Toric Contact Cycles in the Moduli Space of Curves.

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at
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Nottingham
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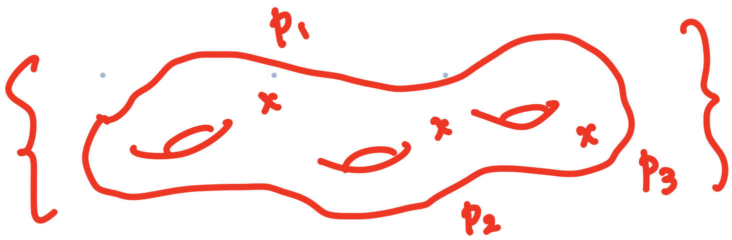
Line bundles on curves & $\mathcal{M}_{g,n}$

$$\begin{tikzcd} \mathrm{Pic}_{g,n} \arrow[r] & \mathcal{M}_{g,n} \arrow[l, bend right, dotted] \end{tikzcd}$$



$(C, p_1, \dots, p_n)$

On each curve <u>two</u> types of line bundles

$$\begin{cases} C \longmapsto \omega_C^{\otimes k} & k \in \mathbb{Z} \\ C \longmapsto \mathcal{O}_C\left(\sum a_i p_i\right) & a_i \in \mathbb{Z} \end{cases}$$

DOUBLE RAMIFICATIONS

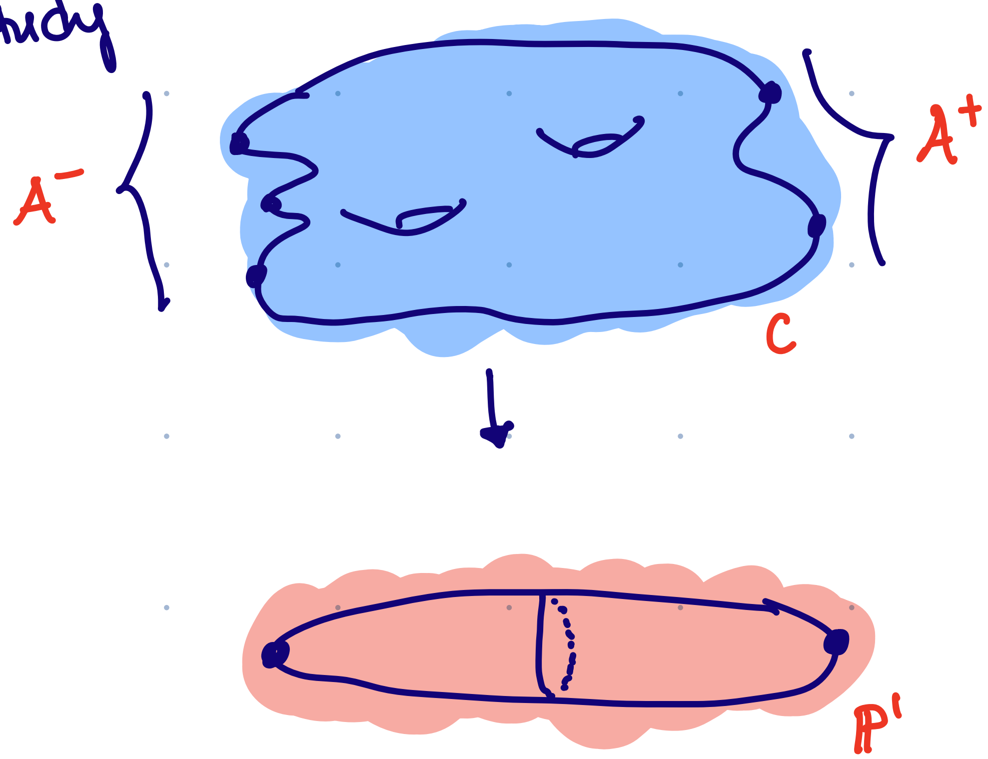
Fix $A = (a_1, \dots, a_n)$ with $\sum a_i = 0$

$$\boxed{DR_g^{\circ}(A) \hookrightarrow \mathcal{M}_{g,n}}$$

CODIM g subset

Curves C with $\mathcal{O}_C(\sum a_i p_i) \simeq \mathcal{O}_C$

GW: study



A word about VERSIONS

COMPACTIFICATIONS:

We have

$$\begin{tikzcd} DR^{\circ}_g(A) \arrow[r, hook] \arrow[d] & \mathcal{M}_{g,n} \arrow[d] \\ DR_g(A) \arrow[r, hook] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

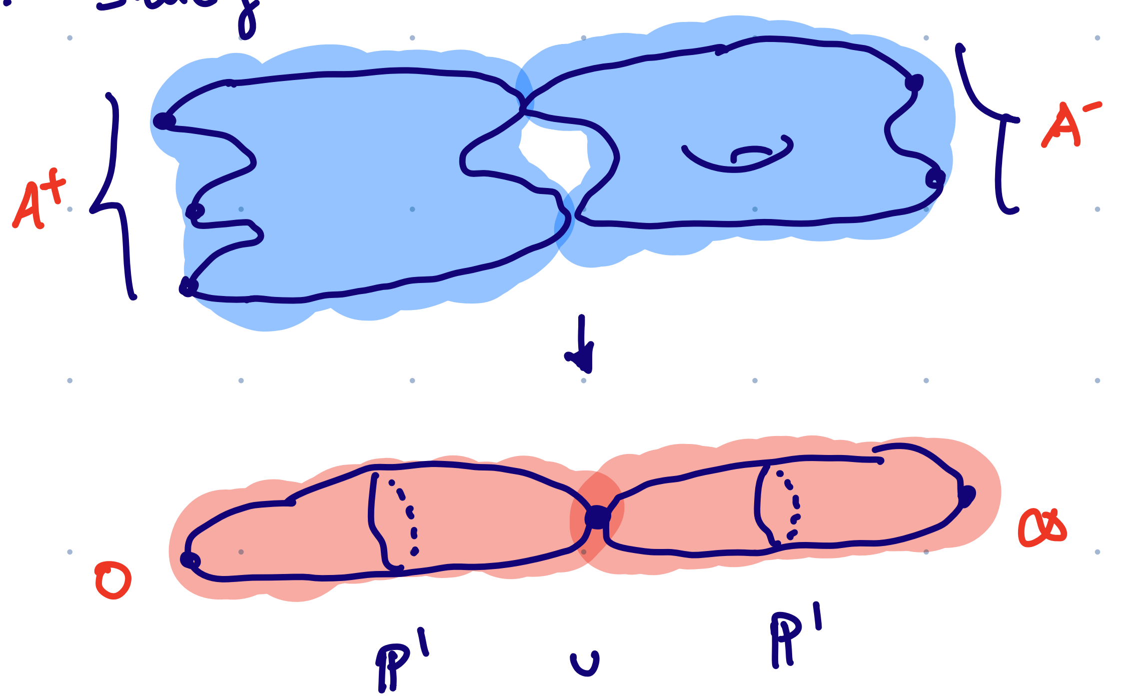
a class in $CH^*(\overline{\mathcal{M}}_{g,n})$. $\leftarrow DR_g(A)$

