}$ are parametrized by $m|a| + n(-b + \sqrt{D})/2 = |a|(m + n\omega)$ for $m, n \in \mathbf{Z}$, and with

$$\alpha = |a|(\varphi + 1/\varphi),\ \ \beta = 2|a|(\omega\varphi + \bar{\omega}/\varphi),\ \text{ and }\ \gamma = |a|(\omega^2\varphi + \bar{\omega}^2/\varphi)$$

we have $\alpha m^2 + \beta mn + \gamma n^2 = \lambda^2\varphi + \bar{\lambda}^2/\varphi$ by matching (m, n)-coefficients, so ([28, (12)])

$$\zeta_K(s) = \sum_{\langle a,b,c\rangle} \frac{\Gamma(s)}{\Gamma(s/2)^2} \int_\eta^{\eta\epsilon_0^2} \sum_{(m,n)\neq(0,0)}\!\!\sum \frac{1}{(\alpha m^2 + \beta mn + \gamma n^2)^s} \frac{\partial\varphi}{\varphi}.$$

In fact, it is convenient to duplicate this calculation with η replaced by $\eta\epsilon_0^2$ to get

$$\zeta_K(s) = \sum_{\langle a,b,c\rangle} \frac{\Gamma(s)/2}{\Gamma(s/2)^2} \int_\eta^{\eta\epsilon_0^4} \sum_{(m,n)\neq(0,0)}\!\!\sum \frac{1}{(\alpha m^2 + \beta mn + \gamma n^2)^s} \frac{\partial\varphi}{\varphi}. \tag{7}$$

Note that the discriminant of the (α, β, γ) quadratic form is

$$\begin{aligned}\beta^2 - 4\alpha\gamma &= 4a^2[\omega^2\varphi^2 + 2\omega\bar{\omega} + \bar{\omega}^2/\varphi^2] - 4a^2[\omega^2\varphi^2 + \omega^2 + \bar{\omega}^2 + \bar{\omega}^2/\varphi^2] \\ &= -4a^2(\omega - \bar{\omega})^2 = -4(\sqrt{D})^2 = -4D.\end{aligned}$$

3.6.3. Goldfeld then writes the Epstein ζ-function in terms of an Eisenstein series as

$$\sum_{(m,n)\neq(0,0)}\!\!\sum \frac{1}{(\alpha m^2 + \beta mn + \gamma n^2)^s} = \frac{f_s(z)}{D^{s/2}} = \frac{1}{D^{s/2}} \sum_{(m,n)\neq(0,0)}\!\!\sum \frac{y^s}{|m + nz|^{2s}}$$

where $z = x + iy = \frac{\bar{\omega}+\omega\varphi^2}{\varphi^2+1} + i\frac{\sqrt{D}}{|a|}\frac{\varphi}{\varphi^2+1}$. Here $f_s(z)$ is z-invariant under $\mathbf{SL}_2(\mathbf{Z})$, since it is $\zeta(2s)\mathrm{Im}(\tau z)^s$ summed over $\tau \in G_\infty\backslash G$ where $G = \mathbf{SL}_2(\mathbf{Z})$ and $G_\infty = \langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle$. As with α, β, γ, here x, y, z are functions of φ, and we can verify the above by noting

$$|m + nz|^{2s} = [(m + nx)^2 + (ny)^2]^s = [m^2 + 2mnx + n^2(x^2 + y^2)]^s,$$

and then equating $\alpha = \sqrt{D}/y$ gives $y = \frac{\sqrt{D}}{|a|}\frac{\varphi}{\varphi^2+1}$, while similarly $\beta = 2x(\sqrt{D}/y)$ gives that $x = (y/2\sqrt{D})\beta = \frac{\varphi(\omega\varphi+\bar{\omega}/\varphi)}{\varphi^2+1}$, and the (m, n)-discriminant of $|m + nz|^2$ is $-4y^2$.

3.6.4. We then split up the φ-region in the above integral in (7), doing so in such a way to allow transformations on z that result in an imaginary part of nearly 1/2 or more. Goldfeld gives a method involving $\eta = |a|/\sqrt{D}$ and M intervals where $M = \mathrm{lcm}[2, k]$ with k the length of the continued fraction period of ω, but I think it is easier to have η be larger, and always traverse two periods.

We use the above notation of §3.5.1 for the continued fraction expansion of ω, so that $\omega = [0, \overline{e_1, \ldots, e_k}]$ and A_v/B_v are convergents to ω.

For $i \geq 2$ we note that $B_i \geq 2$ so that $|a| \leq \sqrt{D}$ implies $(D/a^2)B_i^4 > 16$, and thus

$$H_i = \frac{1}{2}\left[\frac{\sqrt{D}}{|a|}B_i^2 + \left(\frac{D}{a^2}B_i^4 - 4\right)^{1/2}\right] \tag{8}$$

is well-defined, with $H_2 \geq 1$. The point with this is that $\varphi \in [H_i, H_{i+1}]$ is equivalent to $B_i \leq 1/\sqrt{y} \leq B_{i+1}$ since $y = (\sqrt{D}/|a|)\varphi/(\varphi^2 + 1)$. We will make on each φ-interval an $\mathbf{SL}_2(\mathbf{Z})$-transformation of z so that the resulting imaginary part is at least 4/9.

Our choice will be $\eta = H_{4k-1}$. We wish to verify that $\eta\epsilon_0^4 \le H_{6k-1}$ so that in particular $[\eta, \eta\epsilon_0^4] \subset [H_{4k-1}, H_{6k-1}]$; we will then take $H_i^\star = \max(H_i, \eta\epsilon_0^4)$ so (7) becomes

$$\zeta_K(s) = \sum_{\langle a,b,c\rangle} \frac{\Gamma(s)/2}{\Gamma(s/2)^2} \int_\eta^{\eta\epsilon_0^4} \frac{f_s(z)}{D^{s/2}} \frac{\partial\varphi}{\varphi} = \sum_{\langle a,b,c\rangle} \frac{\Gamma(s)/2}{\Gamma(s/2)^2} \sum_{i=4k}^{6k-1} \int_{H_{i-1}^\star}^{H_i^\star} \frac{f_s(z)}{D^{s/2}} \frac{\partial\varphi}{\varphi}. \tag{9}$$

Our verification that $H_{4k-1}\epsilon_0^4 \le H_{6k-1}$ is fairly straightforward (though rather unenlightening), as we will essentially use that $H_{lk-1} \approx (\sqrt{D}/|a|)B_{lk-1}^2 \approx (|a|/\sqrt{D})\epsilon_0^{2l}$, with the approximation having a negative secondary term for lk even that decreases in size as l increases, which is enough to ensure that $H_{6k-1}/H_{4k-1} \ge \epsilon_0^4$.

Lemma 3.6.5. *With notation as above with* (8)*, we have* $H_{4k-1}\epsilon_0^4 \le H_{6k-1}$ *for* $D \ge 5$.

We give the details of the proof in the appendix (Lemma A.4.1).

3.6.6. We next turn to describing unimodular transformations on z that will ensure the imaginary part is large. It is convenient to have a lower bound for φ; in particular, for $i \ge 3$ since $B_i \ge 3$ we have $H_i \ge (9 + \sqrt{77})/2 \ge 8.88$.

For $i \ge 3$ we take m_i and n_i with $-A_i m_i - B_i n_i = 1$ and let $z_{i+1}^\star = \frac{m_i z + n_i}{B_i z - A_i}$, noting here that $f_s(z) = f_s(z_{i+1}^\star)$ since the z-transform is in $\mathbf{SL}_2(\mathbf{Z})$. In particular, for the imaginary part of $z_{i+1}^\star$ the standard transformation rule gives us

$$y_{i+1}^\star = \frac{y}{|B_i z - A_i|^2} = \frac{y}{(B_i x - A_i)^2 + (B_i y)^2}.$$

In the range $B_i \le 1/\sqrt{y} \le B_{i+1}$ we have $(B_i y)^2 \le y$, and also note

$$(B_i x - A_i)^2 = \bigl[(B_i x - B_i\omega) + (B_i\omega - A_i)\bigr]^2 \le \bigl[|B_i x - B_i\omega| + |B_i\omega - A_i|\bigr]^2,$$

where $|B_i\omega - A_i| \le 1/B_{i+1} \le \sqrt{y}$ by the bound for convergents and y-range, while

$$\omega - x = \omega - \frac{\bar\omega + \omega\varphi^2}{\varphi^2+1} = \frac{\omega}{\varphi^2+1} - \frac{\bar\omega}{\varphi^2+1} = \frac{\sqrt{D}}{|a|}\frac{1}{\varphi^2+1} = \frac{y}{\varphi} \le \frac{y}{8.88}$$

so that we have $B_i|x - \omega| \le \sqrt{y}/8.88$ and $(B_i x - A_i)^2 \le 1.238y$. Thus for $i \ge 3$ and in the range $B_i \le 1/\sqrt{y} \le B_{i+1}$, or equivalently $\varphi \in [H_i, H_{i+1}]$, the denominator in the above expression for $y_{i+1}^\star$ is $(B_i x - A_i)^2 + (B_i y)^2 \le 2.25y$, so that $y_{i+1}^\star \ge 4/9$.

We then make a re-writing of this denominator of $y_{i+1}^\star$. In particular we have

$$\begin{aligned}
(B_i x - A_i)^2 + (B_i y)^2 &= B_i^2 \frac{\omega^2\varphi^4 + (2\omega\bar\omega + D/a^2)\varphi^2 + \bar\omega^2}{(\varphi^2+1)^2} - 2A_iB_i\frac{\omega\varphi^2 + \bar\omega}{\varphi^2+1} + A_i^2 \\
&= B_i^2 \frac{\omega^2\varphi^4 + (\omega^2 + \bar\omega^2)\varphi^2 + \bar\omega^2}{(\varphi^2+1)^2} - 2A_iB_i\frac{\omega\varphi^2 + \bar\omega}{\varphi^2+1} + A_i^2 \\
&= \frac{B_i^2(\omega^2\varphi^2 + \bar\omega^2) - 2A_iB_i(\omega\varphi^2 + \bar\omega) + A_i^2(\varphi^2+1)}{\varphi^2+1} \\
&= \frac{\varphi^2(B_i\omega - A_i)^2 + (B_i\bar\omega - A_i)^2}{\varphi^2+1}
\end{aligned}$$

where for the second line we used $(\omega - \bar\omega)^2 = (\sqrt{D}/a)^2 = D/a^2$. This then implies

$$y_{i+1}^\star = \frac{y}{(B_i x - A_i)^2 + (B_i y)^2} = \frac{\sqrt{D}}{|a|} \frac{\varphi}{\varphi^2|B_i\omega - A_i|^2 + |B_i\bar\omega - A_i|^2},$$

and when $a > 0$ (as is true for canonical forms) we write $\tilde\varphi_{i+1} = \frac{|B_i\omega - A_i|}{|B_i\bar\omega - A_i|}\varphi = \lambda_i\varphi$ to get

$$y_{i+1}^\star = \frac{\sqrt{D}/|a|}{|B_i\omega - A_i| \cdot |B_i\bar\omega - A_i|} \frac{\tilde\varphi_{i+1}}{\tilde\varphi_{i+1}^2 + 1} = \frac{\sqrt{D}}{|a_{i+1}|} \frac{\tilde\varphi_{i+1}}{\tilde\varphi_{i+1}^2 + 1}, \tag{10}$$

where we used $|a_{i+1}| = |a| \cdot |B_i\omega - A_i| \cdot |B_i\bar\omega - A_i|$ in terms of the orbit $\langle a_v, b_v, c_v\rangle$ of $\langle a, b, c\rangle$ from §3.5.2 (this uses $a > 0$).

In particular, in correspondence to [28, (17)], since $f_s(z) = f_s(z_i^\star)$ our (9) is

$$\zeta_K(s) = \sum_{\langle a,b,c\rangle} \frac{\Gamma(s)/2}{\Gamma(s/2)^2} \sum_{i=4k}^{6k-1} \int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star} \frac{f_s(z_i^\star)}{D^{s/2}} \frac{\partial\tilde\varphi_i}{\tilde\varphi_i}. \tag{11}$$

This then leads us to consider

$$\frac{f_s(z_i^\star)}{D^{s/2}} = \sum\sum_{(m,n)\neq(0,0)} \frac{1}{(\alpha_i m^2 + \beta_i mn + \gamma_i n^2)^s}$$

where

$$\alpha_i = |a_i|(\tilde\varphi_i + 1/\tilde\varphi_i),\ \ \beta_i = 2|a_i|(\omega_i\tilde\varphi_i + \bar\omega_i/\tilde\varphi_i),\ \text{ and }\ \gamma_i = |a_i|(\omega_i^2\tilde\varphi_i + \bar\omega_i^2/\tilde\varphi_i),$$

which we will to proceed to estimate via Lemma 3.4.1. We can note that for $i \geq 4$ we have $\alpha_i = \sqrt{D}/y_i^\star \leq 2.25\sqrt{D}$ in the range $\tilde\varphi_i \in [\lambda_{i-1}H_{i-1}^\star, \lambda_{i-1}H_i^\star]$.

Lemma 3.6.7. *With notation as above, for $\sigma > 1/2$ we have*

$$\frac{f_s(z_i^\star)}{D^{s/2}} = \frac{2\zeta(2s)}{\alpha_i^s} + \frac{\pi}{\sqrt{D}}\frac{s}{s-1}\Bigl(\frac{D}{\alpha_i}\Bigr)^{1-s} + \Theta\biggl(\frac{4|s|}{\sigma-1/2}\Bigl(\sqrt{\frac{\alpha_i}{4D}} + \frac{1}{\sqrt{\alpha_i}}\Bigr)\Bigl(\frac{\alpha_i}{D}\Bigr)^{\sigma-1/2}\biggr).$$

Note that both sides have a pole at $s = 1$, and indeed the method here gives an analytic continuation of $f_s(z_i^\star)$ to $\sigma > 1/2$ away from this pole.

Proof. We note that $(\alpha_i, \beta_i, \gamma_i)$ has discriminant $-4D$, and the result will follow in a similar manner to Lemma 3.4.2, using Iseki's bound (Lemma 3.4.1).

In particular, the initial accounting gives

$$\frac{f_s(z_i^\star)}{D^{s/2}} = \sum\sum_{(m,n)\neq(0,0)} \frac{1}{(\alpha_i m^2 + \beta_i mn + \gamma_i n^2)^s} = \frac{2\zeta(2s)}{\alpha_i^s} + s\int_{4D/4\alpha_i}^\infty \frac{S_{\alpha_i,\beta_i,\gamma_i}(u)}{u^{s+1}}\partial u,$$

and Lemma 3.4.1 then bounds $S_{\alpha_i,\beta_i,\gamma_i}(u)$ to yield

$$\frac{f_s(z_i^\star)}{D^{s/2}} = \frac{2\zeta(2s)}{\alpha_i^s} + \frac{2\pi}{\sqrt{4D}}\frac{s}{s-1}\Bigl(\frac{D}{\alpha_i}\Bigr)^{1-s} + \Theta\biggl(4|s|\int_{D/\alpha_i}^\infty \Bigl(\sqrt{\frac{\alpha_i}{4D}} + \frac{1}{\sqrt{\alpha_i}}\Bigr)\frac{\partial u}{u^{\sigma+1/2}}\biggr),$$

with the statement of the Lemma then following for $\sigma > 1/2$. □

We consolidate the above into a version of Goldfeld's Theorem 4 (page 636).

Lemma 3.6.8. [28, Theorem 4]. *With notation as below, for $\Delta > 0$ and $D \geq 5$ we have*

$$\zeta_K(s) = \frac{\Gamma(s)/2}{\Gamma(s/2)^2} \sum_{\langle a,b,c\rangle} \sum_{i=4k}^{6k-1} \biggl[\int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star} \Bigl(\frac{2\zeta(2s)}{\alpha_i^s} + \frac{\pi}{\sqrt{D}}\frac{s}{s-1}\Bigl(\frac{D}{\alpha_i}\Bigr)^{1-s}\Bigr)\frac{\partial\tilde\varphi_i}{\tilde\varphi_i}\biggr] + Z_{\mathrm{r}}(s)$$

where

$$|Z_{\mathrm{r}}(s)| \leq \biggl|\frac{\Gamma(s)/2}{\Gamma(s/2)^2}\biggr| \cdot \frac{4|s|}{\sigma-1/2} \sum_{\langle a,b,c\rangle} \sum_{i=4k}^{6k-1} \biggl[\int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star} \Bigl(\frac{1}{2}\Bigl(\frac{\alpha_i}{D}\Bigr)^\sigma + \frac{\alpha_i^{\sigma-1}}{D^{\sigma-1/2}}\Bigr)\frac{\partial\tilde\varphi_i}{\tilde\varphi_i}\biggr].$$

Here the $\langle a,b,c\rangle$-sum runs over the canonical reduced forms of discriminant Δ, while k is the primitive period of $\omega = (-b+\sqrt{D})/2|a|$, with $\lambda_i = |B_i\omega - A_i|/|B_i\bar\omega - A_i|$ in terms of the convergent A_i/B_i for the continued fraction expansion of ω. Moreover, we have that $H_i^\star = \max(H_i, H_{4k-1}\epsilon_0^4)$ with ϵ_0 the fundamental unit and H_i defined in (8), and $\alpha_i = |a_i|(\tilde\varphi_i + 1/\tilde\varphi_i)$ where $\langle a_i, b_i, c_i\rangle$ is the reduced form in the $\mathbf{SL}_2(\mathbf{Z})$-orbit of $\langle a,b,c\rangle$ that corresponds to the ith convergent A_i/B_i as in §3.5.2.

Notably, we have $|\alpha_i| \leq 2.25\sqrt{D}$ for $\tilde\varphi_i \in [\lambda_{i-1}H_{i-1}^\star, \lambda_{i-1}H_i^\star]$.

Proof. This is nothing more than replacing $f_s(z_i^\star)/D^{s/2}$ in (11) using Lemma 3.6.7. □

3.7. We are then left to derive the analogue of Proposition 3.4.6 for the real quadratic case, namely a summation bound for $R_K^{>}(n)$ when weighted by $\sqrt{n}I_j(n/X)$. Here we again have the splitting $R_K^{\star}(n) = R_K^{\leq}(n) + R_K^{>}(n)$, the former supported on $n \leq \sqrt{D}/2$ and the latter on $n > \sqrt{D}/2$.

3.7.1. First we introduce another Mellin transform. We could arrange this somewhat differently, but the given version has some simplifications. For $u > 0$ we define (this being the simplest relevant construction I could find that exhibited nonnegativity)

$$M(u) = \int_{(2)} u^{-s} \frac{\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)}{s-1} \frac{\partial s}{2\pi i} = \int_{(2)} u^{-s} \frac{\Gamma(s-1/2)\Gamma(s-1)}{\Gamma(s/2-1/4)^2} \frac{\partial s}{2\pi i}$$

so that by Mellin inversion we have

$$\frac{\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)}{s-1} = \frac{\Gamma(s-1/2)\Gamma(s-1)}{\Gamma(s/2-1/4)^2} = \int_0^\infty u^s M(u) \frac{\partial u}{u}.$$

Lemma 3.7.2. *We have that M is nonnegative, so that*

$$\int_0^\infty \sqrt{u}|M(u)|\,\partial u = \int_0^\infty \sqrt{u}M(u)\,\partial u = \frac{\Gamma(3/2-1/2)\Gamma(3/2-1)}{\Gamma(3/4-1/4)^2} = \frac{1}{\sqrt{\pi}} \leq 0.56419.$$

The claim that M is nonnegative (which is the main content) essentially follows since $M(u) \sim \sqrt{\pi}/u\Gamma(1/4)^2$ as $u \to 0$ while $M(u) \sim \exp(-u/2)/\sqrt{8\pi u}$ as $u \to \infty$, with more brutish methods sufficing in between. We provide more details in Lemma A.3.2.

3.7.3. The nonnegativity of M plays a critical rôle in the following bound. We recall from §3.3.2 the definition of $I_j(w) = \int_{(2)} w^{-s} \frac{\Gamma(s)^2}{(s-1)^j} \frac{\partial s}{2\pi i}$.

Lemma 3.7.4. *For any measurable set $\mathcal{I} \subset (0,\infty)$, any $j \geq 1$, and any positive a and X we have*

$$2\int_{\mathcal{I}}\int_{(2)} X^s \Big(\frac{1}{a(\varphi+1/\varphi)}\Big)^{s-1/2} \frac{\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)^2}{(s-1)^j} \frac{\partial s}{2\pi i} \frac{\partial\varphi}{\varphi} \leq \sqrt{a}I_j(a/X).$$

Proof. By expanding $\frac{\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)}{(s-1)} = \int_0^\infty u^s M(u)\,\partial u/u$, the left side, call it $J_a(X)$, is

$$\begin{aligned} J_a(X) &= 2\int_{\mathcal{I}}\int_{(2)} X^s \Big(\frac{1}{a(\varphi+1/\varphi)}\Big)^{s-1/2} \int_0^\infty u^s M(u) \frac{\partial u}{u} \frac{\Gamma(s)}{(s-1)^{j-1}} \frac{\partial s}{2\pi i} \frac{\partial\varphi}{\varphi} \\ &= 2\int_{\mathcal{I}} \sqrt{a(\varphi+1/\varphi)} \int_0^\infty M(u) \int_{(2)} \Big(\frac{Xu}{a(\varphi+1/\varphi)}\Big)^s \frac{\Gamma(s)}{(s-1)^{j-1}} \frac{\partial s}{2\pi i} \frac{\partial u}{u} \frac{\partial\varphi}{\varphi} \\ &\leq 2\int_0^\infty \sqrt{a(\varphi+1/\varphi)} \int_0^\infty M(u) \int_{(2)} \Big(\frac{Xu}{a(\varphi+1/\varphi)}\Big)^s \frac{\Gamma(s)}{(s-1)^{j-1}} \frac{\partial s}{2\pi i} \frac{\partial u}{u} \frac{\partial\varphi}{\varphi}. \end{aligned}$$

Here we used the nonnegativity of $M(u)$ from Lemma 3.7.2 and the nonnegativity of the s-integral (for $j \geq 1$), the latter following as with our comments in §3.3.2 regarding the nonnegativity of I_j (here we have $\Gamma(s)$ in place of $\Gamma(s)^2$). It is precisely these that allow the φ-domain to be enlarged. Undoing the Mellin transform manipulations, we then find

$$\begin{aligned} J_a(X) &\leq 2\int_0^\infty \int_{(2)} X^s \Big(\frac{1}{a(\varphi+1/\varphi)}\Big)^{s-1/2} \frac{\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)^2}{(s-1)^j} \frac{\partial s}{2\pi i} \frac{\partial\varphi}{\varphi} \\ &= \int_{(2)} \frac{X^s}{a^{s-1/2}} \frac{\Gamma(s)^2}{(s-1)^j} \frac{\partial s}{2\pi i} = \sqrt{a}I_j(a/X), \end{aligned}$$

where we used (6) to evaluate the φ-integral. This gives the stated inequality. ◻

3.7.5. We now show a summation bound for $R_K^{>}(n)$ when weighted by $\sqrt{n}I_j(n/X)$.

We re-introduce $R_K^{\mathrm{m}}(n)$, which is now the number of times that n appears in the multiset of leading coefficients of reduced forms. In particular, this is *not* the same as the $\langle a,b,c\rangle$ notation, where that requires that the form additionally be canonical. We let $\bar{\mathcal{M}}_\Delta^+$ be the positive submultiset of these leading coefficients, and note Lemma A.1.7 implies the period length of each class satisfies $k \le \log\epsilon_0/\log\phi$ where $\phi=(1+\sqrt{5})/2$, so that $\#\bar{\mathcal{M}}_\Delta^+ \le h_K\log\epsilon_0/\log\phi = \sqrt{D}L_\chi(1)/2\log\phi \le 1.040\sqrt{D}L_\chi(1)$.

As with the definite case, for $n\le\sqrt{D}/2$ we have $R_K^{\star}(n)=R_K^{\mathrm{m}}(n)$, as we show in Lemma A.1.3. Again we write $R_K^{\tilde{\mathrm{m}}}(n)=R_K^{\star}(n)-R_K^{\mathrm{m}}(n)$.

Proposition 3.7.6. *For $X>0$ and $j\ge 3$, when $D\ge 5$ and $\Delta>0$ we have*

$$\sum_{n=1}^{\infty} R_K^{>}(n)\sqrt{n}I_j(n/X) \le (1.194+2.100+0.520)\cdot 2^j X^{3/2}L_\chi(1) = 3.814\cdot 2^j X^{3/2}L_\chi(1).$$

One can presumably make a similar result for $j=2$.

Proof. As with the proof of Proposition 3.4.6, we first replace $R_K^{>}(n)$ by $R_K^{\tilde{\mathrm{m}}}(n)$ and bound the error therein. By Lemma A.1.3 we have $R_K^{\star}(n)=R_K^{\mathrm{m}}(n)$ for $n\le\sqrt{D}/2$, so

$$\begin{aligned}\sum_{n=1}^{\infty} R_K^{>}(n)\sqrt{n}I_j(n/X) &\le \sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\sqrt{n}I_j(n/X) + \sum_{\substack{a\in\bar{\mathcal{M}}_\Delta^+\\ a\ge\sqrt{D}/2}} \sqrt{a}I_j(a/X)\\ &\le \sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\sqrt{n}I_j(n/X) + \#\bar{\mathcal{M}}_\Delta^+\frac{2^jX^{3/2}}{4\sqrt{D}/2},\end{aligned}$$

bounding I_j by Lemma 3.3.4. Since $\#\bar{\mathcal{M}}_\Delta^+\le 1.040\sqrt{D}L_\chi(1)$ this gives the 0.520 term.

3.7.7. We then expand out $I_j(n/X)$ to get

$$\sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\sqrt{n}I_j(n/X) + \sum_{a\in\bar{\mathcal{M}}_\Delta^+}\sqrt{a}I_j(a/X) = \int_{(2)} X^s\frac{\Gamma(s)^2}{(s-1)^j}\frac{\zeta_K(s-1/2)}{\zeta(2s-1)}\frac{\partial s}{2\pi i},$$

and proceed to insert our above expression for $\zeta_K(s)$ from Lemma 3.6.8; we write the resulting decomposition as $V_j^{\mathrm{r}}+T_j^{\mathrm{r}}+U_j^{\mathrm{r}}$, so that

$$\sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\sqrt{n}I_j(n/X) = -\sum_{a\in\bar{\mathcal{M}}_\Delta^+}\sqrt{a}I_j(a/X) + V_j^{\mathrm{r}}+T_j^{\mathrm{r}}+U_j^{\mathrm{r}}. \tag{12}$$

In the first term the factor of $\zeta(2s-1)$ cancels, and with $\alpha_i=|a_i|(\tilde\varphi_i+1/\tilde\varphi_i)$ we have

$$V_j^{\mathrm{r}} = \int_{(2)} X^s\frac{\Gamma(s-1/2)/2}{\Gamma(s/2-1/4)^2}\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star}\frac{2}{\alpha_i^{s-1/2}}\frac{\partial\tilde\varphi_i}{\tilde\varphi_i}\frac{\Gamma(s)^2}{(s-1)^j}\frac{\partial s}{2\pi i}.$$

We then re-arrange this, writing $\mathcal{I}_i=[\lambda_{i-1}H_{i-1}^\star,\lambda_{i-1}H_i^\star]$, and by Lemma 3.7.4 get

$$\begin{aligned}V_j^{\mathrm{r}} &= \sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\mathcal{I}_i}\int_{(2)} X^s\frac{\Gamma(s-1/2)/2}{\Gamma(s/2-1/4)^2}\cdot 2\frac{\sqrt{|a_i|}}{|a_i|^s}\frac{\sqrt{\tilde\varphi_i+1/\tilde\varphi_i}}{(\tilde\varphi_i+1/\tilde\varphi_i)^s}\frac{\partial\tilde\varphi_i}{\tilde\varphi_i}\frac{\Gamma(s)^2}{(s-1)^j}\frac{\partial s}{2\pi i}\\ &\le \sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\frac{1}{2}\sqrt{|a_i|}I_j(|a_i|/X) = \sum_{a\in\bar{\mathcal{M}}_\Delta^+}\sqrt{a}I_j(a/X).\end{aligned}$$

In particular, with (12) we thus have

$$\sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\sqrt{n}I_j(n/X) \le T_j^{\mathrm{r}}+U_j^{\mathrm{r}}.$$

3.7.8. We then turn to T_j^{r}, and writing $\mathcal{I}_i = [\lambda_{i-1}H_{i-1}^\star, \lambda_{i-1}H_i^\star]$ this is

$$T_j^{\mathrm{r}} = \frac{\pi}{\sqrt{D}} \sum_{\langle a,b,c\rangle} \sum_{i=4k}^{6k-1} \int_{\mathcal{I}_i} \int_{(2)} \frac{X^s}{\zeta(2s-1)} \frac{s-1/2}{s-3/2} \frac{\Gamma(s-1/2)/2}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)^2}{(s-1)^j} \Big(\frac{D}{\alpha_i}\Big)^{3/2-s} \frac{\partial s}{2\pi i} \frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}.$$

We write $1/\zeta(2s-1) = \sum_m m\mu(m)/m^{2s}$ and split off one $\Gamma(s)$ in the Γ-quotient, then expand out the Mellin transforms with e^{-u} and the above $M(u)$ to get

$$T_j^{\mathrm{r}} = \frac{\pi/2}{\sqrt{D}} \sum_{\langle a,b,c\rangle} \sum_{i=4k}^{6k-1} \int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star} \Big(\frac{D}{\alpha_i}\Big)^{3/2} \int_0^\infty \int_0^\infty \sum_{m=1}^\infty m\mu(m) \times$$
$$\times \int_{(2)} \Big(\frac{Xu_1u_2\alpha_i}{Dm^2}\Big)^s \frac{s-1/2}{s-3/2} \frac{\partial s/2\pi i}{(s-1)^{j-1}} M(u_1) \frac{\partial u_1}{u_1} e^{-u_2} \frac{\partial u_2}{u_2} \frac{\partial\varphi}{\varphi}.$$

Since $j \ge 3$ the inner s-integral is convergent, and by Lemma 3.3.6 we get

$$|T_j^{\mathrm{r}}| \le \frac{\pi/2}{\sqrt{D}} \sum_{\langle a,b,c\rangle} \sum_{i=4k}^{6k-1} \int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star} \Big(\frac{D}{\alpha_i}\Big)^{3/2} \sum_{m=1}^\infty m|\mu(m)| \times$$
$$\times \int_0^\infty \int_0^\infty 2^{j-1} \Big(\frac{Xu_1u_2\alpha_i}{Dm^2}\Big)^{3/2} M(u_1) \frac{\partial u_1}{u_1} e^{-u_2} \frac{\partial u_2}{u_2} \frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i},$$

where the m-sum is $\sum_m |\mu(m)|/m^2 = \zeta(2)/\zeta(4)$ so that

$$|T_j^{\mathrm{r}}| \le 2^{j-1} \frac{\pi/2}{\sqrt{D}} \frac{\zeta(2)}{\zeta(4)} X^{3/2} \cdot \sum_{\langle a,b,c\rangle} \sum_{i=4k}^{6k-1} \int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star} \frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i} \cdot \Gamma(3/2) \int_0^\infty \sqrt{u}\, M(u)\, \partial u.$$

Lemma 3.7.2 then evaluates the u-integral as $1/\sqrt{\pi}$, while $\Gamma(3/2) = \sqrt{\pi}/2$ and

$$\sum_{\langle a,b,c\rangle} \sum_{i=4k}^{6k-1} \int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star} \frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i} = \sum_{\langle a,b,c\rangle} \sum_{i=4k}^{6k-1} \int_{H_{i-1}^\star}^{H_i^\star} \frac{\partial\varphi}{\varphi} = \sum_{\langle a,b,c\rangle} \int_\eta^{\eta\epsilon_0^4} \frac{\partial\varphi}{\varphi} = h_K \cdot 4\log\epsilon_0,$$

which is $2\sqrt{D}L_\chi(1)$ by Dirichlet's class number formula. We conclude that

$$|T_j^{\mathrm{r}}| \le 2^{j-1} \frac{\pi/2}{\sqrt{D}} \frac{\pi^2/6}{\pi^4/90} X^{3/2} \cdot 2\sqrt{D}L_\chi(1) \cdot 1/2 = (15/4\pi) \cdot 2^j X^{3/2} L_\chi(1).$$

Since $15/4\pi \le 1.194$ this gives the first term in the result.

3.7.9. For the secondary term U_j^{r} with $Z_{\mathrm{r}}(s)$ we move the contour to $\sigma = 3/2$ and bound the integral there, getting

$$|U_j^{\mathrm{r}}| \le \int_{(3/2)} \frac{X^\sigma}{|\zeta(2s-1)|} \left|\frac{\Gamma(s-1/2)/2}{\Gamma(s/2-1/4)^2}\right| \cdot \frac{4|s-1/2|}{\sigma-1} \frac{|\Gamma(s)|^2}{|s-1|^j} \frac{|\partial s|}{2\pi} \times$$
$$\times \sum_{i=4k}^{6k-1} \int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star} \Big(\frac{1}{2}\Big(\frac{\alpha_i}{D}\Big)^{3/2-1/2} + \frac{\alpha_i^{3/2-3/2}}{D^{3/2-1}}\Big) \frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}$$

and from $\alpha_i \le 2.25\sqrt{D}$ the second line is $\le 2\sqrt{D}L_\chi(1)\cdot(2.25/2+1)/\sqrt{D}$ (the $\tilde{\varphi}_i$-integrals yield $2\sqrt{D}L_\chi(1)$ as above). Numerical integration bounds the first line as $\le 0.494\cdot 2^j X^{3/2}$, and multiplying these gives the contribution with 2.100 in the Proposition. $\square$

3.8. We also derive a bound for the summation of $R_K^>(n)/n$, and as this will only appear in a secondary term we make no attempt to reduce the constant, simply using Proposition 3.3.1 in conjunction with positivity,

Lemma 3.8.1. *For $X \ge 1$ and $D \ge 100$ we have*

$$\sum_{n\le X} \frac{R_K^>(n)}{n} \le 141 L_\chi(1)\log X.$$

Note also that the same argument (with a different partial summation) yields our promised [73, Lemma 3.2.1] of an unweighted summation bound for $R_K^{>}$.

Proof. Since I_3 is decreasing, for $n \le X$ we have $I_3(n/X) \ge I_3(1) \ge 0.145$, so by Proposition 3.3.1 and positivity of I_3 we have

$$\sum_{n\le X} \sqrt{n}R_K^{>}(n) \le \sum_{n\le X} \sqrt{n}R_K^{>}(n)\frac{I_3(n/X)}{I_3(1)} \le \frac{2^3\cdot 3.814}{0.145}\cdot X^{3/2}L_\chi(1) \le 211X^{3/2}L_\chi(1).$$

The result follows by partial summation, using that $R_K^{>}(n) = 0$ for $n \le \sqrt{D}/2$. ☐

4. Background on L-functions of elliptic curves and modular forms

We now recall some basic information about L-functions of elliptic curves and modular forms. We give Cremona [17, §2] as a useful basic reference.

For now we examine a simplified setting that suffices for the 6 specific elliptic curves of rank 5 that we shall later use, leaving details of the general case to the appendix (§A.6). If nothing else, this reduces the clutter herein, and the amount of citation chasing needed for a fact-checking reader. After reviewing the basic theory, we will consider an approximation to the Landau product $L_E(s)L_{E\chi}(s)$ involving our exceptional character χ, and show bounds on the approximation error in terms of $L_\chi(1)$.

We recall our notation that χ is a quadratic character corresponding to $\mathbf{Q}(\sqrt{\Delta})$ where Δ is a fundamental discriminant, and $D = |\Delta|$.

4.1. The work of Wiles [77] and others associates to an elliptic curve $E/\mathbf{Q}$ of conductor N_E a weight 2 modular newform f of level N_f, though one can verify this more directly without too much effort for our curves.[14] In particular, we have $L_E(s) = L_f(s)$ in terms of L-functions so we can speak of these interchangeably; similarly we have $N_E = N_f$. (What we call a newform is sometimes termed a primitive normalized eigenform).

The elliptic curves that we shall use are implicitly listed in Table 3. On each line, we give a label of a modular newform g, an associated elliptic curve $A = [a_1, a_2, a_3, a_4, a_6]$ in the standard form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, and a twisting factor B. The desired elliptic curve E of rank 5 from such a line in the table is derived by twisting the given curve A by B (which is a fundamental discriminant), while the associated modular form f is derived by twisting g by B. Since $\gcd(N_g, B) = 1$ in each case, the resulting level of f is N_gB^2, as is the conductor of E.

g	$[a_1,a_2,a_3,a_4,a_6]$	B	$L_{S^2f}(2)$	$\tilde{\Lambda}_f^{(5)}(1)/5!$	$\mathcal{V}(f)$	$\mathcal{U}(f)$
11a	$[0,-1,1,0,0]$	-25351367	1.0576	1525.2	1.009	1.484
17a	$[1,-1,1,-1,0]$	-19502039	0.7850	2379.6	1.005	2.048
19a	$[0,1,1,-9,-15]$	-16763912	0.9279	2918.6	1.583	2.094
91b	$[0,1,1,-7,5]$	6350941	1.2096	2000.4	1.028	2.053
123a	$[0,1,1,-10,10]$	5467960	2.7044	5620.4	1.256	1.286
209a	$[0,1,1,-27,55]$	3217789	1.0517	2743.0	1.012	2.271

Table 3. Data for the six rank 5 curves we use

4.1.1. We will then in turn be interested in twisting f by χ (or by Δ, if one prefers to denote the twist by an integer rather than by a character). This involves some extra care at primes dividing $\gcd(D, B)$.

