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Maximal disjoint Schubert cycles in Rational Homogeneous Spaces

Online Nottingham Algebraic Geometry Seminar

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A Theorem of Tango

A Theorem of Tango

In order to construct indecomposable vector bundles of low rank on the projective space, Tango studied morphisms $\mathbb{P}^m \to \mathbb{G}(r, n)$.

One of his results is the following:

Theorem (Tango, 1974)

If $m > n$ then the only morphisms from the projective space $\mathbb{P}^m$ to a Grassmannian $\mathbb{G}(r, n)$, $0 \leq r \leq n - 1$, are the constant ones.

Proof

The incidence variety $\mathcal{U} \subset \mathbb{P}^n \times \mathbb{G}(r, n)$

$$\mathbb{P}^n \longleftarrow \mathcal{U} \longrightarrow \mathbb{G}(r, n)$$

has a $\mathbb{P}^r$-bundle structure over $\mathbb{G}(r, n)$, which is the projectivization of a vector bundle $\mathcal{Q}$, of rank $r + 1$, called the universal quotient bundle, whose determinant is the ample generator of $\mathrm{Pic}(\mathbb{G}(r, n))$.

This bundle fits in an exact sequence, called universal sequence:

$$0 \to \mathcal{S}^\vee \longrightarrow \mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{Q} \to 0$$

the bundle $\mathcal{S}$, of rank $n - r$ is called the universal subbundle.

The exactness of the sequence provides the relation

$$c_t(\mathcal{Q})c_t(\mathcal{S}^\vee) = 1 \qquad (*)$$

Proof

Let $\varphi : \mathbb{P}^m \to \mathbb{G}(r, n)$ be a morphism; then

$$\varphi^* c_i(\mathcal{Q}) = \lambda_i H^i \quad \text{for} \quad i = 0, \ldots, \operatorname{rk} \mathcal{Q} = r+1$$
$$\varphi^* c_j(\mathcal{S}^\vee) = \mu_j H^j \quad \text{for} \quad j = 0, \ldots, \operatorname{rk} \mathcal{S}^\vee = n-r$$

Using the the relation (*) we get

$$1 = c_t(\varphi^* \mathcal{Q}) c_t(\varphi^* \mathcal{S}^\vee) = \lambda_d \mu_e H^{d+e} t^{d+e} + \cdots + 1$$

where d and e are the maximum integers s.t. $\lambda_d \neq 0$ and $\mu_e \neq 0$.

We thus get $e = d = 0$.

In particular $\varphi^* \mathcal{O}_{\mathbb{G}}(1) = \varphi^* \det \mathcal{Q} = \lambda_1 H = 0$, hence φ is constant. $\square$

Applications and generalizations

A Theorem of Sato

Using Tango's Theorem, Sato proved the following:

Theorem (Sato, 1976)

A uniform vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ of rank $r < n$ is isomorphic to a direct sum of line bundles.

The splitting type $(a_1, \ldots, a_k, \ldots, a_r)$ of $\mathcal{E}$, up to duality and twists, can be written as

$$0 = a_1 = \cdots = a_k < a_{k+1} \leq \cdots \leq a_r$$

For every x we get a morphism $\varphi_x : \mathbb{P}^{n-1} \to \mathbb{G}(k-1, \mathbb{P}(\mathcal{E}_x))$.

If $\mathcal{E}$ is decomposable these morphisms are constant. Sato shows that the condition is also sufficient for the splitting.

Generalizations

To study uniform bundles on other varieties, some Tango-type Theorems were proved, using the following idea:

If $m > n$ every morphism $\varphi : \mathbb{P}^m \to \mathbb{G}(r, n)$ is constant because in the Chow ring of $\mathbb{G}(r, n)$ there are zero divisors in degree $n + 1 \leq m$ and in the Chow ring of $\mathbb{P}^m$ there are no zero divisors in degree $\leq m$.