$= CH^g(\overline{\mathcal{M}}_{g,n})$. via Gromov-Witten theory

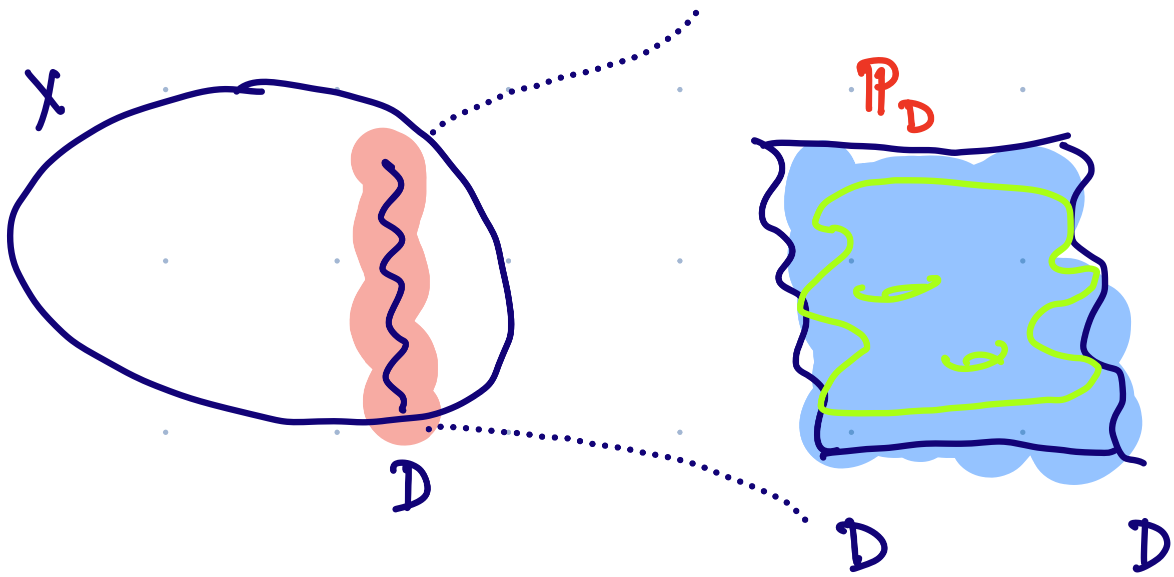
or

Abel-Jacobi resolutions

GW: Study



Aside on GW theory



Roadmap:

GW theory of X $\rightsquigarrow$ Relative GW theory of X|D $\rightsquigarrow$ Double Ramification cycles

Curves in X

Curves in X w/ tangency at D

Local contribution of D.

$DR_g(A)$ is the most basic such class

A TIMELINE: $[DR_g(A)] \in CH^g(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$.

2000 - 2010

- DEF'n OF CLASS (J. Li)
- $[DR_g(A)]$ is TAUTOLOGICAL in Chow (Faber – Pandharipande)
- Applications to $CH^*(\overline{\mathcal{M}}_{g,n})$; THEOREM ★ (Graber – Vakil)
- Partial formulas (Hain, Grushevsky Zakharov).

