Pure codimensionality of the wobbly locus

i/ Ongoing joint work w/ C. Pauly

$x \quad -x$

0. Basics on Higgs bundles & the nilpotent cone

X Riemann surface $g \geq 2$

$N_X(n,d)$ = mod space of rk n deg d v.b

$\cup$ = {iso classes of polystable v.b}

$N^s_X(n,d)$ stable locus (smooth)

Recall: E is (semi) stable if $\forall$ bundle $0 \neq F \subsetneq E$ $\mu(F) = \frac{\deg F}{\operatorname{rk} F} \underset{(=)}{<} \mu(E)$

$M_X(n,d)$ = mod space of rk n deg d Higgs bundles

= { polystable (E, φ) } / iso, with E rk n deg d and $\varphi \in H^0(\operatorname{End} E \otimes K)$

$\cup$ dense

$T^* N^s_X$ Same def of sstability but taking only $F \xrightarrow{\varphi|_F} F \otimes K$

Rk $K^{1/2} \oplus K^{-1/2}$ $\varphi : K^{1/2} \xrightarrow{1} K^{-1/2} \otimes K$ is a stable H.b. w/ unst. underlying bundle. Un $\subset M_X$.

Def: E v.b. is wobbly if $\exists$ $\varphi \in V_E := H^0(\operatorname{End} E \otimes K)$ nilpotent $\varphi \neq 0$

(2)

$\mathbb{W} \subset N_X(n,d)$ wobbly locus

→ First appear in Laumon '88

Drinfeld (81) $\mathbb{W}$ of pure codim 1

→ Pal-Pauly (18) prove ↑ in rk 2

Why wobbly bundles? / Key in understanding the nilpotent cone (in turn key towards $M_X(n,d)$)

\ Proof of Geometric-Langlands in Donagi-Pantev's approach requires understanding $\mathbb{W}$ $= \mathcal{S}$ (shaky)