4.2. Each modular form g in Table 3 has a Fourier expansion about ∞ that is given by $g(z) = \sum_n c_g(n)e^{2\pi inz}$ with $c_g(1) = 1$. In turn we have $f(z) = \sum_n c_f(n)e^{2\pi inz}$ where $c_f(n) = c_g(n)\psi_B(n)$ with ψ_B the Kronecker character of discriminant B. With elliptic curves, we have $c_g(p) = p+1-\#A(\mathbf{F}_p)$ and $c_f(p) = p+1-\#E(\mathbf{F}_p)$ for primes p.

[14]For the minimal twists A that we use, the levels are small enough that the tables of Cremona [17] certainly suffice to show modularity, and this modularity is preserved when taking the quadratic twist from A to E (see [34, Satz 36] or [17, §2.11]).

4.2.1. Following Hecke [34, §8], the L-function of a modular newform has a degree 2 Euler product, so in particular we have

$$L_g(s) = \prod_p (1 - \alpha_p/p^s)^{-1}(1 - \beta_p/p^s)^{-1}$$

where $\alpha_p + \beta_p = c_g(p)$ is the pth coefficient of the Fourier expansion (at ∞) of g, and $\alpha_p\beta_p = p$ for all $p \nmid N_g$. We have $|\alpha_p| = |\beta_p| = \sqrt{p}$ for $p \nmid N_g$, which follows from Hasse's bound [32] in the guise of elliptic curves, and (in weight 2) is due to Eichler [22] on the modular forms side. This implies that $|c_g(p)| \le 2\sqrt{p}$, and thus $|c_g(p)| \le \lfloor 2\sqrt{p} \rfloor$ as the coefficients are integral. We also see that α_p and β_p are complex conjugates, so that $\alpha_p = \bar{\beta}_p$, and as a convention α_p can be taken in the upper half-plane. Meanwhile, when $p|N_g$ we can take $\beta_p = 0$ and have $\alpha_p \in \{\pm 1\}$.

The effect on the L-series when twisting g by a fundamental discriminant is well-understood (see Atkin and Lehner [1, §6]). Upon writing ψ_u for the Kronecker character associated to the fundamental discriminant u, for the modular form $g\psi_u$ we then have $c_{g\psi_u}(n) = c_g(n)\psi_u(n)$, while $N_{g\psi_u} = N_g u^2/\gcd(N_g, u)$ for the level. (These are not true in complete generality, for instance relying on g being a p-minimal twist for all $p|u$).

We then consider the modular forms $f = g\psi_B$ and $f\chi = g[\psi_B\chi]$, where we take $[\psi_B\chi]$ to be the primitive inducing character of $\psi_B\chi$ in the latter. We then list in Table 4 the valuation v_p of N_f and $N_{f\chi}$ at each prime, first for odd primes and then for $p = 2$ (where we note that $2 \nmid N_g$ for our 6 curves, again relegating the general case to §A.6). In the last line, whether $\mathrm{v}_2(N_{f\chi})$ is 0 or 4 depends on whether B and D have the same sign or not. Otherwise, the rightmost 4 columns are a function of the leftmost 3.

$\mathrm{v}_p(N_g)$	$\mathrm{v}_p(B)$	$\mathrm{v}_p(D)$	$\mathrm{v}_p(N_f)$	$\mathrm{v}_p(N_{f\chi})$	$\mathrm{v}_p(N_fD)$	$\mathrm{v}_p(N_{S^2f})$
0	0	0	0	0	0	0
1	0	0	1	1	1	2
0	1	0	2	2	2	0
0	0	1	0	2	1	0
1	0	1	1	2	2	2
0	1	1	2	0	3	0
$\mathrm{v}_2(N_g)$	$\mathrm{v}_2(B)$	$\mathrm{v}_2(D)$	$\mathrm{v}_2(N_f)$	$\mathrm{v}_2(N_{f\chi})$	$\mathrm{v}_2(N_fD)$	$\mathrm{v}_2(N_{S^2f})$
0	0	0	0	0	0	0
0	2	0	4	4	4	0
0	3	0	6	6	6	0
0	0	2	0	4	2	0
0	2	2	4	0	6	0
0	3	2	6	6	8	0
0	0	3	0	6	3	0
0	2	3	4	6	7	0
0	3	3	6	0,4	9	0

TABLE 4. Conductor valuations

The important conclusion here is that $D^2 \le N_f N_{f\chi} \le N_f^2 D^2$ (comparing the sum of the fourth and fifth columns to respectively twice the third column and twice the sixth column), and we will adopt the notation $MD = \sqrt{N_f N_{f\chi}}/4\pi^2$ so that $1 \le 4\pi^2 M \le N_f$.

4.2.2. By Hecke's theory modular L-functions satisfy functional equations,[15] which are most easily described in terms of the scaled completed L-functions as (see [17, §2.8])

$$\tilde{\Lambda}_f(s) = \left(\frac{\sqrt{N_f}}{2\pi}\right)^{s-1} \Gamma(s) L_f(s) \text{ with } \tilde{\Lambda}_f(s) = \epsilon_f \tilde{\Lambda}_f(2-s)$$

[15]The case of level 1 appears in [34, Satz 16], and on page 10 therein he describes how to generalize to higher level, which necessitates a theory of newforms that was only developed by later authors (see [1]).

and

$$\tilde{\Lambda}_{f\chi}(s) = \left(\frac{\sqrt{N_{f\chi}}}{2\pi}\right)^{s-1} \Gamma(s) L_{f\chi}(s) \text{ with } \tilde{\Lambda}_{f\chi}(s) = \epsilon_{f\chi} \tilde{\Lambda}_{f\chi}(2-s),$$

where the root numbers $\epsilon_f, \epsilon_{f\chi}$ are in $\{-1, +1\}$. In §10 we shall have more to say about root number variation in quadratic twist families.

We put $L_f^K(s) = L_f(s)L_{f\chi}(s)$ and $\tilde{\Lambda}_f^K(s) = \tilde{\Lambda}_f(s)\tilde{\Lambda}_{f\chi}(s)$ for product L-functions (using the "induction to K" notation), and similarly $\epsilon_f^K = \epsilon_f \epsilon_{f\chi}$ for root numbers.

4.2.3. Next we turn to symmetric square L-functions. Here the analytic theory is due to Shimura [64] and the automorphic theory to Gelbart and Jacquet [27], while Coates and Schmidt [15] make analogous calculations for elliptic curves, as catalogued in [69, §2].

Firstly, the (motivic) symmetric square L-function is invariant under quadratic twisting [15, Lemma 1.3], so the symmetric square L-function for f is the same as for g. Secondly, since N_g is squarefree in our 6 cases, there are no complications from primes $p^2 | N_g$, and the symmetric square is defined [15, Case 1 & Lemma 1.2] by the Euler product

$$L_{S^2 f}(s) = \prod_p (1 - \alpha_p^2/p^s)^{-1}(1 - \alpha_p\beta_p/p^s)^{-1}(1 - \beta_p^2/p^s)^{-1}. \tag{13}$$

This analytically continues to an entire function [15, Theorem 2.2], and

$$\Lambda_{S^2 f}(s) = (N_{S^2 f}/4\pi^3)^{s/2} L_{S^2 f}(s)\Gamma(s)\Gamma(s/2)$$

satisfies the functional equation $\Lambda_{S^2 f}(s) = \Lambda_{S^2 f}(3 - s)$, with $N_{S^2 f}$ as the symmetric square conductor. In our cases we simply have $N_{S^2 f} = N_g^2$ (see [15, Theorem 2.4]), as catalogued in the rightmost column of Table 4. Note that the edge of the critical strip is $s = 2$, with $L_{S^2 f}(2) > 0$.

The alternating square L-function for f is simply

$$L_{A^2 f}(s) = \prod_p (1 - p/p^s)^{-1} = \zeta(s - 1).$$

4.3. We exploit the fact that small $L_\chi(1)$ implies $\chi(p) = -1$ for almost all small primes p. In particular, the product L-function $L_f(s)L_{f\chi}(s)$ is well-approximated by something independent of χ, with an error depending on the size of $L_\chi(1)$.

Indeed, we will compare the Euler factors of $L_f^K(s)$ to those of $L_{S^2 f}(2s)/L_{A^2 f}(2s)$, and for this we find it convenient to define three types of primes for f. These types are: good, when $p \nmid N_g B$; multiplicative, when $p | N_g$; and potentially good, when $p | B$. (In general, we also need to consider additive primes with $p^2 | N_g$, and potentially multiplicative primes that divide $\gcd(B, N_g)$ and have $p \| N_g$). Table 5 then gives the reciprocal Euler factors for this comparison, where we abbreviated the types by initials.

$\chi(p)$	type	$L_f^K(s)$ reciprocal Euler factor	same for $L_{S^2 f}(2s)/L_{A^2 f}(2s)$
-1	G	$(1 - \alpha_p^2/p^{2s})(1 - \beta_p^2/p^{2s})$	$(1 - \alpha_p^2/p^{2s})(1 - \beta_p^2/p^{2s})$
0	G	$(1 - \alpha_p\psi_B(p)/p^s)(1 - \beta_p\psi_B(p)/p^s)$	$(1 - \alpha_p^2/p^s)(1 - \beta_p^2/p^{2s})$
$+1$	G	$(1 - \alpha_p\psi_B(p)/p^s)^2(1 - \beta_p\psi_B(p)/p^s)^2$	$(1 - \alpha_p^2/p^s)(1 - \beta_p^2/p^{2s})$
-1	M	$(1 - \alpha_p^2/p^{2s})$	$(1 - \alpha_p^2/p^{2s})(1 - p/p^{2s})^{-1}$
0	M	$(1 - \alpha_p\psi_B(p)/p^s)$	$(1 - \alpha_p^2/p^{2s})(1 - p/p^{2s})^{-1}$
$+1$	M	$(1 - \alpha_p\psi_B(p)/p^s)^2$	$(1 - \alpha_p^2/p^{2s})(1 - p/p^{2s})^{-1}$
-1	PG	1	$(1 - \alpha_p^2/p^{2s})(1 - \beta_p^2/p^{2s})$
0	PG	$(1 - \alpha_p[\psi_B\chi](p)/p^s)(1 - \beta_p[\psi_B\chi](p)/p^s)$	$(1 - \alpha_p^2/p^{2s})(1 - \beta_p^2/p^{2s})$
$+1$	PG	1	$(1 - \alpha_p^2/p^{2s})(1 - \beta_p^2/p^{2s})$

TABLE 5. Reciprocal Euler factors

We then have, with $V_p(s)$ as Table 6, that

$$\frac{L_f^K(s)}{L_{S^2f}(2s)/L_{A^2f}(2s)} = \prod_{p|N_f} V_p(s)\cdot\prod_p \frac{1+\alpha'_p/p^s}{1-\alpha'_p\chi(p)/p^s}\frac{1+\beta'_p/p^s}{1-\beta'_p\chi(p)/p^s} \tag{14}$$

where $\alpha'_p=\alpha_p\psi_B(p)$ and $\beta'_p=\beta_p\psi_B(p)$ when $p\nmid B$, while we have $\alpha'_p=\alpha_p[\psi_B\chi](p)$ and $\beta'_p=\beta_p[\psi_B\chi](p)$ when $p|B$ (noting that $\alpha'_p=\beta'_p=0$ when $p|B$ and $p\nmid D$). The point of this comparison is that the final Euler factor in (14) is trivial when $\chi(p)=-1$, while the factor $V_p(s)$ for bad primes is mild.

Indeed, with $Z_p(s)$ for the latter Euler factor in (14) and $\sum_n z_n/n^s=\prod_p Z_p(s)$, we note $|\alpha'_p|,|\beta'_p|\le\sqrt{p}$, and compare to $(\zeta_K(s)/\zeta(2s))^2=\bigl(\sum_n R_K^\star(n)/n^s\bigr)^2$ to get

$$|z_n|\le\sqrt{n}\sum\sum_{n_1n_2=n} R_K^\star(n_1)R_K^\star(n_2). \tag{15}$$

$\chi(p)$	type	$Z_p(s)$	$V_p(s)$
-1	G	1	1
0	G	$(1+\alpha_p\psi_B(p)/p^s)(1+\beta_p\psi_B(p)/p^s)$	1
$+1$	G	$\frac{(1+\alpha_p\psi_B(p)/p^s)}{(1-\alpha_p\psi_B(p)/p^s)}\frac{(1+\beta_p\psi_B(p)/p^s)}{(1-\beta_p\psi_B(p)/p^s)}$	1
-1	M	1	$(1-p/p^{2s})^{-1}$
0	M	$(1+\alpha_p\psi_B(p)/p^s)$	$(1-p/p^{2s})^{-1}$
$+1$	M	$\frac{(1+\alpha_p\psi_B(p)/p^s)}{(1-\alpha_p\psi_B(p)/p^s)}$	$(1-p/p^{2s})^{-1}$
±1	PG	1	$(1-\alpha_p^2/p^{2s})(1-\beta_p^2/p^{2s})$
0	PG	$(1+\alpha_p[\psi_B\chi](p)/p^s)(1+\beta_p[\psi_B\chi](p)/p^s)$	1

TABLE 6. Values of $Z_p(s)$ and $V_p(s)$

For each curve in Table 3 we explicitly list the relevant $V_p(s)$ in Table 7 in §7.2.3.

4.3.1. We are now ready to define our approximant $E_f(s)$, which involves a truncated version of the Euler product quotient in (14). When the class number is sufficiently small, it matters little whether we do so by an Euler product or a sum. As it simplifies some arguments, we opt for the product (rather than the sum, as in [73, §3.3.3]), namely

$$E_f(s)=E_f^{\mathrm{r}}(s)\cdot E_f^{\mathrm{m}}(s)=\prod_{p|N_f}V_p(s)\cdot\prod_{p\le\sqrt{D}/2}Z_p(s). \tag{16}$$

We then define $r_f^K(n)$ from $\sum_n r_f^K(n)/n^s=L_f^K(s)-E_f(s)L_{S^2f}(2s)/L_{A^2f}(2s)$, i.e.

$$\sum_{n=1}^\infty\frac{r_f^K(n)}{n^s}=\frac{L_{S^2f}(2s)}{L_{A^2f}(2s)}E_f^{\mathrm{r}}(s)\cdot\Bigl[\prod_p Z_p(s)-\prod_{p\le\sqrt{D}/2}Z_p(s)\Bigr]. \tag{17}$$

The point will be that sums involving $r_f^K(n)$ can be adequately bounded.

We notate $\sum_l b_l/l^{2s}=L_{S^2f}(2s)/L_{A^2f}(2s)$, noting that $|b_l|\le l\tau_2(l)$ as in Table 5.

Lemma 4.3.2. *With notation as above and $\prod_{p|N_f}V_p(s)=\sum_d v_d/d^{2s}$ we have*

$$|r_f^K(n)|\le\sum_{d|N_f^\infty}|v_d|\sum\sum_{l^2m=n/d^2}|b_l|\cdot C(m)\sqrt{m}$$

where

$$C(m)=\Bigl(2\sum\sum_{w_1w_2=m}R_K^{\le}(w_1)R_K^{>}(w_2)+\sum\sum_{w_1w_2=m}R_K^{>}(w_1)R_K^{>}(w_2)\Bigr). \tag{18}$$

Proof. This is largely accounting with (17) at this point. Indeed, the $|b_l|$ term comes from $L_{S^2f}(2s)/L_{A^2f}(2s)$ and the d-sum from $E_f^{\mathrm{r}}(s)$. The mth coefficient of

$$\Big[\prod_p Z_p(s) - \prod_{p\le\sqrt{D}/2} Z_p(s)\Big]$$

is zero unless m has a prime divisor exceeding $\sqrt{D}/2$, and in general is bounded by $|z_m|$, which in turn is bounded as in (15). To obtain the form of $C(m)$ in the Lemma, we recall the splitting $R_K^\star = R_K^\le + R_K^>$, and note that the putative terms with $R_K^\le(w_1)R_K^\le(w_2)$ can be omitted, since $m = w_1w_2$ would not have a prime divisor exceeding $\sqrt{D}/2$ in this case. This gives the statement in the Lemma. □

4.3.3. It is convenient to define

$$\mathcal{R}(\chi) = \sum_{n=1}^\infty \frac{R_K^\le(n)}{n} = \sum_{n\le\sqrt{D}/2} \frac{R_K^\star(n)}{n} \le \prod_{p\le\sqrt{D}/2} \frac{1+1/p}{1-\chi(p)/p}. \tag{19}$$

We also define $\mathcal{V}_\chi(f) = \sum_d |v_d|/d^3$ and from Table 6 have $\mathcal{V}_\chi(f) \le \mathcal{V}(f)$ where

$$\mathcal{V}(f) = \prod_{p|N_g}\Big(1-\frac{1}{p^2}\Big)^{-1}\prod_{p|B}\Big(1+\frac{|\alpha_p^2+\beta_p^2|}{p^3}+\frac{p^2}{p^6}\Big) \le \prod_{p|N_f}\frac{1+p/p^3}{1-p/p^3} \le \frac{\zeta(2)^2}{\zeta(4)} = 5/2.$$

Similarly, we have

$$\frac{L_{S^2f}(2s)}{\zeta(2s-1)} = \sum_{l=1}^\infty \frac{b_l}{l^{2s}} = \prod_{p|N_g}\frac{1-p/p^{2s}}{1-1/p^{2s}}\prod_{p\nmid N_g}\Big(1-\frac{\alpha_p^2+\beta_p^2}{p^{2s}}+\frac{p^2}{p^{4s}}\Big)^{-1},$$

and upon defining $\mathcal{U}(f) = \sum_l |b_l|/l^3$ and noting $|b_l| \le l\tau_2(l)$ we have

$$\mathcal{U}(f) = \sum_{l=1}^\infty \frac{|b_l|}{l^3} \le \sum_{l=1}^\infty \frac{\tau_2(l)}{l^2} = \zeta(2)^2.$$

These bounds on $\mathcal{U}(f)$ and $\mathcal{V}(f)$ are improvable for individual f. We thus retain them in the stated error term in Proposition 6.1.1, and list bounds in Table 3 for our 6 curves.

4.4. We now show our main bound for a weighted summation of $|r_f^K(n)|$, involving the weighting functions I_j from §3.3.2.

Proposition 4.4.1. *For $X \ge 1$ and $j \ge 3$ and $D \ge 100$ we have*

$$\begin{aligned}J_j(X) &= \sum_{n=1}^\infty |r_f^K(n)| I_j(n/X) = \sum_{n=1}^\infty |r_f^K(n)| \int_{(2)}\Big(\frac{X}{n}\Big)^s \frac{\Gamma(s)^2}{(s-1)^j}\frac{\partial s}{2\pi i}\\ &\le 7.628\cdot\mathcal{U}(f)\mathcal{V}(f)\cdot 2^jX^{3/2}\cdot L_\chi(1)\mathcal{R}(\chi) + 5000\cdot 2^jX^{3/2}L_\chi(1)^2\cdot\log(D^2X^6).\end{aligned}$$

The first term will greatly exceed the second in our application, as $L_\chi(1)$ is small (of size at most $(\log D)^3/\sqrt{D}$) and we will take X of size D, with $D \ge 4\pi^2\exp(10^7)$.

Proof. We use Lemma 4.3.2 to replace $|r_f^K(n)|$ in $J_j(X)$. With the first term appearing in (18) we write $n = l^2md^2 = l^2w_1w_2d^2$ as in that Lemma, and have a contribution $J_j^{\mathrm{a}}(X)$ to $J_j(X)$ bounded as

$$\big|J_j^{\mathrm{a}}(X)\big| \le 2\sum_{d|N_f^\infty}|v_d|\sum_{l=1}^\infty|b_l|\sum_{w_1\le\sqrt{D}/2}\sqrt{w_1}R_K^\le(w_1)\sum_{w_2\ge\sqrt{D}/2}\sqrt{w_2}R_K^>(w_2)I_j\Big(\frac{w_1w_2l^2d^2}{X}\Big).$$

We then apply Proposition 3.3.1 to bound the w_2-sum and get

$$\begin{aligned}\big|J_j^{\mathrm{a}}(X)\big| &\le 2\sum_{d|N_f^\infty}|v_d|\sum_{l=1}^\infty|b_l|\sum_{w_1\le\sqrt{D}/2}\sqrt{w_1}R_K^\le(w_1)\cdot 3.814L_\chi(1)\cdot 2^j\Big(\frac{X}{w_1l^2d^2}\Big)^{3/2}\\ &= 7.628\cdot\mathcal{U}(f)\mathcal{V}(f)\cdot L_\chi(1)\mathcal{R}(\chi)\cdot 2^jX^{3/2}.\end{aligned}$$

4.4.2. With the secondary term from (18), we first make a crude bound for large n, noting that $R_K^\star(w) \le \tau_2(w)$ so that the contribution to $r_f^K(n)$ in Lemma 4.3.2 is

$$\le \sqrt{n} \sum_{l^2} \sum_{w_1} \sum_{w_2} \sum_{d^2=n} \tau_2(l)\tau_2(d)R_K^\star(w_1)R_K^\star(w_2) \le \sqrt{n}\tau_8(n),$$

while we have $\tau_8(n) \le nG(n) \le \binom{13}{6}\binom{10}{3}\binom{8}{1}\binom{8}{1}/(2^6 \cdot 3^3 \cdot 5 \cdot 7) \le 218n$ where G is the multiplicative function with $G(p^k) = \max\bigl(1, \bigl(\!\binom{8}{k}\!\bigr)/p^k\bigr)$. By moving the integral defining I_j to the 3-line we find the $n \ge H = X^6 D^2$ contribute (in this secondary term) to $J_j(X)$ an amount bounded as

$$\le \sum_{n\ge H} 218n^{3/2}\Bigl(\frac{X}{n}\Bigr)^3 \int_{(3)} \frac{|\Gamma(s)|^2}{2^j}\frac{|\partial s|}{2\pi} \le \sum_{n\ge H} \frac{300X^3}{n^{3/2}} \le 300\zeta(3/2)\frac{X^3}{\sqrt{H}} \le \frac{784}{D} \le 80L_\chi(1)^2,$$

since $1/D \le L_\chi(1)^2/\pi^2$ for $D \ge 100$ by Dirichlet's class formula.

For $n < H$ we then imitate the first part, getting a contribution to $J_j(X)$ that is

$$\begin{aligned}
&\le \sum_{d|N_f^\infty} d\tau_2(d) \sum_{l=1}^{\infty} l\tau_2(l) \sum_{\sqrt{D}/2<w_1<H} \sqrt{w_1}R_K^>(w_1) \sum_{w_2\ge\sqrt{D}/2} \sqrt{w_2}R_K^>(w_2)I_j\Bigl(\frac{w_1w_2l^2d^2}{X}\Bigr)\\
&\le 3.814 \cdot 2^j X^{3/2} L_\chi(1) \sum_{d=1}^{\infty} \frac{\tau_2(d)}{d^2} \sum_{l=1}^{\infty} \frac{\tau_2(l)}{l^2} \sum_{\sqrt{D}/2<w_1<H} \frac{R_K^>(w_1)}{w_1}\\
&\le 3.814\zeta(2)^4 \cdot 2^j X^{3/2} L_\chi(1) \cdot 141 L_\chi(1)\log H \le 3938 \cdot 2^j X^{3/2} L_\chi(1)^2 \log H,
\end{aligned}$$

where for the w_1-sum we used Lemma 3.8.1. Adding these gives the result. □

5. Bounds on $E_f(z)$ when $L_\chi(1)$ is small

There are two aspects in our choice of $E_f(z)$. We want a bound like Lemma 4.3.2 on the residual $r_f^K(n)$, but we also want to have control over the size of $E_f(z)$, at least on the central line (and somewhat to the left of it).

Here we give enough results for our Deuring decomposition in §6, while we postpone additional results (regarding bounds for E_f and $L_{S^2 f}$ and their derivatives) until later.

5.1. We first bound the number of small split primes in terms of $L_\chi(1)$.

Lemma 5.1.1. *Suppose there are z primes p with $\chi(p) = +1$ and $p \le X$. Then*

$$1.04\sqrt{D}L_\chi(1) \ge \sum_{n\le\sqrt{D}/2} R_K^\star(n) \ge \sum_{j=0}^{z} 2^j \binom{z}{j}\binom{u}{j} \quad \textit{where} \quad u = \left\lfloor \frac{\log(\sqrt{D}/2)}{\log X} \right\rfloor.$$

In particular, isolating the $j = z$ term in the sum, we have $1.04\sqrt{D}L_\chi(1) \ge 2^z\binom{u}{z}$.

Proof. Denoting primes p with $\chi(p) = +1$ by p_i, the product $\prod_i p_i^{e_i}$ is $\le \sqrt{D}/2$ for every nonnegative integral vector $\vec{e}$ with $\sum_i e_i \le u$, and each such vector gives rise to $n = \prod_i p_i^{e_i}$ with $R_K^\star(n) = 2^{\omega(n)}$ where $\omega(n)$ is the number of distinct prime divisors of n.

We then account as follows. Given a vector $\vec{e}$, let j be the number of coordinates that are nonzero. We then want to distribute v amongst these nonzero coordinates, where $v = \sum e_i - j \le u$ (having already distributed j to ensure these coordinates are indeed nonzero). The number of ways of doing this (across all v) is the multi-choose coefficient $\bigl(\!\binom{j+1}{u-j}\!\bigr)$, namely we have $u - j$ to distribute, and can put each one into either any one of the j coordinates, or simply not distribute it at all. This multi-choose coefficient is equal to $\binom{u}{u-j} = \binom{u}{j}$, and by summing over all j (there being $\binom{z}{j}$ ways of choosing the j nonzero coordinates) we then get

$$\sum_{n\le\sqrt{D}/2} R_K^\star(n) \ge \sum_{j=0}^{z} 2^j \binom{z}{j}\binom{u}{j}$$

and the left side is $\le 1.04\sqrt{D}L_\chi(1)$ by Lemma A.1.9. □

Corollary 5.1.2. *The number of split primes up to $\sqrt{D}/2$ is $\leq 0.52\sqrt{D}L_\chi(1)$.*

Proof. We apply Lemma 5.1.1 with $X = \sqrt{D}/2$, so that $u = 1$, and isolate $j = 1$. $\square$

If we write $\mathcal{F}(u,z) = \sum_j 2^j \binom{z}{j}\binom{u}{j}$ we symmetrically have $\mathcal{F}(u,z) = \mathcal{F}(z,u)$, and indeed these are the Delannoy numbers [3] (or tribonacci triangle, with $\mathcal{F}(0,z) = 1$ and $\mathcal{F}(u,z) = F(u-1,z) + \mathcal{F}(u,z-1) + \mathcal{F}(u-1,z-1)$ for $u,z \geq 1$).

5.2. We then note a bound on the number of split primes when $L_\chi(1)$ is small.

Lemma 5.2.1. *Suppose $D \geq 4\pi^2\exp(10^7)$ and $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$. There are at most 2 split primes up to 10^4 and at most 5 split primes up to $e^{\sqrt{Y}}$ where $Y = \log(\sqrt{D}/2)$.*

Proof. In Lemma 5.1.1 with $X = 10^4$ we have $u \geq 0.054\log D$ for $D \geq 4\pi^2\exp(10^7)$, and with $z = 3$ this would give $1.04\sqrt{D}L_\chi(1) \geq 2^z\binom{u}{z} \geq 8u^3/3! \geq 0.0002(\log D)^3$, contradicting $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$. So there are at most 2 split primes up to 10^4.

Similarly, applying Lemma 5.1.1 with $X_1 = \exp(\sqrt{Y})$ where $Y = \log(\sqrt{D}/2) \geq 50$ gives us $u_1 \geq \sqrt{Y} - 1 \geq 6$, assuming there are l split primes with $p \leq X_1$ implies

$$2^l\frac{(\sqrt{0.5\log D - \log 2} - l)^l}{l!} = 2^l\frac{(\sqrt{Y} - l)^l}{l!} \leq 2^l\binom{u_1}{l} \leq 1.04\sqrt{D}L_\chi(1) \leq 1.04\cdot\frac{(\log D)^3}{10^6},$$

which is a contradiction for $l = 6$ when $D \geq 4e^{100}$ (which is $Y \geq 50$). $\square$

We can also bound the number of ramified primes when $L_\chi(1)$ is small.

Lemma 5.2.2. *Suppose $D \geq 4\pi^2\exp(10^7)$ and $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$. Then there are at most $5\log\log D$ primes dividing D.*

Proof. The assumption on $\sqrt{D}L_\chi(1)$ implies $h_K \leq (\log D)^3/10^6\pi$, so by the theory of genera (§3.2.2) the number of prime divisors of D can be bounded as $2 + (3/\log 2)\log\log D$, which is $\leq 5\log\log D$ when $\log\log D \geq 3$. $\square$

5.3. We then use these restrictions on the number of split and ramified primes to bound E_f, which we recall from §4.3.1 is

$$E_f(s) = E_f^{\mathrm{r}}(s)\cdot E_f^{\mathrm{m}}(s) = \prod_{p|N_f} V_p(s)\cdot\prod_{p\leq\sqrt{D}/2}\frac{1+\alpha_p'/p^s}{1-\alpha_p'\chi(p)/p^s}\frac{1+\beta_p'/p^s}{1-\beta_p'\chi(p)/p^s},$$

where $|\alpha_p'|, |\beta_p'| \leq \sqrt{p}$. First we give a bound for $E_f^{\mathrm{r}}(s)$.

Lemma 5.3.1. *For $\sigma > 1/2$ we have*

$$\left|E_f^{\mathrm{r}}(s)\right| \leq \prod_{p|N_f}\frac{1+p/p^{2\sigma}}{1-p/p^{2\sigma}}.$$

Proof. Table 6 gives the possible Euler factors $V_p(s)$ with $E_f^{\mathrm{r}}(s)$, and each possibility is dominated coefficient-wise by $(1+p/p^{2s})/(1-p/p^{2s})$. The result follows. $\square$

We proceed to bound the remaining $E_f^{\mathrm{m}}(s)$ part of $E_f(s)$. We define

$$\mathcal{P}_s(D) = \prod_{p|D}\Big(1+\frac{\sqrt{p}}{p^s}\Big)^2.$$

Lemma 5.3.2. *Suppose that $D \geq 4\pi^2\exp(10^7)$ and $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$. Then we have $|E_f^{\mathrm{m}}(s)| \leq 7500\mathcal{P}_\sigma(D)$ for $\sigma \geq 4/5$ and $|E_f^{\mathrm{m}}(1+it)| \leq 600\mathcal{P}_1(D)$.*

Proof. The primes that divide D contribute the $\mathcal{P}_\sigma(D)$ term.

For the split primes up to $\exp(\sqrt{Y})$ with $Y = \log(\sqrt{D}/2)$ we use Lemma 5.2.1, and see these contribute to $|E_f^{\mathrm{m}}(s)|$ a factor bounded as

$$\leq\Big(\frac{1+\sqrt{2}/2^\sigma}{1-\sqrt{2}/2^\sigma}\Big)^2\Big(\frac{1+\sqrt{3}/3^\sigma}{1-\sqrt{3}/3^\sigma}\Big)^2\Big(\frac{1+\sqrt{10^4}/(10^4)^\sigma}{1-\sqrt{10^4}/(10^4)^\sigma}\Big)^6,$$

and for $\sigma \geq 4/5$ this is ≤ 7456 while on $\sigma = 1$ it is ≤ 534.

Finally, the number of split primes p with $\exp(\sqrt{Y}) = X_1 < p \leq \sqrt{D}/2$ is bounded as $\leq 0.52\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$ by Corollary 5.1.2, giving a factor with $E_f^{\mathrm{m}}(s)$ that is

$$\leq \exp\Bigl(2 \cdot \frac{(\log D)^3}{10^6} \cdot 3\frac{\sqrt{X_1}}{X_1^\sigma}\Bigr),$$

which is bounded by $1 + 10^{-275}$ for $\sigma \geq 4/5$ and $D \geq 4\pi^2 \exp(10^7)$. Multiplying these together then gives the result. □

We similarly have a somewhat crude bound on $|\log \mathcal{P}_s(D)|$ when $L_\chi(1)$ is small.

Lemma 5.3.3. *Suppose that $D \geq 4\pi^2 \exp(10^7)$ and $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$. Then we have $|\log \mathcal{P}_s(D)| \leq 11(\log\log D)^{3/2-\sigma}$ for $1/2 \leq \sigma \leq 1$.*

Proof. Lemma 5.2.2 implies the number of primes $p|D$ is $\leq 5\log\log D$. We then have

$$\bigl|\log \mathcal{P}_s(D)\bigr| \leq 2\sum_{p|D} \log\Bigl(1 + \frac{\sqrt{p}}{p^\sigma}\Bigr) \leq \sum_{n \leq 5\log\log D} \frac{2\sqrt{n}}{n^\sigma} \leq 2\frac{(5\log\log D)^{3/2-\sigma}}{3/2-\sigma},$$

which is $\leq 10.001(\log\log D)^{3/2-\sigma}$ from $1/2 \leq \sigma \leq 1$ and $D \geq 4\pi^2 \exp(10^7)$. □

5.4. In our derivation of the Deuring decomposition in the next section we will require a bound for L_{S^2f} that follows routinely from convexity. We write $t_\star = |t| + 5$.

Lemma 5.4.1. *In $1 \leq \sigma \leq 2$ we have*

$$\bigl|L_{S^2f}(s)\bigr| \leq 1.65 \cdot (1 + \log N_{S^2f}t_\star^3)^3 \cdot \Bigl(\frac{N_{S^2f}t_\star^3}{8\pi^3}\Bigr)^{1-\sigma/2}.$$

We prove this in the appendix (Lemma A.5.1), demonstrating a standard technique to optimize the exponent on $N_{S^2f}t_\star^3$, though below we do not require too sharp of a result.

6. A Deuring decomposition around $z = 1$ for $\tilde{\Lambda}_f^K(z)$

We next obtain a decomposition of $\tilde{\Lambda}_f^K(z) = L_f(z)L_{f\chi}(z)\Gamma(z)^2(MD)^{z-1}$ suited to a neighborhood of its central point $z = 1$. We state this for a wider class of f than our exposition in §4 properly merits; for instance, for f that have additive or potentially multiplicative primes an appropriate modification in the definition of $V_p(s)$ in §4.3 is needed, and similarly with Table 4, etc. The further details are described in the appendix (§A.6), where we note the bounds of Lemma 5.3.1 on $V_p(s)$ and $E_f(s)$ still apply.[16]

6.1. We write ϵ_f^K for the sign of the functional equation with $\tilde{\Lambda}_f^K(z) = \epsilon_f^K\tilde{\Lambda}_f^K(2-z)$, and with $MD = \sqrt{N_f N_{f\chi}}/4\pi^2$ define

$$T_f(z) = \frac{L_{S^2f}(2z)}{\zeta(2z-1)}E_f(z)\Gamma(z)^2(MD)^{z-1}$$

(noting the removable singularity $T_f(1) = 0$), with $E_f(s)$ as in (16) of §4.3.1 as

$$E_f(s) = E_f^{\mathrm{r}}(s) \cdot E_f^{\mathrm{m}}(s) = \prod_{p|N_f} V_p(s) \cdot \prod_{p \leq \sqrt{D}/2} \frac{1 + \alpha_p'/p^s}{1 - \alpha_p'\chi(p)/p^s}\frac{1 + \beta_p'/p^s}{1 - \beta_p'\chi(p)/p^s}.$$

As noted at the end of §4.2.1, we have $1 \leq 4\pi^2 M \leq N_f$. We also introduce

$$S_f^u(z) = \sum_{b=0}^{u-1} \bigl[1 + \epsilon_f^K(-1)^b\bigr] \cdot \frac{T_f^{(b)}(1)}{b!}(z-1)^b, \tag{20}$$

which corresponds to the lower-order Taylor series terms of $T_f(z) + \epsilon_f^K T_f(2-z)$.

[16] We could just restrict the statement of Proposition 6.1.1 to the 6 curves of Table 3, but we prefer to allow a general lower bound $\tilde{r}$ on the analytic rank of Λ_f^K (rather than just $\tilde{r} = 4$). It is unclear how useful this flexibility really is, but it seems worthwhile to give it explicitly. Note that we can assume $\tilde{r}$ has the same parity as the analytic rank, so that $(-1)^{\tilde{r}} = \epsilon_f^K$.

Proposition 6.1.1. *Suppose that* $D \geq 4\pi^2 \exp(10^7)$ *with* $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$ *and* $N_f^9 \leq D$. *Write* $\tilde{r}$ *for a lower bound for the order of vanishing of* $L_f^K(z)$ *at* $z = 1$ *and assume* $2 \leq \tilde{r} \leq (\log D)/100$ *with* $(-1)^{\tilde{r}} = \epsilon_f^K$. *Then for* $|z-1| \leq 1/99$ *we have*

$$\tilde{\Lambda}_f^K(z) = L_f^K(z)\Gamma(z)^2(MD)^{z-1} = T_f(z) + \epsilon_f^K T_f(2-z) - S_f^{\tilde{r}}(z) + U_f(z) \qquad (21)$$

where

$$\big|U_f(z)\big| \leq 30.53\sqrt{M}\cdot\mathcal{U}(f)\mathcal{V}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\cdot 2^{\tilde{r}}|z-1|^{\tilde{r}}.$$

Here $\mathcal{U}(f)$ *and* $\mathcal{V}(f)$ *are defined in §4.3.3 and are each* ≥ 1 *and* $\leq \zeta(2)^2$, *and* $\mathcal{R}(\chi)$ *is also defined in §4.3.3 (reciprocal sum over small representations, again* ≥ 1*).*

Note that the result is immediate at $z = 1$, for both sides of (21) are zero, as $T_f(1) = 0$.