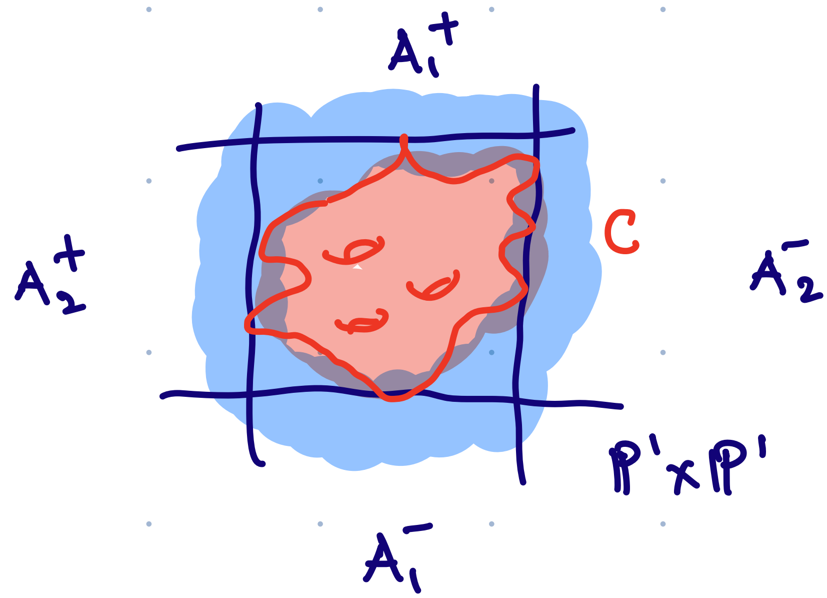
2010 - Now

- Full formula for $[DR(A)]$ (Janda - Pandharipande - Pixton - Zvonkine)
- Logarithmic Abel-Jacobi theory [Marcus, Molcho-Wise, Holmes, Guéré Abreu-Pacini, ...]

SIGNS OF AN INCOMPLETE STORY

• How about maps to $\mathbb{P}^1 \times \mathbb{P}^1$? or a TORIC VARIETY?

Fix A_1 & A_2 length n:



TORIC CONTACT CYCLE

$$TC_g^\circ(A_1|A_2) \subseteq \mathcal{M}_{g,n}$$

CODIM $2g$

$$\| $$

$\{(C, p_1, \ldots, p_n) \mid C \to \mathbb{P}^1 \times \mathbb{P}^1$ with specified contact orders$\}$

TC_g is KEY TO HIGHER GENUS LOG GW THEORY.

PRODUCT FORMULAS

INTERIOR: $\quad [TC_g^\circ(A_1|A_2)] = [DR_g^\circ(A_1)] \cap [DR_g^\circ(A_2)]$

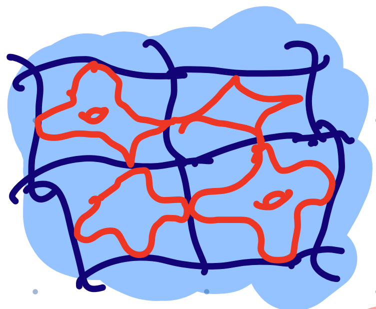
in $\quad CH^*(\mathcal{M}_{g,n})$.

FAILS ON $\overline{\mathcal{M}}_{g,n}$!

$$TC_g(A_1|A_2) \neq DR_g(A_1) \cap DR_g(A_2).$$

in $\quad CH^*(\overline{\mathcal{M}}_{g,n})$.

DEFINED via log GW theory:



maps to broken toric varieties.

Abel–Jacobi: Holmes–Pixton–Schmitt

Gromov-Witten: R

BASIC QUESTIONS about $[TC_g(A_1|A_2)]$ in $CH^*(\overline{\mathcal{M}}_{g,n})$ $\cup$ $R^*(\overline{\mathcal{M}}_{g,n})$

- Does it lie in the TAUTOLOGICAL ring in Chow?
- Why & how does the product rule fail?
- Can we compute top intersections against it?
- Can we find a formula for $TC_g(A_1|A_2)$?

Related: Do GW classes lie in the tautological ring? [Levine–Pandharipande]

Note: I didn't define $TC_g(A_1|A_2)$: if this bothers you ask now!

TAUTOLOGIES ON ARTIN STACKS & TROPICAL MODULI

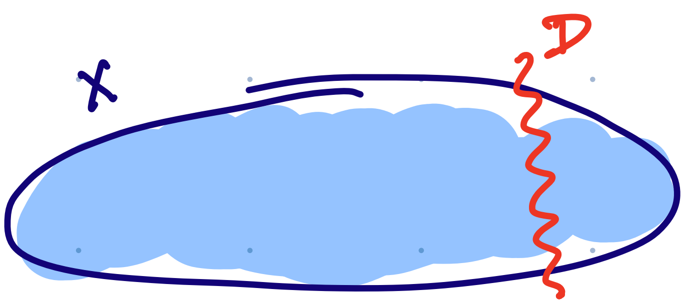
- $D \subseteq X$ a Cartier divisor [really a pair (L,s)]

