Pencils of plane cubics revisited

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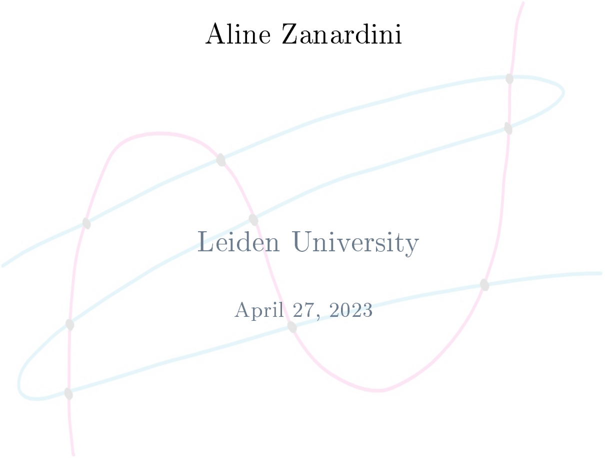


Leiden University

April 27, 2023

Plan for the talk:

- Miranda's work
- GIT for pencils of curves
    - a stability criterion
- Recovering Miranda's result

Miranda's work (1980)

His main theorem

A pencil of plane cubics $\mathcal{P}$ is stable if and only if $\mathcal{P}$ contains a smooth member and every fiber of the corresponding rational elliptic surface $X_{\mathcal{P}}$ is reduced.

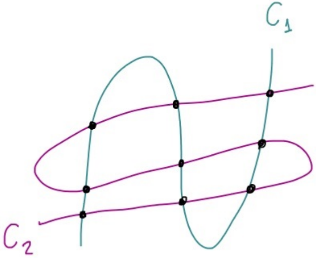
Miranda's work (1980)

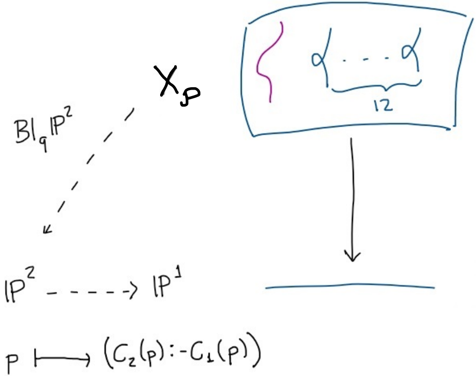
His main theorem

A pencil of plane cubics $\mathcal{P}$ is stable if and only if $\mathcal{P}$ contains a smooth member and every fiber of the corresponding rational elliptic surface $X_{\mathcal{P}}$ is reduced. Moreover, if $\mathcal{P}$ contains a smooth member, then $\mathcal{P}$ is semistable if and only if $X_{\mathcal{P}}$ does not contain a fiber of type II^*, III^* or IV^*.

Example (Pencils of cubics)

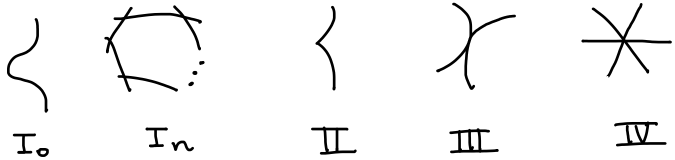
$\mathcal{P}: \lambda C_1 + \mu C_2 = 0$



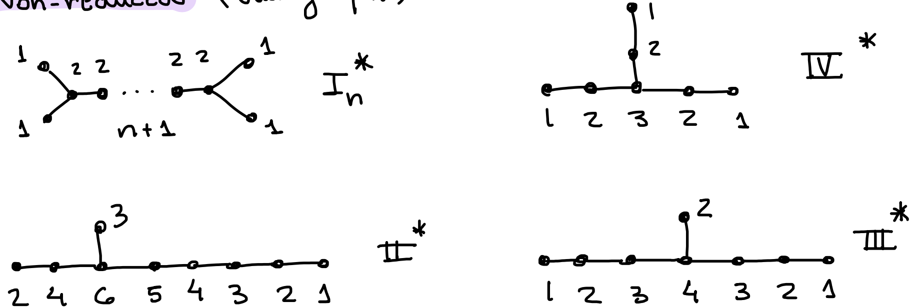


Kodaira's classification

Reduced



Non-reduced (dual graph)



Can we recover Miranda's result via a (slightly) different approach?

A classification problem

Goal

We are interested in classifying pencils of plane curves of degree d up to projective equivalence.

GIT stability of pencils of plane curves

- $V = H^0(\mathbb{P}^2, \mathcal{O}(1))$
- $G = SL(V)$
- $\mathscr{P}_d$: the space of pencils of plane curves of degree d
- $G \curvearrowright \mathscr{P}_d \simeq Gr(2, S^d V^*) = \underbrace{Gr(2,n)}_{\doteq X_d} \hookrightarrow \mathbb{P}(\Lambda^2 S^d V^*) = \mathbb{P}^N$

  (Plücker embedding)

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  (Plücker embedding)

- Here $n = \binom{d+2}{2}$ and $N = \binom{n}{2} - 1$
- A pencil $\mathcal{P} \in \mathscr{P}_d$, with generators $C_f : \sum f_{ij} x^i y^j z^{d-i-j} = 0$ and $C_g : \sum g_{ij} x^i y^j z^{d-i-j} = 0$, has Plücker coordinates given by all the 2×2 minors:

$$m_{ijkl} \doteq \begin{vmatrix} f_{ij} & f_{kl} \\ g_{ij} & g_{kl} \end{vmatrix}$$