Proof. By applying Cauchy's residue theorem, and then substituting $s \to 2-s$ and using the functional equation for $\tilde{\Lambda}_f^K$ in the second integral, we have

$$\begin{aligned}\frac{\tilde{\Lambda}_f^K(z)}{(z-1)^{\tilde{r}}} &= \left(\int_{(2)} - \int_{(0)}\right)\frac{\tilde{\Lambda}_f^K(s)}{(s-z)(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i}\\ &= \int_{(2)}\frac{(MD)^{s-1}\,\Gamma(s)^2 L_f^K(s)}{(s-z)(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i} - \epsilon_f^K\,(-1)^{\tilde{r}}\int_{(2)}\frac{(MD)^{s-1}\,\Gamma(s)^2 L_f^K(s)}{(2-s-z)(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i},\end{aligned}$$

and we then insert the definitions of $E_f(s)$ and $r_f^K(n)$ from (16) in §4.3.1 as

$$L_f^K(s) = E_f(s)\frac{L_{S^2 f}(2s)}{\zeta(2s-1)} + \sum_{n=1}^{\infty}\frac{r_f^K(n)}{n^s}$$

to get

$$\begin{aligned}\frac{\tilde{\Lambda}_f^K(z)}{(z-1)^{\tilde{r}}} &= \int_{(2)}\frac{L_{S^2 f}(2s)}{\zeta(2s-1)}(MD)^{s-1}\frac{E_f(s)\Gamma(s)^2}{(s-z)(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i} +\\ &\quad - \epsilon_f^K(-1)^{\tilde{r}}\int_{(2)}\frac{L_{S^2 f}(2s)}{\zeta(2s-1)}(MD)^{s-1}\frac{E_f(s)\Gamma(s)^2}{(2-s-z)(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i} +\\ &\quad + \int_{(2)}\frac{(MD)^s}{MD}\sum_{n=1}^{\infty}\frac{r_f^K(n)}{n^s}\left[\frac{1}{s-z} - \epsilon_f^K\frac{(-1)^{\tilde{r}}}{2-s-z}\right]\frac{\Gamma(s)^2}{(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i}. \qquad (22)\end{aligned}$$

The main terms in 21 will come from residues arising from moving the first two integrals to the left, with the error therein (on the new contour) being quite negligible due to the decay of D^σ. Meanwhile, the integral involving r_f^K gives an error that can be directly estimated in terms of $L_\chi(1)$ by Proposition 4.4.1. We handle the latter first.

6.1.2. For the integral that involves $r_f^K(n)$ in (22) we proceed to substitute[17] the series expansion $1/(s-z) = \sum_l (z-1)^l/(s-1)^{l+1}$ in powers of $1/(s-1)$ and consider

$$\begin{aligned}G_{\tilde{r}}(z) &= \int_{(2)}\sum_{n=1}^{\infty}\frac{r_f^K(n)}{n^s}(MD)^{s-1}\left[\frac{1}{s-z} - \frac{\epsilon_f^K(-1)^{\tilde{r}}}{2-s-z}\right]\frac{\Gamma(s)^2}{(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i}\\ &= \int_{(2)}\sum_{n=1}^{\infty}\frac{r_f^K(n)}{n^s}\frac{(MD)^s}{MD}\left[\sum_{l=0}^{\infty}\frac{(z-1)^l}{(s-1)^{l+1}} + \epsilon_f^K(-1)^{\tilde{r}}\sum_{l=0}^{\infty}\frac{(1-z)^l}{(s-1)^{l+1}}\right]\frac{\Gamma(s)^2}{(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i}\\ &= \sum_{n=1}^{\infty}\frac{r_f^K(n)}{MD}\sum_{l=0}^{\infty}(z-1)^l\big[1+\epsilon_f^K(-1)^{l+\tilde{r}}\big]\int_{(2)}\left(\frac{MD}{n}\right)^s\frac{\Gamma(s)^2}{(s-1)^{l+\tilde{r}+1}}\frac{\partial s}{2\pi i}.\end{aligned}$$

[17] It may be possible to avoid this series expansion; e.g. right before (20) in [71] we instead used that the Mellin transform of $\Gamma(s)/(s-1/2)$ exceeds that of $\Gamma(s)(s-1/2)/\big((s-1/2)^2+\xi_0^2\big)$, as the latter just has an additional cosine term that is bounded by 1.

The integral here is $I_{l+\tilde{r}+1}(n/MD)$ and thus positive, and as $1+\epsilon_f^K(-1)^{l+\tilde{r}}=0$ for l odd, with $\tilde{l}=l/2$ we have

$$|G_{\tilde{r}}(z)| \le \frac{2}{MD}\sum_{\tilde{l}=0}^{\infty}|z-1|^{2\tilde{l}}\sum_{n=1}^{\infty}|r_f^K(n)|I_{2\tilde{l}+\tilde{r}+1}(n/MD).$$

We then use Proposition 4.4.1 with $X=MD$ (here $\tilde{r}\ge 2$ ensures "$j\ge 3$"), where the negligible secondary term is $\le 5000\cdot 2^j X^{3/2}L_\chi(1)\big((\log D)^3/\sqrt{D}\big)\log(M^6D^8)$, which is $\le \exp(-4999900)\cdot 2^j X^{3/2}L_\chi(1)$ as $M\le N_f\le D^{1/9}$ and $D\ge 4\pi^2\exp(10^7)$. We thus find that

$$\begin{aligned}|G_{\tilde{r}}(z)| &\le \frac{2}{MD}\sum_{\tilde{l}=0}^{\infty}|z-1|^{2\tilde{l}}\cdot 7.629\cdot\mathcal{U}(f)\mathcal{V}(f)\cdot L_\chi(1)\mathcal{R}(\chi)\cdot 2^{2\tilde{l}+\tilde{r}+1}(MD)^{3/2}\\ &\le 30.516\sqrt{M}\cdot\mathcal{U}(f)\mathcal{V}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\cdot 2^{\tilde{r}}\sum_{\tilde{l}=0}^{\infty}2^{\tilde{2}l}|z-1|^{2\tilde{l}},\end{aligned}$$

and since $|z-1|\le 1/99$ the final $\tilde{l}$-sum converges and is $\le 9801/9797$. Replacing into (22) and multiplying $G_{\tilde{r}}(z)$ by $|z-1|^{\tilde{r}}$ gives a contribution to the stated bound for $U_f(z)$ with 30.53 replaced by 30.529.

6.1.3. We are left to consider the other parts with (22), namely

$$\begin{aligned}&(z-1)^{\tilde{r}}\int_{(2)}\frac{L_{S^2f}(2s)}{\zeta(2s-1)}(MD)^{s-1}\frac{E_f(s)\Gamma(s)^2}{(s-z)(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i}+\\ &\qquad -\epsilon_f^K(-1)^{\tilde{r}}(z-1)^{\tilde{r}}\int_{(2)}\frac{L_{S^2f}(2s)}{\zeta(2s-1)}(MD)^{s-1}\frac{E_f(s)\Gamma(s)^2}{(2-s-z)(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i}.\end{aligned}$$

We first truncate the integrals at height $H_2=(\log D)^2$. This truncation induces a negligible error due to decay of the Γ-function, with everything else in the integrands adequately bounded on the $\sigma=2$ line. Indeed, we have the bound $|L_{S^2f}(4+2it)|\le\zeta(3)^3$ and also $|E_f(2+it)|\le\zeta(3)^2\zeta(3/2)^4$, so by using $\pi\le\sqrt{D}L_\chi(1)$ with $D\ge 4\pi^2\exp(10^7)$ and $M\le N_f\le D^{1/9}$ we obtain a contribution to $\tilde{\Lambda}_f^K(z)$ that is

$$\le |z-1|^{\tilde{r}}\cdot 10^9(MD)(\log D)^3\exp\big(-\pi(\log D)^2\big)\le\exp(-10^{13})\cdot\sqrt{D}L_\chi(1)\cdot|z-1|^{\tilde{r}},$$

which is adequate for $U_f(z)$.

We then move the integrals to the left, doing so in such a way that the poles at $s=z$, $s=2-z$, and $s=1$ are all crossed. We describe the new contour below, and for now compute the residue contributions.

The pole at $s=z$ with the first integral gives a residue of

$$(z-1)^{\tilde{r}}\frac{L_{S^2f}(2z)}{\zeta(2z-1)}(MD)^{z-1}\frac{E_f(z)\Gamma(z)^2}{(z-1)^{\tilde{r}}}=T_f(z).$$

Similarly the pole at $s=2-z$ with the second integral gives a residue of $\epsilon_f^K T_f(2-z)$.

The first integral is $(z-1)^{\tilde{r}}\int\frac{T_f(s)}{(s-z)(s-1)^{\tilde{r}}}\frac{\partial s}{2\pi i}$, so from $T_f(s)=\sum_b T_f^{(b)}(1)(s-1)^b/b!$ and $1/(s-z)=-\sum_c(s-1)^c/(z-1)^{c+1}$ the pole at $s=1$ has a residue therein of

$$(z-1)^{\tilde{r}}\sum\sum_{b+c=\tilde{r}-1}\frac{T_f^{(b)}(1)}{b!}\frac{-1}{(z-1)^{c+1}}=-\sum_{b=0}^{\tilde{r}-1}\frac{T_f^{(b)}(1)}{b!}(z-1)^b.$$

The calculation is similar for the second integral, where we obtain a residue at $s=1$ of

$$-(z-1)^{\tilde{r}}\epsilon_f^K(-1)^{\tilde{r}}\cdot\sum\sum_{b+c=\tilde{r}-1}\frac{T_f^{(b)}(1)}{b!}\frac{-(-1)^c}{(z-1)^{c+1}}=-\epsilon_f^K\sum_{b=0}^{\tilde{r}-1}(-1)^b\frac{T_f^{(b)}(1)}{b!}(z-1)^b.$$

Adding these two contributions gives $-S_f^{\tilde{r}}(z)$ as in the statement of the Lemma.

6.1.4. We are left to consider the integral on the new contour. As noted above, the contribution will be negligibly small due to the decay of D^σ when moving to the left, though the details of showing this are quite tedious when including explicit constants.

The new contour has: the vertical segment from $4/5 - iH_1$ to $4/5 + iH_1$; the vertical segments $\sigma_0 \pm iH_1$ to $\sigma_0 \pm iH_2$ where $\sigma_0 = 1 - 1/140 \log\log D$ and we take $H_1 = 500$ and $H_2 = (\log D)^2$; and the associated horizontal connecting segments. As previously, the horizontal segments at height H_2 contribute negligibly as $|\Gamma(\sigma + iH_2)|$ is small.

The lower bulge to $\sigma = 4/5$ ensures that the poles at $s = z$ and $s = 2 - z$ are to the right of the new contour since $|z - 1| \le 1/99$. Moreover, by the computations of Titchmarsh [67] (up to height 1468) and the zero-free region for ζ due to de La Vallée Poussin [41, §31],[18] the above contour shift does not encounter any zeros of $\zeta(2z - 1)$, and indeed we have adequate control over $1/\zeta(2z - 1)$ on the new contour. Specifically, from [41] we have $1 - \sigma \ge 0.0328214/(\log t - 3.806)$ for $t > 574$, while for the ζ-reciprocal we have $|1/\zeta(\sigma + it)| \le 10^7 \log t$ for $\sigma \ge 1 - 1/8 \log t$ and $t \ge 45$ as given by Trudgian [68] (using the superior zero-free region of Kadiri, and computations of Platt),[19] as follows by applying the lemma of Borel and Carathéodory.[20] Additionally, by numerical computation we have $|\zeta(3/5 + it)| \ge 0.076$ for $|t| \le 1000$, and $|\zeta(\sigma + 1000i)| \ge 0.86$ for $3/5 \le \sigma \le 1$.

We start with the contribution from the segment on $\sigma = 4/5$, which is bounded as

$$\le \frac{2|z-1|^{\tilde r}}{(MD)^{1/5}} \int_{-H_1}^{H_1} \left| \frac{L_{S^2f}(8/5 + 2it)}{\zeta(3/5 + 2it)} \frac{E_f(4/5 + it)\,\Gamma(4/5 + it)^2}{(1 - 4/5 - 1/99 + it)(1/5 + it)^{\tilde r}} \right| \frac{\partial t}{2\pi}.$$

The convexity bound for the symmetric-square (Lemma 5.4.1) and $N_{S^2f} \le N_f^2$ yield

$$\begin{aligned} \left|L_{S^2f}(8/5 + 2it)\right| &\le 1.65(2N_f^2 t_\star^3)^{1/5}(1 + \log 2N_f^2 t_\star^3)^3 \\ &\le 5N_f^{2/5} t_\star^{3/5} (\log N_f^2)^3 (\log t_\star^3)^3 \le 135 \cdot 87000 N_f^{2/5 + 1/20} \cdot t_\star^{3/5} (\log t_\star)^3 \end{aligned}$$

(writing $t_\star = |t| + 5$ and using $(\log N_f^2)^3 \le 87000 N_f^{1/20}$), while Lemma 5.3.1 yields

$$\left|E_f^{\mathrm{r}}(4/5 + it)\right| \le \prod_{p \mid N_f} \frac{1 + 1/p^{3/5}}{1 - 1/p^{3/5}} \le 233 N_f^{1/20}$$

since the multiplicands are $\le p^{1/20}$ for $p \ge 49$. Our assumption that $D \ge N_f^9$ then implies that the product of these two terms is

$$\le 10^{10} \sqrt{N_f} \cdot t_\star^{3/5} (\log t_\star)^3 \le 10^{10} D^{1/18} \cdot t_\star^{3/5} (\log t_\star)^3.$$

Moreover, by combining Lemmata 5.3.2 and 5.3.3 we have

$$|E_f^{\mathrm{m}}(4/5 + it)| \le 7500 \cdot \mathcal{P}_{4/5}(D) \le 7500 \cdot \exp\bigl(11(\log\log D)^{7/10}\bigr) \le 10^{-228500} D^{1/19},$$

using $\log D \ge 10^7$. Our assumption of $\tilde r \le (\log D)/100$ implies that $5^{\tilde r} \le D^{(\log 5)/100}$, and upon pulling this out we get an overall contribution to $U_f(z)$ bounded as

$$\le \frac{2|z-1|^{\tilde r}}{(MD)^{1/5}} \cdot 10^{-228490} D^{1/18 + 1/19 + (\log 5)/100} \int_{-H_1}^{H_1} \left| \frac{\Gamma(4/5 + it)^2 \cdot t_\star^{3/5} (\log t_\star)^3}{0.076 \cdot (0.189 + it)} \right| \frac{\partial t}{2\pi},$$

where direct numerical integration then gives a bound for the integral as ≤ 131. We then note that $1/18 + 1/19 + \log(5)/100 \le 1/8$, so the error contribution is bounded as

$$\le \frac{|z-1|^{\tilde r}}{(D/4\pi^2)^{1/5}} \cdot 10^{-228487} D^{1/8} \le \frac{10^{-228486}}{D^{3/40}} |z-1|^{\tilde r} \le \exp(-10^6)|z-1|^{\tilde r} \cdot \sqrt{D} L_\chi(1),$$

[18] We could obtain (as in [73]) a much superior zero-free region under our assumption of small $L_\chi(1)$, but this seems rather arduous to do explicitly; as we can make due otherwise, we do not include it.

[19] We had originally wanted to highlight the utility of ancient results like [41, 67], but even though the method to pass from a zero-free region to a bound on $1/\zeta$ is old, it seems [68] is the first to actually write down an explicit constant; thus, for expediency, we have intertwined [41, 67] with modern developments.

[20] As Trudgian notes, one can use an idea of Tenenbaum to optimize the usage the usage of said lemma, improving the implicit constant from Titchmarsh's book by a factor of roughly 3. He also notes that both Gronwall and Landau describe methods that could be made explicit if desired.

where we used $\log D \geq 10^7$ and $\sqrt{D}L_\chi(1) \geq \pi$ in the last step. This suffices for $U_f(z)$.

Similarly, we find that the horizontal segments at height H_1 contribute

$$\leq 4|z-1|^{\tilde{r}} \int_{4/5}^{\sigma_0} \left| \frac{L_{S^2f}(2\sigma+2iH_1)}{\zeta(2\sigma-1+2iH_1)} \frac{E_f(\sigma+iH_1)(MD)^\sigma}{MD} \frac{\Gamma(\sigma+iH_1)^2}{(H_1-1)^{1+\tilde{r}}} \right| \frac{\partial\sigma}{2\pi}.$$

The Γ-factor is bounded as $\leq |\Gamma(1+500i)|^2 \leq 1000\pi\exp(-500\pi)$, and Lemma 5.4.1 bounds the L_{S^2f}-term as $\leq 5(1+\log 2N_f^2H_1^3)^3(N_f^2H_1^3)^{1-\sigma}$, while $1/\zeta$ is bounded by $1/0.86$, and $|E_f^{\mathrm{m}}(s)| \leq 7500\exp\bigl(11(\log\log D)^{3/2-\sigma}\bigr)$ as before, with Lemma 5.3.1 implying

$$\bigl|E_f^{\mathrm{r}}(s)\bigr| \leq \exp\Bigl(2.5\sum_{p|N_f}\frac{p}{p^{2\sigma}}\Bigr) \leq \exp\Bigl(2.5\sum_{n\leq 3\log N_f}\frac{n}{n^{2\sigma}}\Bigr) \leq \exp\Bigl(2.5\frac{(3\log N_f)^{2-2\sigma}}{2-2\sigma}\Bigr)$$

for $3/4 < \sigma < 1$. The integrand is thus maximized at $\sigma = \sigma_0 = 1 - 1/140\log\log D$ as

$$\leq \frac{10^{10}}{e^{-500\pi}}(\log 10^9 D^{2/9})^3(500^3D^{2/9})^{1-\sigma_0}(\log D)^{178}\exp\bigl(11(\log\log D)^{3/2-\sigma_0}\bigr)\Bigl(\frac{D}{4\pi^2}\Bigr)^{\sigma_0-1}$$

where we used $N_f^9 \leq D$ and noted $|E_f^{\mathrm{r}}(\sigma_0+it)| \leq (\log D)^{178}$. Since $D \geq 4\pi^2\exp(10^7)$ this integrand is $\leq 10^{-884}$, so the $U_f(z)$-contribution is $\leq 10^{-883}|z-1|^{\tilde{r}}\cdot\sqrt{D}L_\chi(1)$.

Finally, the contribution from the vertical segments on the σ_0-line is bounded as

$$\leq 4|z-1|^{\tilde{r}}\int_{H_1}^{H_2}(MD)^{\sigma_0-1}\left|\frac{L_{S^2f}(2\sigma_0+2it)}{\zeta(2\sigma_0-1+2it)}E_f(\sigma_0+it)\frac{\Gamma(\sigma_0+it)^2}{(H_1-1)^{1+\tilde{r}}}\right|\frac{\partial t}{2\pi}.$$

The Γ-factor is again $\leq |\Gamma(1+500i)|^2 \leq 1000\pi\exp(-500\pi)$ and similarly we again have $|E_f(s)| \leq 7500(\log D)^{178}\exp\bigl(11(\log\log D)^{3/2-\sigma_0}\bigr)$, while the L_{S^2f} -termis bounded by Lemma 5.4.1 as $\leq 5(1+\log 2N_f^2H_2^3)^3(N_f^2H_2^3)^{1-\sigma_0}$ where $H_2 = (\log D)^2$, and $1/\zeta$ is bounded as $\leq 10^7\log 2t \leq 10^8\log\log D$ from the zero-free region as noted above. Upon using $N_f^9 \leq D$ to note $2N_f^2H_2^3 \leq D^{1/4}$ for $D \geq 4\pi^2\exp(10^7)$, the integrand is thus

$$\leq 10^{-600}(\log\log D)(\log D^{1/4})^3(D^{1/4})^{1-\sigma_0}(\log D)^{178}\exp\bigl(11(\log\log D)^{3/2-\sigma_0}\bigr)\Bigl(\frac{D}{4\pi^2}\Bigr)^{\sigma_0-1},$$

and multiplying this by $(\log D)^2$ for the path length, the bound $D \geq 4\pi^2\exp(10^7)$ implies the contribution to $U_f(z)$ is $\leq 10^{-743}|z-1|^{\tilde{r}}\cdot\sqrt{D}L_\chi(1)$. □

7. Restricting zeros by computing $L_f'''(1)$

We will later use the above Deuring decomposition (21) specifically when $\tilde{r} = 4$ and we know there is at least a triple central zero of $L_f(s)$ along with exactly two additional zeros of $L_f(s)$ nearby. Such an f will be associated to an elliptic curve of rank 5.

In this section we will describe a method (to try) to deduce such facts about the zeros of $L_f(s)$, via computing its third central derivative to a large precision. We then specialize to our 6 curves in Table 3, and refer to [74] for the computational details therein.

7.1. Firstly, given an elliptic curve of rank 5 whose L-function has odd parity, a lower bound of 3 for the analytic rank can be obtained either: by combining the results of Gross and Zagier [31] with those of Kolyvagin [40, Theorem A, Corollary C], which together imply that if the analytic rank is 1 then the algebraic rank is also 1; or by computing $L_f'(1)$ to sufficient precision and using explicit bounds on the quantities in the formula of Gross and Zagier (see Delaunay and Roblot [21, Proposition 3.1] for a special case).

7.1.1. It turns out to be easier to work with $\tilde{\Lambda}_f(s)$ rather than $L_f(s)$. Recall that our scaling is $\tilde{\Lambda}_f(s) = (\sqrt{N_f}/2\pi)^{s-1}\Gamma(s)L_f(s)$. Assuming f has odd parity and a central zero of order 3 or more, by the Taylor expansion about $s = 1$ we have

$$\tilde{\Lambda}_f(s) = \frac{\tilde{\Lambda}_f'''(1)}{3!}(s-1)^3 + \frac{\tilde{\Lambda}_f^{(5)}(1)}{5!}(s-1)^5 + \Theta\Bigl(\frac{\hat{\Lambda}_f^{(7)}}{7!}|s-1|^7\Bigr)$$

for $|s-1| \leq 1/10^5$ where $\hat{\Lambda}_f^{(7)}$ is a bound for the seventh derivative in this region and Θ is analogous to Landau's O-notation with an implicit constant of 1.

In our cases of interest, we can computationally show that $\tilde{\Lambda}_f'''(1)$ is zero to a (large) desired precision, while simultaneously a computation will show that the fifth central derivative is not close to zero. By choosing a circle small enough about $s = 1$ so that

$$\left|\frac{\tilde{\Lambda}_f^{(5)}(1)}{5!}(s-1)^5\right| > \left|\frac{\tilde{\Lambda}_f'''(1)}{3!}(s-1)^3\right| + \left|\frac{\hat{\Lambda}_f^{(7)}}{7!}(s-1)^7\right|$$

on the boundary of the circle, by Rouché's theorem we can then conclude that $\tilde{\Lambda}_f(s)$ has exactly five zeros inside such a circle. As we know that at least three of these zeros are at the central point $s = 1$, by symmetry of the functional equation for $\tilde{\Lambda}_f(s)$ the other two must lie on the central line or the real axis (or indeed, possibly both).

We formulate this precisely, writing $b_5 > 0$ as a lower bound for $|\tilde{\Lambda}_f^{(5)}(1)/5!|$ and B_7 for an upper bound for $|\tilde{\Lambda}_f^{(7)}(s)/7!|$ in the disk $|s-1| \leq 1/10^5$.

Lemma 7.1.2. *Suppose f has odd parity and $\tilde{\Lambda}_f'(1) = 0$, and that $|\tilde{\Lambda}_f'''(1)/3!| \leq B_3$ where $|3B_3/b_5| \leq \min(b_5/3B_7, 1/10^{10})$. Then $\tilde{\Lambda}_f(s)$ has exactly five zeros (counting multiplicity) in the disk $|s-1|^2 \leq 3B_3/b_5$.*

Proof. On the circle $|s-1|^2 = 3B_3/b_5$ we have

$$\left|\frac{\hat{\Lambda}_f^{(7)}}{7!}(s-1)^7\right| \leq B_7|s-1|^7 = \frac{3B_3}{b_5}B_7|s-1|^5 \leq \frac{b_5}{3}|s-1|^5$$

and

$$\left|\frac{\tilde{\Lambda}_f'''(1)}{3!}(s-1)^3\right| \leq B_3|s-1|^3 = \frac{b_5}{3}|s-1|^5,$$

while $|\tilde{\Lambda}_f^{(5)}(1)(s-1)^5/5!| \geq b_5|s-1|^5$, so that the result follows by the above Taylor expansion for $\tilde{\Lambda}_f(s)$ in conjunction with Rouché's theorem [61]. □

We can compute $\tilde{\Lambda}_f^{(5)}(1)$ to (say) five digits by the method below, and thereby get a lower bound for it; these appear in the 5th column of Table 3 for our 6 curves.

A suitable bound on the 7th derivative, which is for a region rather than just at one point, can easily be obtained by crude estimates (and indeed, this bound only plays a minor technical rôle in any case). For instance, we can bound $\tilde{\Lambda}_f$ on the σ-line, apply the functional equation to get a bound on the $(2-\sigma)$-line, and use convexity. Likely the optimal choice is $\sigma = 3/2$ (in analogue to Lemma 5.4.1), though $\sigma = 2$ seems easier to use, getting $|\tilde{\Lambda}_f(2+it)| \leq \zeta(3/2)^2(\sqrt{N_f}/2\pi)$ and indeed the same for $0 \leq \sigma \leq 2$, whence Cauchy's bound on derivatives implies the 7th derivative is bounded in $|s-1| \leq 1/10^5$ as $\leq 7! \cdot 1.001^7 \cdot 1.087\sqrt{N_f}$. For each of our 6 curves we have $N_f \leq 10^{16}$ so that $B_7 \leq 10^{12}$, so in particular we have $b_5/3B_7 \geq 1525/(3 \cdot 10^{12}) \geq 10^{-10}$ in all cases.

7.1.3. Each of the 6 curves in Table 3 (provably) has rank 5. For instance, the computer algebra system Magma [8] takes only a few seconds to both find 5 independent points on each curve and show an upper bound of 5 for each rank via a 2-Selmer computation.

Let us state an explicit Lemma for our 6 curves.

Lemma 7.1.4. *Suppose that f is associated to one of the curves in Table 3, and that we have $|\tilde{\Lambda}_f'''(1)/3!| \leq \lambda \leq 1/10^8$. Then $\tilde{\Lambda}_f(s)$ has (at least) a triple central zero and exactly two other zeros inside $|s-1|^2 \leq 3\lambda/1525$.*

Proof. We use Kolyvagin's result to show each curve has analytic rank at least 3, and then apply Lemma 7.1.2 with the explicitly tabulated lower bounds for the fifth central derivative of the curves in question, with $b_5 \geq 1525$ for each. □

7.2. We recall that $\tilde\Lambda_f^{(l)}(1)$ can be approximated by a method detailed by Buhler, Gross, and Zagier [10, §4] (see also Cremona [17, §2.13]). For integral $l \geq 1$ we have

$$\frac{\tilde\Lambda_f^{(l)}(1)}{l!} = [1+\epsilon_f(-1)^l]\cdot\sum_{n=1}^{\infty}\frac{c_f(n)}{n}G_l\Bigl(\frac{2\pi n}{\sqrt{N_f}}\Bigr)$$

where

$$G_l(x) = \frac{1}{(l-1)!}\int_1^\infty e^{-xy}(\log y)^{l-1}\frac{\partial y}{y}$$

satisfies $G_l'(x) = -G_{l-1}(x)/x$, with $G_0 = e^{-x}$.

It is critical here that $G_l(x) \sim e^{-x}/x^l$ decays exponentially as $x \to \infty$, implying that we can calculate to d digits using approximately $(\sqrt{N_f}/2\pi)\cdot\log 10^d$ coefficients of the series, which is thus linear in d. The $c_f(p)$ can be computed in time polynomial in $\log p$ via Schoof's algorithm [63], though in practice using baby/giant steps (taking roughly $p^{1/4}$ time) is likely just as good for our range.

For $l = 1$ this G_l is a familiar exponential integral, and in general (e.g.) since G_l is holonomic its values can be computed in quasi-linear time (in the precision) via a binary splitting method given by van der Hoeven [35]. However, as noted to us by A. R. Booker, there is also the possibility to exploit the equi-spacedness of the evaluation points of G_l and use a multi-point polynomial evaluation scheme, again giving a quasi-quadratic running time overall. We analyze the situation more fully in our companion paper [74], finding that neither of these particularly displays the asymptotic quadratic behavior in our target range of 1000 digits. Meanwhile, the best method appears to be to compute batches of G_3-values by local power series, using the differential equation satisfied by G_3 to efficiently compute high-order derivatives (via recursion) at suitable demarcation points.[21]

7.2.1. As noted in the Introduction, for the purposes of the main application of our Theorem 10.3.2 it would suffice to verify that $L_f'''(1) \approx 0.0000\ldots$ to merely 30 digits. While this cannot be said to be a trivial computation, it is still fairly routine; for instance, the computation for the curve 11a twisted by -25351367 (the example of largest conductor) takes about 4.5 hours using the off-the-shelf Magma implementation [8].

7.2.2. The value of $L_{S^2f}(2)$ can be computed as in [69]. In Table 3, for each curve we list an approximation to this value in the fourth column, given as a lower bound. Additionally we list an approximation for the fifth central derivative (again as a lower bound). We also list upper bounds for the expressions $\mathcal{U}(f)$ and $\mathcal{V}(f)$ that appear in §4.3.3.

Remark. We can note that $L_{S^2f}(2) \geq 0.78$ for each of the 6 curves. In general, we could use an explicit version of the lower bound $L_{S^2f}(2) \gg 1/\log N_f$ of Goldfeld, Hoffstein, and Lieman [29] as we gave in a technical report [72, Lemma 3.4] (similar calculations also appear in work of Rouse [62, Proposition 11]).

7.2.3. In Table 7 we list $E_f^{\mathrm{r}}(s)$ for each of the 6 curves, indeed listing $V_p(s)$ for each bad prime. Note that with potentially good primes we have $\alpha_p^2 + \beta_p^2 = c_g(p)^2 - 2p$, and the listed Euler factors $V_p(s)$ are only applicable when $p \nmid D$. One fact we can note is that $E_f^{\mathrm{r}}(1) \geq 1$ in all cases. Indeed, there are only two instances where $V_p(1) < 1$, namely $p = 191$ with $g = 91b$ and $p = 5$ with $g = 123a$, and in both cases the contribution from the multiplicative primes outweighs this.

8. More bounds on E_f and L_{S^2f} and their derivatives

We now wish to give assorted technical preparations that will allow us to bound various secondary terms in our usage of the Deuring decomposition of §6.

[21]Note that this method is only quasi-cubic complexity even in theory, and in practice the growth exponent is more likely to be behave like $2+\rho$ where ρ is the realistic exponent for multiplication (for instance $\log(3)/\log(2) \approx 1.585$ in the range where Karatsuba multiplication applies).

g	B	p	$V_p(s)$	V_p'/V_p bound
11a	-25351367	11	$(1-11/11^{2s})^{-1}$	0.4822
		73	$(1+130/73^{2s}+73^2/73^{4s})$	0.2131
		269	$(1+438/269^{2s}+269^2/269^{4s})$	0.0686
		1291	$(1+2518/1291^{2s}+1291^2/1291^{4s})$	0.0220
17a	-19502039	17	$(1-17/17^{2s})^{-1}$	0.3563
		47	$(1+94/47^{2s}+47^2/47^{4s})$	0.3375
		53	$(1+70/53^{2s}+53^2/53^{4s})$	0.2002
		7829	$(1+10758/7829^{2s}+7829^2/7829^{4s})$	0.0033
19a	-16763912	19	$(1-19/19^{2s})^{-1}$	0.3293
		2	$(1+4/2^{2s}+2^2/2^{4s})$	2.7803
		11	$(1+13/11^{2s}+11^2/11^{4s})$	0.5355
		197	$(1+70/197^{2s}+197^2/197^{4s})$	0.0198
		967	$(1+334/967^{2s}+967^2/967^{4s})$	0.0051
91b	6350941	7,13	$(1-7/7^{2s})^{-1}(1-13/13^{2s})^{-1}$	0.6516+0.4299
		41	$(1+46/41^{2s}+41^2/41^{4s})$	0.2079
		191	$(1-194/191^{2s}+191^2/191^{4s})$	0.0568
		811	$(1+1222/811^{2s}+811^2/811^{4s})$	0.0253
123a	5467960	3,41	$(1-3/3^{2s})^{-1}(1-41/41^{2s})^{-1}$	1.103+0.1871
		2	$(1+0/2^{2s}+2^2/2^{4s})$	0.9277
		5	$(1-6/5^{2s}+5^2/5^{4s})$	0.8304
		223	$(1+190/223^{2s}+223^2/223^{4s})$	0.0421
		613	$(1+550/613^{2s}+613^2/613^{4s})$	0.0191
209a	3217789	11,19	$(1-11/11^{2s})^{-1}(1-19/19^{2s})^{-1}$	0.4822+0.3293
		53	$(1+70/53^{2s}+53^2/53^{4s})$	0.2002
		109	$(1+214/109^{2s}+109^2/109^{4s})$	0.1721
		557	$(1+214/557^{2s}+557^2/557^{4s})$	0.0090

TABLE 7. Values for $V_p(s)$ for the six rank 5 curves we use

8.1. First we give bounds on the symmetric-square L-functions. Similar to the previous section, it seems easier to simply state computationally verifiable results for our specific 6 curves, as opposed to working in more generality.

Lemma 8.1.1. *For each of the 6 curves in Table 3 for $|s-2|\le 2/10^5$ we have*

$$0.999L_{S^2f}(2)\le \left|L_{S^2f}(s)\right|\le 1.001L_{S^2f}(2),$$

and for $l\le 6$ in $|s-2|\le 2/10^5$ we have

$$|L^{(l)}_{S^2f}(s)|\le 1.001L_{S^2f}(2)\cdot 100^l.$$

Remark. Note that the given bounds on $L_{S^2f}(2)$ will *not* hold in general; e.g., when the logarithmic derivative at $s=2$ is large the first statement is prone to fail. We really only need a bound of this sort in terms of the conductor, but I find it convenient to instead give a version with explicit constants for the given curves. (One should generally expect the derivatives to grow roughly as $(\log N_{S^2f})^l$).

Proof. We bound high derivatives of $L_{S^2f}(s)$ in $|s-2|\le 2/10^5$ by convexity (using Lemma 5.4.1, or an analogy for L_{S^2f} of our comments after Lemma 7.1.2). and approximate smaller derivatives by computation. Combining via Taylor series, the result follows.

For instance, with f as the indicated twist of 209a (so that $N_{S^2f}=209^2$) we can computationally estimate (as with [69]) the zeroth through sixth derivatives at $s=2$ as

$$\approx (1.0517,\ -0.7960,\ 2!\cdot 1.4184,\ -3!\cdot 1.7968,\ 4!\cdot 1.9123,\ -5!\cdot 1.7157,\ 6!\cdot 1.2856),$$

and the seventh derivative is $\approx -7!\cdot 0.7927$. As with Lemma 5.4.1 we have a bound from convexity of $|L_{S^2f}(s)|\le 5\sqrt{N_{S^2f}}(1+5\log N_{S^2f})^3$ in $|s-2|\le 1$, so that Cauchy's bound on derivatives implies that the kth derivative is bounded in $|s-2|\le 2/10^5$ as $\le (1.001)^k k!\cdot 5\sqrt{N_{S^2f}}(1+5\log N_{S^2f})^3$, and upon using $k=8$ the bounds stated in the Lemma readily follows. Indeed, for $|s-2|\le 2/10^5$ we have

$$L^{(l)}_{S^2f}(s)=\sum_{j=l}^{7} L^{(j)}_{S^2f}(2)\frac{(s-2)^{j-l}}{(j-l)!}+\Theta\Big(\frac{8!(1.001)^8}{(8-l)!}\cdot 5\sqrt{N_{S^2f}}(1+5\log N_{S^2f})^3|s-2|^{8-l}\Big),$$

where the error is adequately small due to this constraint $|s-2|\le 2/10^5$ – the occurrence of 100^l is gratuitously large, as for instance for $l=6$ and g as 209a we have

$$\begin{aligned}|L^{(6)}_{S^2f}(s)| &\le 6!\cdot 1.286+7!\cdot 0.793\cdot\frac{2}{10^5}+\frac{8!}{2!}(1.001)^8\cdot 5\cdot 209\cdot\big(1+10\log 209\big)^3\cdot\Big(\frac{2}{10^5}\Big)^2\\ &\le 926+0.1+1370\le 1.052(100)^6\le 1.001L_{S^2f}(2)\cdot 100^6,\end{aligned}$$

and similarly with the other curves and derivatives. □

8.1.2. We also give a result for the effect on E_f from primes $p|N_f$.

Lemma 8.1.3. *For each of the curves in Table 3 we have that* $|E^{\mathrm{r}}_f(s)|\le 1.0037E^{\mathrm{r}}_f(1)$ *for* $|s-1|\le 1/1000$, *and also* $\big|\log(E^{\mathrm{r}}_f(s)/E^{\mathrm{r}}_f(1))\big|\le 3.67|s-1|$ *in this range.*

Proof. We note

$$\log E^{\mathrm{r}}_f(s)-\log E^{\mathrm{r}}_f(1)=\int_1^s\frac{(E^{\mathrm{r}}_f)'}{E^{\mathrm{r}}_f}(z)\,\partial z$$

and proceed to bound the logarithmic derivative of E^{r}_f for $\sigma\ge 999/1000$.

The definition of $E^{\mathrm{r}}_f(s)$ is

$$E^{\mathrm{r}}_f(s)=\prod_{p|N_f}V_p(s)=\prod_{p|N_g}\Big(1-\frac{p}{p^{2s}}\Big)^{-1}\prod_{p|B,p\nmid D}\Big(1-\frac{\alpha_p^2}{p^{2s}}\Big)\Big(1-\frac{\beta_p^2}{p^{2s}}\Big)$$

where α_p,β_p depend on f (as does B). We then note each $(V_p'/V_p)(s)$ has period $2\pi i/\log p^2$ and each maximum modulus for $\sigma\ge 0.999$ occurs on the boundary, reducing the problem to calculus. We record said maximum in the final column of Table 7, as an upper bound.