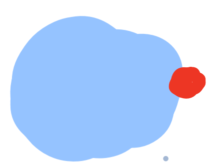
$\rightsquigarrow X \longrightarrow [\mathbb{A}^1/\mathbb{G}_m]$

"Artin fan"

- $D = D_1 + \cdots + D_k \subseteq X$

$\rightsquigarrow X \longrightarrow [\mathbb{A}^k/\mathbb{G}_m^k]$

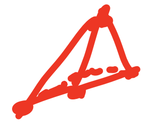




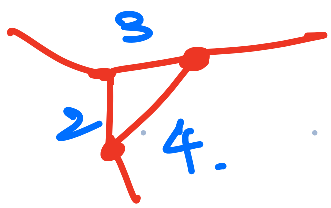
$[\mathbb{A}^1/\mathbb{G}_m]$

Moduli of Tropical Curves

$M^{trop}_{g,n} \approx$ Glued from Cones σ_G

$G =$  $\sigma_G =$ 

"FAN" OF $\overline{M}_{g,n}$

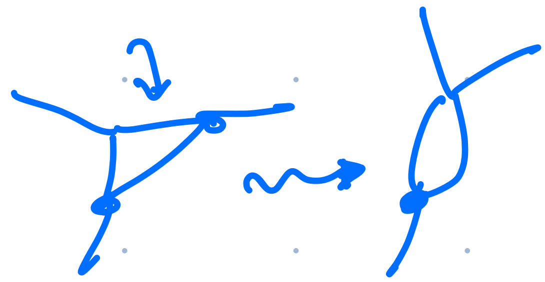


Moduli of Tropical Curves.

$$\equiv \varinjlim_{G} \sigma_G$$

$$\downarrow$$

$$\varinjlim A_G$$



If $\sigma_G = \mathbb{R}^n_{\geq 0}$

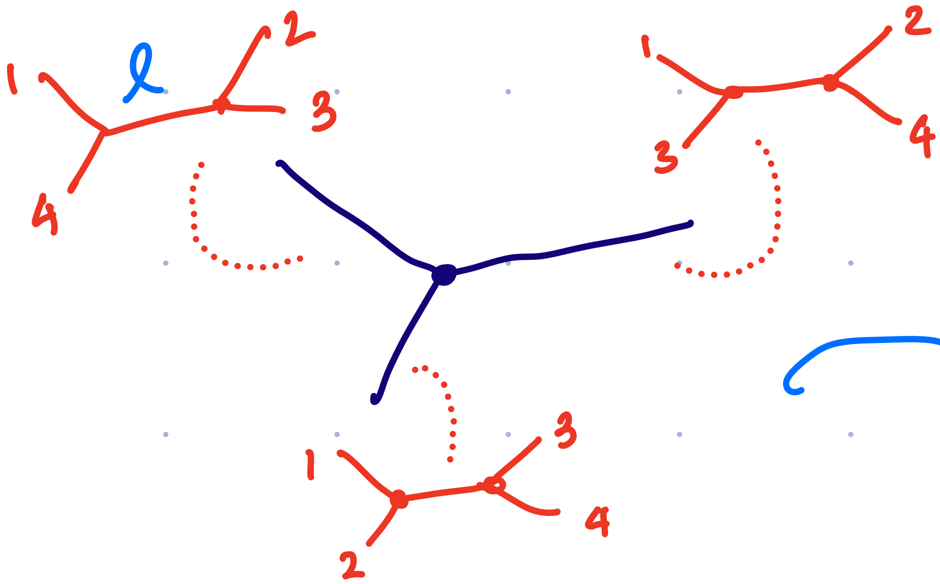
$A_G = [\mathbb{A}^n / \mathbb{G}_m^n]$

Abramovich-Wise; Caporaso-Payne; Ulirsch; Olsson; Chan-Cavalieri

$$\overline{M}_{g,n} \longrightarrow \mathcal{A}_{g,n}$$

THE ARTIN FAN.