(0.1) The nilpotent cone

$$h: M_X(n,d) \longrightarrow B := \bigoplus_{i=1}^{n} H^0(X, K^i)$$

$$(E,\varphi) \longmapsto \det(x\,\mathrm{Id} - \varphi)$$

Hitchin map

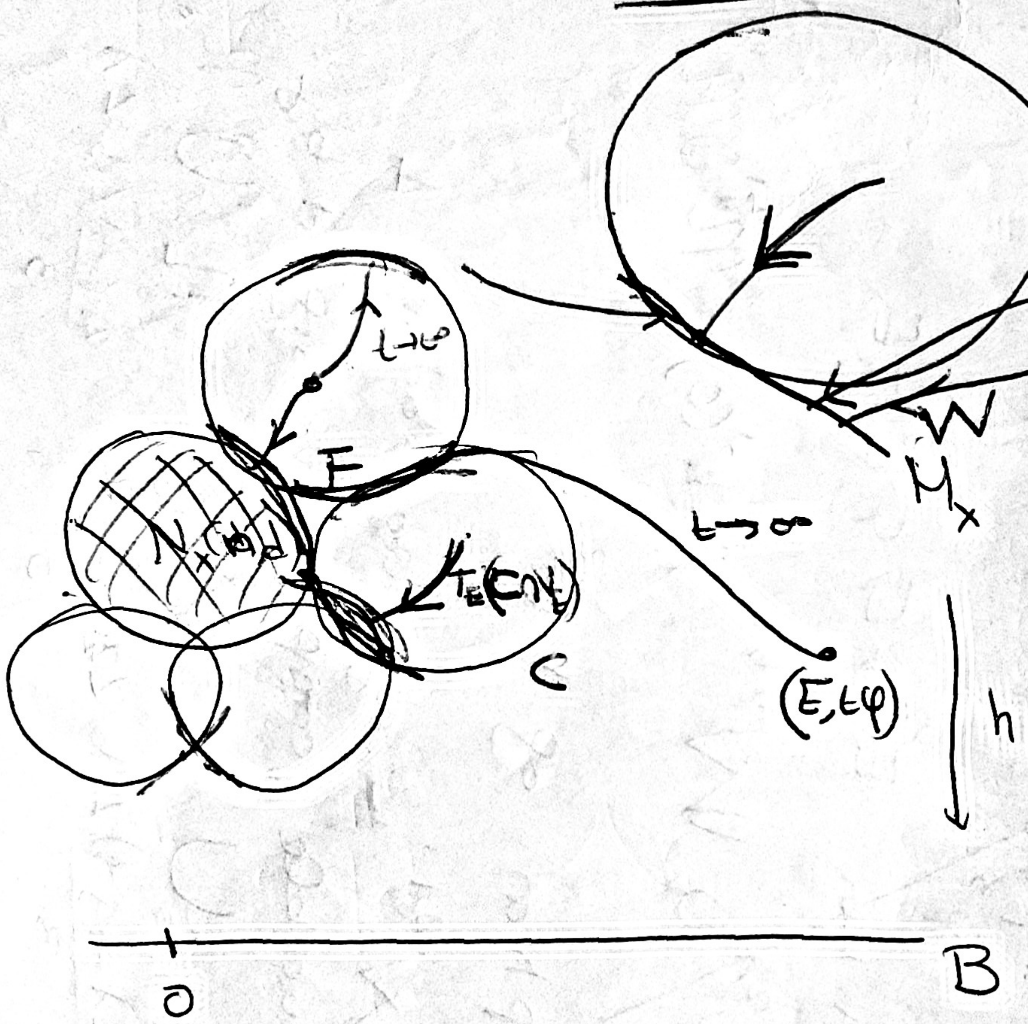
Faltings

Laumon

Ginzburg $\mathcal{N} = h^{-1}(0)$ complete intersection

Lagrangian.

ZOOM



Zar. loc. trivial fibr. w/ Lagr. fiber

II

Rks: $N_X(n,d) \longrightarrow h^{-1}(0)$

$E \longmapsto (E,0)$

(reduced scheme underlying a component)

ii) $\mathbb{C}^\times \curvearrowright M_X(n,d)$

$t\cdot(E,\varphi)=(E,t\varphi)$

and $\lim_{t\to 0} t(E,\varphi) \in h^{-1}(0)^{\mathbb{C}^\times}$.

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iii) Also $(E,\varphi)\in h^{-1}(0)$

$\lim_{t\to\infty}(E,t\varphi) \in h^{-1}(0)^{\mathbb{C}^\times}$

$F^- = \{(E,\varphi) \mid \lim t(E,\varphi) \in F\}$

In fact

$W = \bigcup_{F^-\neq N_X} \overline{F^-} \cap N_X =$

$=$

iv) If W^s pure codim 1

$\Rightarrow \forall$ irred comp of W^s $\exists (E,\varphi)\in T^*N^s$ and irred comp C of $h^{-1}(0)$ s.t. $V_E \cap C \cong \mathbb{C}$

v)

(0.2) Geometric Langlands & wobbly bundles

<u>GLC</u> for GL_n: every rk n local system $\mathbb{L} \to X$ extends uniquely to a perverse sheaf

$$\begin{array}{ccc} \mathbb{L} & \to & X \\ \downarrow \wr & & \downarrow AJ \\ \mathbb{L} & \to & \mathrm{Pic}(X) \end{array}$$

Higgs bundles: dominate N_X by $\mathrm{Jac}(X_b)^{N_X}$ ← spectral curve

$$\mathrm{Jac}(X_b) \cong h^{-1}(b) \overset{r_b}{\dashrightarrow} N_X$$

generic

$\mathbb{V} \to X$ g^ric local system $\xrightarrow{NAHC}$ $(E,\varphi) \in h^{-1}(b)$ g^ric

$$\tilde{\mathbb{L}} \to \mathrm{Jac}(X_b) \leftarrow \mathbb{L} \to$$

<u>Idea</u>: pushforward to N_X. First need to solve r_b

$$\begin{array}{ccc} & \widehat{\mathrm{Jac}} & \\ \pi \swarrow & & \searrow r_b \\ \mathrm{Jac} & \dashrightarrow & N_X \end{array}$$

<u>Theorem</u> (PN '20) if N_X is smooth, $r_b(E_X) \in \mathbb{W}$. $\exists\, \mathfrak{X} \mathbb{W}$ BNR

$\cup\, r_b(E_X) = \mathbb{W}$.
$\quad = \mathfrak{S}$

Moreover, not quite $\tilde{\mathbb{L}}$ but $\tilde{\mathbb{L}}(\mathbb{W})$ ← finer geom needed

[DP] $\mathbb{P}^1 \setminus \{5\text{ pts}\}$

1 Wobbly bundles

$(E, \varphi) \in M_X(n,d) \cap h^{-1}(0)$

$E_i = \ker \varphi^{i+1}$ $\qquad E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{r-1} = E$