The worst case is then with $(g,B)=(19a,-16763912)$, for which the logarithmic derivative is bounded by 3.67, whereupon integrating gives the second statement of the lemma. The first then follows since $\exp(3.67/1000)\le 1.0037$. □

8.2. Next we turn to bounding the effect of noninert primes on E_f. We introduce the notation $\tilde D=D/4\pi^2$, noting that $\tilde D\le MD$ as seen at the end of §4.2.1.

Lemma 8.2.1. *Suppose that* $D\ge 4\pi^2\exp(10^7)$ *and* $\sqrt{D}L_\chi(1)\le(\log D)^3/10^6$. *Then for* $|s-1|\le 1000/\log\tilde D$ *we have* $|E^{\mathrm{m}}_f(s)|\le 1.039E^{\mathrm{m}}_f(1)$.

Remark. It is this result in particular that is a bottleneck against taking D too much smaller in our main Theorem 10.3.2; on the other hand, since Proposition 12.3.1 extends to $\log D\le 10^8$ in any event, at various junctures we have preferred simplicity in our bounds as opposed to sharpness.

Proof. Similar to the previous proof, we will use

$$\log E^{\mathrm{m}}_f(s)-\log E^{\mathrm{m}}_f(1)=\int_1^s\frac{(E^{\mathrm{m}}_f)'}{E^{\mathrm{m}}_f}(z)\,\partial z,$$

and bound the logarithmic derivative.

We recall (§4.3.1) that $E^{\mathrm{m}}_f(s)$ is given by

$$E^{\mathrm{m}}_f(s)=\prod_{p\le\sqrt{D}/2}\frac{1+\alpha'_p/p^s}{1-\alpha'_p\chi(p)/p^s}\frac{1+\beta'_p/p^s}{1-\beta'_p\chi(p)/p^s}.$$

Taking logarithms, we have

$$\log E_f^{\mathrm{m}}(s) = \sum_{p\le\sqrt{D}/2}\sum_{l=1}^{\infty}\big[\chi(p)^l-(-1)^l\big]\frac{(\alpha_p')^l+(\beta_p')^l}{lp^{ls}},$$

and taking the derivative of this multiplies the right side by $-l\log p$.

As with Lemma 5.2.1, there are at most 2 split primes up to 10000, and at most 5 split primes up to $e^{\sqrt{Y}}$ where $Y=\log(\sqrt{D}/2)$. For $\sigma\ge 999/1000$ these small split primes contribute to the logarithmic derivative no more than

$$F(2)+F(3)+3F(10^4)\le 8.86 \quad\text{where } F(p)=\sum_{\substack{l=1\\ l\text{ odd}}}^{\infty}4\frac{p^{l/2}\log p}{p^{0.999l}}=\frac{4\sqrt{p}\log p}{p^{0.999}-p^{0.001}}.$$

The split primes exceeding $X_1=e^{\sqrt{Y}}$ contribute for $D\ge 4\pi^2\exp(10^7)$ an amount

$$\le 0.52\frac{(\log D)^3}{10^6}\sum_{l=1}^{\infty}4\frac{X_1^{l/2}\log X_1}{X_1^{0.999l}}\le 10^{-465},$$

where we used Corollary 5.1.2 to bound the number of split primes.

For the primes $p|D$, we can employ the bound $|\log\mathcal{P}_{\tilde{s}}(D)|\le 11(\log\log D)^{3/2-\tilde{\sigma}}$ from Lemma 5.3.3 for $\tilde{\sigma}\ge 1/2$. Taking the $\tilde{s}$-derivative of $P_{\tilde{s}}(D)$ in a circle of radius 0.499, we find that for $\sigma\ge 0.999$ the ramified primes contribute to the logarithmic derivative of $E_f^{\mathrm{m}}(s)$ no more than $23\log\log D$. (Of course one can improve this by taking a circle of radius $1/\log\log\log D$ if desired, but this makes no difference for us).

Integrating the logarithmic derivative then implies for $|s-1|\le 1/1000$ we have

$$\left|\frac{E_f^{\mathrm{m}}(s)}{E_f^{\mathrm{m}}(1)}\right|\le\exp\Big(|s-1|\cdot\big[8.87+23\log\log D\big]\Big). \tag{23}$$

For $|s-1|\le 1000/\log\tilde{D}$ and $D\ge 4\pi^2\exp(10^7)$ this is $\le\exp(0.038)\le 1.039$. □

Now we can state our desired bound on the derivatives of E_f.

Lemma 8.2.2. *Suppose $D\ge 4\pi^2\exp(10^7)$ and $\sqrt{D}L_\chi(1)\le(\log D)^3/10^6$ and f is associated to one of the curves in Table 3. Then for $|s-1|\le 2/\log\tilde{D}$ we have*

$$|E_f^{(l)}(s)|\le 1.043E_f(1)\cdot l!\Big(\frac{\log\tilde{D}}{998}\Big)^l.$$

Proof. For $|s-1|\le 1000/\log\tilde{D}$ and $l=0$ we can combine Lemmata 8.1.3 and 8.2.1, and then in $|s-1|\le 2/\log\tilde{D}$ for the higher derivatives we can apply Cauchy's bound on derivatives in a circle about s of radius $998/\log\tilde{D}$. □

We also have a version that bounds $E_f(s)$ for $|s-1|\le 1/1000$.

Lemma 8.2.3. *Suppose $D\ge 4\pi^2\exp(10^7)$ and $\sqrt{D}L_\chi(1)\le(\log D)^3/10^6$ and f is associated to one of the curves in Table 3. Then for $|s-1|\le 1/1000$ we have*

$$|E_f(s)|\le E_f(1)\cdot\exp\big(25|s-1|\cdot(\log\log D)\big)$$

and

$$|E_f(s)|\ge E_f(1)\cdot\exp\big(-25|s-1|\cdot(\log\log D)\big).$$

Proof. By Lemma 8.1.3 and (23), for $\sigma\ge 999/1000$ we have

$$\left|\log\Big(\frac{E_f(s)}{E_f(1)}\Big)\right|=\left|\log\Big(\frac{E_f^{\mathrm{r}}(s)}{E_f^{\mathrm{r}}(1)}\frac{E_f^{\mathrm{m}}(s)}{E_f^{\mathrm{m}}(1)}\Big)\right|\le\left|\log\Big(\frac{E_f^{\mathrm{r}}(s)}{E_f^{\mathrm{r}}(1)}\Big)\right|+\left|\log\Big(\frac{E_f^{\mathrm{m}}(s)}{E_f^{\mathrm{m}}(1)}\Big)\right|$$
$$\le|s-1|\cdot\big[3.67+8.87+23\log\log D\big],$$

and the assumption that $D\ge 4\pi^2\exp(10^7)$ then gives the result. □

Finally, we can improve Lemma 8.2.2 for $l=0$ when considering a smaller circle.

Corollary 8.2.4. *Suppose $D \geq 4\pi^2\exp(10^7)$ and $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$ and f is from Table 3. Then for $|s-1| \leq 2/\log\tilde{D}$ we have $|E_f(s)| \leq 1.0001E_f(1)$.*

Proof. This follows from Lemma 8.2.3 and our assumption of $D \geq 4\pi^2\exp(10^7)$, since we have $\exp(50\log(10^7)/10^7) \leq 1.0001$. ☐

9. Consequences of the above Deuring decomposition

We next use the Deuring decomposition in Proposition 6.1.1 to show a lower bound for $\sqrt{D}L_\chi(1)$ under an assumption about zeros of $\tilde{\Lambda}_f^K(s)$.

9.1. Let us first record the derivatives of $T_f(z)$, which we recall itself is

$$T_f(z) = \Gamma(z)^2\frac{L_{S^2f}(2z)}{\zeta(2z-1)}E_f(z)(MD)^{z-1}.$$

In general, with $F(z) = \Gamma(z)^2/\zeta(2z-1)$ we have that the lth derivative of $T_f(z)$ is

$$T_f^{(l)}(z) = \sum\sum\sum\sum_{a+b+c+d=l}\frac{l!}{a!b!c!d!}F^{(a)}(z)L_{S^2f}^{(b)}(2z)E_f^{(c)}(z)(\log MD)^d(MD)^{z-1}.$$

In particular, we will apply this formula to the fourth derivative ($l=4$) at $z=1$, and upon noting that $T_f(1) = F(1) = 0$, we wish for the $(a,b,c,d) = (1,0,0,3)$ term to be dominant. This will follow from bounding the derivatives of E_f by our assumption of small $L_\chi(1)$, and by also bounding the derivatives of L_{S^2f}. We do the latter computationally for our specific 6 elliptic curves from Table 3, though it could be done in terms of N_f and in turn bounding this in terms of D.

We can note that $F'(1) = 2$ and $F''(1) = -16\gamma \approx -9.23545$, while

$$|F''(1)| \leq 10,\ |F'''(1)| \leq 57,\ |F^{(4)}(1)| \leq 368,\ |F^{(5)}(1)| \leq 2731, \text{ and } |F^{(6)}(1)| \leq 22510,$$

so in particular $|F^{(i)}(1)| \leq 6^i$ for $i \leq 6$. Similar bounds hold in $|z-1| \leq 1/10^5$ upon being multiplied by 1.001, while $|F'(z)| \leq 2.001$ and $|F(z)| \leq 2.001|z-1|$ in this range. (All of these can be routinely proven by bounding high derivatives of F by Cauchy's theorem, and using a Taylor expansion about $z=1$, similar to the proof of Lemma 8.1.1).

9.2. We next show that (for D large) the expected term is dominant in the 2nd and 4th derivatives of $T_f(z)$ at $z=1$, while the 6th derivative is adequately bounded near this central point. We recall M is defined by $4\pi^2MD = \sqrt{N_fN_{f\chi}}$ so that $1 \leq 4\pi^2M \leq N_f$ as noted at the end of §4.2.1, and also our notation $\tilde{D} = D/4\pi^2$.

Lemma 9.2.1. *Suppose that $D \geq 4\pi^2\exp(10^7)$ and $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$, while f is associated to one of the 6 curves in Table 3. Then with Θ representing a factor bounded by 1, we have*

$$T_f''(1) = \frac{2!}{1!}\cdot 2L_{S^2f}(2)E_f(1)(\log MD)\cdot\bigl[1+\Theta(1/750)\bigr]. \tag{\textbf{A}}$$

and

$$T_f^{(4)}(1) = \frac{4!}{3!}\cdot 2L_{S^2f}(2)E_f(1)(\log MD)^3\cdot\bigl[1+\Theta(1/200)\bigr], \tag{\textbf{B}}$$

while for $|z-1| \leq (5/4)/\log MD$ we have

$$\bigl|T_f^{(6)}(z)\bigr| \leq 4.5\cdot\frac{6!}{5!}\cdot 2L_{S^2f}(2)E_f(1)(\log MD)^5. \tag{\textbf{C}}$$

Proof. We explicitly have that (using $F'(1) = 2$ and $F''(1) = -16\gamma$)

$$T_f''(1) = \frac{2!}{1!}\cdot 2L_{S^2f}(2)E_f(1)\Bigl[\log MD - 4\gamma + 2\frac{L'_{S^2f}}{L_{S^2f}}(2) + \frac{E'_f}{E_f}(1)\Bigr],$$

where $|E'_f(1)/E_f(1)| \leq (\log\tilde{D})/955$ from Lemma 8.2.2 and $|L'_{S^2f}(2)/L_{S^2f}(2)| \leq 101$ from Lemma 8.1.1. Thus the bracketed term in $T_f''(1)$ is $\log MD + \Theta\bigl(205 + (\log\tilde{D})/955\bigr)$, which by $D \geq 4\pi^2\exp(10^7)$ is $(\log MD)\cdot\bigl[1+\Theta(1/936)\bigr]$. This shows (**A**).

With the fourth derivative, the main term is $B_4 = \frac{4!}{3!} \cdot 2L_{S^2f}(2)E_f(1)(\log MD)^3$ (from $d = 3$), and the terms for $d = 2$ are bounded as

$$\le \frac{4!}{2!2!} \cdot 10 \cdot L_{S^2f}(2) \cdot E_f(1) \cdot (\log MD)^2 + \frac{4!}{2!} \cdot 2 \cdot 202 L_{S^2f}(2) \cdot E_f(1) \cdot (\log MD)^2 + \\ + \frac{4!}{2!} \cdot 2 \cdot L_{S^2f}(2) \cdot E_f(1) \frac{\log \tilde{D}}{955} \cdot (\log MD)^2 \le B_4 \cdot \Big(\frac{15/2 + 3 \cdot 202}{10^7} + \frac{3}{955}\Big),$$

which is $\le B_4/312$. The terms with $d \le 1$ contribute considerably less, e.g. the bound for the $(1,0,2,1)$-term is $\approx 2/998$ of the bound for the $(1,0,1,2)$-term. Indeed, by Lemmata 8.2.2 and 8.1.1 and the bounds on derivatives of F, the terms with $d \le 1$ contribute

$$\le 1.05 L_{S^2f}(2)E_f(1) \sum\sum_{\substack{a+b+c+d=4\\ a\ge 1,\, d\le 1}}\sum\sum \frac{4!}{a!b!c!d!} 6^a (200)^b c! \Big(\frac{\log \tilde{D}}{998}\Big)^c (\log MD)^d$$

$$\le 1.05 L_{S^2f}(2)E_f(1)(\log MD)^3 \sum\sum_{\substack{a+b+c+d=4\\ a\ge 1,\, d\le 1}}\sum\sum \frac{4!}{a!b!c!d!} \frac{6^a (200)^b}{(10^7)^{(a-1)+b}} \frac{c!}{998^c},$$

which is $\le 0.0001 L_{S^2f}(2)E_f(1)(\log MD)^3 = B_4/80000$, giving the error in (**B**).

Finally, for the bound on the sixth derivative, the contribution from $d = 6$ in the region $|z-1| \le (5/4)/\log MD \le 1/10^5$ is (using Corollary 8.2.4)

$$\le \frac{6!}{6!} \cdot \big|F(z)L_{S^2f}(2z)E_f(z)(\log MD)^6 \cdot (MD)^{z-1}\big| \\ \le 2.001|z-1| \cdot 1.001 L_{S^2f}(2) \cdot 1.001 E_f(1) \cdot \exp(5/4)(\log MD)^6 \\ \le 8.75 \cdot L_{S^2f}(2)E_f(1)(\log MD)^5 \le 0.73 \cdot \frac{6!}{5!} \cdot 2L_{S^2f}(2)E_f(1)(\log MD)^5,$$

which we denote as $0.73B_6$, while the principal contribution with $d = 5$ is

$$\le \frac{6!}{5!} \cdot \big|F'(z)L_{S^2f}(2z)E_f(z)(\log MD)^5 \cdot (MD)^{z-1}\big| \\ \le 6 \cdot 2.001 \cdot 1.001 L_{S^2f}(2) \cdot 1.001 E_f(1) \cdot \exp(5/4)(\log MD)^5 \\ \le 3.50 \cdot \frac{6!}{5!} \cdot 2L_{S^2f}(2)E_f(1)(\log MD)^5 = 3.50 B_6.$$

The other $d = 5$ terms give a contribution

$$\le \frac{6!}{5!} \cdot 2.001|z-1| \cdot L_{S^2f}(2)E_f(1)\exp(5/4)(\log MD)^5 \cdot \Big(202 + \frac{\log \tilde{D}}{955}\Big) \le B_6/271,$$

while the main $d = 4$ term from $(1,0,1,4)$ is $\le 3.5 \cdot (4/955)B_6$, etc. Adding these various contributions up, we get the stated bound $\le 4.5B_6$ in (**C**). ☐

9.3. Now we state and prove our main consequence of the Deuring decomposition.

Lemma 9.3.1. *Suppose that f and $f\chi$ are of odd parity and f has analytic rank at least 3, and that $L_f(s)$ has an additional pair of zeros $1 \pm i\kappa$ with κ either real or imaginary (with possibly $\kappa = 0$) and $|\kappa| \le 1/10^5$. Suppose that $D \ge 4\pi^2 \exp(10^7)$ with $D \ge N_f^9$ and $\sqrt{D}L_\chi(1) \le (\log D)^3/10^6$, and the conditions* (**A**)*,* (**B**)*, and* (**C**) *above are met, along with the bounds of Lemmata 8.1.1 and 8.1.3 for f (these bounds are met for the 6 curves in Table 3). Then with $MD = \sqrt{N_f N_{f\chi}}/4\pi^2$ so that $1 \le 4\pi^2 M \le N_f$ we have*

$$117\sqrt{N_f} \cdot \mathcal{W}(f) \cdot \sqrt{D}L_\chi(1)\mathcal{R}(\chi) \ge L_{S^2f}(2)E_f(1) \cdot \min\Big(\frac{(\log MD)^3}{(5/4)^2}, \frac{\log MD}{|\kappa|^2}\Big).$$

Here $\mathcal{W}(f) = \mathcal{U}(f)\mathcal{V}(f)$, with $\mathcal{U}(f)$ and $\mathcal{V}(f)$ defined in §4.3.3 (each being $\le \zeta(2)^2$), and $\mathcal{R}(\chi)$ is also defined in §4.3.3 (as a reciprocal sum over small representations).

Proof. We will apply the Deuring decomposition of Proposition 6.1.1 with $\tilde{r} = 4$.

9.3.2. We first consider the case where $|\kappa| \le (5/4)/\log MD$. In this range we will essentially use that

$$\sin\xi - \xi = -\frac{\xi^3}{3!}\Bigl[1 + \Theta\Bigl(\frac{|\xi|^2}{5!/3!}\Bigr) + \Theta\Bigl(\frac{|\xi|^4}{7!/3!}\Bigr) + \Theta\Bigl(\frac{|\xi|^6}{9!/3!}\Bigr) + \cdots\Bigr]$$

where $\xi = \kappa \log MD$ has $|\xi| \le 5/4$, so that the bracketed term is $[1+\Theta(0.0811)]$. Indeed, here it does not matter whether κ is on either the real or imaginary axis or not (also, $\kappa = 0$ can be seen to be allowable by continuity).

Upon replacing $0 = \tilde\Lambda_f^K(1+i\kappa)$ by the Deuring decomposition of Proposition 6.1.1 and using the Taylor expansion of T_f about 1 we obtain

$$0 = 2\frac{T_f^{(4)}(1)}{4!}|\kappa|^4 + \Theta\Bigl(2\cdot\frac{\hat T_f^{(6)}}{6!}|\kappa|^6\Bigr) + \Theta\bigl(2^4\cdot 30.53\sqrt{M}\cdot\mathcal{W}(f)\sqrt{D}L_\chi(1)\mathcal{R}(\chi)|\kappa|^4\bigr)$$

where $\hat T_f^{(6)}$ is a bound for the sixth derivative of T_f in $|z-1| \le (5/4)/\log MD$. By the above bounds from (**B**) and (**C**) we find that the fourth derivative dominates between the two first terms, and thus the final term must be large, namely

$$\Bigl(2\cdot\frac{1-1/200}{3!} - 2\cdot\frac{4.5}{5!}\cdot(5/4)^2\Bigr)\cdot 2L_{S^2f}(2)E_f(1)(\log MD)^3|\kappa|^4$$
$$\le (16\cdot 30.53/2\pi)\sqrt{N_f}\cdot\mathcal{W}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)|\kappa|^4.$$

The prepending term in parentheses is ≥ 0.214, so by rearrangement we achieve the first part of the minimum in the Lemma, since $(16\cdot 30.53/2\pi)/0.428/(5/4)^2 \le 117$.

9.3.3. When $|\kappa| \ge (5/4)/\log MD$ we split into cases depending on whether κ is real or not (note that $|\kappa| \le 1/10^5$ still). In the former we will use $|\sin\xi| \le 4\xi/5$ for $\xi \ge 5/4$, and in the latter (essentially) that $\sinh\xi \ge 1.28\xi$ in this range.

For κ real we can take $\kappa \ge 0$ by symmetry and we have

$$|T_f(1+i\kappa)| = |\Gamma(1+i\kappa)|^2\cdot\Bigl|\frac{L_{S^2f}(2+2i\kappa)}{\zeta(1+2i\kappa)}\Bigr|\cdot|E_f(1+i\kappa)|.$$

We have $|\Gamma(1+i\kappa)| \le 1$ and $|1/\zeta(1+2i\kappa)| \le 2\kappa$, while $|L_{S^2f}(2+2i\kappa)| \le 1.001L_{S^2f}(2)$ by Lemma 8.1.1, and by Lemma 8.2.3 we have

$$\frac{|E_f(1+i\kappa)|}{E_f(1)} \le \exp\Bigl(\int_0^\kappa\Bigl|\frac{E_f'}{E_f}(1+it)\Bigr|\,\partial t\Bigr) \le \exp\bigl(25\kappa\log\log D\bigr).$$

When $\kappa \le 1/\sqrt{\log D}$ we use $D \ge 4\pi^2\exp(10^7)$ to see that this is bounded as

$$|E_f(1+i\kappa)|/E_f(1) \le 1 + 27\kappa\log\log D \le 1 + 0.00005\kappa\log D.$$

Otherwise, when $\kappa \ge 1/\sqrt{\log D}$ we use $\kappa \le 1/10^5$ and from $D \ge 4\pi^2\exp(10^7)$ have

$$|E_f(1+i\kappa)|/E_f(1) \le \exp\Bigl(\frac{25}{10^5}(\log\log D)\Bigr) \le 0.00032\sqrt{\log D} \le 0.00032\kappa\log D.$$

We have that $1 \le (4/5)\kappa\log MD$, and thus

$$|E_f(1+i\kappa)|/E_f(1) \le 0.800\kappa\log MD + 0.00032\kappa\log D \le 0.801\kappa\log MD.$$

Putting these together, we find that

$$\bigl|T_f(1+i\kappa)\bigr| \le 2.002\kappa L_{S^2f}(2)E_f(1)\cdot 0.801\kappa\log MD.$$

We have $0 = \tilde\Lambda_f^K(1+i\kappa) = T_f(1+i\kappa) + T_f(1-i\kappa) - S_f^4(1+i\kappa) + U_f(1+i\kappa)$, with $S_f^4(1+i\kappa) = T_f''(1)(i\kappa)^2$ as in (20). Using (**A**) we have

$$|T_f''(1)| \ge 3.994L_{S^2f}(2)E_f(1)\log MD.$$

From this, since $3.994 - 2(2.002\cdot 0.801) \ge 0.786$ we get

$$|U_f(1+i\kappa)| \ge |T_f''(1)|\kappa^2 - 2|T_f(1+i\kappa)| \ge 0.786\kappa^2L_{S^2f}(2)E_f(1)\log MD,$$

and since $|U_f(1+i\kappa)| \le 78\sqrt{N_f}\cdot\mathcal{W}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\kappa^4$ as above, the second part of the minimum in the Lemma statement follows as before since $78/0.786 \le 100 < 117$.

9.3.4. Finally, if the pair of zeros of $L_f(s)$ with $|\kappa| \geq (5/4)/\log MD$ is on the real axis, we can take $i\kappa \geq 0$ by symmetry. By the second part of Lemma 8.2.3 we have

$$\begin{aligned}|T_f(1+i\kappa)| &= (MD)^{|\kappa|} \cdot \left|\Gamma(1+i\kappa)^2 \frac{L_{S^2 f}(2+2i\kappa)}{\zeta(1+2i\kappa)} E_f(1+i\kappa)\right| \\ &\geq 0.999^2 \cdot 1.999|\kappa| \cdot 0.999 L_{S^2 f}(2) \cdot (MD)^{|\kappa|} \cdot E_f(1) \exp\bigl(-25|\kappa| \log\log D\bigr) \\ &\geq 1.993|\kappa| L_{S^2 f}(2) \cdot E_f(1) \cdot e^{5/4} \frac{|\kappa| \log MD}{5/4} \Bigl[1 - \frac{20 \log\log D}{\log \tilde{D}}\Bigr],\end{aligned}$$

where with $\lambda = (|\kappa| \log MD)/(5/4)$ we used that for $\lambda \geq 1$ we have

$$(MD)^{|\kappa|} e^{-25|\kappa| \log\log D} = e^{5\lambda/4} \exp\Bigl(-25 \frac{4\lambda}{5} \frac{\log\log D}{\log MD}\Bigr) \geq e^{5/4} \lambda \Bigl[1 - 25 \frac{4}{5} \frac{\log\log D}{\log MD}\Bigr].$$

Similarly, we have an upper bound for $|T_f(1-i\kappa)|$ of

$$\begin{aligned}|T_f(1-i\kappa)| &\leq 1.001^2 \cdot 2.001|\kappa| \cdot 1.001 L_{S^2 f}(2) \cdot (MD)^{-|\kappa|} \cdot E_f(1) \exp\bigl(25|\kappa| \log\log D\bigr) \\ &\leq 2.008|\kappa| \cdot L_{S^2 f}(2) E_f(1) \cdot e^{-5/4} \frac{|\kappa| \log MD}{5/4} \Bigl[1 + \frac{21 \log\log D}{\log \tilde{D}}\Bigr].\end{aligned}$$

Subtracting the second from the first and using $D \geq 4\pi^2 \exp(10^7)$ gives us

$$|T_f(1+i\kappa)| - |T_f(1-i\kappa)| \geq (5.565 - 0.461)|\kappa|^2 (\log MD) \cdot L_{S^2 f}(2) E_f(1).$$

Meanwhile, from (**A**) we have

$$|S_f^4(1+i\kappa)| \leq 4.006|\kappa|^2 \cdot L_{S^2 f}(2) E_f(1) \log MD,$$

so that $\tilde{\Lambda}_f(1+i\kappa) = 0$ and $(5.565 - 0.461) - 4.006 = 1.098$ together imply

$$|U_f(1+i\kappa)| \geq 1.098|\kappa|^2 (\log MD) \cdot L_{S^2 f}(2) E_f(1),$$

and we conclude as before since $78/1.098 \leq 72 < 117$. □

10. Modular forms root numbers and an effective lower bound for $L_\chi(1)$

We now describe various conditions that ensure that $f\chi$ has odd parity, so that we can then apply the above Lemma 9.3.1.

10.1. Let us first recall Oesterlé's explicit result [56] for class numbers of imaginary quadratic fields, which he deduces from the rank 3 elliptic curve of conductor 5077.

There are three possibilities for $\chi(5077)$. When this is $+1$ the class number is bounded below (by Lemma 5.1.1) as $h_K \geq 1 + 2\lfloor (\log \sqrt{D/4})/\log 5077 \rfloor \geq (\log D)/55$. When it is -1 the twist $f\chi$ has odd parity and he obtains[22]

$$h_K \geq \frac{\log D}{55} \prod_{p|D} \Bigl(1 - \frac{\lfloor 2\sqrt{p} \rfloor}{p+1}\Bigr).$$

However, when $5077|D$ we have limited control over the root number, and it appears difficult to obtain a similar result. The above can be replicated for any rank 3 curve whose conductor has an odd number of prime factors (counting multiplicity), but will generically have a condition about D being coprime to the conductor.

[22]One can be more clever and include the explicit a_p for the rank 3 curve rather than $2\sqrt{p}$, but this has little effect on the worst case when there are $\approx (\log\log D)/(\log 2)$ primes dividing D.

10.1.1. One way to get a uniform result is to exploit the greater control over the root number that exists in quadratic twist families. For instance, Gross and Zagier point out that one can use the -139th quadratic twist of the elliptic curve 37b (this has rank 3).[23] As above, when $\chi(37)=+1$ we are done, and when $f\chi$ has odd parity we have [56, §5.1]

$$h_K \geq \frac{\log D}{7000}\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

Moreover, a calculation with root numbers [56, §4.3] shows that this condition of odd parity holds whenever $\chi(37)\neq +1$ (including when $37|D$ or $139|D$). The smallness here of $1/7000$ compared to $1/55$ is principally because of the larger conductor.

10.1.2. Our lower bound will be much bigger than $(\log\sqrt{D/4})/(\log N_f)$, and so we must necessarily employ at least two such curves (rather than just one) in exploiting the possible splitting of small primes; we actually will utilize three curves (in each case) to make cleaner statements of our results. We also need our curves to have rank 5, forcing the conductors to be larger. The case of real quadratic fields flips the parity of the number of prime factors of the conductor of the curve that is twisted, but is otherwise analogous.

10.2. We will ultimately consider the modular forms f associated to the 6 curves in Table 3. Our goal is to show that $f\chi$ has odd parity under suitable conditions, so as to place ourselves in the situation where either: there are three small split primes; or $f\chi$ indeed has odd parity and we can use Lemma 9.3.1. However, we will derive the results below about root numbers in more generality than just these f.

10.2.1. We assemble various lemmata concerning root numbers of L-functions of modular forms, most of which are initially due to Atkin and Lehner [1]. Alternatively, as we only work in weight 2, this could be done in terms of root numbers of elliptic curves. We readily admit our presentation is somewhat specific to our case of interest where f itself has odd parity and we wish $f\chi$ to also have odd parity.

We write $\epsilon_p(F)$ for the local root number of a modular newform F at p, noting that $\epsilon_p(F)=+1$ when F has good reduction at p. For the global root number $\epsilon(F)$ there is a product formula over all bad primes and the place at infinity; as we are in weight 2 we have $\epsilon_\infty(F)=i^2=-1$ for the latter. More precisely, for each bad prime q there is an involution W_q such that $W_q(F)=\pm F$ (see [1, Theorem 3]); multiplying these for all $q|N_F$ gives the sign of the Fricke involution on F; and then in Hecke's derivation of the functional equation there is an additional factor of i^k where k is the weight. We will mainly be interested in how the root number varies in quadratic twist families, which will simplify some aspects of this.

For odd primes p we define $p^\star$ as $p\cdot(-1)^{(p-1)/2}$ so that $p^\star$ is always a fundamental discriminant. In the context of an even fundamental discriminant t, we define $2^\star$ by dividing t out by $p^\star$ for all its odd prime factors p. We write ψ_u for the quadratic character corresponding to $\mathbf{Q}(\sqrt{u})$, and note $\psi_u(-1)=+1$ for $u>0$ while $\psi_u(-1)=-1$ for $u<0$.

Lemma 10.2.2. *Let g be a weight 2 newform of level l, and t be a fundamental discriminant. Then $\epsilon_p(g\psi_t)=\psi_{p^\star}(-1)$ for $p|t$, while $\epsilon_p(g\psi_t)=\epsilon_p(g)\psi_t(p)$ for $p|l$ with $p\nmid t$.*

Proof. This appears in Atkin and Lehner [1, Lemma 30, Theorem 6]. □

Lemma 10.2.3. *Let g be a newform of weight 2 and squarefree level l. Let B be a fundamental discriminant coprime to l with the same sign as D, and let $f=g\psi_B$. With $G=\gcd(D,l)$ and $[\psi_B\chi]$ the primitive inducing character of $\psi_B\chi$, we have*

$$\epsilon(f\chi)=\epsilon(g\psi_B\chi)=[\psi_B\chi](-l/G)\frac{\epsilon(g)}{\prod_{p|G}\epsilon_p(g)}.$$

[23] This curve also has some very specific additional properties which ease the proof that the Heegner point is torsion (and thus the analytic rank is 3).

Proof. When $p|B$ and $p|D$ and p is odd we have $\epsilon_p(f\chi) = \epsilon_p(g\psi_B\chi) = +1$ since $f\chi$ has good reduction at p. When B and D are both even, we use the above Lemma 10.2.2 to get $\epsilon_2(f\chi) = \epsilon_2(g\psi_B\chi) = \psi_u(-1)$ where $u \in \{1, 8\}$ (since B and D have the same sign), so that $\epsilon_2(f\chi) = +1$ also in this case.

When $p|B$ or $p|D$ (but not both) we have $\epsilon_p(f\chi) = \epsilon_p(g\psi_B\chi) = \psi_{p^\star}(-1)$, where this is suitably interpreted for $p = 2$ as above. Otherwise when $p|l$ and $p \nmid BD$ we have $\epsilon_p(f\chi) = \epsilon_p(g)[\psi_B\chi](p)$. These again follow from the previous Lemma 10.2.2.

The primes with $p|BD$ thus give a factor of $\psi_{BD}(-1) = [\psi_B\chi](-1)$, and the primes with $p|l$ and $p \nmid BD$ yield $\prod_p \epsilon_p(g) \cdot [\psi_B\chi](l/G)$ where the product is over $p|l$ with $p \nmid D$. Upon noting $\epsilon_\infty(f\chi) = \epsilon_\infty(g) = -1$ the result follows. □

10.2.4. We now want to set up situations where we know that various twists will have odd parity. The case of imaginary quadratic K is slightly easier.

Lemma 10.2.5. *Let f be the Bth quadratic twist of a weight 2 newform g of prime level q with $\epsilon(g) = +1$, where B is a negative fundamental discriminant with $\psi_B(-q) = -1$. Suppose $\chi(q) \neq +1$ and $\chi(-1) = -1$. Then $f\chi$ has odd parity.*

Proof. There are two cases. When $\chi(q) = -1$ then we have $G = \gcd(D, q) = 1$ and the previous Lemma 10.2.3 gives

$$\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-q)\epsilon(g) = -\chi(-q)\epsilon(g) = \chi(q)\epsilon(g) = -\epsilon(g) = -1.$$

When $\chi(q) = 0$ we have $G = \gcd(D, q) = q$ and again by Lemma 10.2.3 we get

$$\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-1)\frac{\epsilon(g)}{\epsilon_q(g)} = (-1)^2 \cdot \epsilon_\infty(g) = -1,$$

since $\epsilon_q(g) = -1$ for g to originally have even parity. □

Our cases are $(g, B) \in \{(11a, -25351367), (17a, -19502039), (19a, -16763912)\}$.

Lemma 10.2.6. *Let f be the Bth quadratic twist of a newform g of weight 2 and level m with $m = p_1p_2$ a product of 2 distinct odd primes, with $\epsilon_{p_1}(g) = \epsilon_{p_2}(g) = -1$, and B a positive fundamental discriminant with $\psi_B(p_1) = \psi_B(p_2) = +1$. Suppose that $\chi(p_1) \neq +1$ and $\chi(p_2) \neq +1$ and $\chi(-1) = +1$. Then $\epsilon(f\chi) = -1$.*

Proof. There are four possibilities for $G = \gcd(D, p_1p_2)$. When this is trivial, we then have $\chi(p_1) = \chi(p_2) = -1$ so that $\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-p_1p_2)\epsilon(g) = -1$, where we used Lemma 10.2.3 and $\epsilon(g) = -\epsilon_{p_1}(g)\epsilon_{p_2}(g) = -1$.

When $G = p_1$ we again use Lemma 10.2.3 and have

$$\begin{aligned}\epsilon(f\chi) = \epsilon(g\psi_B\chi) &= [\psi_B\chi](-p_2) \cdot \epsilon(g)/\epsilon_{p_1}(g) \\ &= \psi_B(p_2)\chi(p_2) \cdot (-1)/(-1) = \chi(p_2) = -1,\end{aligned}$$

and by symmetry the same calculation suffices when $G = p_2$.

Finally, when $G = p_1p_2$ we have $\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-1) \cdot \epsilon_\infty(g) = -1$. □

Our cases here are $(g, B) \in \{(91b, 6350941), (123a, 5467960), (209a, 3217789)\}$.

10.3. Using the calculations of §7 as done in [74], we can show our effective lower bound for $L_\chi(1)$. In particular, we have $|\tilde{\Lambda}_f'''(1)/3!| \leq 10^{-1025}$ for each f in Table 3, so that Lemma 7.1.4 implies $|\kappa|^2 \leq 10^{-1027}$ for the additional two zeros $1 \pm i\kappa$ of $L_f(s)$. Thus by Lemma 9.3.1 when $f\chi$ has odd parity we have

$$117\sqrt{N_f} \cdot \mathcal{W}(f) \cdot \sqrt{D}L_\chi(1)\mathcal{R}(\chi) \geq L_{S^2f}(2)E_f(1) \cdot \min\Bigl(\frac{(\log MD)^3}{(5/4)^2}, \frac{\log MD}{10^{-1027}}\Bigr).$$

10.3.1. We wish to remove the effect of split primes from this, in order to write a final result in terms of $p|D$. For this we need to consider both $E_f(1)$ and $\mathcal{R}(\chi)$.

Firstly, we can note from Table 7 that $E_f^{\mathrm{r}}(1) \geq 1$ in all cases. Meanwhile, the effect on $E_f^{\mathrm{m}}(1)$ from split primes is bounded via Lemma 5.2.1, with there again being at most 2 split primes up to 10^4 and 5 up to $\exp(\sqrt{Y})$ where $Y = \log(\sqrt{D}/2)$; by using the explicit bounds $c_f(2) \geq -2$ and $c_f(3) \geq -3$; we see the split primes contribute a factor

$$\geq 0.999 \cdot \frac{1 - 2/2 + 2/2^2}{1 + 2/2 + 2/2^2} \frac{1 - 3/3 + 3/3^2}{1 + 3/3 + 3/3^2} \biggl(\Bigl(\frac{1 - 1/\sqrt{10^4}}{1 + 1/\sqrt{10^4}} \Bigr)^2 \biggr)^3 \geq \frac{1}{39.51}.$$

This implies

$$E_f^{\mathrm{m}}(1) \geq \frac{1}{39.51} \prod_{p|D} \Bigl(1 + \frac{c_f(p)}{p} + \frac{p}{p^2} \Bigr).$$

Meanwhile, as in (19) of §4.3.3 we have

$$\mathcal{R}(\chi) \leq \prod_{p \leq \sqrt{D}/2} \frac{1 + 1/p}{1 - \chi(p)/p} \leq 6.01 \prod_{p|D} \Bigl(1 + \frac{1}{p} \Bigr),$$

where we bounded the effect of the split primes as ≤ 6.01. We can then proceed to include the primes with $p|D$ in $\mathcal{R}(\chi)$ with the analogous bound for $E_f^{\mathrm{m}}(1)$ upon noting the identity $(1 + c_f(p)/p + p/p^2)/(1 + 1/p) = 1 + c_f(p)/(p + 1)$, while $|c_f(p)| \leq \lfloor 2\sqrt{p} \rfloor$.