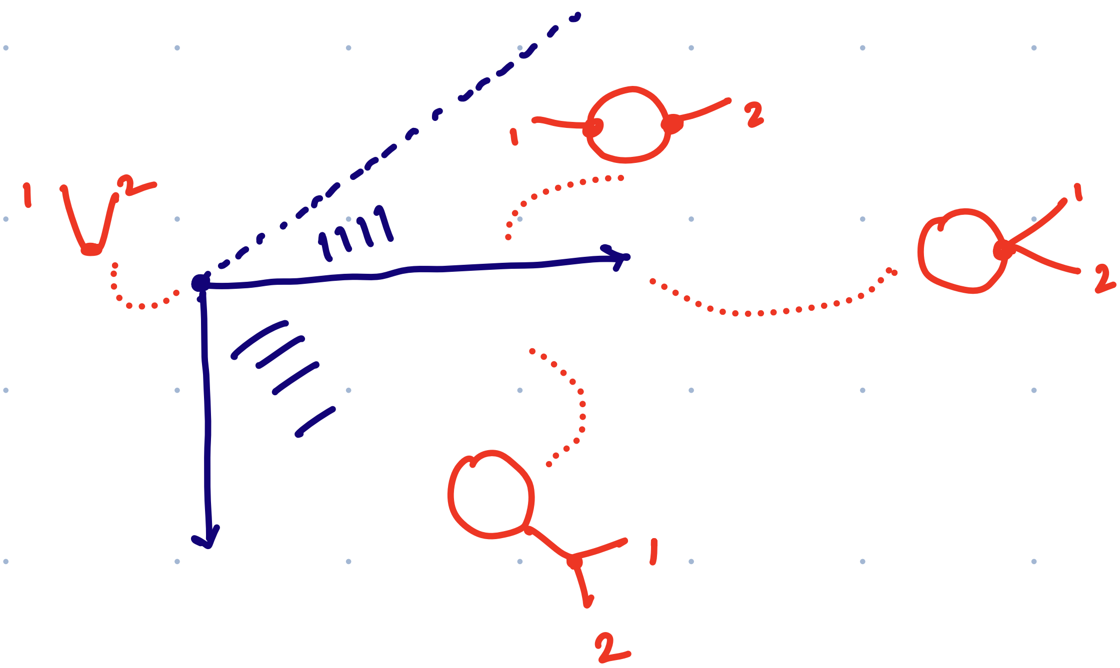
THE MODULI SPACE $\mathcal{M}_{0,4}^{trop.}$



Artin fan

THE MODULI SPACE $\mathcal{M}_{1,2}^{trop}$

$\mathcal{M}_{1,2}^{trop}$



THE TAUTOLOGICAL RING (Mumford, ...)

$$R^*(\overline{M}_{g,n}) \subseteq CH^*(\overline{M}_{g,n})$$

SUBRING

$$\overline{M}_{g,n} \xrightarrow{\pi} A_{g,n} \simeq M^{trop}_{g,n}$$

$$\pi^* : CH^*(A_{g,n}) \longrightarrow CH^*(\overline{M}_{g,n}).$$

and

cotangent line classes $\psi_1, \dots, \psi_n$

$$\overline{M}_{g,n} \hookrightarrow \overline{M}_{g,n+1}.$$

FACT: (Molcho-Pandharipande-Schmitt)

$\mathrm{im}(\pi^*)$ & $\psi_1, \dots, \psi_n$ give elements in the tautological ring.

What is $CH^*(\mathcal{A}_{g,n})$?

THEOREM (Molcho – R ; Molcho - Pandharipande – Schmitt)

$$CH^*(\mathcal{A}_{g,n}) = PP^*(M_{g,n}^{trop})$$

RING OF PIECEWISE POLYNOMIALS

More generally, if $\mathcal{A}$ is an ARTIN FAN

$$CH^*(\mathcal{A}) = PP^*(trop(\mathcal{A}))$$

[After Payne, Brion, ...]