Then $\mu(E_i) < \mu(E) < \mu(Q_i) = \quad \forall\, i < r-1$ $\qquad Q_i = E/E_i$

$n_i = \mathrm{rk}\, \frac{E_i}{E_{i-1}}$

$d_i = \deg \frac{E_i}{E_{i-1}}$

$\underline{\text{Moreover}}$ $\boxed{\varphi|_{E_i} \text{ is nilpotent}}$

Also, for $i < r-1$ $\quad \overline{\varphi}_i : Q_i \longrightarrow E_{r-1}K \subset Q_i K$ $/ E_i$

$q_\xi \longmapsto \varphi(q_\xi + E_i)$

Laumon: (n_0, d_0) construct on a dense open set of ired

Idea: use recursion to show pure codim 2 (identify which ones appear) construct components

$$(E_i, \varphi_i) \hookrightarrow (E, \varphi) \longrightarrow (Q_i, \overline{\varphi}_i)$$

such extensions parametrised by $\mathbb{H}^1(C_0)$

(Thaddeus)

$$C_0 : Q_i^* E_i \longrightarrow Q_i^* E_i K$$

$$s \longmapsto s \circ \overline{\varphi}_i + \varphi_i \circ s$$

EXPLAIN here

HOPE: generic E_i, Q_i appearing like that are semist.

$\rightsquigarrow (E_i, Q_i) \in W^{n_0 \cdots n_i}_{d_0 \cdots d_i} \times W^{n_{i+1} \cdots n_{r-1}}_{d_{i+1} \cdots d_{r-1}}$

$\rightsquigarrow$ Recursion: $\varphi_i, \overline{\varphi}_i$ unique.

<u>Consequences</u> : $\rightsquigarrow \mathbb{H}^1(C_\bullet)$ indep of $\varphi_i, \bar{\varphi}_i$

$\rightsquigarrow$ Under the right conditions

$$\mathcal{H}^1 \longrightarrow W_{\bar{d}}^{\bar{n}} \times W_{\bar{\delta}}^{\bar{m}}$$

bundle parametrising $W_{\bar{d},\bar{\delta}}^{\bar{n},\bar{m}}$

~~To~~ recover C $\rightsquigarrow$ add ... Higgs fields

$$\mathcal{H}_{\bar{d},\bar{\delta}}^{\bar{n},\bar{m}} \longrightarrow W_{\bar{d},\bar{\delta}}^{\bar{n},\bar{m}} \qquad t\varphi_{(t,s)} = \begin{pmatrix} t\varphi_i & * \\ & s\bar{\varphi}_i \end{pmatrix}$$

all extensions of els of $\mathcal{H}_{\bar{\delta}}^{\bar{m}}$ by els of $\mathcal{H}_{\bar{d}}^{\bar{n}}$

$$\begin{tikzcd} W_{\bar{d},\bar{\delta}}^{\bar{n},\bar{m}} \arrow[d, "p"] \\ W_{\bar{d}}^{\bar{n}} \times W_{\bar{\delta}}^{\bar{m}} \end{tikzcd} \qquad t\varphi = t\begin{pmatrix} \varphi_i & \\ & t\bar{\varphi}_i \end{pmatrix}$$

$$\mathcal{H}_{\bar{d}}^{\bar{n}} \nearrow \qquad \nwarrow \mathcal{H}_{\bar{\delta}}^{\bar{m}}$$

~~$\rightsquigarrow$ Take $\bar{m} = n_{r-1}$ $\bar{n} = (n_0 \ldots n_{r-2})$ if~~

<u>Rks</u> :

- Best case scenario: $* = 0$ (nilpotency order $\bar{n},\bar{m}$
- If $\varphi_i \neq 0$ or $\bar{\varphi}_i \neq 0$ $\Rightarrow$ each $E \in W_{\bar{d},\bar{\delta}}$ will have at least 2 Higgs fields

$\Rightarrow$ $\underbrace{\text{smooth components}}_{\text{of } \mathbb{W}}$ of given by $\bar{m} = (n_1, \ldots n_{r-1})$
$\bar{n} = n_0$

(2) Rank 3

$E \in W \subset N_X(3, d) \longrightarrow \in \{0, 1\}$ — 0: singular, 1: smooth moduli

