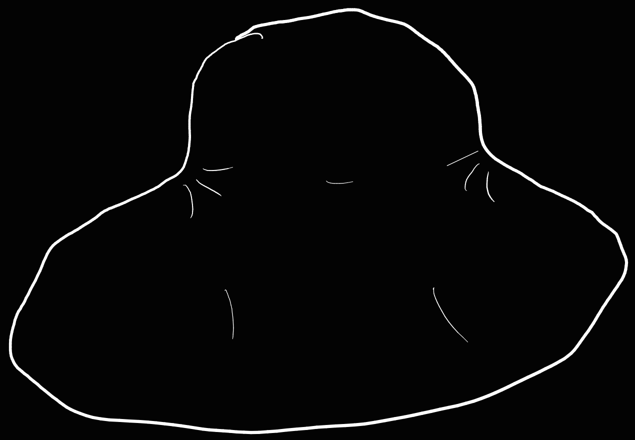
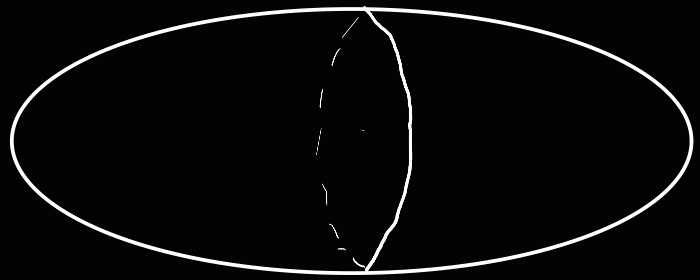
Choice of a metric

$S^2 \hookrightarrow \mathbb{R}^3$

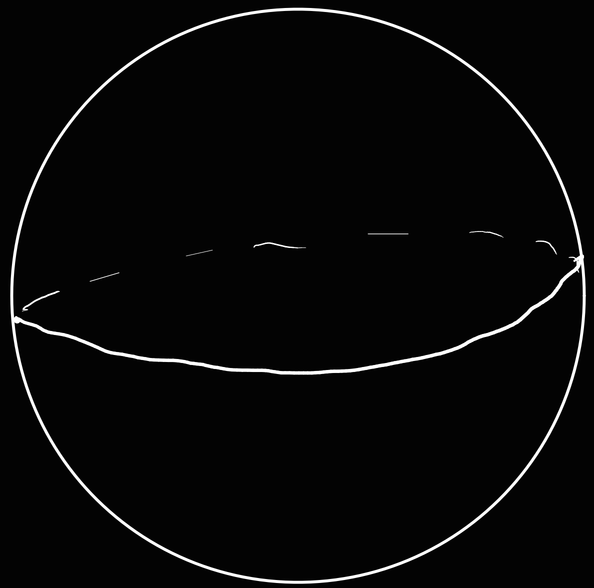




- constant curvature

$\hat{=}$

- minimal surface volume for fixed enclosed volume



Choice of a grading

$$R = \bigoplus_{\ell \geq 0} R_\ell$$

- f.g. $\mathbb{C}$-alg,
- Gorenstein, isolated sing log terminal
- $R_0 = \mathbb{C}$

$\mathbb{Z}$-grading $\simeq \xi$

$$\mathrm{vol}(\xi) = \lim_{k\to\infty} \frac{\dim_{\mathbb{C}}(R/\mathfrak{m}_k)}{(k)^d}$$

$$\mathfrak{m}_k = \bigoplus_{\ell > k} R_\ell$$

$d = \dim R$

Minimisation in the space of gradings

$$R = \bigoplus_{u \in M = \mathbb{Z}^n} R_u \qquad \xi \in N_{\mathbb{R}} = M^* \otimes \mathbb{R}$$

positive $\langle \xi, u \rangle > 0$

$u \neq 0 \,,\; R_u \neq 0$

$$\mathrm{vol}(\xi) = \lim_{k \to \infty} \frac{\dim_{\mathbb{C}}\left(R/m_k\right)}{(k)^d}$$

$$m_k = \bigoplus_{\langle \xi, u \rangle > k} R_u$$

<u>Remark</u>: $\xi \in N = M^*$ $\qquad R_\ell = \bigoplus_{\langle u, \xi \rangle = \ell} R_u$