[Uses Kresch, Kimura, Bae-Park]

TAUTOLOGICAL CLASSES FROM $M_{g,n}^{trop}$

THEOREM (Molcho – R '21 ; Holmes-Schwarz '21)

- The classes $TC_g(A_1|A_2)$ are TAUTOLOGICAL

What is the geometry? Why does product rule fail?

If $\tilde{X} \xrightarrow{\pi} X$ is a blowup

then $\pi_*(\alpha \cdot \beta) \neq \pi_*(\alpha) \cdot \pi_*(\beta)$

But if $\tilde{X} \longrightarrow X$ is a "tropical blowup"

the failure is controlled via $PP^*(X^{trop})$.

TROPICAL DOUBLE RAMIFICATION:

$A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ w/ $\sum a_i = 0$

$$DR_g^{trop}(A) \subseteq \mathcal{M}_{g,n}^{trop}$$

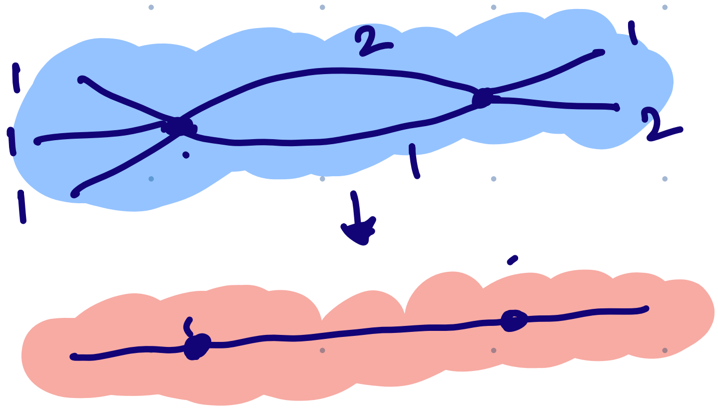
$\begin{cases} \text{Ulirsch - Zakharov} \\ \text{Cavalieri - Markwig - R} \end{cases}$

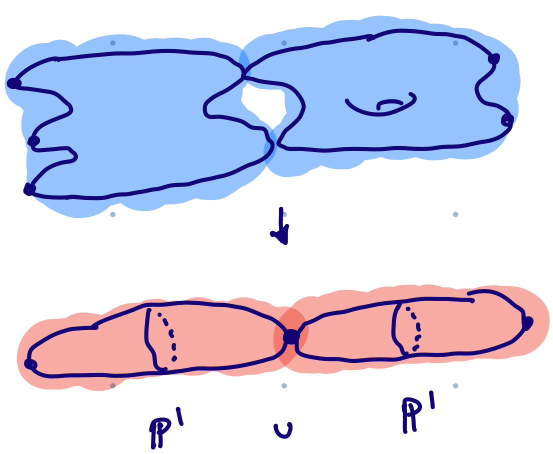
$$\| $$

$$\{ \Gamma \mid \sum a_i p_i \sim 0 \}$$

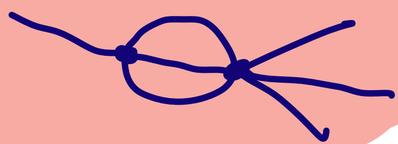
Defined by analogy

BALANCED & CONTINUOUS MAP.

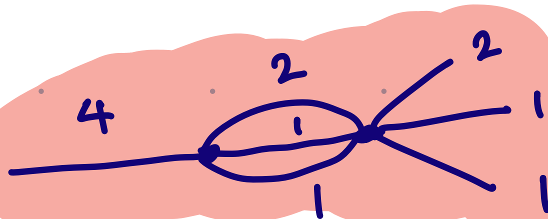




WHAT DOES $DR^{trop}(A)$ look like?