Case 1 $\exists \varphi \in V_E : \varphi^2 = 0$ $\qquad E_0 \subsetneq E_1 = E$, E_0: rk 2

$W^{2,1}$

Case 2 $\exists \varphi \in V_E : \varphi^3 = 0 \quad \varphi^2 \neq 0$ $\qquad E_0 \subsetneq E_1 \subsetneq E_2 = E$, E_0: rk 1

$W^{1,1,1}$

(2.1) $W^{2,1}$ & $\mathcal{N}^{2,1}$

Theorem (Pauly - P. N.) (1) $\mathbb{W}^{2,1}$ is of pure codim 1 w/ irreducible components

$$\bigcup_{\frac{2\lambda - (4g-4)}{3} < d_0 < \frac{2\lambda - (g-2)}{3}} \mathbb{W}^{2,1}_{d_0}$$

(2) The irred comps of $\mathcal{N}^{2,1}$ are classified by d_0 with

$$\bigcup_{\frac{2\lambda - (4g-4)}{3} < d_0 < \frac{2\lambda}{3}} \mathcal{N}^{2,1}_{d_0}$$

Rk: ~~for~~ $\frac{2\lambda}{3} > d_0 \geq \frac{2-(2g-2)}{3} \rightsquigarrow E \in$ "$\mathbb{W}^{2,1}_{d_0}$" has > 1 Higgs field

• ~~So~~ These ... $\mathcal{N}^{2,1}$ ... $\to W^{2,1}$ contract to singular

Moreover, let

$$Z^{2,1}_{d_0} = B^0(2, \overbrace{4g-4+3d_0-2\lambda}^{\delta}) \times Pic^{\lambda-d_0}$$

Brill-Noether $= \{E \in N_X(2,\delta) \mid h^0(E) \neq 0\}$

$$(E_0, Q_0) \rightsquigarrow (E_0 Q_0^* K, Q_0)$$

$$\in N_X(2,\delta) \times Pic^{\lambda-d_0}$$

$$\frac{2\lambda-(2g-2)}{3} \leq d_0 < \frac{2}{3}\lambda$$

There exist rational bundles

$$\mathcal{E}^{2,1}_{d_0} \longrightarrow Z^{2,1}_{d_0}$$

$$H^1(E_0 Q_0^*) \longmapsto (E_0 Q_0^* K, Q_0)$$

$$\mathcal{H}^{2,1}_{d_0} \longrightarrow Z^{2,1}_{d_0}$$

$$H^0(E_0 Q_0^* K) \longmapsto (\quad)$$

and rational maps

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E} \oplus \mathcal{H}) & \overset{\tilde{f}}{\dashrightarrow} & M_X(3,\lambda) \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{E}) & \overset{\bar{f}}{\dashrightarrow} & N_X(3,\lambda) \end{array}$$

With $\mathrm{Im}\, \tilde{f} = \mathcal{N}^{2,1}_{d_0}$ ired

$\mathrm{Im}\, \bar{f} = \mathbb{W}^{2,1}_{d_0}$ ired for $d_0 < \frac{(2g-2)+2\lambda}{3}$

3.2 $W^{1,1,1}$

$E_0 \subsetneq E_1 \subsetneq E_2 = E$

$E_0 \in Pic^{d_0}(X)$

ASSUMPTION

"$Q_0 \in$" $W^{1,1}_{d_1,d_2}$

unique $/ \mathbb{C}^\times$ if $2d_2 - d_1 \leq 2g-1$

$\varphi_0 \neq 0$ nilp

[Pal-Pauly]

Quasi-prop 1) $W^{1,1,1}_{\cancel{d_0,d_1,d_2}}$ is of codim 1

$\cup$

$\bigcup W^{1,1,1}_{d_0,d_1,d_2}$

$d_i \in$ same range

~~Ensuring~~ • generic stability of $Ext^1(E_0, Q_0^*)$

• wobbliness of Q_0 & E_1

• uniqueness of ext. of Higgs fields from φ_0

• good dimension

2) There exist $\mathcal{N}^{1,1,1}_{d_0,d_1,d_2} \subset \mathcal{N}^{1,1,1}$

line

$W^{1,1,1}_{d_0,d_1,d_0}$

for $d_i \in$ the range above.

$\boxed{\text{STRATEGY VALID } \forall \text{ ~~types~~ \& ranks}}$

rk 3 → we conv. it is all

rk n → needs further work