Combining all this, when $f\chi$ has odd parity (and $D \geq 4\pi^2 \exp(10^7)$) we have

$$\sqrt{D} L_\chi(1) \geq \frac{1/238}{117} \frac{L_{S^2 f}(2)}{\mathcal{W}(f)\sqrt{N_f}} \cdot \prod_{p|D} \Bigl(1 - \frac{\lfloor 2\sqrt{p} \rfloor}{p + 1} \Bigr) \cdot \min \Bigl(\frac{(\log MD)^3}{(5/4)^2}, \frac{\log MD}{10^{-1027}} \Bigr).$$

Theorem 10.3.2. *Suppose $D \geq 4\pi^2 \exp(10^7)$. Then*

$$\sqrt{D} L_\chi(1) \geq \min\bigl(10^{1000} \log D, (\log D)^3/10^{13} \bigr) \cdot \prod_{p|D} \Bigl(1 - \frac{\lfloor 2\sqrt{p} \rfloor}{p + 1} \Bigr).$$

Proof. First we consider the case where K imaginary. If $f\chi$ has even parity for the first 3 curves in Table 3, by Lemma 10.2.5 we have $\chi(11) = \chi(17) = \chi(19) = +1$, and thus $h_K \geq \lfloor (\log \sqrt{D/4}) / \log 19 \rfloor^3$ by Lemma 5.1.1, which suffices for the statement here.

Similarly, when K is real and $f\chi$ has even parity for each of the last 3 curves in Table 3, Lemma 10.2.6 then implies $\chi(p) = +1$ for at least one p in each of the three sets $\{7, 13\}$, $\{3, 41\}$, and $\{11, 19\}$, so $\sqrt{D} L_\chi(1) \geq (\log D)^3/10^{13}$ by Lemma 5.1.1.

Meanwhile, independent of whether K is real or imaginary, when $f\chi$ has odd parity for (at least) one of the 3 applicable curves we can use Lemma 9.3.1 and get

$$\sqrt{D} L_\chi(1) \geq \frac{1/238}{117} \frac{L_{S^2 f}(2)}{\mathcal{W}(f)\sqrt{N_f}} \cdot \prod_{p|D} \Bigl(1 - \frac{\lfloor 2\sqrt{p} \rfloor}{p + 1} \Bigr) \cdot \min \Bigl(\frac{(\log \tilde{D})^3}{(5/4)^2}, 10^{1027} \log \tilde{D} \Bigr).$$

For each curve in Table 3 except the twist of 19a, the quotient $L_{S^2 f}(2)/\mathcal{W}(f)\sqrt{N_f}$ is sufficiently large to yield the statement of the Theorem (the worse case is 17a, when the comparison is $9.18 \cdot 10^{12} < 10^{13}$). Moreover, as the twist of 19a is by the even discriminant -16763912, therein we can explicitly use $c_f(p) = (0, -2, -3)$ for $p = (2, 3, 5)$ to improve the lower bound on $E_f^{\mathrm{m}}(1)$, thus obtaining the claimed constant of 10^{13}. □

11. Error bounds for degree 1 Deuring approximations

The previous Section essentially ends the first part of the paper, having shown an improved effective lower bound on $L_\chi(1)$. We now turn to consequences of this. Our first task to give a version of Proposition 3.3.1 for the case of degree 1 L-functions, with a different Mellin transform involved in the weighting.

11.1. For $x>0$ we define the positive function

$$\tilde{I}(x)=\int_{(2)} x^{-s}\frac{\Gamma(s)}{s-1/2}\frac{\partial s}{2\pi i}=\frac{1}{\sqrt{x}}\int_{(2)} x^{-s+1/2}\frac{\Gamma(s)}{s-1/2}\frac{\partial s}{2\pi i}$$
$$=\frac{1}{\sqrt{x}}\int_x^\infty\int_{(2)} u^{-s-1/2}\Gamma(s)\frac{\partial s}{2\pi i}\partial u=\frac{1}{\sqrt{x}}\int_x^\infty e^{-u}\frac{\partial u}{\sqrt{u}}.$$

We split $R_K^\star(n)=R_K^{\leq}(n)+R_K^{>}(n)$ at $\sqrt{D}/2$, with $R_K^\star(n)=R_K^{\mathrm{m}}(n)+R_K^{\tilde{\mathrm{m}}}(n)$ where $R_K^{\mathrm{m}}(n)$ is the multiplicity of n in the multiset of leading coefficients of reduced forms.

11.1.1. We first consider the imaginary case.

Lemma 11.1.2. *For $X>0$ and $\Delta<0$ with $D>4$ we have*

$$\sum_{m=1}^\infty\sum_{n=1}^\infty R_K^{>}(n)\tilde{I}(m^2n/X)\leq(2+2.318+0.492)\cdot XL_\chi(1)=4.810\cdot XL_\chi(1).$$

Proof. Rather than directly consider $R_K^{>}(n)$, we find it is easier to work with $R_K^{\tilde{\mathrm{m}}}(n)$, and therein $\zeta(2s)\sum_n R_K^{\tilde{\mathrm{m}}}(n)/n^s=\zeta_K(s)-\sum_{\langle a\rangle}\zeta(2s)/a^s$.

By numerical integration on the 1-line we have $|\tilde{I}(x)|\leq 0.469/x$, so that

$$\sum_{m=1}^\infty\sum_{n=1}^\infty R_K^{>}(n)\tilde{I}(m^2n/X)\leq\sum_{m=1}^\infty\sum_{n=1}^\infty R_K^{\tilde{\mathrm{m}}}(n)\tilde{I}(m^2n/X)+\sum_{m=1}^\infty\sum_{\substack{\langle a,b,c\rangle\\ a\geq\sqrt{D}/2}}\tilde{I}(am^2/X)$$
$$\leq\sum_{m=1}^\infty\sum_{n=1}^\infty R_K^{\tilde{\mathrm{m}}}(n)\tilde{I}(m^2n/X)+0.469\zeta(2)h_K\frac{X}{\sqrt{D}/2},$$

while by Dirichlet's class number formula $h_K/\sqrt{D}=L_\chi(1)/\pi$, so the second term here gives the 0.492 contribution in the result.

We have

$$\sum_{m=1}^\infty\sum_{n=1}^\infty R_K^{\tilde{\mathrm{m}}}(n)\tilde{I}(m^2n/X)=\int_{(2)}X^s\Bigl[\zeta_K(s)-\sum_{\langle a,b,c\rangle}\frac{\zeta(2s)}{a^s}\Bigr]\frac{\Gamma(s)}{s-1/2}\frac{\partial s}{2\pi i}.$$

11.1.3. We then use Lemma 3.4.2 to decompose $\zeta_K(s)$, and find the contribution T_{i} from the main term therein is

$$T_{\mathrm{i}}=\frac{\pi}{\sqrt{D}}\int_{(2)}X^s\sum_{\langle a\rangle}\Bigl(\frac{D}{4a}\Bigr)^{1-s}\frac{s}{s-1}\frac{\Gamma(s)}{s-1/2}\frac{\partial s}{2\pi i}.$$

We replace $s\Gamma(s)=\Gamma(s+1)$ and expand this as $\int_0^\infty u^{s+1}e^{-u}\partial u/u$ to get

$$T_{\mathrm{i}}=\frac{\pi}{\sqrt{D}}\sum_{\langle a\rangle}\frac{D}{4a}\int_{(2)}\int_0^\infty u\Bigl(Xu\frac{4a}{D}\Bigr)^s\frac{\partial u}{e^u u}\frac{\partial s/2\pi i}{(s-1)(s-1/2)}.$$

We switch the order of integration and note

$$\int_{(2)}\frac{y^s\,\partial s/2\pi i}{(s-1)(s-1/2)}=\begin{cases}2y-2\sqrt{y} & \text{for } y\geq 1,\\ 0 & \text{for } y\leq 1,\end{cases}$$

so that this integral is nonnegative and $\leq 2y$. The contribution here is thus bounded as

$$|T_{\mathrm{i}}|\leq\frac{2\pi}{\sqrt{D}}\sum_{\langle a,b,c\rangle}\int_0^\infty u(Xu)\frac{\partial u}{e^u u}=\frac{2\pi h_K}{\sqrt{D}}X=2L_\chi(1)X.$$

For the secondary term we move the contour to $\sigma=1$ and it contributes

$$\leq X\int_{(1)}\frac{|s|}{\sigma-1/2}\left|\frac{\Gamma(s)}{s-1/2}\right|\frac{|\partial s|}{2\pi}\cdot\sum_{\langle a,b,c\rangle}\Bigl(1+\frac{\sqrt{D}}{a}\Bigr)\Bigl(\frac{4a}{D}\Bigr)\leq 1.154X\cdot\frac{h_K}{\sqrt{D}}\Bigl(\frac{4}{\sqrt{3}}+4\Bigr),$$

and as $1.154\cdot 6.31/\pi\leq 2.318$ this gives the second term in the result. ∎

11.1.4. Next we handle the real case, with the proof following that for Proposition 3.7.6.

Lemma 11.1.5. *For $X > 0$ and $\Delta > 0$ with $D \geq 5$ we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_K^{>}(n)\tilde{I}(m^2n/X) \leq (2 + 3.830 + 1.604)\cdot XL_\chi(1) = 7.434\cdot XL_\chi(1).$$

Proof. We again commence by replacing $R_K^{>}(n)$ by $R_K^{\tilde{m}}(n)$, and bound the error therein via $|\tilde{I}(x)| \leq 0.469/x$, so that by Lemmata A.1.3 and A.1.9 we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_K^{>}(n)\tilde{I}(m^2n/X) &\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_K^{\tilde{m}}(n)\tilde{I}(m^2n/X) + \sum_{m=1}^{\infty}\sum_{\substack{a\in\bar{\mathcal{M}}_\Delta^+\\ a\geq\sqrt{D}/2}} \tilde{I}(am^2/X)\\ &\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_K^{\tilde{m}}(n)\tilde{I}(m^2n/X) + 0.469\zeta(2)h_K\frac{\log\epsilon_0}{\log\bigl(\frac{1+\sqrt{5}}{2}\bigr)}\frac{X}{\sqrt{D}/2},\end{aligned}$$

while by Dirichlet's class number formula $\sqrt{D}L_\chi(1) = 2h_K\log\epsilon_0$, so the second term here gives the 1.604 contribution in the result.

We then expand out $\tilde{I}(m^2n/X)$ to get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_K^{\tilde{m}}(n)\tilde{I}(m^2n/X) + \sum_{m=1}^{\infty}\sum_{a\in\bar{\mathcal{M}}_\Delta^+} \tilde{I}(m^2a/X) = \int_{(2)} X^s\frac{\Gamma(s)}{(s-1/2)}\zeta_K(s)\frac{\partial s}{2\pi i},$$

and proceed to insert our above expression for $\zeta_K(s)$ from Lemma 3.6.8, and we write the resulting decomposition as $V_{\mathrm{r}} + T_{\mathrm{r}} + U_{\mathrm{r}}$, so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_K^{\tilde{m}}(n)\tilde{I}(m^2n/X) = -\sum_{m=1}^{\infty}\sum_{a\in\bar{\mathcal{M}}_\Delta^+} \tilde{I}(m^2a/X) + V_{\mathrm{r}} + T_{\mathrm{r}} + U_{\mathrm{r}}. \tag{24}$$

11.1.6. We first note V_{r} is smaller than $\sum_m\sum_a \tilde{I}(m^2a/X)$. Indeed we have

$$V_{\mathrm{r}} = \int_{(2)} X^s\frac{\Gamma(s)/2}{\Gamma(s/2)^2}\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star}\frac{2\zeta(2s)}{\alpha_i^s}\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}\frac{\Gamma(s)}{(s-1/2)}\frac{\partial s}{2\pi i},$$

and can initially use the duplication formula to obtain the nonnegativity result (for $w > 0$)

$$\begin{aligned}\int_{(2)} w^s\frac{\Gamma(s)^2}{\Gamma(s/2)^2}\frac{\partial s/2\pi i}{s-1/2} &= \int_{(2)}\frac{(4w)^s}{4\pi}\Gamma(s/2+1/2)^2\frac{\partial s/2\pi i}{s-1/2}\\ &= \int_{(2)}\frac{(4w)^s}{4\pi}\frac{\Gamma(s/2+1/2)\Gamma(s/2+3/2)}{(s-1/2)(s/2+1/2)}\frac{\partial s}{2\pi i}\\ &= \int_0^\infty\int_0^\infty\int_{(2)}\frac{u_1^{s/2}u_2^{s/2}(4w)^s/4\pi}{(s-1/2)(s/2+1/2)}\frac{\partial s}{2\pi i}\frac{\partial u_1}{e^{u_1}\sqrt{u_1}}\frac{\partial u_2}{e^{u_2}/\sqrt{u_2}} \geq 0,\end{aligned}$$

as the s-integral $\int_{(2)}\frac{u^s\partial s/2\pi i}{(s-1/2)(s/2+1/2)}$ is 0 for $u \leq 1$ and is $4\sqrt{u}/3 - 4/3u$ for $u \geq 1$.

We then re-arrange V_{r}, writing $\mathcal{I}_i = [\lambda_{i-1}H_{i-1}^\star, \lambda_{i-1}H_i^\star]$, and have

$$\begin{aligned}V_{\mathrm{r}} &= \sum_{m=1}^{\infty}\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\mathcal{I}_i}\int_{(2)}\Bigl(\frac{X}{m^2}\Bigr)^s\frac{\Gamma(s)/2}{\Gamma(s/2)^2}\cdot\frac{2}{|a_i|^s}\frac{1}{(\tilde{\varphi}_i+1/\tilde{\varphi}_i)^s}\frac{\Gamma(s)}{(s-1/2)}\frac{\partial s}{2\pi i}\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}\\ &\leq \sum_{m=1}^{\infty}\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_0^\infty\int_{(2)}\Bigl(\frac{X}{m^2}\Bigr)^s\frac{\Gamma(s)/2}{\Gamma(s/2)^2}\cdot\frac{2}{|a_i|^s}\frac{1}{(\tilde{\varphi}_i+1/\tilde{\varphi}_i)^s}\frac{\Gamma(s)}{(s-1/2)}\frac{\partial s}{2\pi i}\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}\\ &= \sum_{m=1}^{\infty}\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\frac{1}{2}\tilde{I}(|a_i|m^2/X) = \sum_{m=1}^{\infty}\sum_{a\in\bar{\mathcal{M}}_\Delta^+}\tilde{I}(am^2/X),\end{aligned}$$

as the nonnegativity of the s-integral implies that the $\tilde{\varphi}$-domain can be enlarged, and then we evaluated the $\tilde{\varphi}$-integral by (6). In particular, from (24) we thus have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_K^{\tilde{\mathrm{m}}}(n)\tilde{I}(m^2n/X) \le T_{\mathrm{r}} + U_{\mathrm{r}}.$$

11.1.7. We next consider the term

$$T_{\mathrm{r}} = \frac{\pi}{\sqrt{D}}\int_{(2)} X^s \sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star}\Bigl(\frac{D}{\alpha_i}\Bigr)^{1-s}\frac{\Gamma(s)/2}{\Gamma(s/2)^2}\frac{s}{s-1}\frac{\Gamma(s)}{s-1/2}\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}\frac{\partial s}{2\pi i}.$$

We use the duplication formula $\Gamma(s)/\Gamma(s/2) = 2^{s-1}\Gamma(s/2+1/2)/\sqrt{\pi}$ and divide out a factor of $(s/2+1/2)$ from $\Gamma(s/2+1/2) = \Gamma(s/2+3/2)/(s/2+1/2)$, so that

$$T_{\mathrm{r}} = \frac{1/8}{\sqrt{D}}\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\mathcal{I}_i}\int_{(2)}(4X)^s\Bigl(\frac{D}{\alpha_i}\Bigr)^{1-s}\frac{\Gamma(s/2+1/2)}{s-1/2}\frac{\Gamma(s/2+3/2)}{s/2+1/2}\frac{s}{s-1}\frac{\partial s}{2\pi i}\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}.$$

We then expand both Γ-functions to get

$$T_{\mathrm{r}} = \frac{1/8}{\sqrt{D}}\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\mathcal{I}_i}\frac{D}{\alpha_i}\int_0^\infty u_1^{1/2}\int_0^\infty u_2^{3/2}\times$$
$$\times\int_{(2)}\Bigl(4X\sqrt{u_1u_2}\frac{\alpha_i}{D}\Bigr)^s\frac{s/(s/2+1/2)}{(s-1)(s-1/2)}\frac{\partial s}{2\pi i}\frac{\partial u_1}{e^{u_1}u_1}\frac{\partial u_2}{e^{u_2}u_2}\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}.$$

We then note that

$$0 \le \int_{(2)} w^s\frac{s/(s/2+1/2)}{(s-1)(s-1/2)}\frac{\partial s}{2\pi i} \le 2w.$$

for all $w>0$. Indeed, when $w \le 1$ the integral is 0 by moving the contour to the right, while when $w \ge 1$ it is $2w - 4\sqrt{w}/3 - 2/3w$ by moving to the left. Thus this term is bounded as

$$|T_{\mathrm{r}}| \le \frac{1/8}{\sqrt{D}}\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\mathcal{I}_i}\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}\cdot\int_0^\infty u_1^{1/2}\int_0^\infty u_2^{3/2}\bigl(8X\sqrt{u_1u_2}\bigr)\frac{\partial u_1}{e^{u_1}u_1}\frac{\partial u_2}{e^{u_2}u_2},$$

and the u-integrals (with the factor of $8X$ pulled out) give $\Gamma(1)\Gamma(2)=1$, while

$$\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\int_{\mathcal{I}_i}\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i} = \sum_{\langle a,b,c\rangle}\int_\eta^{\eta\epsilon_0^4}\frac{\partial\varphi}{\varphi} = 4h_K\log\epsilon_0 = 2\sqrt{D}L_\chi(1).$$

This gives the first term (with “2”) in the Lemma.

11.1.8. For the term with $Z_{\mathrm{r}}(s)$ we move the integral to $\sigma=1$ for a bound

$$\le X\int_{(1)}\left|\frac{\Gamma(s)/2}{\Gamma(s/2)^2}\frac{\Gamma(s)}{s-1/2}\right|\frac{4|s|}{\sigma-1/2}\frac{|\partial s|}{2\pi}\cdot\sum_{\langle a,b,c\rangle}\sum_{i=4k}^{6k-1}\left[\int_{\lambda_{i-1}H_{i-1}^\star}^{\lambda_{i-1}H_i^\star}\Bigl(\frac{1}{2}\frac{\alpha_i}{D}+\frac{1}{\sqrt{D}}\Bigr)\frac{\partial\tilde{\varphi}_i}{\tilde{\varphi}_i}\right],$$

and using $\alpha_i \le 2.25\sqrt{D}$ this is

$$\le 0.901X\cdot(2.25/2+1)\cdot 2L_\chi(1) \le 3.830XL_\chi(1),$$

which is the second term in the result. ☐

We conclude by combining the above two results from Lemmata 11.1.2 and 11.1.5.

Lemma 11.1.9. *For $X>0$ and $\Delta>0$ with $D\ge 5$ we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_K^{>}(n)\tilde{I}(m^2n/X) \le 7.434\cdot XL_\chi(1).$$

11.2. We conclude the preliminary material for this part by introducing $E_\psi^{\mathbf{P}}(s)$ for a Dirichlet character ψ and set of primes $\mathbf{P}$, which is the analogue of $E_f(s)$ from §4.3.1.

11.2.1. We let ψ be a real primitive Dirichlet character of odd conductor k, and then write $[\psi\chi]$ for the primitive character inducing $\psi\chi$. Comparing Euler products gives

$$\frac{L_\psi^K(s)}{\zeta(2s)} = \prod_p \frac{1-1/p^{2s}}{(1-\psi(p)/p^s)(1-[\psi\chi](p)/p^s)} = \prod_{p\nmid k} \frac{1+\psi(p)/p^s}{1-(\psi\chi)(p)/p^s} \prod_{p|k} \frac{1-1/p^{2s}}{1-[\psi\chi](p)/p^s}$$
$$= \prod_{p\nmid k} \frac{1+\psi(p)/p^s}{1-(\psi\chi)(p)/p^s} \prod_{p|g}\Bigl(1+\frac{[\psi\chi](p)}{p^s}\Bigr) \prod_{p|(k/g)}\Bigl(1-\frac{1}{p^{2s}}\Bigr) = G(s)P_{k/g}(2s)$$

where $g=\gcd(k,D)$, while $P_u(s)=\prod_{p|u}\bigl(1-1/p^s\bigr)$ and

$$G(s)=\prod_p G_p(s) = \prod_{p\nmid k} \frac{1+\psi(p)/p^s}{1-(\psi\chi)(p)/p^s} \prod_{p|g}\Bigl(1+\frac{[\psi\chi](p)}{p^s}\Bigr) = \sum_{n=1}^\infty \frac{S(n)}{n^s},$$

where we note that $|S(n)|\le R_K^\star(n)$.

Given a set $\mathbf{P}$ of primes, we define $\bar{\mathbf{P}}_\le^\star$ to be the set containing all $n\le\sqrt{D}/2$ with no prime factor in $\mathbf{P}$ (including $n=1$), and $E_\psi^{\mathbf{P}}(s)$ and $r_\psi^{\mathbf{P}}(m)$ by[24]

$$E_\psi^{\mathbf{P}}(s) = \prod_{p\in\mathbf{P}} G_p(s) \sum_{n\in\bar{\mathbf{P}}_\le^\star} \frac{S(n)}{n^s} \quad\text{and}\quad \frac{L_\psi^K(s)}{\zeta(2s)} = \Bigl(E_\psi^{\mathbf{P}}(s) + \sum_{m=1}^\infty \frac{r_\psi^{\mathbf{P}}(m)}{m^s}\Bigr) \prod_{p|(k/g)}\Bigl(1-\frac{1}{p^{2s}}\Bigr).$$

Our initial choice of $\mathbf{P}$ will be the set of all primes $\le\sqrt{D}/2$, when the n-sum is just 1, and $E_\psi^{\mathbf{P}}(s)$ is a pure Euler product truncation as with $E_f(s)$ in §4.3.1. However, in §13.3.2 we employ an alternative selection for $\mathbf{P}$, namely all $p\le 10^4$ with $p|D$ (in particular avoiding split primes), in which case $E_\psi^{\mathbf{P}}(s)$ is essentially a sum-based truncation.

11.2.2. The crucial point (valid for any choice of $\mathbf{P}$) is that $r_\psi^{\mathbf{P}}(m)=0$ for $m\le\sqrt{D}/2$ and $|r_\psi^{\mathbf{P}}(m)|\le R_K^\star(m)$ in general, so that $|r_\psi^{\mathbf{P}}(m)|\le R_K^>(m)$. Indeed, we have

$$\sum_{m=1}^\infty \frac{r_\psi^{\mathbf{P}}(m)}{m^s} = G(s)-E_\psi^{\mathbf{P}}(s) = \prod_{p\in\mathbf{P}} G_p(s)\cdot\Bigl[\prod_{p\notin\mathbf{P}} G_p(s) - \sum_{n\in\bar{\mathbf{P}}_\le^\star} \frac{S(n)}{n^s}\Bigr],$$

where in the bracketed term the sum over n is the series truncation to $n\le\sqrt{D}/2$ of the preceding Euler product over $p\notin\mathbf{P}$, and upon calling this bracketed term $\sum_l b(l)/l^s$, we thus have $b(l)=0$ for $l\le\sqrt{D}/2$ while $|b(l)|\le|S(l)|\le R_K^\star(l)$ in general, whereupon multiplication by the product over $p\in\mathbf{P}$ implies the desired $|r_\psi^{\mathbf{P}}(m)|\le R_K^\star(m)$.

12. Using auxiliary moduli to bound $L_\chi(1)$ for mid-sized discriminants

We now proceed to show a lower bound on $L_\chi(1)$ for D with $10^3\le\log D\le 10^8$, namely that $L_\chi(1)\ge 100\log D$ in this range. As noted in the Introduction, this is largely a matter of generating enough Dirichlet L-functions that possess a zero of abnormally small height. The theory is similar to §5 of [71], though here we also have included the real quadratic case. (One could alternatively phrase some of the results herein in terms of a Deuring decomposition for $L_\psi^K(s)$, but we simply work at a known zero of $L_\psi(s)$).

12.1. We copy over Lemma 7 from [71], and its attendant upper bound.

Lemma 12.1.1. *Suppose $\xi\ge 0$ and $x>0$. Then*

$$\Bigl|\int_{(2)} x^{-s}\Gamma(s)\frac{s-1/2}{(s-1/2)^2+\xi^2}\,\frac{\partial s}{2\pi i}\Bigr| = \Bigl|\int_x^\infty e^{-u}\sqrt{\frac{u}{x}}\cos\Bigl(\xi\log\frac{u}{x}\Bigr)\,\frac{\partial u}{u}\Bigr|$$
$$\le \int_x^\infty e^{-u}\sqrt{\frac{u}{x}}\,\frac{\partial u}{u} = \int_{(2)} x^{-s}\frac{\Gamma(s)}{s-1/2}\,\frac{\partial s}{2\pi i} = \tilde{I}(x).$$

[24]One can note that our convention with r_ψ is somewhat different than with r_f in §4.3.1, as indeed $\zeta(2s)\sum_m r_\psi^{\mathbf{P}}(m)/m^s$ here corresponds to $\sum_n r_f^K(n)/n^s$ there.

As noted in the proof in [71], there is a minor issue with convergence but otherwise this follows by unravelling the Γ-function and swapping the integration order.

12.2. We let ψ be an auxiliary primitive real[25] Dirichlet character of odd conductor k with $\psi(-1) = -1$, and put $g = \gcd(k, D)$. We write $1/2 + i\xi_0$ for a low-height zero of $L_\psi(s)$ on the half-line.

We define $\zeta_u(s) = \zeta(s)\prod_{p|u}(1 - 1/p^s)$ and

$$T_\psi^{\mathbf{P}}(s) = \Gamma(s)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{s-1/2}\zeta_{k/g}(2s)E_\psi^{\mathbf{P}}(s),$$

with $E_\psi^{\mathbf{P}}(s)$ as in §11.2, where $\mathbf{P}$ is an arbitrary set of primes.

Lemma 12.2.1. *For $D \geq 5$ we have $M_1 + M_2 + M_3 = 0$ where*

$$M_1 = 2|T_\psi^{\mathbf{P}}(1/2 + i\xi_0)|\sin\big(\arg iT_\psi^{\mathbf{P}}(1/2 + i\xi_0)\big),$$

while

$$|M_2| \leq 14.868 L_\chi(1)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{1/2} \quad \textit{and} \quad |M_3| \leq 3.322 \cdot \tilde{E}_\psi^{\mathbf{P}}(1/4)\Big(\frac{2\pi g}{k\sqrt{D}}\Big)^{1/4}\prod_{p|(k/g)}\Big(1 + \frac{1}{\sqrt{p}}\Big),$$

where $\tilde{E}_\psi^{\mathbf{P}}(1/4)$ is a bound for $E_\psi^{\mathbf{P}}$ on the 1/4-line.

Proof. We consider the integral

$$\begin{aligned} 0 &= \Big(\int_{(2)} - \int_{(-1)}\Big)\Gamma(s)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{s-1/2}L_\psi(s)L_{\psi\chi}(s)\frac{(s-1/2)}{(s-1/2)^2 + \xi_0^2}\frac{\partial s}{2\pi i} \\ &= 2\int_{(2)}\Gamma(s)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{s-1/2}L_\psi^K(s)\frac{(s-1/2)}{(s-1/2)^2 + \xi_0^2}\frac{\partial s}{2\pi i}, \end{aligned}$$

with the first step by Cauchy's integral theorem since the integrand is entire, and the second step following by the symmetry of the functional equation for $L_\psi^K(s)$ (here we use that ψ is odd to obtain the stated Γ-factor, and by convention $L_{\psi\chi}$ is the L-function of the primitive inducing character).

We then use the notation of §11.2 to replace

$$L_\psi^K(s) = \zeta(2s)\Big(E_\psi^{\mathbf{P}}(s) + \sum_{n=1}^{\infty}\frac{r_\psi^{\mathbf{P}}(n)}{n^s}\Big)\prod_{p|(k/g)}\Big(1 - \frac{1}{p^{2s}}\Big).$$

We then move the contour to the left with $E_\psi^{\mathbf{P}}(s)$ for the main term, while bounding the contribution from the sum with $r_\psi^{\mathbf{P}}(n)$ by Mellin transforms.

12.2.2. The term M_1 induced from residues at $s = 1/2 \pm i\xi_0$ when moving the contour to the left is

$$\begin{aligned} &2\frac{i\xi_0}{2i\xi_0}\Gamma(1/2 + i\xi_0)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{i\xi_0}\zeta_{k/g}(1 + 2i\xi_0)E_\psi^{\mathbf{P}}(1/2 + i\xi_0) \\ &\qquad + 2\frac{-i\xi_0}{-2i\xi_0}\Gamma(1/2 - i\xi_0)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{-i\xi_0}\zeta_{k/g}(1 - 2i\xi_0)E_\psi^{\mathbf{P}}(1/2 - i\xi_0), \end{aligned}$$

which is $T_\psi^{\mathbf{P}}(1/2 + i\xi_0) + T_\psi^{\mathbf{P}}(1/2 - i\xi_0)$, and by symmetry this is then the real part of $2T_\psi^{\mathbf{P}}(1/2 + i\xi_0)$. Going further, it is thus twice its absolute value times the cosine of its argument. Multiplying by i changes the cosine to the sine, and so this becomes

$$M_1 = 2|T_\psi^{\mathbf{P}}(1/2 + i\xi_0)|\sin\big(\arg iT_\psi^{\mathbf{P}}(1/2 + i\xi_0)\big).$$

[25] It is possible to use nonreal characters, but using them seems unnecessarily complicated. (Similarly, we fix the Γ-factor by using only odd characters). It might be noted, however, that Biró's improvement [5] over a precursor method of Beck was largely due to utilizing nonreal characters.

12.2.3. The term M_2 coming from $r_\psi^{\mathbf{P}}(n)$ is

$$M_2 = 2\int_{(2)} \Gamma(s)\zeta(2s)\prod_{p|(k/g)}\Bigl(1-\frac{1}{p^{2s}}\Bigr)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{s-1/2}\sum_{n=1}^\infty \frac{r_\psi^{\mathbf{P}}(n)}{n^s}\frac{(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i}$$

and by expanding $\zeta(2s)$ this is

$$M_2 = 2\sqrt{\frac{2\pi g}{k\sqrt{D}}}\int_{(2)}\sum_{\substack{m=1\\(m,k/g)=1}}^\infty \frac{1}{m^{2s}}\sum_{n=1}^\infty \frac{r_\psi^{\mathbf{P}}(n)}{n^s}\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^s\frac{\Gamma(s)\,(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i}.$$

We then recall that $|r_\psi^{\mathbf{P}}(n)|\le R_K^{>}(n)$ from §11.2.2 and write the integral in terms of Mellin transforms, using Lemmata 12.1.1 and 11.1.9 to obtain the bound

$$\begin{aligned}|M_2| &\le 2\sqrt{\frac{2\pi g}{k\sqrt{D}}}\sum_{m=1}^\infty\sum_{n=1}^\infty R_K^{>}(n)\tilde{I}\Bigl(nm^2\frac{2\pi g}{k\sqrt{D}}\Bigr)\\ &\le 2\sqrt{\frac{2\pi g}{k\sqrt{D}}}\cdot\Bigl(7.434L_\chi(1)\frac{k\sqrt{D}}{2\pi g}\Bigr) = 14.868\cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2}.\end{aligned}$$

12.2.4. Finally, the term M_3 from the complementary integral with $E_\psi^{\mathbf{P}}(s)$ is

$$2\int_{(1/4)}\Gamma(s)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{s-1/2}E_\psi^{\mathbf{P}}(s)\zeta(2s)P_{k/g}(2s)\frac{(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i},$$

and since (using numerical integration)

$$2\int_{(1/4)}\biggl|\frac{\Gamma(s)\zeta(2s)(s-1/2)}{(s-1/2)^2+\xi_0^2}\biggr|\frac{|\partial s|}{2\pi}\le 2\int_{(1/4)}\biggl|\frac{\Gamma(s)\zeta(2s)}{(s-1/2)}\biggr|\frac{|\partial s|}{2\pi}\le 3.322,$$

the stated bound for M_3 readily follows.[26] □

12.2.5. We expand $2|T_\psi^{\mathbf{P}}(1/2+i\xi_0)|\sin\bigl(\arg iT_\psi^{\mathbf{P}}(1/2+i\xi_0)\bigr)$ in the notation of [71] as

$$\xi_3|E_\psi^{\mathbf{P}}(1/2+i\xi_0)|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathbf{P}}(1/2+i\xi_0)\bigr]$$

where $\xi_3 = 2|\Gamma(1/2+i\xi_0)\zeta_{k/g}(1+2i\xi_0)|$ and

$$\xi_2 = \xi_0\log\frac{k/g}{2\pi}+\arg\bigl[i\Gamma(1/2+i\xi_0)\zeta_{k/g}(1+2i\xi_0)\bigr]. \tag{25}$$

In particular, since $M_1+M_2+M_3=0$ this gives

$$\begin{aligned}&\xi_3|E_\psi^{\mathbf{P}}(1/2+i\xi_0)|\cdot\Bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathbf{P}}(1/2+i\xi_0)\bigr]\Bigr|\\ &\quad\le 14.868\cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2}+3.322\cdot\tilde{E}_\psi^{\mathbf{P}}(1/4)\Bigl(\frac{2\pi g}{k\sqrt{D}}\Bigr)^{1/4}\cdot\prod_{p|(k/g)}\Bigl(1+\frac{1}{\sqrt{p}}\Bigr).\end{aligned} \tag{26}$$

We can note that $\xi_3\approx\sqrt{\pi}/\xi_0$ in terms of the zero $1/2+i\xi_0$ of $L_\psi(s)$.

When $L_\chi(1)$ is small, say $\ll(\log D)/\sqrt{D}$, the second term on the right of (26) will dominate, with its $1/D^{1/8}$ behavior. We can improve this by moving the contour for M_3 further to the left, but that entails switching to a sum-based form of $E_\psi^{\mathbf{P}}$, as the Euler factors from split primes have poles on the 0-line. We revisit this in §13.3.

[26] In [71, (23)] I forgot to include the square root with the fraction involving $t^2+1/16$, and so the "11" derived there essentially used $(s-1/2)^2$ as the denominator, rather than our $(s-1/2)$.

12.3. We are now set to imitate §6 of [71], particularly the end of that section from (25) onwards. Our goal is the following result.

Proposition 12.3.1. *We have* $\sqrt{D}L_\chi(1) \geq 100\log D$ *when* $10^3 \leq \log D \leq 10^8$.

We first state a variant of Lemma 5.1.1 that uses Lemma A.1.10 (with $u = 5$) instead of Lemma A.1.9 for the real case (the proof is the same as before).

Lemma 12.3.2. *Suppose there are z primes p with $\chi(p) = +1$ and $p \leq X$. Then*

$$\frac{1}{\pi}\sqrt{D}L_\chi(1) \geq \sum_{n\leq\sqrt{D}/10} R_K^\star(n) \geq \sum_{j=0}^{z} 2^j\binom{z}{j}\binom{u}{j} \quad \textit{where} \quad u = \left\lfloor \frac{\log(\sqrt{D}/10)}{\log X}\right\rfloor.$$

Also, when K is real we can replace $1/\pi$ on the left side by $1/2\log\bigl(\frac{5+\sqrt{29}}{2}\bigr) \leq 0.304$.

We also recall our notation $\mathcal{F}(u,z) = \sum_j 2^j\binom{z}{j}\binom{u}{j}$.

12.3.3. We let E_ψ^+ be $E_\psi^{\mathbf{P}}$ with the choice (§11.2) of $\mathbf{P}$ being all primes up to $\sqrt{D}/2$. We will show a Lemma that adequately bounds E_ψ^+. For convenience, we define

$$Y_{\mathrm{r}}(n) = 1 + 1/n^{1/4},\ \ Y_{\mathrm{s}}(n) = \frac{1+1/n^{1/4}}{1-1/n^{1/4}},$$

$$V_{\mathrm{r}}(n) = 1 - 1/\sqrt{n},\ \ V_{\mathrm{s}}(n) = \frac{1-1/\sqrt{n}}{1+1/\sqrt{n}},\ \text{ and } W(n) = \frac{\log n}{\sqrt{n}-1}.$$

Lemma 12.3.4. *Suppose that $\sqrt{D}L_\chi(1) \leq 100\log D$, and also $10^3 \leq \log D \leq 10^8$. Then we have the bounds of $|E_\psi^+(1/2+it)| \geq 6\cdot 10^{-7}$ and $|E_\psi^+(1/4+it)| \leq 10^{13}$ for all real t, while $|\arg E_\psi^+(1/2+it)| \leq 43.166|t|$.*

Proof. We first note that if there are 34 primes dividing D, then by genus theory (§3.2.2) the class number is divisible by 2^{32}, and thus is at least this large. The class number formula then implies $\sqrt{D}L_\chi(1) \geq 2^{32}\pi > 10^{10} \geq 100\log D$ in our range of D, contradicting our assumption. Thus there are at most 33 primes dividing D.

With Lemma 12.3.2, taking $X = 487624$ we have $u = \lfloor\log(\sqrt{D}/10)/\log X\rfloor \geq 38$ for our range of $D \geq \exp(1000)$, and then with $z = 7$ we see that this Lemma implies $\sqrt{D}L_\chi(1) \geq \pi\mathcal{F}(38,7) \geq 1.01\cdot 10^{10} > 100\log D$. Thus there are at most 6 split primes up to 487624. Repeating the argument with $X = 10^{43}$ has $u \geq 5$, and $z = 104$ implies $\sqrt{D}L_\chi(1) \geq \pi\mathcal{F}(5,104) \geq 1.04\cdot 10^{10} > 100\log D$.