I. $\Gamma =$  $A = (4, -2, -1, -1)$

II. Assign slopes by balancing



(not necessarily unique)

III. Solve for edge lengths that allow continuity

$$2l_1 = l_2 = l_3 \subseteq M^{trop}_{2,4}$$

JUST LIKE TORIC GEOMETRY:

$$\mathrm{DR}_g^{trop}(A) \subseteq \mathcal{M}_{g,n}^{trop}$$

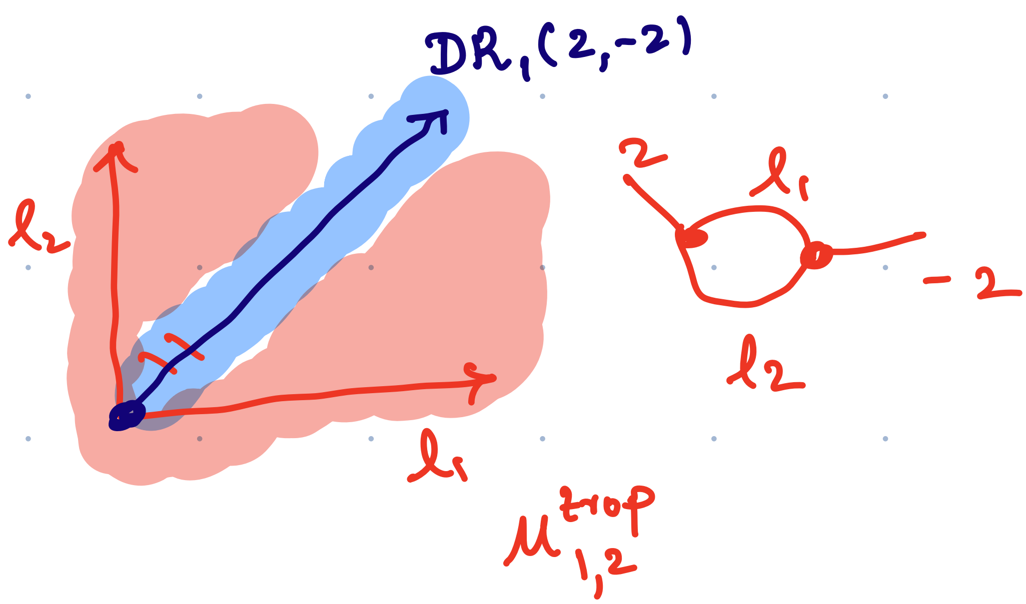
SUBFAN

$\updownarrow$

BLOWUPS OF $\overline{\mathcal{M}}_{g,n}$

(OPEN SUBSETS IN)

$A = (2, -2)$



RECIPE

~~THEOREM~~ (Molcho–R '21): To calculate

$TC_g(A_1|A_2)$:

1. Intersect $DR_g^{trop}(A_1) \cap DR_g^{trop}(A_2) \subseteq M_{g,n}^{trop}$

2. Blowup accordingly

$$\widetilde{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

3. Calculate the (VIRTUAL) strict transform $\widetilde{DR}_g(A_1)$ & $\widetilde{DR}_g(A_2)$.

4. Intersect there, pushforward. Get answer via $PP^*(\mathcal{M}_{g,n}^{trop})$

THE VIRTUAL STRICT TRANSFORM:

Molcho – R. '21:

We explain what this is using:

- Fulton's blowup formula.
- Aluffi's formulas for Segre classes.
- Picewise polynomials.

No formula yet...

Instance of a more general phenomenon.

The ring $\log CH^*(\overline{M}_{g,n}) := \varinjlim_{\substack{M^+ \to \overline{M}_{g,n} \\ \text{a blowup}}} CH^*(M^+)$

is RICH!

PLEASE SEE WORK OF

Molcho - Pandharipande - Schmitt

Holmes - Pixton - Schmitt

Holmes - Schwarz

Nabijou - R

Molcho - R

+ Cavalieri - Gross - Markwig

THANKS!