- $\mathrm{vol}(\lambda \xi) = \frac{1}{\lambda^d} \mathrm{vol}(\xi)$

Normalisation

$A(\xi) = \langle \xi, v \rangle$ $\quad v \in M \quad$ "canonical weight"

log discrepancy

Strategy: find positive $\xi \in N_{\mathbb{R}}$, with $A(\xi) = 1$
such that $\mathrm{vol}(\xi)$ is minimal

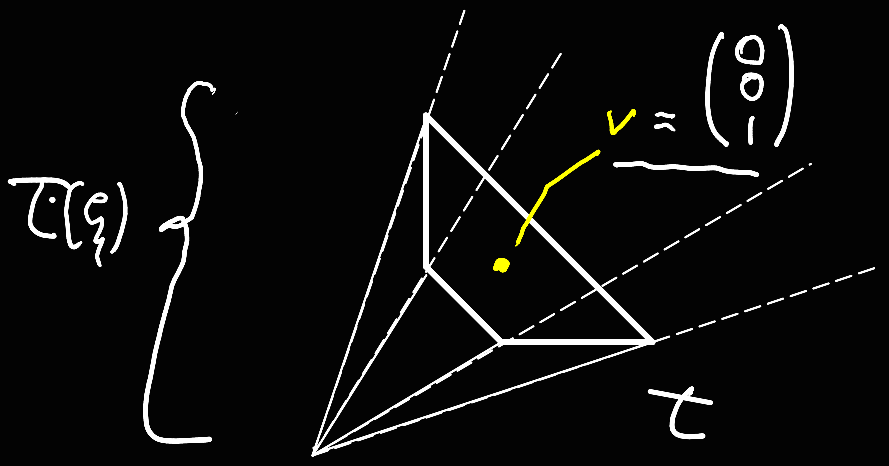
Toric case

$\tau \subseteq M_{\mathbb{R}}$ rational, polyhedral

$$R = \bigoplus_{u \in \tau \cap M} \mathbb{C}\chi^u \qquad \xi \in N_{\mathbb{R}} \text{ positive}$$

$$\xi \in \operatorname{int} \tau^\vee$$

$\xi = (0,0,1)$



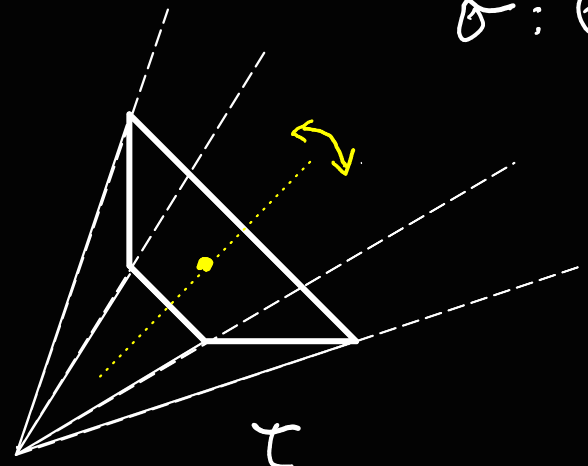
$$\mathrm{vol}_\xi = \mathrm{Vol}\,\tau(\xi)$$

$$\tau(\xi) = \tau \cap [\xi \leq 1]$$

$$1 = A(\xi) = \langle \xi, v \rangle$$

$$v \in [\xi = 1]$$

Exploiting a symmetry



$\sigma : e_1 \leftrightarrow e_2$

$\underline{\xi} = \begin{pmatrix} s \\ s \\ 1 \end{pmatrix}$

vol is of degree 3 in s

$s = \dfrac{4 \pm \sqrt{13}}{3}$ positive

Remark: • $\mathbb{R}\xi \cap \mathbb{N} = \{0\}$ irrational

•

A geometric interpretation

$T = (\mathbb{C}^*)^n \curvearrowright X = \operatorname{Spec} R \subseteq \mathbb{C}^N$ $\qquad R = \bigoplus_{u \in \mathbb{Z}^n} R_u$