We also still have Corollary 5.1.2 that says the number of split primes up to $\sqrt{D}/2$ is $\leq 0.52\sqrt{D}L_\chi(1) \leq 10^{10}$.

Thus, somewhat crudely, under our assumptions on D and $L_\chi(1)$ we have

$$|E_\psi^+(1/2+it)| \geq \prod_{p\leq 137} V_{\mathrm{r}}(p)\prod_{p\leq 13} V_{\mathrm{s}}(p)\cdot V_{\mathrm{s}}(487624)^{97}\cdot V_{\mathrm{s}}(10^{43})^{10^{10}} \geq 7.99\cdot 10^{-7}$$

(where we noted the 33rd prime is 137, and the 6th is 13), and

$$\tilde{E}_\psi^+(1/4) \leq \prod_{p\leq 137} Y_{\mathrm{r}}(p)\prod_{p\leq 13} Y_{\mathrm{s}}(p)\cdot Y_{\mathrm{s}}(487624)^{97}\cdot Y_{\mathrm{s}}(10^{43})^{10^{10}} \leq 3.61\cdot 10^{12}.$$

Meanwhile, we have

$$\begin{aligned}|\arg E_\psi^+(1/2+it)| &\leq |t|\max_u\left|\frac{(E_\psi^+)'}{E_\psi^+}(1/2+iu)\right|\\ &\leq |t|\Bigl(\sum_{p\leq 137} W(p) + 2\sum_{p\leq 13} W(p) + 194\cdot W(487624) + 2\cdot 10^{10}W(10^{43})\Bigr)\\ &\leq |t|(24.165 + 15.357 + 3.644) = 43.166|t|.\end{aligned}$$

Of course one can make various improvements, but these shall suffice for now. □

Lemma 12.3.5. *Suppose that* $\sqrt{D}L_\chi(1) \le 100\log D$*, and also* $10^3 \le \log D \le 10^8$*. Let* ψ *be a real primitive odd Dirichlet character of odd conductor* k *with* $k \le 2^{32}$ *and* $g = \gcd(k, D) = 1$*, such that* $L_\psi(s)$ *has a zero* $1/2 + i\xi_0$ *satisfying* $0 \le \xi_0 \le 0.00223$*. Then* $\xi_0 \log\sqrt{D} + \xi_2$ *is within 0.097 of a multiple of* π*, with* ξ_2 *defined as in* (25)*.*

Proof. This essentially follows from (26) and Lemma 12.3.4. Indeed, under our assumptions we have

$$|M_2| \le 14.868 \cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2} \le 5.932 \cdot 2^{16}\frac{100\log D}{D^{1/4}} \le 10^{-97}$$

and

$$|M_3| \le 3.322 \cdot \tilde{E}^+_\psi(1/4)\Bigl(\frac{2\pi g}{k\sqrt{D}}\Bigr)^{1/4} \cdot \prod_{p|(k/g)}\Bigl(1 + \frac{1}{\sqrt{p}}\Bigr) \le \frac{5.26 \cdot 10^{13}}{D^{1/8}} \le 10^{-40}.$$

Thus from (26) we have (using the crude $\xi_3 \ge 1$)

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D} + \xi_2 + \arg E^+_\psi(1/2 + i\xi_0)\bigr]\Bigr| \le \frac{10^{-39}}{\xi_3|E^+_\psi(1/2 + i\xi_0)|} \le 10^{-31},$$

where

$$\xi_2 = \xi_0\log\frac{k}{2\pi} + \arg\bigl[i\Gamma(1/2 + i\xi_0)\zeta_k(1 + 2i\xi_0)\bigr].$$

Lemma 12.3.4 also says $|\arg E^+_\psi(1/2 + it)| \le (43.166)(0.00223) \le 0.097 - 10^{-30}$, which then implies that $\xi_0\log\sqrt{D} + \xi_2$ is within 0.100 of a multiple of π. □

One can thus note that each such character ψ misses approximately $2\cdot 0.097/\pi \approx 1/15$ of the D values (on a logarithmic scale), and this fraction is dependent on the height of the lowest zero multiplied by the bound on the logarithmic derivative of $E^{\mathbf{P}}_\psi(1/2 + it)$.

12.3.6. We complete the proof that $\sqrt{D}L_\chi(1) \ge 100\log D$ for $10^3 \le \log D \le 10^8$.

Proof. (Proposition 12.3.1). We list in Tables 8 and 9 a selection of 60 auxiliary real primitive characters, all of them with prime conductor (for convenience). In addition to giving ξ_0 (correct to the precision given), these Tables list a "miss period" $p = 2\pi/\xi_0$ and a "shift" interval $[l, h]$ for each character, the point being that when there is no integer b with $(\log D) \notin [bp + l, bp + h]$, then $\sqrt{D}L_\chi(1)$ is large by Lemma 12.3.5, assuming $\gcd(k, D) = 1$ (and $10^3 \le \log D \le 10^8$). We need to have $l \le (2/\xi_0)(-\xi_2 - 0.097)$ and $h \ge (2/\xi_0)(-\xi_2 + 0.097)$; in fact, we actually did the tabulations with 0.097 replaced here by 0.103 (due to the various re-writing stages of this paper), and we rounded our listed values for l down and h up to the nearest integer for convenience.

We then routinely verify that for any D with $10^3 \le \log D \le 10^8$ there are at most 23 auxiliary characters[27] which miss D. Thus every D is hit by at least 37 of them; so either (at least) one of these 37 is coprime to D and Lemma 12.3.5 suffices, or D is divisible by 37 primes, when the result follows by genus theory. □

Remark. The main features of these moduli are that they are prime (for convenience), not too large, and have a zero of relatively small height.

With the class number 100 problem for imaginary quadratic fields, we generated the list in [71, Table 1] from difficult examples that were found with an earlier investigation (concerning real zeros of real odd[28] Dirichlet L-functions [70]). These used what we termed a "Low score" [49] given by $\sum_a \log(8\pi a/e^\gamma\sqrt{k})$ summed over minima for $\mathbf{Q}(\sqrt{-k})$, whose small values are weakly correlated to $L_\chi(1/2)$, whose smallness is in turn correlated to a low-height zero. While it perhaps might be more trenchant to compute $L_\chi(1/2)$ directly as a surrogate, for Tables 8 and 9 we generated Low scores for all $k \le 2^{32}$ in

[27] The worst case is for $\log D \approx 90290383$, when there are 19 that miss.

[28] Chua [14] has a version of Low's method for the even (real quadratic) case, albeit it takes time linear in k, so for our purposes approximating $L_\chi(1/2)$ in time $O(\sqrt{k}\log k)$ would be superior.

k	ξ_0	$p = 2\pi/\xi_0$	$[l, h]$	$2\pi/\xi_0 \log k$
2798913571	0.0020159837308	3116.684530266	$[-141, 64]$	143.28
275971211	0.0022262707952	2822.291574196	$[-127, 59]$	145.21
2517922283	0.0019927337950	3153.048000104	$[-142, 66]$	145.66
3631268243	0.0019573945201	3209.973892727	$[-144, 67]$	145.82
3985600643	0.0019423166624	3234.892347205	$[-145, 68]$	146.34
1020839059	0.0020654615960	3042.024755763	$[-136, 64]$	146.65
2440122943	0.0019755475206	3180.477939203	$[-143, 67]$	147.14
162173551	0.0022455507400	2798.059823538	$[-125, 60]$	148.01
428171663	0.0021335173353	2944.989104767	$[-132, 63]$	148.18
1166402099	0.0020092504342	3127.129003042	$[-139, 67]$	149.79
921190087	0.0020296897410	3095.638303892	$[-138, 66]$	149.97
1983309763	0.0019530030558	3217.191744014	$[-144, 68]$	150.28
2045178127	0.0018840103617	3335.005706363	$[-147, 72]$	155.56
3117865243	0.0018420866501	3410.906488477	$[-151, 74]$	156.03
3853296127	0.0018155165579	3460.825118719	$[-153, 75]$	156.80
3057192779	0.0018184878465	3455.170360006	$[-152, 75]$	158.20
171459523	0.0020725257522	3031.656084643	$[-133, 67]$	159.90
3334368203	0.0017792427678	3531.381675836	$[-155, 78]$	161.05
3373766047	0.0017744236857	3540.972405743	$[-155, 78]$	161.40
2297320183	0.0018049859833	3481.016121660	$[-152, 77]$	161.49
2534311019	0.0017925518774	3505.162325566	$[-153, 77]$	161.88
754366559	0.0018955224975	3314.751112449	$[-145, 74]$	162.16
3088483259	0.0017397870859	3611.467953745	$[-157, 80]$	165.28
478282543	0.0018954710810	3314.841028219	$[-144, 75]$	165.86
1176200099	0.0018134452223	3464.778108469	$[-151, 78]$	165.89
1819421063	0.0017755545685	3538.717096501	$[-154, 79]$	165.97
1796138467	0.0017747751706	3540.271134703	$[-154, 79]$	166.14
1038929107	0.0017477665191	3594.979786178	$[-155, 82]$	173.16
2231381419	0.0016649086342	3773.891959061	$[-162, 86]$	175.32
1927312571	0.0016182273277	3882.758126501	$[-165, 90]$	181.61

TABLE 8. 30 auxiliary characters with low-height zeros

about 2 days using 71 threads on a E5-2699.[29] We then calculated the lowest-height zero for those with prime conductor and a sufficiently negative Low score,[30] and for the Tables took the 60 resulting zeros of lowest height, relative to the expected $2\pi/\log k$ (where we record the ratio therein in the fifth column of the table).

The zeros themselves were then calculated by using GP/PARI [57] and Weinberger's method [76]. The Table data are listed to sufficient precision for the Proposition.

There is also some sense that we should avoid characters for which the zeros are *too* close to each other. For instance, if two ξ_0 are within 0.00001 of each other, then they will track each other rather closely for $\log\sqrt{D} \leq 10^4$ (say), in particular having their miss ranges likely overlapping therein. However, this is not a worry if we ensure (e.g.) that there are not 23 auxiliary characters all with such zeros close to each other.

[29]The choice of $k \leq 2^{32}$ has nothing to do with 32-bit arithmetic, but rather was a convenient demarcation point time-wise. A more relevant limitation in our code would be $k \leq 3 \cdot 821641^2 \approx 2 \cdot 10^{12}$, where 821641 is the 65536th prime (our factor table entries would then need more than 2 bytes each).

[30]We computed zeros for all prime k with negative Low scores (approximately 12500 cases); the cases with L nearer to 0 mostly found zeros whose height was only about (say) 20 times less than expected.

Indeed, to some extent "false positives" dominated our calculations. Even extreme Low scores L did not match up too well to zeros of abnormally small height. For instance $k = 3032758559$ has the best $L \approx -0.058889$, but has only $2\pi/\xi_0 \log k \approx 136.94$. Of the best 30 Low scores only $k = 3846105671$ appears ($L \approx -0.043480$) in the tables. The worst Low score (with $L \approx -0.002370$) that appears is for $k = 171459523$; however $k = 31129723$ has $L \approx -0.001319$ and $2\pi/\xi_0 \log k \approx 176.64$, while also $k = 12461947$ has $L \approx -0.001404$ and $2/\pi\xi_0 \log k \approx 154.00$, so these could appear – either I had an artificial restriction (like $k \geq 2^{25}$) during table preparation, or chose not to replicate examples from [71].

k	ξ_0	$p = 2\pi/\xi_0$	$[l, h]$	$2\pi/\xi_0 \log k$
1433103107	0.0016300436112	3854.611780831	$[-164, 90]$	182.83
2031817663	0.0016017153799	3922.785150249	$[-167, 92]$	183.03
306079643	0.0017229424570	3646.776061335	$[-154, 86]$	186.64
3311502587	0.0015031181550	4180.100736790	$[-176, 99]$	190.69
4180567099	0.0014694732560	4275.807866085	$[-180, 102]$	193.01
2115632171	0.0015005148074	4187.353084586	$[-175, 100]$	195.01
1295145091	0.0015275253107	4113.310112187	$[-172, 99]$	196.04
2534772307	0.0014603339974	4302.567301978	$[-180, 104]$	198.70
1816676227	0.0014239803531	4412.410110538	$[-183, 108]$	206.96
649059211	0.0014651912369	4288.303908111	$[-176, 106]$	211.34
325757807	0.0014883725980	4221.513696044	$[-173, 105]$	215.37
3846105671	0.0013115357307	4790.708449754	$[-196, 119]$	217.07
823946743	0.0013826133868	4544.426784071	$[-185, 114]$	221.36
4238508763	0.0012770498657	4920.078280441	$[-201, 123]$	221.95
1821908299	0.0013118861223	4789.428899602	$[-195, 120]$	224.61
4112591003	0.0012621636157	4978.106823052	$[-203, 125]$	224.87
1922096731	0.0013059026683	4811.373358421	$[-196, 121]$	225.08
3025845331	0.0012740560429	4931.639657700	$[-201, 124]$	225.91
2067321943	0.0012283784751	5115.023939573	$[-206, 131]$	238.47
2168243683	0.0012039049287	5219.004555611	$[-209, 134]$	242.78
3281873687	0.0011104240748	5658.365528584	$[-225, 147]$	258.24
1409931443	0.0011370089969	5526.064722502	$[-219, 145]$	262.31
648760127	0.0011433888013	5495.230756376	$[-216, 145]$	270.83
3697896911	0.0010004324567	6280.469276116	$[-245, 168]$	285.07
1386038791	0.0009748661671	6445.177316702	$[-249, 175]$	306.19
4244770963	0.0009141780988	6873.042917593	$[-265, 187]$	310.03
2473259119	0.0009111772528	6895.678406981	$[-265, 189]$	318.82
3497154523	0.0008466775057	7420.990005328	$[-282, 205]$	337.70
1526873947	0.0008089056296	7767.513387796	$[-292, 218]$	367.32
800703083	0.0007648066394	8215.390640349	$[-306, 234]$	400.73

TABLE 9. 30 more auxiliary characters with low-height zeros

Undoubtedly the type of result we showed here can be improved in various ways, but ultimately one needs an upper bound (here $\log D \le 10^8$) to finitize the problem.

13. HANDLING SMALLER-SIZED DISCRIMINANTS IN OUR SPECIAL CASES

The above Proposition 12.3.1 reduces our considerations to $\log D \le 1000$. For these smaller D, we find it convenient to use additional information available in our situations of real quadratic fields with a small fundamental unit. We could work somewhat more generally (with Mollin's Lemma below), but largely proceed to our cases of interest.

13.1. As an application of the methods of this paper, we shall show a class number result for the family of real quadratic fields having $D = 4u^2+1$ squarefree with $u > 0$. This was considered by Chowla, and it is not much more difficult to consider in parallel $D = u^2+4$ with $u > 0$ odd (Yokoi's family), and $D = 4(u^2+1)$ with $u > 0$ odd.[31] Assuming D is a fundamental discriminant, Chowla's family has fundamental unit $2u+\sqrt{D}$ when $u > 1$, while Yokoi's has $(u+\sqrt{D})/2$, and for $D = 4(u^2+1)$ it is $u+\sqrt{D}/2$ (all have norm -1).

We shall also consider a family already mentioned by Euler, which can be defined by $D = (3u+1)^2+(4u+1)^2 = 25u^2+14u+2 = (5u+7/5)^2+1/25$ when u is odd, and 4 times this (to make D fundamental) when u is even. When $25u^2+14u+2$ is squarefree and $u \notin \{0,-1\}$, this has fundamental unit $|25u+7|+5\sqrt{D}/w$ where $w = 1$ for u odd and $w = 2$ for u even, and this again always has norm -1.

[31]One can note that u^2+4 with $2\|u$ yields the $4(u^2+1)$ family, and with $4|u$ the $4u^2+1$ family.

13.1.1. For the convenience of the reader, we copy here [51, Lemma 1.1], and its corollary that we generalize from [52, Lemma 1]. As noted by Mollin, this is due to Davenport, Ankeny, and Hasse, with Yokoi providing the case where t is a square.

An initial version of this was shown[32] by Chowla and Friedlander [13, §4 (C)], namely that when $\mathbf{Q}(\sqrt{4u^2+1})$ has class number 1, we have $\chi(p) = -1$ for $p < u$.

We follow Mollin in defining a solution to $x^2 - Dy^2 = \pm 4t$ (with $t > 0$) to be *trivial* when $t = m^2$ with $m|\gcd(x, y)$, so that solutions with $t = 1$ are trivial. Other than this, primitive representations by the principal form give rise to nontrivial solutions.

Lemma 13.1.2. [51, Lemma 1.1]. *Let $D > 0$ be fundamental, with $(A + B\sqrt{D})/2$ the fundamental unit of $\mathbf{Q}(\sqrt{D})$, so that $A^2 - DB^2 = 4\delta$ with $\delta = \pm 1$.*

If there exists a nontrivial solution to $x^2 - Dy^2 = \pm 4t$, then $t \geq (A - \delta - 1)/B^2$.

Note that Mollin's phrasing takes D to be squarefree, while ours takes it to be a fundamental discriminant, and thereby in his notation we have $\sigma = 2$ in all cases.

Proof. Let $t > 1$ be given, and suppose (u, v) is a nontrivial solution to $x^2 - Dy^2 = \pm 4t$ with $u \geq 0$ and $v > 0$ minimal. Multiplying by the norm of the fundamental unit we have

$$\pm 4t\delta = (u^2 - Dv^2)(A^2 - DB^2)/4 = (uA - DvB)^2/4 - D(uB - vA)^2/4,$$

where $\big((uA - DvB)/2, (uB - vA)/2\big)$ can be seen to provide another nontrivial solution to $x^2 - Dy^2 = \pm 4t$, so that by minimality we have $|uB - vA| \geq 2v$.

Breaking into cases with u, we find that either $u \geq v(A + 2)/B$ whence

$$\begin{aligned} 4t = u^2 - Dv^2 \geq v^2(A+2)^2/B^2 - Dv^2 &= v^2(A^2 - DB^2 + 4A + 4)/B^2 \\ &= 4v^2(\delta + A + 1)/B^2 \geq 4(\delta + A + 1)/B^2, \end{aligned}$$

or $u \leq v(A - 2)/B$ when

$$\begin{aligned} -4t = u^2 - Dv^2 \leq v^2(A-2)^2/B^2 - Dv^2 &= v^2(A^2 - 4A + 4 - DB^2)/B^2 \\ &= 4v^2(\delta - A + 1)/B^2 \leq 4(1 + \delta - A)/B^2, \end{aligned}$$

and these give the stated bound on t. □

Corollary 13.1.3. *Suppose that $\chi(p) = +1$. Then $p^{h_K} \geq (A - \delta - 1)/B^2$.*

Proof. This follows because $\chi(p) = +1$ implies there is a prime ideal $\mathfrak{p}$ of norm p, and then $\mathfrak{p}^{h_K}$ is principal, giving a nontrivial solution to $x^2 - Dy^2 = \pm 4\delta p^{h_K}$. □

One can note that this result is only really useful when B is quite small. Indeed, we have $A \approx B\sqrt{D}$, so that $A/B^2 \approx \sqrt{D}/B$, while generically B is of size $\exp(\sqrt{D})$.

13.1.4. In all our cases we have $\delta = -1$, and Table 10 lists the relevant A/B^2. Its smallest value $\sqrt{D/10^2 - 1/50^2}$ is in case 4. We also have $\epsilon_0 \leq 10\sqrt{D}$ for our cases.

case	D	u	ϵ_0	A/B^2
(1)	$4u^2+1$	$u > 1$	$2u + \sqrt{D}$	$4u/2^2$
(2)	u^2+4	$u > 0$, odd	$(u + \sqrt{D})/2$	$u/1^2$
(3)	$4(u^2+1)$	$u > 0$, odd	$u + \sqrt{D}/2$	$2u/1^2$
(4)	$25u^2 + 14u + 2$	$u > 0$, odd	$(25u + 7) + 5\sqrt{D}$	$(50u + 14)/10^2$
	$25u^2 - 14u + 2$	$u > 1$, odd	$(25u - 7) + 5\sqrt{D}$	$(50u - 14)/10^2$
(5)	$4(25u^2 + 14u + 2)$	$u > 0$, even	$(25u + 7) + 5\sqrt{D}/2$	$(50u + 14)/5^2$
	$4(25u^2 - 14u + 2)$	$u > 0$, even	$(25u - 7) + 5\sqrt{D}/2$	$(50u - 14)/5^2$

TABLE 10. The five cases with small regulator that we consider

In Table 11 we additionally give the principal form and the continued fraction expansion of $1/\omega$ (various excluded cases for u correspond to non-primitive periods).

[32] They actually thank the referee for improving their initial version of this. Also, as noted by Mollin, for the case of the Chowla conjecture it is easy to show that u must be non-composite for $\mathbf{Q}(\sqrt{4u^2+1})$ to

case	D	u	principal $\langle a,b,c\rangle$	$1/\omega$
(1)	$4u^2+1$	$u>1$	$\langle 1, 2u-1, -u\rangle$	$[\overline{1,1,2u-1}]$
(2)	u^2+4	$u>0$, odd	$\langle 1, u, -1\rangle$	$[\overline{u}]$
(3)	$4(u^2+1)$	$u>0$, odd	$\langle 1, 2u, -1\rangle$	$[\overline{2u}]$
(4)	$25u^2+14u+2$	$u>0$, odd	$\langle 1, 5u, -(14u+2)/4\rangle$	$[\overline{1,2,2,1,5u}]$
	$25u^2-14u+2$	$u>1$, odd	$\langle 1, 5u-2, -(3u-1)/2\rangle$	$[\overline{3,3,5u-2}]$
(5)	$4(25u^2+14u+2)$	$u>0$, even	$\langle 1, 10u+2, -(4u+1)\rangle$	$[\overline{2,2,10u+2}]$
	$4(25u^2-14u+2)$	$u>0$, even	$\langle 1, 10u-4, -(6u-2)\rangle$	$[\overline{1,1,1,1,10u-4}]$

Table 11. More information with the 5 cases of small regulator

13.2. We now handle the range $10^{26} \le D \le \exp(10^3)$ for our five special cases, which reduces each to a feasible sieving problem then considered in §14.

13.2.1. We split the D-range at $\exp(75)$ or $\exp(80)$, and for the upper range are a bit lazy in bounding the various error terms. Here we use the auxiliary character of conductor 12461947, which (as noted in [71, Table 1]) has a zero with $\xi_0 \approx 0.0024972078778$.

For the lower range we make a more concerted effort with the error term, indeed using a different set $\mathbf{P}$ in our choice of $E_\psi^{\mathbf{P}}$. Here we use the character of conductor 17923 (with $\xi_0 \approx 0.0309857994985$) whose smallness is convenient.

We state both results together, though the proofs take somewhat different paths.

Lemma 13.2.2. *Suppose that D is fundamental with $75 \le \log D \le 1000$, and we are in one of the following cases.*

1. *$D = 4u^2+1$ with $u>0$,*
2. *$D = u^2+4$ with u odd and $u>0$,*
3. *$D = 4(u^2+1)$ with u odd and $u>0$,*
4. *$D = 25u^2+14u+2$ with u odd,*
5. *$D = 4(25u^2+14u+2)$ with u even.*

Then $h_K \ge 6$.

Lemma 13.2.3. *With D as in Lemma 13.2.2 but $10^{26} \le D \le e^{80}$ we have $h_K \ge 6$.*

We proceed to the proof for the range $75 \le \log D \le 1000$.

Proof. (Lemma 13.2.2). We assume that $h_K \le 5$ and show a contradiction. In all five cases the fundamental unit has norm -1, and thus the theory of genera (§3.2.2) implies that D has at most 3 prime divisors (otherwise 8 divides h_K). The largest of these must be at least $D^{1/3} \ge \exp(25)$.

Also, no D under consideration has a prime divisor that is 3 mod 4.

13.2.4. By using Mollin's result (Corollary 13.1.3), when $h_K \le 5$ we see that any split prime must be at least $(D/10^2 - 1/50^2)^{1/10}$, which we denote by Z_0. As we are now purely in the real case, as with Lemma 12.3.2 we have

$$\mathcal{F}(u,z) = \sum_{j=0}^{z} 2^j \binom{z}{j}\binom{u}{j} \le \sum_{n \le \sqrt{D}/10} R_K^\star(n) \le \frac{\sqrt{D}L_\chi(1)}{2\log\left(\frac{5+\sqrt{29}}{2}\right)} \le 0.304\sqrt{D}L_\chi(1),$$

where $u = \lfloor \log(\sqrt{D}/10)/\log X\rfloor$ and there are z split primes up to X.

Since $\epsilon_0 \le 10\sqrt{D}$ and we assume $h_K \le 5$, the class number formula implies that $\sqrt{D}L_\chi(1) = 2h_K \log\epsilon_0 \le 10\log(10\sqrt{D})$, so $0.304\sqrt{D}L_\chi(1) \le 3.04\log(10e^{500}) < 1527$. Since $\mathcal{F}(4,7) = 2241$ there are at most 6 split primes up to $Z_4 = (\sqrt{D}/10)^{1/4}$, while similarly $\mathcal{F}(3,10) = 1561$ implies at most 9 split primes up to $Z_3 = (\sqrt{D}/10)^{1/3}$, and $\mathcal{F}(2,28) = 1625$ implies at most 27 split primes up to $Z_2 = (\sqrt{D}/10)^{1/2}$.

For the remaining primes up to $\sqrt{D}/2$, we use Corollary 5.1.2 to see that no more than $0.52\sqrt{D}L_\chi(1) \le 0.52 \cdot 10\log(10\sqrt{D})$ of them are split.

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have class number 1. Indeed, for prime $p|u$ we have $4u^2+1 \equiv 1 \pmod{4p}$, so that by quadratic reciprocity we have $\chi(4p) = \chi(p) = +1$, and if $p < u$ this contradicts the result of Chowla and Friedlander.

In the notation of §12.3.3, this gives us

$$|E_\psi^+(1/2+it)| \ge V_{\rm r}(2)V_{\rm r}(5)V_{\rm r}(D^{1/3})V_{\rm s}(Z_0)^6V_{\rm s}(Z_4)^3V_{\rm s}(Z_3)^{18}V_{\rm s}(Z_2)^{5.2\log(10\sqrt{D})} \ge 0.089$$

for $D \ge \exp(75)$, while similarly

$$|E_\psi^+(1/4+it)| \le Y_{\rm r}(2)Y_{\rm r}(5)Y_{\rm r}(D^{1/3})Y_{\rm s}(Z_0)^6Y_{\rm s}(Z_4)^3Y_{\rm s}(Z_3)^{18}Y_{\rm s}(Z_2)^{5.2\log(10\sqrt{D})} \le 53086,$$

and

$$|\arg E_\psi^+(1/2+it)| \le |t|\big[W(2)+W(5)+W(D^{1/3})+12W(Z_0)+6W(Z_4)+ \\ +36W(Z_3)+10.4\log(10\sqrt{D})W(Z_2)\big] \le 8.509|t|.$$

(We could improve these slightly by noting subcases when $2\nmid D$, but do not bother).

13.2.5. We can then apply (26) with $k = 12461947$, noting that $\gcd(k,D)=1$ since k is 3 mod 4. This ψ has $\xi_0 \approx 0.0024972078778$, which is sufficiently small so that we never encounter a "miss range" for $75 \le \log D \le 1000$. Explicitly, with (26) we have

$$|M_2| \le 14.868\cdot L_\chi(1)\Big(\frac{k\sqrt{D}}{2\pi}\Big)^{1/2} \le 5.932\sqrt{12461947}\cdot\frac{10\log(10\sqrt{D})}{D^{1/4}} \le 0.060,$$

and

$$|M_3| \le 3.322\cdot\tilde{E}_\psi^+(1/4)\Big(\frac{2\pi}{k\sqrt{D}}\Big)^{1/4}\cdot\prod_{p|k}\Big(1+\frac{1}{\sqrt{p}}\Big) \le \frac{4701}{D^{1/8}} \le 0.399.$$

Thus when $h_K \le 5$ and $75 \le \log D \le 1000$ we have

$$\Big|\sin\big[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\big]\Big| \le \frac{0.060+0.399}{\xi_3|E_\psi^+(1/2+i\xi_0)|} \le \frac{0.459}{709\cdot 0.089} \le 0.008,$$

where

$$\xi_2 = \xi_0\log\frac{k}{2\pi}+\arg\big[i\Gamma(1/2+i\xi_0)\zeta_k(1+2i\xi_0)\big] \approx 0.034189907.$$

13.2.6. On the other hand, we have

$$\big|\arg E_\psi^+(1/2+i\xi_0)\big| \le 8.509\xi_0 \le 0.022,$$

while $0.093 \le \xi_0\log\sqrt{D} \le 1.249$. These combine to imply

$$\sin\big[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\big] \\ \ge \min\big(\sin[0.093+0.034-0.022],\sin[1.249+0.035+0.022]\big) \ge 0.104,$$

contradicting the above inequality, so our assumption of $h_K \le 5$ must be wrong. $\Box$

13.3. We next provide a version of Lemma 12.2.1 that applies when we take $\mathbf{P}$ to have no split primes with our definition (§11.2) of $E_\psi^{\mathbf{P}}(s)$.

Let us recall that $P_u(s) = \prod_{p|u}(1-1/p^s)$, while

$$G(s) = \prod_p G_p(s) = \prod_{p\nmid k}\frac{1+\psi(p)/p^s}{1-(\psi\chi)(p)/p^s}\prod_{p|g}\Big(1+\frac{[\psi\chi](p)}{p^s}\Big) = \sum_{n=1}^\infty\frac{S(n)}{n^s},$$

where we note that $|S(n)| \le R_K^\star(n)$. Given a set $\mathbf{P}$ of primes, we defined $\bar{\mathbf{P}}_\le^\star$ as the set containing all $n \le \sqrt{D}/2$ with no prime factor in $\mathbf{P}$ (including $n=1$), and

$$E_\psi^{\mathbf{P}}(s) = \prod_{p\in\mathbf{P}}G_p(s)\sum_{n\in\bar{\mathbf{P}}_\le^\star}\frac{S(n)}{n^s}.$$

Note that $G_p(s)\equiv 1$ when $p|(k/g)$, and also when p is inert.

Lemma 13.3.1. *When* $\mathbf{P}$ *has no split primes and* $\gcd(k,D)=1$, *in Lemma 12.2.1 we have*

$$|M_3| \le 1.340\frac{\tau_2(k)}{k^{1/4}}\cdot\prod_{p\in\mathbf{P}}\big(1+p^{1/4}\big)\cdot L_\chi(1)D^{1/4}.$$

Proof. Recall from §12.2.4 that M_3 as

$$M_3 = 2\int_{(1/4)} \Gamma(s)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{s-1/2} E_\psi^{\mathbf{P}}(s)\zeta(2s)P_{k/g}(2s)\frac{(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i},$$

Here $g=\gcd(k,D)=1$, so $P_{k/g}(2s)=\prod_{p|k}(1-1/p^{2s})$ has a zero at $s=0$, to cancel the pole of $\Gamma(s)$. We then move the contour to $\sigma=-1/4$ and get

$$|M_3| \le 2\Big(\frac{2\pi}{k\sqrt{D}}\Big)^{3/4}\prod_{p|k}\big(1+\sqrt{p}\big)\prod_{p\in\mathbf{P}}\big(1+p^{1/4}\big)\sum_{n\in\bar{\mathbf{P}}^\star_\le} R_K^\star(n)\, n^{1/4}\times$$
$$\times\int_{(-1/4)}\left|\frac{\Gamma(s)\zeta(2s)(s-1/2)}{(s-1/2)^2+\xi_0^2}\right|\frac{|\partial s|}{2\pi}.$$

The product over $p|k$ then is $\le \sqrt{k}\tau_2(k)$, while we see that the sum over n in $\bar{\mathbf{P}}^\star_\le$ is bounded by that over $n\le\sqrt{D}/2$, which is $\le 1.04(\sqrt{D}/2)^{1/4}\cdot\sqrt{D}L_\chi(1)$ by Lemma 5.1.1. By numerical computation we find the integral on $\sigma=-1/4$ is ≤ 0.193, and the Lemma follows since we have $2(2\pi)^{3/4}\cdot(1.04/2^{1/4})\cdot 0.193\le 1.340$. □

13.3.2. We proceed to show Lemma 13.2.3, which we restate below for convenience.

Herein we let $\mathbf{P}$ be the set of all primes with $p|D$ and $p\le 10^4$ (there are at most two such primes when $h_K\le 5$ and $D\ge 10^{26}$), and with this choice we write E_ψ^{m} for $E_\psi^{\mathbf{P}}$. What we shall show is that the $n=1$ term dominates the n-sum in E_ψ^{m}, and thus

$$\Bigg|\sum_{n\in\bar{\mathbf{P}}^\star_\le}\frac{S(n)}{n^{1/2+i\xi_0}}\Bigg| \ge 1-\sum_{n\in\bar{\mathbf{P}}^\star_\le\setminus\{1\}}\frac{R_K^\star(n)}{\sqrt{n}} \ge 2-\prod_{\substack{p\le\sqrt{D}/2\\ p\notin\mathbf{P}}}\frac{1+1/\sqrt{p}}{1-\chi(p)/\sqrt{p}}$$

is not small, and this will suffice to control $E_\psi^{\mathrm{m}}(1/2+i\xi_0)$.

Lemma 13.2.3. *With D as in Lemma 13.2.2 but $10^{26}\le D\le e^{80}$ we have $h_K\ge 6$.*

Proof. We again assume that $h_K\le 5$ and show a contradiction.

We will use the auxiliary ψ with $k=17923$, which has $\xi_0\approx 0.0309857994985$. As before we have $\gcd(k,D)=1$ since k is 3 mod 4 and prime.

By Lemmata 12.2.1 and 13.3.1 we have $M_1+M_2+M_3=0$ where

$$|M_2|+|M_3|\le 14.868L_\chi(1)\Big(\frac{k\sqrt{D}}{2\pi}\Big)^{1/2}+\,1.340\frac{\tau_2(k)}{k^{1/4}}\cdot 11^2\cdot L_\chi(1)D^{1/4}\le 851L_\chi(1)D^{1/4}$$

and we used that $\mathbf{P}$ has at most 2 primes each of which is $\le 10^4$, while

$$\begin{aligned} M_1 &= 2|T_\psi^{\mathrm{m}}(1/2+i\xi_0)|\sin\big(\arg iT_\psi^{\mathrm{m}}(1/2+i\xi_0)\big)\\ &= \xi_3|E_\psi^{\mathrm{m}}(1/2+i\xi_0)|\cdot\sin\big[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathrm{m}}(1/2+i\xi_0)\big]\end{aligned}$$

with $\xi_3=2|\Gamma(1/2+i\xi_0)|\cdot|\zeta_k(1+2i\xi_0)|\ge 57.084$ and

$$\xi_2=\xi_0\log\frac{k}{2\pi}+\arg[i\Gamma(1/2+i\xi_0)\zeta_k(1+2i\xi_0)]\approx 0.221562909.$$

Our assumptions imply $851L_\chi(1)D^{1/4}\le 851\cdot 10(\log 10\sqrt{D})/D^{1/4}\le 0.087$, so

$$\Big|\sin\big[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathrm{m}}(1/2+i\xi_0)\big]\Big|\le\frac{0.087}{57.084}\frac{1}{\big|E_\psi^{\mathrm{m}}(1/2+i\xi_0)\big|}. \tag{27}$$

13.3.3. We are left to bound the quantities in (27) involving $E_\psi^{\mathrm{m}}(1/2+i\xi_0)$.

From Corollary 13.1.3 every split prime is $\ge Z_0=(D/10^2-1/50^2)^{1/10}$, while under our assumptions $\log D\le 80$ and $h_K\le 5$ we see that Lemma 12.3.2 implies

$$\mathcal{F}(u,z)=\sum_{j=1}^{z}2^j\binom{z}{j}\binom{u}{j}\le\sum_{n\le\sqrt{D}/10}R_K^\star(n)\le 0.304\sqrt{D}L_\chi(1)\le 3.04\log(10\sqrt{e^{80}})<129,$$

where $u = \lfloor \log(\sqrt{D}/10)/\log X \rfloor$ and there are z split primes up to X. We then note that $\mathcal{F}(4,3) = \mathcal{F}(3,4) = 129$ while $\mathcal{F}(2,8) = 145$; thus we find the third smallest split prime is $\geq Z_4 = (\sqrt{D}/10)^{1/4}$, while the fourth smallest is $\geq Z_3 = (\sqrt{D}/10)^{1/3}$, and the eighth smallest is $\geq Z_2 = (\sqrt{D}/10)^{1/2}$. Similarly, there are at most 63 split primes up to $Z_1 = \sqrt{D}/10$, and for the remaining primes up to $\sqrt{D}/2$ we use Corollary 5.1.2 and $D \leq \exp(80)$ to see that no more than $0.52\sqrt{D}L_\chi(1) \leq 0.52 \cdot 10 \log(10\sqrt{D}) < 220$ of them are split.

Upon writing $H_{\mathrm{r}}(u) = 1 + 1/\sqrt{p}$ and $H_{\mathrm{s}}(u) = \frac{1+1/\sqrt{p}}{1-1/\sqrt{p}}$ we thus have

$$\sum_{n \in \bar{\mathbf{P}}^\star_\leq} \frac{R_K^\star(n)}{\sqrt{n}} \leq H_{\mathrm{r}}(10^4)^3 H_{\mathrm{s}}(Z_0)^2 H_{\mathrm{s}}(Z_4) H_{\mathrm{s}}(Z_3)^4 H_{\mathrm{s}}(Z_2)^{56} H_{\mathrm{s}}(Z_1)^{156}$$

where we recalled the three possible ramified primes, each of which is $\geq 10^4$. As the right side is ≤ 1.713 for $D \geq 10^{26}$, we find that $\sum_{n \in \bar{\mathbf{P}}^\star_\leq} S(n)/n^s = 1 + \Theta(0.713)$ for $\sigma \geq 1/2$.