A normalized one-parameter subgroup $\lambda : t \mapsto \begin{pmatrix} t^{a_x} & 0 & 0 \\ 0 & t^{a_y} & 0 \\ 0 & 0 & t^{a_z} \end{pmatrix}$ acts on the Plücker coordinates m_{ijkl} of a pencil $\mathcal{P}$ as follows:

$$m_{ijkl} \mapsto t^{r_{ijkl}} \cdot m_{ijkl}$$

where $r_{ijkl} \doteq a_x(i+k) + a_y(j+l) + a_z(2d-i-k-j-l)$.

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Definition

$$\omega(\mathcal{P}, \lambda) \doteq \min\{(a_x - a_z)(i+k) + (a_y - a_z)(j+l) \,:\, m_{ijkl} \neq 0\}$$

Hilbert-Mumford Criterion

A pencil $\mathcal{P} \in \mathscr{P}_d$ is unstable (resp. nonstable) if and only if there exists λ such that

$$\frac{\omega(\mathcal{P}, \lambda)}{(a_x - a_z) + (a_y - a_z)} > \frac{2d}{3} \quad \text{(resp. } \geq\text{)}$$

A key idea

Similarly, we can define

$$\omega(f,\lambda) \doteq \min\{(a_x - a_z)i + (a_y - a_z)j \ : \ f_{ij} \neq 0\}$$

and we can compare $\omega(\mathcal{P},\lambda)$ and $\omega(f,\lambda)$ for $C_f \in \mathcal{P}$

i.e. we also consider $G \curvearrowright S^d V^*$

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Proposition 1 (Z.)

Given $\mathcal{P}$, any two (distinct) curves $C_f, C_g \in \mathcal{P}$ and $\lambda : \mathbb{C}^\times \to G$ we have that

$$\omega(f,\lambda) + \omega(g,\lambda) \leq \omega(\mathcal{P},\lambda)$$

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Proposition 2 (Z.)

Given $\mathcal{P}$, any curve $C_f \in \mathcal{P}$ and $\lambda : \mathbb{C}^\times \to G$ there exists a curve $C_g \in \mathcal{P}$ such that

$$\omega(\mathcal{P},\lambda) = \omega(f,\lambda) + \omega(g,\lambda)$$

The stability criterion

Another key idea

Besides considering the action of G on S^dV^* we can further consider the action of G on $S^{2d}V^*$

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Theorem (joint w/ M. Hattori)

A pencil $\mathcal{P} \in X_d$ is GIT stable (resp. semistable) if and only if for any choice of generators $C_f, C_g \in \mathcal{P}$ the curve $C_f + C_g$ of degree $2d$ is GIT stable (resp. semistable).

Case $d = 3$ and Shah's work

A pencil $\mathcal{P} \in X_3$ is stable if and only if for any two cubics C_f, C_g in $\mathcal{P}$ the sextic curve $C_f + C_g$ satisfies the following conditions:

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A pencil $\mathcal{P} \in X_3$ is stable if and only if for any two cubics C_f, C_g in $\mathcal{P}$ the sextic curve $C_f + C_g$ satisfies the following conditions:

(i) it does not contain a multiple line as a component,

(ii) it does not have consecutive triple points, and

(iii) it does not have a point with multiplicity ≥ 4

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(i) it does not contain a multiple line as a component,

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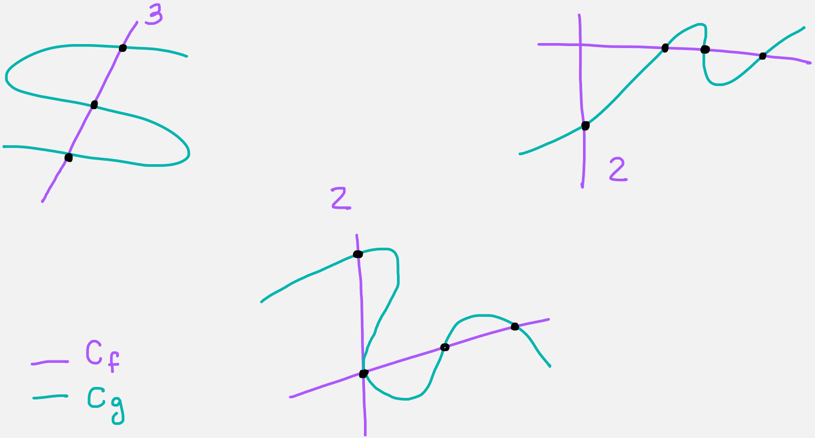
Therefore, $\mathcal{P} \in X_3$ is stable if and only if the following conditions hold:

(i') $\mathcal{P}$ contains a smooth member,

(ii') any curve in $\mathcal{P}$ is reduced, and

(iii') at a base point of $\mathcal{P}$ any curve in $\mathcal{P}$ is either smooth or it has at worst a node as singularity.

Similarly, we can show $\mathcal{P} \in X_3$ is unstable if and only if we can find two cubics C_f, C_g in $\mathcal{P}$ such that (up to relabeling):



Thank you!