Definition - Theorem (Pan, 2013)

A variety M has good divisibility up to degree s if, given $x_i \in \mathrm{A}^i(M)$, $x_j \in \mathrm{A}^j(M)$ with $i + j = s$ and $x_i x_j = 0$, we have $x_i = 0$ or $x_j = 0$.

If M has good divisibility up to degree $m > n$ then the only morphisms from M to a Grassmannian $\mathbb{G}(r, n)$ are the constant ones.

Good divisibility of quadrics

$\mathbb{Q}^m$ smooth quadric hypersurface

If m is odd then $\mathrm{A}^i(\mathbb{Q}^m)$ is one dimensional $\forall\ i = 0, \ldots, m$, so the good divisibility is m.

If m is even, then, given two linear spaces $\Lambda_\alpha, \Lambda_\beta$ of dimension $m/2$ belonging to different families, we have

$$H \cdot (\Lambda_\alpha - \Lambda_\beta) = 0$$

so the good divisibility is $m/2$.

However Kachi and Sato proved that, for m even, every morphism $\mathbb{Q}^m \to \mathbb{G}(r, n)$ is constant if $m > n + 1$.

This suggests that the notion of good divisibility could be improved.

Effective good divisibility

Alternative proof of Tango's Theorem

Let n be the minimum integer s.t. $\exists \varphi : \mathbb{P}^m \twoheadrightarrow M' \subseteq \mathbb{G}(r,n)$ nonconstant. We want to show that $n \geq m$.

Pick $p \in \mathbb{P}^n$ point, $H \subset \mathbb{P}^n$ hyperplane not containing p.

- $\Sigma_p \subset \mathbb{G}(r,n)$, parametrizing linear spaces passing by p.
- $\Sigma_H \subset \mathbb{G}(r,n)$, parametrizing linear spaces contained in H.

Computing dimensions we see that

- $\operatorname{codim} \Sigma_p = n + 1 - r$
- $\operatorname{codim} \Sigma_H = r$,

therefore

$$\operatorname{codim} \Sigma_p + \operatorname{codim} \Sigma_H = n + 1$$

Since $[\Sigma_p] \cdot [\Sigma_H] = 0$ we have

$$0 = \varphi^*[\Sigma_p] \cdot \varphi^*[\Sigma_H] = a[H^{n+1-r}] \cdot b[H^r] = ab[H^{n+1}]$$

Alternative proof

$$0 = \varphi^*[\Sigma_p] \cdot \varphi^*[\Sigma_H] = a[H^{n+1-r}] \cdot b[H^r] = ab[H^{n+1}]$$

If $n < m$ then necessarily $ab = 0$, that is $\varphi^*[\Sigma_p] = 0$ or $\varphi^*[\Sigma_H] = 0$.

Either M' does not contain points parametrizing linear spaces by p or M' does not contain points parametrizing linear spaces contained in H. Up to duality in $\mathbb{P}^n$ we can assume that we are in the first case.

The linear projection from p onto H induces a morphism

$$\pi_p : \mathbb{G}(r, n) \setminus \Sigma_p \to \mathbb{G}(r, n-1)$$

whose fibers are affine spaces $\mathbb{A}^r$.

Since $M' \cap \Sigma_p = \emptyset$ then $\pi_p \circ \varphi : \mathbb{P}^m \to \mathbb{G}(r, n-1)$ is a morphism which is constant by the minimality of n; therefore φ is constant.

We conclude that $n \geq m$.

Effective good divisibility

In the alternative proof we pulled back effective (disjoint) Schubert cycles, so the idea of Tango's proof can be refined as follows:

Every morphism $\varphi : \mathbb{P}^m \to \mathbb{G}(r, n)$ is constant because in $A^\bullet(\mathbb{G}(r, n))$ there are effective zero divisors in degree $n + 1 \leq m$ and in $A^\bullet(\mathbb{P}^m)$ there are no effective zero divisors in degree $\leq m$.