This implies that

$$|E_\psi^{\mathrm{m}}(1/2 + i\xi_0)| \geq |(1 - 1/2^{1/2+i\xi_0})| \cdot |(1 - 1/5^{1/2+i\xi_0})| \cdot (1 - 0.713) \geq 0.046$$

where we recalled that no prime divisor of D is 3 mod 4 (so $3 \notin \mathbf{P}$), and

$$\begin{aligned} |\arg E_\psi^{\mathrm{m}}(1/2 + i\xi_0)| &\leq \arg(1 - 1/2^{1/2+i\xi_0}) + \arg(1 - 1/5^{1/2+i\xi_0}) + \\ &\qquad + \max_t \arg[1 - 0.713\cos(t) + 0.713 i \sin(t)] \\ &\leq 0.052 + 0.041 + 0.794 = 0.887. \end{aligned}$$

From (27) the first implies $\left|\sin\left[\xi_0 \log \sqrt{D} + \xi_2 + \arg E_\psi^{\mathrm{m}}(1/2 + i\xi_0)\right]\right| \leq 0.032$, while the second combined with $0.927 \leq \xi_0 \log \sqrt{D} \leq 1.240$ and $\xi_2 \approx 0.221562909$ yields

$$\begin{aligned} &\sin\left[\xi_0 \log \sqrt{D} + \xi_2 + \arg E_\psi^{\mathrm{m}}(1/2 + i\xi_0)\right] \\ &\qquad \geq \min(\sin[0.927 + 0.221 - 0.887], \sin[1.240 + 0.222 + 0.887]) \geq 0.258. \end{aligned}$$

This is a contradiction, and thus $h_K \geq 6$ for D in this range. □

14. Sieving out the smallest range of discriminants

We complete our proof of Theorem 1.2.4 by handling $D \leq 10^{28}$ computationally. There are roughly 10^{14} u-values to consider. For each u we will show there are sufficiently many small split primes so as to imply that the class number is large, using that the regulator $\epsilon_0 \leq 10\sqrt{D}$ is small in our five families.

For the bulk of the range, the problem reduces to showing that there are (at least) four small split primes that do not exceed 316. This can then be handled by a computational sieve, for instance by splitting the 60 smallest odd primes into 4 groups of 15, and showing for each u-value there is at least one split prime in each group. Of course this does not work quite as described, as each group fails to have a split prime for approximately $1/2^{15}$ of the u-values, but the residual set is then much smaller, and can be suitably handled.

The sieving problem for the Chowla conjecture was briefly considered by Mollin and Williams [53] in their context of proving the conjecture under GRH, handling $D \leq 10^{13}$ in a few minutes with a 1986 Fortran program. They had some simplifications due to only being concerned with cases of $h_K = 1$.

We will use Lemma 12.3.2, and recall that $\mathcal{F}(2,6) = 85$ and $\mathcal{F}(3,4) = 129$.

14.1. For $D \leq 10^8$ we can simply compute the class numbers directly. There are only around 10^4 relevant u-values in each case, and each computation is quite fast, with the whole range taking a few minutes.

14.1.1. For $10^8 \le D \le 10^{12}$ we again handled each u-value individually, using a lookup table of Kronecker symbol values (for small primes) to find at least 30 split primes up to $1000 \le \sqrt{D}/10$, so that Lemma 12.3.2 implies

$$2h_K \log \epsilon_0 = \sqrt{D}L_\chi(1) \ge 60\pi, \text{ so that } h_K \ge 30\pi/\log(10\sqrt{D}) > 5.84.$$

The computation for each u-value involves Kronecker lookups for about 60 primes on average (the worst case is $u = 400310$ for $D = 4u^2 + 1$ which takes 119 primes, that is, up to 653), and our code took less than a second in total.

14.1.2. In the range $10^{12} \le D \le 10^{17}$ we again process each u-value individually. By Lemma 12.3.2 it suffices to find 6 split primes up to $316 \le (\sqrt{D}/10)^{1/2}$, since their existence implies $h_K \ge \mathcal{F}(2,6)\pi/2\log(10\sqrt{D}) \ge 6.10$. Here we have $\approx 10^9$ total u-values to process, and each on average takes about 12 Kronecker lookups (the worst case, needing 57 primes, is $u = 49998347$ for $D = u^2 + 4$) and our code took under a minute.

14.2. Finally we consider the range $10^{17} \le D \le 10^{28}$. We first partition the odd primes $5 \le p \le 293$ into 4 groups of size 15. If for a given u there is at least one corresponding split prime in each of the groups, then we see there are at least 4 split primes $\le 316 \le (\sqrt{D}/10)^{1/3}$, and thus $h_K \ge \mathcal{F}(3,4)\pi/2\log(10\sqrt{D}) > 5.86$.

Whereas in the lower ranges we processed each u individually via Kronecker lookups, here we use bitwise XOR logic (and thus cannot tell how many split primes there are within a given group, but instead only determine whether or not there are any split primes or not), with a 64-bit register thus containing data for 64 u-values, which thereby allows the $\approx 10^{14}$ u-values to be traversed reasonably quickly. For each prime p, we kept a list of p 64-bit stamps, and cycled through these. The sieving of each batch of 64 u-values takes 15 XOR operations, and 15 updates of which stamp to apply next. Our code went through approximately $2.5 \cdot 10^9$ u-values per second for each group, and the entire run took a few core-days.[33]

Each group of 15 primes misses $\approx 1/2^{15}$ of the u-values, and those that remain are handled individually. The worst examples that we found were $u = 37136775445867$ for $D = 4u^2 + 1$ and $u = 48033835914287$ for $D = u^2 + 4$, both of which have the 64th prime 311 as their fourth smallest split prime.

14.3. By the above computations, we find the lists of cases of $h_K \le 5$ for the five families as given in Tables 1 and 2, and that there are no other cases with $D \le 10^{28}$.

By Lemma 13.2.3 we know $h_K \ge 6$ in these families when $10^{28} \le D \le \exp(80)$, and Lemma 13.2.2 implies $h_K \ge 6$ for $75 \le \log D \le 1000$. For $1000 \le \log D \le 10^8$, Proposition 12.3.1 says $\sqrt{D}L_\chi(1) \ge 100\log D > 199\log(10\sqrt{D}) > 199\log\epsilon_0$ so that $h_K > 99$.

Finally, for $\log D \ge 10^8$ we use Theorem 10.3.2 to get that

$$\sqrt{D}L_\chi(1) \ge \min\Bigl(10^{1000}\log D, \frac{(\log D)^3}{10^{14}}\Bigr)\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr) \ge (100\log D)\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

When $h_K \le 5$ and D is in one of our families, the theory of genera implies there are at most three primes that divide D. Moreover, no prime that is 3 mod 4 divides D. Thus (under the assumptions $h_K \le 5$ and $\log D \ge 10^8$) we find that

$$h_K = \frac{\sqrt{D}L_\chi(1)}{2\log\epsilon_0} \ge \frac{100\log D}{2\log(10\sqrt{D})}\Bigl(1-\frac{2}{2+1}\Bigr)\Bigl(1-\frac{4}{5+1}\Bigr)\Bigl(1-\frac{2D^{1/6}}{D^{1/3}}\Bigr) \ge 11.$$

This is a contradiction, and we thereby conclude as in Theorem 1.2.4.

[33] Probably the choice of "15" primes for the size of the sieving set is not an optimal balance (perhaps 10 or so is best), but our code development took various paths through parallelization schemes, etc., and as this choice was reasonably fast, I saw no reason to fiddle with it.

15. A more general class number 1 result for real quadratic fields

We conclude with our other computational result, namely classifying all real quadratic K of class number 1 with a fundamental unit $(A+B\sqrt{D})/2$ with $B \le D^{1/4}$. We do not aim for much generality, but instead concentrate on obtaining this specific result.

The D-exponent of $1/4$ is chosen here to make things workable without too much difficulty. In theory one could go up to $\approx (10^{1000}-1/2)$ from Theorem 10.3.2, though $1/2$ is already a natural barrier for the use of Mollin's Lemma (Corollary 13.1.3).

15.1. We begin by giving a variant of Proposition 12.3.1. The method of proof is mostly the same, though it turns out to be more convenient to work with just one auxiliary character, and handle the gcd-condition separately.

The constant 1.51 is chosen here to be slightly larger than 3/2, as that will be the natural bound coming from $B \le D^{1/4}$ when $h_K = 1$.

Lemma 15.1.1. *We have* $\sqrt{D}L_\chi(1) > 1.51\log D$ *when* $200 \le \log D \le 1000$.

Proof. We suppose $\sqrt{D}L_\chi(1) \le 1.51\log D$ and proceed to show a contradiction.

15.1.2. By the theory of genera (§3.2.2), there are no more than 10 prime divisors of D, as else 2^9 divides the class number, whence $\sqrt{D}L_\chi(1) \ge \pi h_K > 1608 > 1.51\log D$.

With $X = 1151727$ in Lemma 12.3.2 we find that $u = \lfloor \log(\sqrt{D}/10)/\log X\rfloor \ge 7$ for $D \ge \exp(200)$; and as $\mathcal{F}(7,3) = 575$, if there are 3 split primes up to this X then we have $1.51\log D \le 1510 < 1806 < \mathcal{F}(7,3)\pi \le \sqrt{D}L_\chi(1)$; this contradicts our assumption on $\sqrt{D}L_\chi(1)$, so there are at most 2 split primes up to 1151727. Similarly, with $X = 10^{14}$ we have $u \ge 3$, so there are at most 6 split primes up to 10^{14} (using $\mathcal{F}(3,7) = 575$) while (with $u = 1$) the Lemma implies there are at most $(1/\pi)\sqrt{D}L_\chi(1)/2 < 0.241\log D \le 241$ split primes up to $\sqrt{D}/10 \ge 10^{42}$, and finally by Corollary 5.1.2 we see there are at most $0.52\sqrt{D}L_\chi(1) \le 0.52\cdot 1.51\log D < 786$ split primes up to $\sqrt{D}/2$.

In the notation of §12.3.3 this gives us (somewhat crudely) that

$$|E_\psi^+(1/2+it)| \ge \prod_{p\le 29} V_{\mathrm{r}}(p)\cdot V_{\mathrm{s}}(2)V_{\mathrm{s}}(3)V_{\mathrm{s}}(1151727)^4 V_{\mathrm{s}}(10^{14})^{234} V_{\mathrm{s}}(10^{42})^{545} \ge 0.00036$$

and

$$\tilde{E}_\psi^+(1/4) \le \prod_{p\le 29} Y_{\mathrm{r}}(p)\cdot Y_{\mathrm{s}}(2)Y_{\mathrm{s}}(3)Y_{\mathrm{s}}(1151727)^4 Y_{\mathrm{s}}(10^{14})^{234} Y_{\mathrm{s}}(10^{42})^{545} \le 11922,$$

while we find $|\arg E_\psi^+(1/2+it)| \le 17.510|t|$ as it is

$$\le |t|\Bigl(\sum_{p\le 29} W(p) + 2W(2) + 2W(3) + 8W(1151727) + 468W(10^{14}) + 1090W(10^{42})\Bigr).$$

15.1.3. Next we imitate Lemma 12.3.5 and apply (26) with $k = 12461947$, breaking into two cases depending on whether $k|D$ or not. (Here $\xi_0 \approx 0.0024972078778$). Independent of whether $\gcd(k,D)$ is 1 or k, when $\sqrt{D}L_\chi(1) \le 1.51\log D$ we have

$$|M_2| \le 14.868\cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2} \le 20939\frac{1.51\log D}{D^{1/4}} \le 10^{-14}$$

and

$$|M_3| \le 3.322\cdot \tilde{E}_\psi^+(1/4)\Bigl(\frac{2\pi g}{k\sqrt{D}}\Bigr)^{1/4}\cdot \prod_{p|(k/g)}\Bigl(1+\frac{1}{\sqrt{p}}\Bigr) \le \frac{5.26\cdot 11922}{D^{1/8}} \le 10^{-6}.$$

When $\gcd(k,D) = 1$, since $\xi_3 = 2|\Gamma(1/2+i\xi_0)|\cdot|\zeta_k(1+2i\xi_0)| \approx 709.76$ we have

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\bigr]\Bigr| \le \frac{1.01\cdot 10^{-6}}{709|E_\psi^+(1/2+i\xi_0)|} \le 10^{-5}.$$

Our range of D implies that $0.249 \le \xi_0 \log\sqrt{D} \le 1.249$, while $\xi_2 \approx 0.034189907$ and $|\arg E^+_\psi(1/2+i\xi_0)| \le 17.510\xi_0 \le 0.044$, so that

$$\begin{aligned}&\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E^+_\psi(1/2+i\xi_0)\bigr]\\ &\qquad\qquad \ge \min\bigl(\sin[0.249+0.034-0.044],\sin[1.249+0.035+0.044]\bigr)\ge 0.234.\end{aligned}$$

This is a contradiction, so our assumption $\sqrt{D}L_\chi(1)\le 1.51\log D$ must be incorrect.

When $g=\gcd(k,D)=k$ we can simply re-use the same initial bounds, while now we have $\xi_2=\xi_0\log(1/2\pi)+\arg\bigl[i\Gamma(1/2+i\xi_0)\zeta(1+2i\xi_0)\bigr]\approx -0.0066$ so that

$$\begin{aligned}&\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E^+_\psi(1/2+i\xi_0)\bigr]\\ &\qquad\qquad \ge \min\bigl(\sin[0.249-0.007-0.044],\sin[1.249-0.006+0.044]\bigr)\ge 0.196.\end{aligned}$$

This again is a contradiction, and we conclude the Lemma. □

15.2. Next we specialize to the real quadratic case, using our assumption about the size of the fundamental unit. For the larger D-range we can still use $k=12461947$. Other than using Mollin's Lemma, the proof is the same as above. It will be slightly more convenient to have the wider condition $B\le D^{1/4}+1/\sqrt{D}$ rather than $B\le D^{1/4}$; thus in particular, when $B>D^{1/4}+1/\sqrt{D}$ we have $A^2\ge B^2D-4>(D^{3/4}-1)^2$ so that $\epsilon_0=(A+B\sqrt{D})/2>[(D^{3/4}-1)+(D^{3/4}+1)]/2=D^{3/4}$.

Lemma 15.2.1. *Suppose that we have $\Delta>0$ with $h_K=1$, while $\epsilon_0=(A+B\sqrt{D})/2$ has $B\le D^{1/4}+1/\sqrt{D}$. Then for $50\le\log D\le 200$ we have $\sqrt{D}L_\chi(1)>1.51\log D$.*

The assumption on B also implies $\sqrt{D}L_\chi(1)=2h_K\log\epsilon_0\le 1.501\log D$, and thus one could further conclude that no such K exists, but we do this later in §15.5.

Proof. We again suppose $\sqrt{D}L_\chi(1)\le 1.51\log D$ and will get a contradiction. First we note that $h_K=1$ implies D has at most 2 prime divisors by the theory of genera (§3.2.2).

By Corollary 13.1.3 and the condition $B\le D^{1/4}+1/\sqrt{D}$, the smallest split prime is

$$\ge \frac{(A-\delta-1)}{B^2}\ge\frac{(\sqrt{DB^2+4\delta}-\delta-1)}{B^2}\ge D^{1/4}-1\ge 268000, \tag{28}$$

and as $0.304\sqrt{D}L_\chi(1)<0.46\log D\le 92$ there are at most 45 split primes up to $\sqrt{D}/10$ by Lemma 12.3.2, and similarly at most $0.52\sqrt{D}L_\chi(1)<158$ up to $\sqrt{D}/2$ by Corollary 5.1.2.

Thus we find (in the notation of §12.3.3) that

$$|E^+_\psi(1/2+it)|\ge V_{\mathrm{r}}(2)V_{\mathrm{r}}(\sqrt{D})V_{\mathrm{s}}(268000)^{45}V_{\mathrm{s}}(\sqrt{D}/10)^{112}\ge 0.245$$

and

$$\tilde{E}^+_\psi(1/4)\le Y_{\mathrm{r}}(2)Y_{\mathrm{r}}(\sqrt{D})Y_{\mathrm{s}}(268000)^{45}V_{\mathrm{s}}(\sqrt{D}/10)^{112}\le 209,$$

while $|\arg E^+_\psi(1/2+it)|$ is bounded as

$$|\arg E^+_\psi(1/2+it)|\le|t|\bigl(W(2)+W(\sqrt{D})+90W(268000)+224W(\sqrt{D}/10)\bigr)\le 3.911|t|.$$

15.2.2. As before, we use (26) with $k=12461947$, which has $\xi_0\approx 0.0024972078778$. When $\sqrt{D}L_\chi(1)\le 1.51\log D$ we get (independent of $\gcd(k,D)$ for now)

$$|M_2|\le 14.868\cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2}\le 20939\frac{1.51\log D}{D^{1/4}}\le 5.892$$

and

$$|M_3|\le 3.322\cdot\tilde{E}^+_\psi(1/4)\Bigl(\frac{2\pi g}{k\sqrt{D}}\Bigr)^{1/4}\cdot\prod_{p|(k/g)}\Bigl(1+\frac{1}{\sqrt{p}}\Bigr)\le\frac{5.26\cdot 209}{D^{1/8}}\le 2.123.$$

For $\gcd(k,D)=1$, as $\xi_3=2|\Gamma(1/2+i\xi_0)|\cdot|\zeta_k(1+2i\xi_0)|\approx 709.76$, from (26) we have

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E^+_\psi(1/2+i\xi_0)\bigr]\Bigr|\le\frac{5.892+2.123}{709\cdot|E^+_\psi(1/2+i\xi_0)|}\le 0.047.$$

Our range of D implies that $0.062 \leq \xi_0 \log\sqrt{D} \leq 0.250$, while $\xi_2 \approx 0.034189907$ and $|\arg E_\psi^+(1/2+i\xi_0)| \leq 3.911\xi_0 \leq 0.010$, so that

$$\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr] \\ \geq \min\bigl(\sin[0.062+0.034-0.010], \sin[0.250+0.035+0.010]\bigr) \geq 0.085.$$

This is a contradiction, so our assumption $\sqrt{D}L_\chi(1) \leq 1.51\log D$ must be incorrect.

When $\gcd(k,D)=k$ we improve the first bound from 5.892 to 0.002, so that

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr]\Bigr| \leq \frac{2.125}{709\cdot|E_\psi^+(1/2+i\xi_0)|} \leq 0.013,$$

while the shifted $\xi_2 = \xi_0\log(1/2\pi)+\arg\bigl[i\Gamma(1/2+i\xi_0)\zeta(1+2i\xi_0)\bigr] \approx -0.00661$ gives us

$$\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr] \\ \geq \min\bigl(\sin[0.062-0.007-0.010], \sin[0.250-0.006+0.010]\bigr) \geq 0.044.$$

Again this is a contradiction, and we conclude $\sqrt{D}L_\chi(1) \geq 1.51\log D$. □

15.3. The remaining range $D \leq \exp(50) \approx 5.18\cdot 10^{21}$ is already quite feasible for sieving.[34] However, we can also handle the range $37 \leq \log D \leq 53$ via $k=17923$, with the proof *mutatis mutandis* of the previous Lemma.[35]

Lemma 15.3.1. *Suppose that we have $\Delta > 0$ with $h_K = 1$, while $\epsilon_0 = (A+B\sqrt{D})/2$ has $B \leq D^{1/4}+1/\sqrt{D}$. Then for $37 \leq \log D \leq 53$ we have $\sqrt{D}L_\chi(1) \geq 1.51\log D$.*

Proof. We again suppose $\sqrt{D}L_\chi(1) \leq 1.51\log D$ and will get a contradiction. First we note that $h_K=1$ implies D has at most 2 prime divisors by the theory of genera.

Next, by Corollary 13.1.3 the smallest split prime is

$$\geq \frac{(A-\delta-1)}{B^2} \geq \frac{(\sqrt{DB^2+4\delta}-\delta-1)}{B^2} \geq D^{1/4}-1 \geq 10^4,$$

and as $0.304\sqrt{D}L_\chi(1) \leq 0.46\log D \leq 24.4$ by Lemma 12.3.2 there are at most 11 split primes up to $\sqrt{D}/10$, and by Corollary 5.1.2 at most $0.52\sqrt{D}L_\chi(1) < 42$ up to $\sqrt{D}/2$.

Thus we find that

$$|E_\psi^+(1/2+it)| \geq V_{\mathrm{r}}(2)V_{\mathrm{r}}(\sqrt{D})V_{\mathrm{s}}(10^4)^{11}V_{\mathrm{s}}(\sqrt{D}/10)^{30} \geq 0.230$$

and

$$\tilde{E}_\psi^+(1/4) \leq Y_{\mathrm{r}}(2)Y_{\mathrm{r}}(\sqrt{D})Y_{\mathrm{s}}(10^4)^{11}Y_{\mathrm{s}}(\sqrt{D}/10)^{30} \leq 48.111$$

while $|\arg E_\psi^+(1/2+it)|$ is bounded as

$$\leq |t|\bigl(W(2)+W(\sqrt{D})+22W(10^4)+60W(\sqrt{D}/10)\bigr) \leq 4.018|t|.$$

15.3.2. This time we use (26) with $k=17923$ which has $\xi_0 \approx 0.0309857994985$, and when $\sqrt{D}L_\chi(1) \leq 1.51\log D$ we have

$$|M_2| \leq 14.868\cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2} \leq 795\frac{1.51\log D}{D^{1/4}} \leq 4.269$$

and

$$|M_3| \leq 3.322\cdot\tilde{E}_\psi^+(1/4)\Bigl(\frac{2\pi g}{k\sqrt{D}}\Bigr)^{1/4}\cdot\prod_{p|(k/g)}\Bigl(1+\frac{1}{\sqrt{p}}\Bigr) \leq \frac{5.26\cdot 48.111}{D^{1/8}} \leq 2.481.$$

For $\gcd(D,k)=1$, as $\xi_3 = 2|\Gamma(1/2+i\xi_0)|\cdot|\zeta_k(1+2i\xi_0)| \approx 57.084$, from (26) we have

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr]\Bigr| \leq \frac{4.269+2.481}{57\cdot|E_\psi^+(1/2+i\xi_0)|} \leq 0.515.$$

[34] For instance, for pseudosquares (where all the χ-values are $+1$ instead of -1), Wooding and Williams [78] reached $120120\cdot 2^{64} \approx 2.2\cdot 10^{24}$ (later extended by Sorenson).

[35] While we could derive a version of Lemma 13.3.1 when $k|D$ (involving accounting the residue from $s=0$), it turns out this is not necessary.

Our range of D implies that $0.573 \le \xi_0 \log\sqrt{D} \le 0.822$, while $\xi_2 \approx 0.221562909$ and $\bigl|\arg E_\psi^+(1/2+i\xi_0)\bigr| \le 4.018\xi_0 \le 0.125$, so that

$$\begin{aligned}\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr]&\\ \ge \min\bigl(\sin[0.573+0.221-0.125],\sin[0.822+0.222+0.125]\bigr) &\ge 0.620.\end{aligned}$$

This is a contradiction, so our assumption $\sqrt{D}L_\chi(1) \le 1.51\log D$ must be incorrect.

When $\gcd(k,D)=k$ we improve the first bound from 4.269 to 0.032, so that

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr]\Bigr| \le \frac{2.513}{57\cdot|E_\psi^+(1/2+i\xi_0)|} \le 0.192,$$

while the shifted $\xi_2 = \xi_0\log(1/2\pi)+\arg\bigl[i\Gamma(1/2+i\xi_0)\zeta(1+2i\xi_0)\bigr] \approx -0.0819$ gives us

$$\begin{aligned}\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr]&\\ \ge \min\bigl(\sin[0.573-0.082-0.125],\sin[0.822-0.081+0.125]\bigr) &\ge 0.357.\end{aligned}$$

Again this is a contradiction, and we conclude $\sqrt{D}L_\chi(1) \ge 1.51\log D$. □

15.4. We thus have the range $D \le \exp(37) \approx 1.17\cdot 10^{16}$ left to sieve. As with (28), by Corollary 13.1.3 for each D we only need to find one split prime up to $(D^{1/4}-1)$ to show our desired conclusion that $h_K > 1$.

In particular, it is relatively easy to precondition our sieving modulo small primes, say up to 20, thus dividing into 77597520 congruence classes, of which only 114984 are viable (being fundamental, plus having no split primes and at most one ramified prime up to 20). We then used the 12 primes $23 \le p \le 71$ with 64-bit XOR stamps as in §14.2, and processed the unsieved D individually. This computation took about 50 minutes, with the worst example being $D = 947147572030805$, for which $p = 251$ was needed.

The range $D \le 72^4 \le 27\cdot 10^6$ is still left to handle, where we simply computed the class group for $D \le 10^4$, and again found small split primes for the remainder.

15.5. We recapitulate our proof of Theorem 1.2.6.

We sieved the range $D \le \exp(37)$, finding the 22 listed examples with $B \le D^{1/4}$ and $h_K = 1$. (Note that $D = 69$ has $B = 3 \le D^{1/4}+1/\sqrt{D}$, but $B > D^{1/4}$).

For $37 \le \log D \le 10^8$ we have that $2h_K\log\epsilon_0 = \sqrt{D}L_\chi(1) \ge 1.51\log D$ by the combination of Lemmata 15.3.1, 15.2.1, and 15.1.1, and Proposition 12.3.1. When $h_K = 1$ and $B \le D^{1/4}+1/\sqrt{D}$ we have

$$2h_K\log\epsilon_0 \le 2\log\Bigl(\frac{A+B\sqrt{D}}{2}\Bigr) \le 2\log\Bigl(\frac{\sqrt{(D^{3/4}+1)^2+4}+(D^{3/4}+1)}{2}\Bigr),$$

and as this is $\le 1.501\log D$ for $D \ge e^{37}$ there are no such D in this range.

For $\log D \ge 10^8$ we use Theorem 10.3.2, noting that D has at most 2 prime factors when $h_K = 1$ by the theory of genera. Since $(\log D)^3/10^{13} \ge 1000\log D$ this readily implies

$$\sqrt{D}L_\chi(1) = 2h_K\log\epsilon_0 \ge (1000\log D)\Bigl(1-\frac{3}{4}\Bigr)\Bigl(1-\frac{2D^{1/4}}{D^{1/2}}\Bigr) \ge 249\log D,$$

and as above this is a contradiction when $B \le D^{1/4}+1/\sqrt{D}$ and $h_K = 1$.

15.5.1. Finally, let us collate the above into a quotable bound on $\sqrt{D}L_\chi(1)$

Lemma 15.5.2. *When* $1253 < D \le e^{200}$ *and* $\Delta > 0$ *we have* $\sqrt{D}L_\chi(1) \ge 1.5\log D$. *When* $907 < D \le e^{200}$ *and* $\Delta < 0$ *we have* $\sqrt{D}L_\chi(1) \ge 1.5\log D$.

Proof. The real quadratic case follows as above. Indeed, when $B > D^{1/4}+1/\sqrt{D}$ we have $\epsilon_0 > D^{3/4}$ which implies $2h_K\log\epsilon_0 = \sqrt{D}L_\chi(1) \ge 1.5\log D$, while when $h_K = 1$ and $B \le D^{1/4}+1/\sqrt{D}$ we use the above sieving computation for $1253 < D \le e^{37}$ and Lemmata 15.3.1 and 15.2.1 for $37 \le \log D \le 200$.

Meanwhile, the imaginary quadratic case follows similarly, using the classification in [71, Table 4] for class numbers up through 100. In particular, said classification shows for $2383747 < D \le e^{200}$ we have $\sqrt{D}L_\chi(1) = \pi h_K \ge 101\pi > 300 \ge 1.5 \log D$, while the range $907 < D \le 2383747$ can be handled on a case-by-case basis. □

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Appendix A. Various technical results

In this appendix we give some additional details for some proofs where we felt it would be distracting to do so in the main text, and expand some details with §4 to include elliptic curves that have potentially multiplicative or additive primes.

A.1. We first give some results regarding continued fractions and real quadratic fields.

This material could logically appear after §§3.5.1-3.5.2 in the main text (and we borrow freely from the notation introduced there), but it seems more expeditious to break it off separately. In particular, there seems to be an error (missing a factor of 2) in the proof of Goldfeld [28, Lemma 4], and our Lemma §A.1.10 gives a version of this.

Meanwhile, our Lemma §A.1.3 improves the statement given by Goldfeld and Schinzel in [30, page 578]. This seems like it should be classical,[36] but I was only able to find a rather recent (2010) version of Jun and Lee [38, Lemma 2.1, Lemma 2.4(1)], who exploit an interpretation in terms of ideals. We instead, perhaps in a bit of a long-winded manner, give a proof using continued fractions, along the lines suggested by [30].

A.1.1. We introduce some notation. We let $\bar{\mathcal{Q}}_\Delta$ be the set of reduced forms of discriminant $\Delta > 0$, and $\bar{\mathcal{Q}}^{\mathrm{c}}_\Delta$ the canonical reduced forms. We write $U_v = \left(\begin{smallmatrix} A_{v-1} & (-1)^v A_v \\ B_{v-1} & (-1)^v B_v \end{smallmatrix}\right)$ for the vth orbit operator on reduced forms (§3.5.2), and this is with respect to an initial ω. For a form $F = \langle a, b, c\rangle$ we write $F^- = \langle -a, b, -c\rangle$. In the case of fundamental unit of norm -1 we have $F^- = FU_k$ where k is the period length of ω_F (and otherwise we have $FU_k = F$). Finally, we write $\bar{\mathcal{Q}}^{\mathrm{c},\pm}_\Delta$ as corresponding to a set of $\mathbf{SL}_2(\mathbf{Z})$-representatives, so that this is equal to $\bar{\mathcal{Q}}^{\mathrm{c}}_\Delta$ when the fundamental unit has norm -1, and otherwise includes both F and F^- as F runs over $\bar{\mathcal{Q}}^{\mathrm{c}}_\Delta$.

We then have

$$\bar{\mathcal{Q}}_\Delta = \bigcup_{F\in\bar{\mathcal{Q}}^{\mathrm{c}}_\Delta} \bigcup_{v=0}^{k-1} \{FU_v, F^-U_v\} = \bigcup_{F\in\bar{\mathcal{Q}}^{\mathrm{c},\pm}_\Delta} \bigcup_{v=0}^{k'-1} \{FU_v\}$$

where $k' = \mathrm{lcm}[2,k]$. As before, the period length k can depend on the class of F, and as above U_v depends on F. We define the multiset (with $\dot{\{}\,\dot{\}}$ as notation) of leading coefficients of reduced forms via

$$\bar{\mathcal{M}}_\Delta = \dot{\{}\, F(1,0) : F \in \bar{\mathcal{Q}}_\Delta \,\dot{\}} = \bigcup_{F\in\bar{\mathcal{Q}}^{\mathrm{c}}_\Delta} \bigcup_{v=0}^{k-1} \dot{\{}\, a_v(F), a_v(F^-) \,\dot{\}} = \bigcup_{F\in\bar{\mathcal{Q}}^{\mathrm{c},\pm}_\Delta} \bigcup_{v=0}^{k'-1} \dot{\{}\, a_v(F) \,\dot{\}}; \quad (29)$$

this contains both positive and negative elements, and indeed n always appears with the same multiplicity as $-n$, so we define the positive submultiset therein as

$$\bar{\mathcal{M}}^+_\Delta = \bigcup_{F\in\bar{\mathcal{Q}}^{\mathrm{c}}_\Delta} \bigcup_{v=0}^{k-1} \dot{\{}\, |a_v(F)| \,\dot{\}} = \bigcup_{\langle a,b,c\rangle} \dot{\{}\, |a_v| : 1 \le v \le k \,\dot{\}}.$$

[36] Dickson's *Introduction to the Theory of Numbers* (1929) gives a related result (Theorem 85) ascribed to Lagrange, but this fails to account representations as primary (and also skirts multiplicity issues).

In the final expression, we switched back to the $\langle a,b,c\rangle$-notation (running over $\bar{\mathcal{Q}}^{\mathrm{c}}_{\Delta}$) and consider a_v to depend on said form (similarly with k of course).

A.1.2. We have the following property for $n \le \sqrt{D}/2$.

Lemma A.1.3. *For $n \le \sqrt{D}/2$ the multiplicity of n in $\mathcal{M}^+_\Delta$ is equal to $R^\star_K(n)$.*

Proof. Recall that $R^\star_K(n)$ is the number of primitive primary representations $F(p,q)=n$, where F runs over representatives of $\mathbf{SL}_2(\mathbf{Z})$-classes of forms of discriminant Δ and "primary" indicates (p,q) is in a certain region of the plane [19, §6 (10,11)]), namely that we have $p-q\omega_F>0$ and $1\le |p-q\bar\omega_F|/|p-q\omega_F|<\tilde\epsilon_0^2$ where $\tilde\epsilon_0$ is the smallest totally positive unit, so that $\tilde\epsilon_0=\epsilon_0$ when ϵ_0 has norm $+1$ and $\tilde\epsilon_0=\epsilon_0^2$ when it has norm -1.

We can re-interpret $R^\star_K(n)$ as half the combined number of primitive primary representations of $F(p,q)=n$ and $F(p,q)=-n$, where F runs over $\mathbf{SL}_2(\mathbf{Z})$-representatives,[37] which we can take for instance as $F\in\bar{\mathcal{Q}}^{\mathrm{c},\pm}_{\Delta}$.

A.1.4. We first show that the multiplicity of $|n|$ in $\bar{\mathcal{M}}^+_\Delta$ is no more than $R^\star_K(|n|)$ (this fact is indeed true independent of the condition $|n|\le\sqrt{D}/2$). For this we note that

$$\begin{aligned}\bar{\mathcal{M}}_\Delta = \dot{\{}\, F(1,0) : F\in\bar{\mathcal{Q}}_\Delta \,\dot{\}} &= \bigcup_{F\in\bar{\mathcal{Q}}^{\mathrm{c},\pm}_{\Delta}} \bigcup_{v=0}^{k'-1} \dot{\{}\, FU_v(1,0)\,\dot{\}}\\ &= \bigcup_{F\in\bar{\mathcal{Q}}^{\mathrm{c},\pm}_{\Delta}} \bigcup_{v=0}^{k'-1} \dot{\{}\, F\big((-1)^vA_{v-1},(-1)^vB_{v-1}\big)\,\dot{\}}\end{aligned}$$

and we will show each of these representations from $(p,q)=\big((-1)^vA_{v-1},(-1)^vB_{v-1}\big)$ is primary (and clearly primitive). First we can note that the question of whether (p,q) is primary is the same for F and F^-, as these both use the same ω. Thus we can assume that $a>0$ for F.

We then proceed: the sign $(-1)^v$ ensures that $p-q\omega>0$, while the other condition follows by noting (see §3.5.1) that the quotients $|A_l-B_l\bar\omega|/|A_l-B_l\omega|$ are increasing in l, while at the same time

$$\frac{|p-q\bar\omega|}{|p-q\omega|}=\frac{|A_{v-1}-B_{v-1}\bar\omega|}{|A_{v-1}-B_{v-1}\omega|}=\frac{(A_{v-1}-B_{v-1}\bar\omega)^2}{|(A_{v-1}-B_{v-1}\omega)(A_{v-1}-B_{v-1}\bar\omega)|}=\frac{(A_{v-1}-B_{v-1}\bar\omega)^2}{|a_v|/|a|}$$

with the latter expression equal to $\tilde\epsilon_0^2$ for $v=k'=\mathrm{lcm}[k,2]$. We thus see that

$$1=\frac{|A_{-1}-B_{-1}\omega|}{|A_{-1}-B_{-1}\bar\omega|}\le\frac{|p-q\omega|}{|p-q\bar\omega|}<\frac{|A_{k'-1}-B_{k'-1}\omega|}{|A_{k'-1}-B_{k'-1}\bar\omega|}=\tilde\epsilon_0^2,$$

so that the representation is indeed primary.

Thus the combined multiplicity of $-n$ and n in $\bar{\mathcal{M}}_\Delta$ is no more than the combined number of primitive primary representations of $-n$ and n, which is twice $R^\star_K(n)$. Due to $-n$ and n appearing with the same multiplicity in $\bar{\mathcal{M}}_\Delta$, this in turn implies the multiplicity of $|n|$ in $\bar{\mathcal{M}}^+_\Delta$ is no more than $R^\star_K(n)$.

A.1.5. For the other direction, we want to show that when $0\ne|n|\le\sqrt{D}/2$ each primary primitive representation $F(p,q)$ of n with $F\in\bar{\mathcal{Q}}^{\mathrm{c},\pm}_{\Delta}$ yields a continued fraction convergent p/q to ω_F, and moreover this p/q is A_{v-1}/B_{v-1} for some $0\le v<k'$. This will then imply that twice $R^\star_K(n)$ is bounded above by the combined multiplicity of n and $-n$ in the multiset union over canonical reduced forms F of the $\{a_v(F)\}$ over $0\le v<k'$, which as in (29) is the combined multiplicity of n and $-n$ in the multiset $\bar{\mathcal{M}}_\Delta$ of leading coefficients of reduced forms. Again this in turn will then imply for $n>0$ that $R^\star_K(n)$ is bounded above by multiplicity of n in $\bar{\mathcal{M}}^+_\Delta$.

[37] A beneficial example to have in mind is for $\mathbf{Q}(\sqrt{7\cdot 43})$, where one $\mathbf{SL}_2(\mathbf{Z})$-orbit has $(1,17,-3)$, $(-3,13,11)$, $(11,9,-5)$, $(-5,11,9)$, $(9,7,-7)$, $(-7,7,9)$, $(9,11,-5)$, $(-5,9,11)$, $(11,-13,-3)$, $(-3,17,1)$; here -3 is represented twice, while 3 is represented by $(3,13,-11)$ and $(3,17,-1)$ in the other orbit.