$\cup$

$H = (S^1)^n \qquad \underline{\underline{\xi}} \in N_{\mathbb{R}} = \operatorname{Lie}(H)$

$H \curvearrowright L = \underline{\underline{X}} \cap S^{2N-1}$ (link of X), ξ vector field

- $\underline{\text{Sasakian metric}}$ $g : TL$
  - $|\xi| = 1$, $\xi \perp \underline{\mathcal{D}} = TL \cap J(TL)$
  - $g(v,w) = g(Jv, Jw) \qquad v, w \in \mathcal{D}$

...

Properties of Sasakian metrics

- irregular

there is an orbit of the flow of ξ which is not closed

$\Longleftrightarrow$

$\mathbb{R}\cdot\xi$ is irrational

- Einstein condition

$$\mathrm{Ric}_g = g$$

$\rightsquigarrow$ constant scalar curvature

- Volume

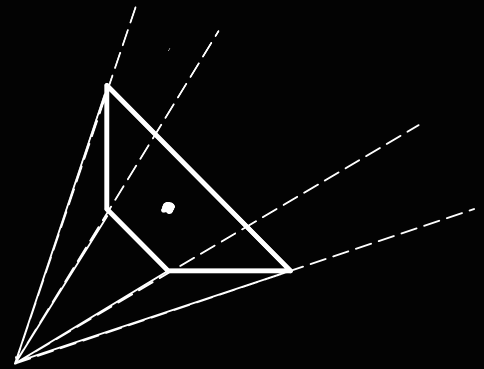
$$\mathrm{Vol}_g(L) = \mathrm{Vol}(\xi)$$

Theorem (Martelli–Sparks–Yau, Futaki–Ono–Wang)

(i) If there exists a SE metric on L with Reeb field ξ, then $\mathrm{vol}(\xi) = \mathrm{vol}(L)$ is minimal for $A(\xi) = 1$

(ii) in the toric case the converse is true

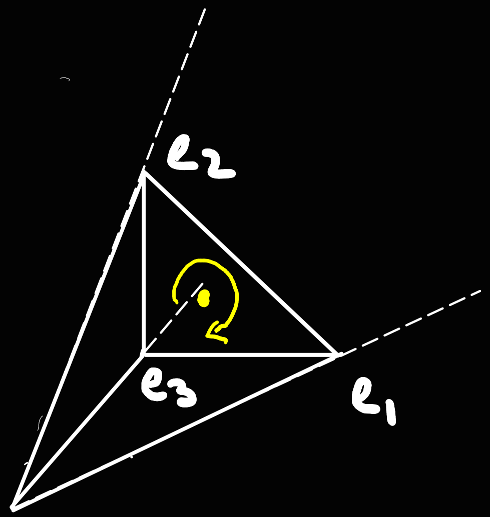
Example:



irregular Sasaki-Einstein metric on $L = \mathcal{A}ff(\cancel{dP_6}) \cap S^{2N-1}$ wrong in the talk! dP_7

$\cong S^3 \times S^2$ //

$S^3 \times S^2 \# \ldots \# S^3 \times S^2$ //



$S_3 \curvearrowright \{e_1, e_2, e_3\} \rightsquigarrow S_3 \curvearrowright \mathbb{Z}^3$

$\xi = s \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ spans a rational subspace

$A^3 \cap S^{2\cdot 3 - 1} \approx S^5$ //

Thm (——)

There are no irregular SE metrics on S^5.

Proof:

K-stability (Tian, Li)

- ξ defines a valuation on R: $\nu_\xi(\sum f_u) = \min\{\langle \xi, u\rangle \mid f_u \neq 0\}$

ξ = vector field

- (L, ξ) is K-semistable if $\mathrm{vol}(\xi)$ is minimal among all valuations $\xi \in \mathcal{V}_R$.

link

Theorem (Colins-Székelyhidi)

(L, ξ) admits a Sasaki-Einstein metric

$$\Longleftrightarrow$$

(L, ξ) is K-stable

constant curvature

$\hat{=}$

volume minimisation