First we note that we can restrict to $n > 0$. Indeed when the fundamental unit has norm -1 a primary representation $F(p, q) = n$ gives a representation $FU_{\pm k}(p, q) = -n$, with the $\pm$ chosen to ensure this is primary. Meanwhile, in the case of norm $+1$ the conditions for $F(p, q) = n$ and $F^-(p, q) = -n$ to be primary are identical, since F and F^- have the same ω.

We then consider the case where $a > 0$, following the calculation with [30, (12)]. We write $F(p, q) = ap^2 + bpq + cq^2 = n$, and the primary condition $p - q\omega > 0$ combined with $n = a(p - q\omega)(p - q\bar{\omega})$ implies that $p - q\bar{\omega} > 0$ due to $a > 0$ and $n > 0$. The other primary condition then implies $p - q\bar{\omega} \geq p - q\omega$, which yields $q\omega \geq q\bar{\omega}$ so that $q \geq 0$. We also find that $p - q\bar{\omega} > q(\omega - \bar{\omega}) = q\sqrt{D}/a$. When $q = 0$ we simply have $p = 1$ and the convergent is $p/q = 1/0 = A_{v-1}/B_{v-1}$ for $v = 0$. Otherwise we compute that[38]

$$|p - q\omega| = \frac{n/a}{|p - q\bar{\omega}|} \leq \frac{n/a}{q\sqrt{D}/a} = \frac{n/\sqrt{D}}{q} < \frac{1}{2q}$$

since $n \leq \sqrt{D}/2$, so that p/q is a convergent to $\omega = (-b + \sqrt{D})/2a$, and thus we have $p/q = A_{v-1}/B_{v-1}$ for some $v \geq 1$. Moreover, we see that the condition that (p, q) is primary is equivalent to $1 \leq v < k'$ where $k' = \mathrm{lcm}[2, k]$, which again follows by noting that the quotients $|A_l - B_l\bar{\omega}|/|A_l - B_l\omega|$ are increasing in l, while

$$\frac{p - q\bar{\omega}}{p - q\omega} = \frac{A_{v-1} - B_{v-1}\bar{\omega}}{A_{v-1} - B_{v-1}\omega} = \frac{(A_{v-1} - B_{v-1}\bar{\omega})^2}{(A_{v-1} - B_{v-1}\omega)(A_{v-1} - B_{v-1}\bar{\omega})} = \frac{(A_{v-1} - B_{v-1}\bar{\omega})^2}{|a_v|/|a|} \quad (30)$$

with the latter expression equal to $\tilde{\epsilon}_0^2$ for $v = k'$. This shows the desired result when $a > 0$.

When $a < 0$ we instead consider $FU_1(U_1^{-1}\vec{w}) = n$ where $\vec{w} = (p, q)$ so that we have $U_1^{-1}\vec{w} = (-B_1p + A_1q, -B_0p + A_0q) = (-B_1p + q, -p) = (p', q')$. Here the leading coefficient of FU_1 is positive, while $\frac{|p'-q'U_1\bar{\omega}|}{|p'-q'U_1\omega|} = \frac{|\omega|}{|\bar{\omega}|}\frac{|p-q\bar{\omega}|}{|p-q\omega|}$ where $U_1\omega = 1/\omega_2 = 1/\omega - B_1$.

When this ratio is ≥ 1, then (p', q') is primary and the first part of the above argument implies $U_1^{-1}\vec{w} = \big(A_{v-1}(U_1\omega), B_{v-1}(U_1\omega)\big)$ for some convergent $A_{v-1}(U_1\omega)/B_{v-1}(U_1\omega)$ to $U_1\omega$ with $v \geq 0$. Undoing the effect of U_1 we find that $\vec{w} = (p, q) = \big(A_v(\omega), B_v(\omega)\big)$ for some $v \geq 0$ with respect to the convergents of ω. Using that (p, q) is primary, the second part (30) of the above argument applied to F^- (so that $a > 0$) then shows that $v < k' - 1$.

On the other hand, when $\frac{|p'-q'U_1\bar{\omega}|}{|p'-q'U_1\omega|} < 1$ we instead consider $(p'', q'') = U_{k'}(p', q')$, which is primary for $U_1\omega$, and the argument follows as before. $\square$

A.1.6. We give a utility result that we will use below; in particular it bounds the period of a purely periodic continued fraction in terms of the associated fundamental unit.

We first recall that a quadratic surd κ has a unique representation $(P + \sqrt{m^2D})/Q$ where m, P, Q are coprime integers with $m, Q > 0$ and D is a fundamental discriminant. We call m^2D the discriminant of κ, and when $m = 1$ we say that κ is fundamental. A reduced quadratic surd has $\kappa > 1$ and $-1 < \bar{\kappa} < 0$, and Galois showed these are precisely the quadratic surds whose continued fraction expansions are purely periodic [25].

Lemma A.1.7. *Suppose that κ is a reduced quadratic surd of fundamental discriminant D, and the continued fraction expansion $1/\kappa = [0, \overline{e_1, \ldots, e_k}]$ has z instances of $e_v \geq u$. Then the fundamental unit ϵ_0 of $\mathbf{Q}(\sqrt{D})$ satisfies $\epsilon_0 \geq [(u + \sqrt{u^2 + 4})/2]^z$, so in particular $z \leq \log \epsilon_0 / \log[(u + \sqrt{u^2 + 4})/2]$.*

With $u = 1$ this gives a bound for the period length of $k \leq \log \epsilon_0 / \log\big(\frac{1+\sqrt{5}}{2}\big)$.

Proof. As with §3.5.1, writing A_v/B_v for the convergents to $1/\kappa$, for $l \geq 1$ we have

$$\begin{aligned} \epsilon_0^l = A_{lk-1} - B_{lk-1}\bar{\kappa} &= B_{lk-1}\kappa - B_{lk-1}\bar{\kappa} - (B_{lk-1}\kappa - A_{lk-1}) \\ &= B_{lk-1}(\kappa - \bar{\kappa}) - (B_{lk-1}\kappa - A_{lk-1}), \end{aligned}$$

[38] I am not sure why Goldfeld and Schinzel [30] only have $|p - q\omega| \leq n/(2\sqrt{D}q)$ here, but it could be for brevity. In any case, our adaptation of [30] herein is notably burdened by chasing signs and sectors around to ensure representations are primary, while this inequality $|p - q\omega| < 1/2q$ is the main point.

and as $|B_{lk-1}\kappa - A_{lk-1}| \leq 1/B_{lk}$ we then have $B_{(l+1)k-1}/B_{lk-1} \to \epsilon_0$ in the limit $l \to \infty$. Writing $\tau = [\overline{u}]$, the fact that there are z instances of $e_v \geq u$ implies the above limiting ratio is at least as big as that for $B_{l+z}(\tau)/B_l(\tau)$ as $l \to \infty$ (where $A_i(\tau)/B_i(\tau)$ are the convergents to τ), and this limit is $\tau^z = \bigl[(u+\sqrt{u^2+4})/2\bigr]^z$. □

A.1.8. We next give a version of Goldfeld's bound for the sum of $R_K^{\leq}(n)$. Unfortunately, it seems that [28, Lemma 4, page 637] is wrong by a factor of 2, as we explain in the footnote below. Moreover, we have chosen $\sqrt{D}/2$ instead of $\sqrt{D}/4$ as our cutoff; this has two aspects to it, the first of which seems to be the inoptimality we noted above in Lemma A.1.3, but the second of which allows slightly superior results (in the real quadratic case) when the cutoff is reduced. As these variant bounds are convenient in some of our numerics, we additionally list such results.

Lemma A.1.9. *For $\Delta < 0$ and $D > 4$ we have*

$$\sum_n R_K^{\leq}(n) = \sum_{n \leq \sqrt{D}/2} R_K^{\star}(n) \leq \frac{1}{\pi}\sqrt{D}L_\chi(1) \leq 0.319\sqrt{D}L_\chi(1).$$

For $\Delta > 0$ we have

$$\sum_n R_K^{\leq}(n) = \sum_{n \leq \sqrt{D}/2} R_K^{\star}(n) \leq \#\bar{\mathcal{M}}_\Delta^+ \leq h_K \frac{\log \epsilon_0}{\log\bigl(\frac{1+\sqrt{5}}{2}\bigr)} \leq 1.040\sqrt{D}L_\chi(1).$$

Proof. When $\Delta < 0$, for $n \leq \sqrt{D}/2$ the multiplicity of n in the multiset of minima is $R_K^{\star}(n)$, so $\sum_n R_K^{\leq}(n) \leq h_K = \sqrt{D}L_\chi(1)/\pi \leq 0.319\sqrt{D}L_\chi(1)$.

When $\Delta > 0$, by Lemma A.1.3 for $n \leq \sqrt{D}/2$ the multiplicity of n in $\bar{\mathcal{M}}_\Delta^+$ is $R_K^{\star}(n)$. Summing over all classes, by Lemma A.1.7 we thus have[39]

$$\sum_{n \leq \sqrt{D}/2} R_K^{\star}(n) = \sum_{\substack{a \in \bar{\mathcal{M}}_\Delta^+ \\ a \leq \sqrt{D}/2}} 1 = \sum_{\langle a,b,c\rangle} \sum_{\substack{v=1 \\ |a_v| \leq \sqrt{D}/2}}^{k} 1 \leq \sum_{\langle a,b,c\rangle} \sum_{v=1}^{k} 1 \leq h_K \frac{\log \epsilon_0}{\log\bigl(\frac{1+\sqrt{5}}{2}\bigr)}.$$

Since $h_K \log \epsilon_0 = \sqrt{D}L_\chi(1)/2$ and $1/2\log\bigl(\frac{1+\sqrt{5}}{2}\bigr) \leq 1.040$, the result follows. □

Finally we have our variant of the above Lemma A.1.9 in the real case.

Lemma A.1.10. *For $\Delta > 0$ we have*

$$\sum_{n \leq \sqrt{D}/4} R_K^{\star}(n) \leq \frac{\sqrt{D}L_\chi(1)}{2\log(1+\sqrt{2})} \leq 0.568\sqrt{D}L_\chi(1).$$

More generally, for $\Delta > 0$ and integral $u \geq 1$ we have

$$\sum_{n \leq \sqrt{D}/2u} R_K^{\star}(n) \leq \frac{\sqrt{D}L_\chi(1)}{2\log\bigl(\frac{u+\sqrt{u^2+4}}{2}\bigr)}.$$

Goldfeld's bound for $n \leq \sqrt{D}/4$ should be $\sqrt{D}L_\chi(1)/\log 4 \approx 0.721\sqrt{D}L_\chi(1)$.

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[39]Goldfeld seems to be wrong by a factor of 2 in his analysis.

What I think has happened is that he has written (near the bottom of page 638) the sum over $\mathbf{SL}_2$-equivalent forms rather than $\mathbf{GL}_2$ (see the "properly unimodular transformation" on page 577 of [30]), and when the fundamental unit has norm $+1$ his sum thus has $2h_K$ such forms rather than the asserted h_K. Moreover, when the fundamental unit has norm -1, so the primitive period k is odd, the citation to [30, (17)] has ϵ_0 as the smallest totally positive unit rather than the fundamental unit, again inducing a factor of 2.

Actually finding an example where Goldfeld's bound is wrong appears difficult, as one would want forms with many 2's in the continued fraction expansion, and likely in more than one class.

Proof. As in the proof of Lemma A.1.9, for integral $u \geq 1$ we have

$$\sum_{n \leq \sqrt{D}/2u} R_K^\star(n) = \sum_{\langle a,b,c\rangle} \sum_{\substack{v=1 \\ |a_v| \leq \sqrt{D}/2u}}^{k} 1,$$

but now we additionally recall (§3.5.2) that $|a_v| = Q_v/2$ in terms of the complete quotients $\omega_v = (P_v + \sqrt{D})/Q_v$, so that $\omega_v \geq \sqrt{D}/Q_v \geq u$ and thus $e_v = \lfloor \omega_v \rfloor \geq u$.

This implies

$$\sum_{n \leq \sqrt{D}/2u} R_K^\star(n) \leq \sum_{\langle a,b,c\rangle} \sum_{\substack{v=1 \\ e_v \geq u}}^{k} 1 \leq h_K \frac{\log \epsilon_0}{\log\left(\frac{u+\sqrt{u^2+4}}{2}\right)} = \frac{\sqrt{D} L_\chi(1)}{2\log\left(\frac{u+\sqrt{u^2+4}}{2}\right)},$$

where we used Lemma A.1.7 and Dirichlet's class number formula. ☐

A.2. We next demonstrate a residue calculus exercise to calculate $\int_{-\infty}^{\infty} |\Gamma(3/2+it)|^2 \frac{\partial t}{2\pi}$ as with Lemma 3.3.4. (We also could just bound this computationally as ≤ 0.2501).

Lemma A.2.1. *We have* $\int_{-\infty}^{\infty} |\Gamma(3/2+it)|^2 \frac{\partial t}{2\pi} = 1/4$

Note that 5.13.2 of the DLMF asserts this is indeed $\Gamma(3)/2^3 = 1/4$.

Proof. We first recall Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$, which in turn implies $|\Gamma(1/2+it)|^2 = \Gamma(1/2+it)\Gamma(1/2-it) = \pi/\cosh \pi t$, so that by the recursion relation $\Gamma(3/2 \pm it) = (1/2 \pm it)\Gamma(1/2 \pm it)$ we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Gamma(3/2+it)|^2 \frac{\partial t}{2\pi} &= \int_{-\infty}^{\infty} \frac{\pi(1/4+t^2)}{\cosh \pi t} \frac{\partial t}{2\pi} = 2\int_0^\infty \frac{\pi}{(\varphi+1/\varphi)} \left[\frac{1}{4} + \frac{(\log\varphi)^2}{\pi^2}\right] \frac{\partial\varphi/\pi}{2\pi\varphi} \\
&= \int_{-\infty}^{\infty} \frac{\pi}{\varphi^2+1} \left[\frac{1}{4} + \frac{(\log\varphi)^2}{\pi^2}\right] \frac{\partial\varphi}{2\pi^2} - \int_0^\infty \frac{\pi}{\varphi^2+1} \frac{(\log\varphi+i\pi)^2 - (\log\varphi)^2}{\pi^2} \frac{\partial\varphi}{2\pi^2} \\
&= \frac{\pi}{8\pi^2}\frac{2\pi i}{2i} + \frac{\pi}{2\pi^2}\frac{(\log i)^2}{\pi^2}\frac{2\pi i}{2i} + \frac{\pi}{2\pi^2}\int_0^\infty \frac{\partial\varphi}{\varphi^2+1} = \frac{1}{8} - \frac{1}{8} + \frac{1}{2\pi}\frac{\pi}{2} = \frac{1}{4},
\end{aligned}$$

where we substituted $\varphi = e^{\pi t}$, extended the integral to the negative real axis via the relation $\log(-\varphi) = \log\varphi + \pi i$ for $\varphi > 0$, noted $\int_0^\infty \frac{\log\varphi}{\varphi^2+1}\partial\varphi = 0$ by $\varphi \to 1/\varphi$ symmetry, and computed the residue at $\varphi = i$. (With $(\log\varphi)^2$ the branch cut is from 0 to $-i\infty$, and the singularity about $\varphi = 0$ does not contribute). ☐

A.3. We next prove the positivity of the Mellin transform in Lemma 3.7.2.

A.3.1. We recall the Mellin transform of §3.7.1, namely that

$$M(u) = \int_{(2)} u^{-s} \frac{\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)}{s-1} \frac{\partial s}{2\pi i} = \int_{(2)} u^{-s} \frac{\Gamma(s-1/2)\Gamma(s-1)}{\Gamma(s/2-1/4)^2} \frac{\partial s}{2\pi i}$$

so that by Mellin inversion we have

$$\frac{\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)}{s-1} = \frac{\Gamma(s-1/2)\Gamma(s-1)}{\Gamma(s/2-1/4)^2} = \int_0^\infty u^s M(u) \frac{\partial u}{u}.$$

Lemma A.3.2. *We have that M is nonnegative, so that*

$$\int_0^\infty \sqrt{u}|M(u)|\,\partial u = \int_0^\infty \sqrt{u} M(u)\,\partial u = \frac{\Gamma(3/2-1/2)\Gamma(3/2-1)}{\Gamma(3/4-1/4)^2} = \frac{1}{\sqrt{\pi}} \leq 0.56419.$$

Proof. The main claim is that M is nonnegative, as the other parts follow readily.

First we consider small u. Here we first use Legendre's duplication formula to note that $\Gamma(s-1/2)\Gamma(s-1) = \sqrt{\pi}\Gamma(2s-2)2^{3-2s}$, and then we move the contour to the left to pick up residues at $s = 1-k/2$ for $k \geq 0$, finding

$$M(u) = \int_{(2)} u^{-s} \frac{\sqrt{\pi}\Gamma(2s-2)2^{3-2s}}{\Gamma(s/2-1/4)^2} \frac{\partial s}{2\pi i} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^\infty u^{k/2-1} \frac{(-1)^k}{k!} \frac{2^{1+k}}{\Gamma(1/4-k/4)^2}.$$

This series expansion suffices to conclude via computation that $M(u) > 0$ for $u \leq 100$ (say), using a rudimentary bound such as $1/\Gamma(1/4 - k/4)^2 \leq \Gamma(1 + k/4)^2 \leq \lceil k/4 \rceil !^2$ to bound the error when truncating the k-series.

For large u, we use the general theory of asymptotics for Meijer G-functions, as presented by Braaksma [9] (see also Luke [50, (5.7.13)]). This yields (as $u \to \infty$)

$$M(u) = \frac{e^{-u/2}}{\sqrt{8\pi u}}\big[1 + O(1/u)\big],$$

and we wish to obtain an explicit error term. This is turn is dependent on an explicit version of Stirling's approximation, where we adapt [9, §3]. We shall not give full details, but simply a sketch.

We begin by codifying Braaksma's notation for our case of $\frac{\Gamma(s-1/2)\Gamma(s-1)}{\Gamma(s/2-1/4)^2}$. Firstly we note that we have s for $-s$ throughout, and in his (1.3) we have $(m, n, p, q) = (2, 0, 2, 2)$ with $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1/2, 1/2, 1, 1)$ and $(a_1, a_2, b_1, b_2) = (-1/4, -1/4, -1/2, -1)$. We then have $\mu = 1$ in (1.8) and $\beta = 1/2$ in (1.10), while $\alpha = 3/2$ in (3.24) and we have $A_0 = \sqrt{2\pi}(1/2)^{3/2}$ in (3.28). The Mellin transform of interest is then defined in (2.24) and (2.25) and estimated (with $N = 1$) in (2.27) and (2.28) as

$$M(2w) = A_0(2\pi)^{-1}w^{1-3/2}\exp(-w) + O\big(w^{1-5/2}\exp(-w)\big),$$

which is equivalent to the above claim for $M(u)$ with $u = 2w$.

To make the error explicit, for the Stirling approxmation in (3.32) we have

$$\rho(s) = (2\pi)\frac{(1/2)^s}{\Gamma(1+s-3/2-1)}\frac{\Gamma(s-1/2)\Gamma(s-1)}{\Gamma(s/2-1/4)} - A_0\frac{\Gamma(1+s-3/2)}{\Gamma(1+s-3/2-1)}$$

where $|\rho(s)|$ is uniformly bounded as $|s| \to \infty$ away from a sector containing the negative real axis. We need only be interested in this for $\sigma \geq 2$ (say), and a mechanical application of explicit Stirling bounds (see [55] for instance) shows $|\rho(s)| \leq 0.125$. This in turn gives the asymptotic for $M(u)$ with an explicit relative error bounded by $0.3/u$, from which we deduce the asserted nonnegativity of M. □

A.4. We recall the definition (8) of $H_i = (U + \sqrt{U^2 - 4})/2$ where $U = (\sqrt{D}/|a|)B_i^2$, and prove Lemma A.4.1 regarding its growth.

Lemma A.4.1. *We have $H_{4k-1}\epsilon_0^4 \leq H_{6k-1}$ for $D \geq 5$.*

Proof. We write out the formula in §3.5.1 for ϵ_0^4 in terms of convergents and $\bar{\omega}$, and use the bound $|A_l - B_l\omega| \geq 1/(B_l + B_{l+1})$, while $A_{4k-1} - B_{4k-1}\omega > 0$ since $4k-1$ is odd, so

$$\begin{aligned}\epsilon_0^4 = A_{4k-1} - \bar{\omega}B_{4k-1} &= \big(A_{4k-1} - B_{4k-1}\omega\big) + B_{4k-1}(\omega - \bar{\omega}) \\ &\geq \frac{1}{B_{4k} + B_{4k-1}} + B_{4k-1}\frac{\sqrt{D}}{|a|}.\end{aligned}$$

In particular we have

$$B_{4k-1} \leq \frac{|a|}{\sqrt{D}}\Big(\epsilon_0^4 - \frac{1}{B_{4k} + B_{4k-1}}\Big),$$

and as $H_{4k-1} \leq \frac{\sqrt{D}}{|a|}B_{4k-1}^2$ this implies

$$H_{4k-1} \leq \frac{|a|}{\sqrt{D}}\Big(\epsilon_0^8 - \frac{2\epsilon_0^4}{B_{4k} + B_{4k-1}} + \frac{1}{(B_{4k} + B_{4k-1})^2}\Big).$$

We easily have $1/(B_{4k} + B_{4k-1})^2 \leq \epsilon_0^4/(B_{4k} + B_{4k-1})$, and since $\epsilon_0^4 \geq B_{4k-1}\sqrt{D}/|a|$ and $B_{4k}/B_{4k-1} = e_{4k} + B_{4k-2}/B_{4k-1} \leq \sqrt{D} + 1$ this implies

$$H_{4k-1} \leq \frac{|a|}{\sqrt{D}}\Big(\epsilon_0^8 - \frac{\epsilon_0^4}{B_{4k} + B_{4k-1}}\Big) \leq \frac{|a|}{\sqrt{D}}\epsilon_0^8 - \frac{B_{4k-1}}{B_{4k} + B_{4k-1}} \leq \frac{|a|}{\sqrt{D}}\epsilon_0^8 - \frac{1}{2 + \sqrt{D}}.$$

Meanwhile, we similarly have

$$\epsilon_0^6 \leq \frac{1}{B_{6k-1}} + B_{6k-1}\frac{\sqrt{D}}{|a|} \quad \text{so that} \quad B_{6k-1} \geq \frac{|a|}{\sqrt{D}}\Big(\epsilon_0^6 - \frac{1}{B_{6k-1}}\Big),$$

where with $U = (\sqrt{D}/|a|)B_{6k-1}^2 \ge 4$ we note $(U/2)[1+\sqrt{1-4/U^2}] \ge U(1-2/U^2)$, which in conjunction with $|a| \le \sqrt{D}$ gives

$$H_{6k-1} = \frac{U+\sqrt{U^2-4}}{2} \ge \frac{\sqrt{D}}{|a|}B_{6k-1}^2 - 2\frac{|a|/\sqrt{D}}{B_{6k-1}^2} \ge \frac{|a|}{\sqrt{D}}\Bigl(\epsilon_0^{12} - \frac{2\epsilon_0^6}{B_{6k-1}} + \frac{1}{B_{6k-1}^2}\Bigr) - 2$$
$$\ge \frac{|a|}{\sqrt{D}}\epsilon_0^{12} - \frac{|a|}{\sqrt{D}}\frac{2}{B_{6k-1}}\Bigl(\frac{1}{B_{6k-1}} + B_{6k-1}\frac{\sqrt{D}}{|a|}\Bigr) - 2 \ge \frac{|a|}{\sqrt{D}}\epsilon_0^{12} - 6,$$

where we dropped the $1/B_{6k+1}^2$ term by positivity and re-used the upper bound for ϵ_0^6.

We combine these bounds and use $\epsilon_0 \ge (\sqrt{D}+\sqrt{D-4})/2$, finding for $D \ge 8$ that

$$\epsilon_0^4 H_{4k-1} \le \frac{|a|}{\sqrt{D}}\epsilon_0^{12} - \frac{\epsilon_0^4}{2+\sqrt{D}} \le \frac{|a|}{\sqrt{D}}\epsilon_0^{12} - 6 \le H_{6k-1}$$

as desired. Moreover, when $D=5$ we have $H_3 = (3^2\sqrt{5}+\sqrt{5\cdot 3^4-4})/2 \approx 20.075$ so that $H_3\epsilon_0^4 = H_3\bigl(\frac{1+\sqrt{5}}{2}\bigr)^4 \approx 137.595$ while $H_5 = (8^2\sqrt{5}+\sqrt{5\cdot 8^4-4})/2 \approx 143.101$. □

A.5. Next we give an explicit convexity result for L_{S^2f}, as with Lemma 5.4.1.

Lemma A.5.1. *Writing $t_\star = |t|+5$, in the range $1 \le \sigma \le 2$ we have*

$$\bigl|L_{S^2f}(s)\bigr| \le 1.65\cdot(1+\log N_{S^2f}t_\star^3)^3\cdot\Bigl(\frac{N_{S^2f}t_\star^3}{8\pi^3}\Bigr)^{1-\sigma/2}.$$

Proof. Recalling that the 2-line is the edge of the critical strip for $L_{S^2f}(s)$, we thereby consider $\alpha = 2+1/(\log N_{S^2f}t_\star^3)$, noting $2<\alpha\le 2+1/\log(5^3)$ so as to confine its range.[40]

By the functional equation (see §4.2.3) for Λ_{S^2f}, for any real u we have

$$\frac{L_{S^2f}(3-\alpha-iu)}{L_{S^2f}(\alpha+iu)} = \frac{(\sqrt{N_{S^2f}}/2\pi^{3/2})^{\alpha+iu}}{(\sqrt{N_{S^2f}}/2\pi^{3/2})^{3-\alpha-iu}}\cdot\frac{\Gamma(\alpha+iu)\Gamma(\alpha/2+iu/2)}{\Gamma(3-\alpha-iu)\Gamma\bigl((3-\alpha-iu)/2\bigr)},$$

and we wish to bound the quotient of Γ-factors, calling it $G(\alpha+iu)$. By Stirling's formula we have $|G(\alpha+iu)| \sim u^{2\alpha-3}(u/2)^{\alpha-3/2}$ as $u\to\infty$, but we must make this explicit. For this we can use (for instance) the bound of Nemes [55, Theorem 1.3], which states that for $|\arg z| \le \pi/2$ we have $\Gamma(z) = \sqrt{2\pi}z^{z-1/2}e^{-z}\bigl[1+\frac{1/12}{z}+R_2(z)\bigr]$ where

$$|R_2(z)| \le \frac{1+\zeta(2)}{(2\pi)^{2+1}}\frac{\Gamma(2)}{|z|^2}\frac{2\sqrt{2}+1}{2} < \frac{0.021}{|z|^2}.$$

It is then a tedious and rather unenlightening exercise (using the above confinement of α) to show that $|G(\alpha+iu)| \le (u_\star^3/2)^{\alpha-3/2}$ here.[41]

Since $|L_{S^2f}(3-\alpha-iu)| = |L_{S^2f}(3-\alpha+iu)|$ this gives

$$\bigl|L_{S^2f}(3-\alpha+iu)\bigr| \le |L_{S^2f}(\alpha+iu)|\cdot\Bigl(\frac{N_{S^2f}u_\star^3}{8\pi^3}\Bigr)^{\alpha-3/2}.$$

Upon comparing the degree 3 Euler product of $L_{S^2f}(s)$ to that for $\zeta(s-1)^3$ we obtain that $|L_{S^2f}(\alpha+iu)| \le |\zeta(\alpha-1)^3| \le \bigl[1+1/(\alpha-2)\bigr]^3 = (1+\log N_{S^2f}t_\star^3)^3$, so that

$$\bigl|L_{S^2f}(3-\alpha+iu)\bigr| \le (1+\log N_{S^2f}t_\star^3)^3\cdot\Bigl(\frac{N_{S^2f}u_\star^3}{8\pi^3}\Bigr)^{\alpha-3/2}.$$

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[40] For a cruder version, we could simply work on $\sigma = 5/2$ (rather than $\sigma=\alpha$) using

$$L_{S^2f}(1/2-it) = L_{S^2f}(5/2+it)\frac{(N_{S^2f}/4\pi^3)^{5/4+it/2}\Gamma(5/2+it)\Gamma(5/4+it/2)}{(N_{S^2f}/4\pi^3)^{1/4-it/2}\Gamma(1/2-it)\Gamma(1/4-it/2)}.$$

Then we note $|L_{S^2f}(5/2+it)| \le \zeta(3/2)^3$ while $\frac{\Gamma(5/2+it)\Gamma(5/4+it/2)}{\Gamma(1/2-it)\Gamma(1/4-it/2)} \sim t^3/2$ as $t\to\infty$, with an explicit bound for the Γ-quotient as $\le (|t|+5)^3$, so convexity implies for $1/2\le\sigma\le 5/2$ that

$$|L_{S^2f}(\sigma+it)| \le 20(N_{S^2f}/4\pi^3)^{5/4-\sigma/2}(|t|+5)^{3(5/2-\sigma)/2}.$$

[41] This is rather easy to verify for $|u| \le 100$ (say) due to distinction between $|u|$ and $u_\star = |u|+5$, while asymptotically this same distinction ensures that the secondary term is negative.

Applying the convexity theorem of Phragmén and Lindelöf [60] for $3-\alpha\le\sigma\le\alpha$ gives

$$\bigl|L_{S^2f}(\sigma+iu)\bigr|\le(1+\log N_{S^2f}t_\star^3)^3\cdot\Bigl(\frac{N_{S^2f}u_\star^3}{8\pi^3}\Bigr)^{(\alpha-\sigma)/2}.$$

This is true in particular for $u=t$. Finally, replacing α by 2 in this bound induces a factor of $(N_{S^2f}t_\star^3/8\pi^3)^{1/2\log N_{S^2f}t_\star^3}\le\exp(1/2)\le 1.65$, giving the Lemma. □

A.6. Finally, we consider §4 when g has potentially multiplicative or additive primes.

A.6.1. We recall the setting of §4. We have a weight 2 modular newform g with integral coefficients that is associated to an elliptic curve A, and its twist by the fundamental discriminant B (allowing $B=1$) gives the modular form f and the elliptic curve E. Therein we essentially assumed that N_g is odd and squarefree, with $\gcd(N_g,B)=1$.

Now we only assume that g is globally twist-minimal, or equivalently that it is twist-minimal at all p. For our purposes, for odd primes p this means that $\mathrm{v}_p(N_g)\le\mathrm{v}_p(N_{g\psi_{p^\star}})$ where $p^\star=p\cdot(-1)^{(p-1)/2}$, and at $p=2$ we have $\mathrm{v}_p(N_g)\le\mathrm{v}_p(N_{g\psi_u})$ for $u\in\{-4,-8,8\}$. Here ψ_u is the Kronecker character of discriminant u, and v_p the p-valuation. (One can further distinguish twist-minimality when $\mathrm{v}_p(N_g)=\mathrm{v}_p(N_{g\psi_{p^\star}})$ via the discriminant valuations of the associated elliptic curves, but we need not do this here).

We can note (using either Tate's algorithm [66] for elliptic curves, or the theory of Atkin and Lehner [1] and/or Atkin and Li [2] for modular forms) that when g is twist-minimal that for $p\ge5$ we have $\mathrm{v}_p(N_{g\psi_u})=2$ when $p|u$. Meanwhile, for $p=3$ we have $\mathrm{v}_3(N_{g\psi_{-3}})=\max(\mathrm{v}_3(N_g),2)$, and for $p=2$ we have $\mathrm{v}_2(N_{g\psi_{-4}})=\max(\mathrm{v}_2(N_g),4)$ and $\mathrm{v}_2(N_{g\psi_{\pm8}})=\max(\mathrm{v}_2(N_g),6)$. We also have $\mathrm{v}_2(N_g)\notin\{4,6\}$ for twist-minimal g.

In Table 4a we give the analogue of Table 4 for the additional cases we now consider. We recall (see [48]) that for elliptic curves over $\mathbf{Q}$ we have $\mathrm{v}_p(N_g)\le2$ for $p\ge5$, and also $\mathrm{v}_3(N_g)\le5$ and $\mathrm{v}_2(N_g)\le8$. The entries in the table involving conductors of elliptic curves can again be determined by Tate's algorithm. For the symmetric-square conductor we refer to Coates and Schmidt [15], as catalogued in [69, §2.2].

We can consider first three columns of this table to be "inputs", with the other four then giving the indicated values as functions therein. However, there are two ambiguities: first, when $\mathrm{v}_2(B)=\mathrm{v}_2(D)=3$ with $1\le v=\mathrm{v}_2(N_g)\le3$, where $\mathrm{v}_2(N_{f\chi})$ is v or 4 depending on whether B and D have the same sign; and second when $\mathrm{v}_2(N_g)\ge7$ we can have $\mathrm{v}_2(N_{S^2f})$ as either 6 or 8, with the cases described in [69, Theorem 2.3].

The main conclusions from this are that $2\mathrm{v}_p(D)\le\mathrm{v}_p(N_f)+\mathrm{v}_p(N_fN_{f\chi})\le2\mathrm{v}_p(N_fD)$ and $\mathrm{v}_p(N_{S^2f})\le2\mathrm{v}_p(N_g)\le2\mathrm{v}_p(N_f)$, so that $D^2\le N_fN_{f\chi}\le(N_fD)^2$ and $N_{S^2f}\le N_f^2$.

A.6.2. In Table 5a we list the reciprocal Euler factors for $L_f^K(s)$ and $L_{S^2f}(2s)/L_{A^2f}(2s)$ for potentially multiplicative and additive primes. For the latter, we have $\gamma_p\in\{-p,0,p\}$ and a recipe is given in [69, Corollary 2.2, Theorem 2.3, Theorem 2.4] (involving nothing more than congruence and divisibility conditions) to determine it in any given case.

A.6.3. Finally, in Table 6a we list the corresponding values of $Z_p(s)$ and $V_p(s)$ at potentially multiplicative and additive primes.

As with (14), we then have

$$\frac{L_f^K(s)}{L_{S^2f}(2s)/L_{A^2f}(2s)}=\prod_{p|N_f}V_p(s)\cdot\prod_p\frac{1+\alpha_p'/p^s}{1-\alpha_p'\chi(p)/p^s}\frac{1+\beta_p'/p^s}{1-\beta_p'\chi(p)/p^s},$$

where we thus take $\alpha_p'=\alpha_p[\psi_B\chi](p)$ when p is potentially multiplicative with $p|D$, while $\alpha_p'=0$ otherwise, and $\beta_p'=0$ in all cases. We then note that the resulting $V_p(s)$ satisfies the bound used in Lemma 5.3.1, and also with $\mathcal{V}(f)$ in §4.3.3.

$v_p(N_g)$	$v_p(B)$	$v_p(D)$	$v_p(N_f)$	$v_p(N_{f\chi})$	$v_p(N_f D)$	$v_p(N_{S^2f})$
1	1	0	2	2	2	2
1	1	1	2	1	3	2
2	0,1	0	2	2	2	2
2	0,1	1	2	2	3	2

$v_3(N_g)$	$v_3(B)$	$v_3(D)$	$v_3(N_f)$	$v_3(N_{f\chi})$	$v_3(N_f D)$	$v_3(N_{S^2f})$
1	1	0	2	2	2	2
1	1	1	2	1	3	2
$v \geq 2$	0,1	0	v	v	v	$2\lceil v/2 \rceil$
$v \geq 2$	0,1	1	v	v	$v+1$	$2\lceil v/2 \rceil$

$v_2(N_g)$	$v_2(B)$	$v_2(D)$	$v_2(N_f)$	$v_2(N_{f\chi})$	$v_2(N_f D)$	$v_2(N_{S^2f})$
$1 \leq v \leq 3$	0	0	v	0	v	$2\lceil v/2 \rceil$
$1 \leq v \leq 3$	2	0	4	4	4	$2\lceil v/2 \rceil$
$1 \leq v \leq 3$	3	0	6	6	6	$2\lceil v/2 \rceil$
$1 \leq v \leq 3$	0	2	v	4	$v+2$	$2\lceil v/2 \rceil$
$1 \leq v \leq 3$	2	2	4	v	6	$2\lceil v/2 \rceil$
$1 \leq v \leq 3$	3	2	6	6	8	$2\lceil v/2 \rceil$
$1 \leq v \leq 3$	0	3	v	6	$v+3$	$2\lceil v/2 \rceil$
$1 \leq v \leq 3$	2	3	4	6	7	$2\lceil v/2 \rceil$
$1 \leq v \leq 3$	3	3	6	v,4	9	$2\lceil v/2 \rceil$
5	0,2	0	5	5	5	6
5	3	0	6	6	6	6
5	0,2	2	5	5	7	6
5	3	2	6	6	8	6
5	0,2	3	5	5	8	6
5	3	3	6	5	9	6
$v \geq 7$	0,2,3	0	v	v	v	6,8
$v \geq 7$	0,2,3	2	v	v	$v+2$	6,8
$v \geq 7$	0,2,3	3	v	v	$v+3$	6,8

TABLE 4A. Conductor valuations

$\chi(p)$	type	$L_f^K(s)$ reciprocal Euler factor	same for $L_{S^2f}(2s)/L_{A^2f}(2s)$
± 1	PM	1	$(1-\alpha_p^2/p^{2s})(1-p/p^{2s})^{-1}$
0	PM	$(1-\alpha_p[\psi_B\chi](p)/p^s)$	$(1-\alpha_p^2/p^{2s})(1-p/p^{2s})^{-1}$
any	A	1	$(1-\gamma_p/p^{2s})(1-p/p^{2s})^{-1}$

TABLE 5A. Reciprocal Euler factors

$\chi(p)$	type	$Z_p(s)$	$V_p(s)$
± 1	PM	1	$(1-\alpha_p^2/p^{2s})(1-p/p^{2s})^{-1}$
0	PM	$(1+\alpha_p[\psi_B\chi](p)/p^s)$	$(1-p/p^{2s})^{-1}$
any	A	1	$(1-\gamma_p/p^{2s})(1-p/p^{2s})^{-1}$

TABLE 6A. Values of $Z_p(s)$ and $V_p(s)$