Definition - Theorem (Muñoz, __, Solá Conde, 2020)

M has effective good divisibility up to degree s if, given $x_i \in A^i(M)$, $x_j \in A^j(M)$, effective, with $i + j = s$ and $x_i x_j = 0$, then $x_i = 0$ or $x_j = 0$.

If M has effective good divisibility up to degree $m > n$ then the only morphisms from M to a Grassmannian $\mathbb{G}(r, n)$ are the constant ones.

Example

To apply the Theorem we need to be able to compute effective good divisibility of varieties, so the natural candidates are varieties with well known Chow rings.

Theorem (Naldi, _ 2022)

The effective good divisibility of $\mathbb{G}(l, m)$ is m.

The proof was done using Schubert calculus on Grassmannians.

Corollary

If $m > n$ then the only morphisms from a Grassmannian $\mathbb{G}(l, m)$ to a Grassmannian $\mathbb{G}(r, n)$ are the constant ones.

Goal

Compute the effective good divisibility of rational homogeneous varieties of classical type and describe the effective cycles with zero product of maximal (sum of) dimensions, called maximal disjoint pairs.

- In $\mathbb{G}(r, n)$ the pair $([\Sigma_p], [\Sigma_H])$ is a maximal disjoint pair.

Flag varieties

Flag varieties of linear spaces in $\mathbb{P}^n$

For a fixed n, set

- $E_i = (0 : \cdots : 1 : \cdots : 0) \in \mathbb{P}^n$;
- $I = \{a_1, \ldots, a_k\} \subseteq \{1, \ldots, n\} := \Delta$;
- $\Lambda_{a_i} = \langle E_1, \ldots, E_{a_i} \rangle$

and consider the (partial) flag

$$\Lambda_{a_1} \subset \Lambda_{a_2} \subset \cdots \subset \Lambda_{a_k} \qquad (*)$$

The group $G = \mathrm{PGL}_{n+1}$ acts transitively on flags indexed by I.
The stabilizer of $(*)$ is the subgroup P_I of classes of matrices

$$\left[\begin{pmatrix} B_1 & * & * & * \\ 0 & B_2 & * & * \\ 0 & 0 & \ldots & * \\ 0 & 0 & 0 & B_{k+1} \end{pmatrix}\right]$$

where the $B_j's$ are square matrices of orders determined by I.

Flag varieties

The quotient G/P_I is then the variety parametrizing flags

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_k$$

of linear subspaces in $\mathbb{P}^n$ with $\dim \Lambda_i = a_i - 1$; we denote it by $\mathrm{A}_n(I)$.

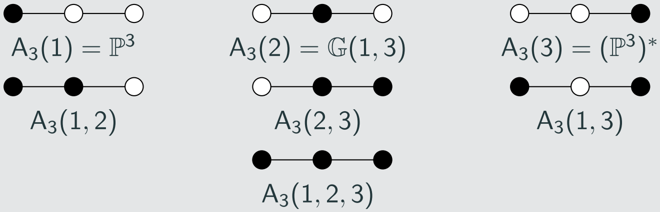
- $\mathrm{A}_n(I)$ is a Fano manifold of Picard number $\# I$
- For every $J \subsetneq I$ there is a smooth fibration $\mathrm{A}_n(I) \to \mathrm{A}_n(J)$ whose fibers are flag varieties or products of flag varieties.

If we consider $I = \Delta$, i.e., full flags of linear subspaces, then P_Δ is the Borel subgroup of upper triangular matrices and the variety $\mathrm{A}_n(\Delta) := G/P_\Delta$ is called variety of complete flags of $\mathbb{P}^n$.

Flag varieties

We represent $A_n(I)$ with the Dynkin diagram of the Lie algebra of PGL_{n+1} - which is $\mathfrak{sl}_{n+1}$ - marked in the nodes indexed by I.

$G = PGL_4$



Using marked diagrams it is easy to understand morphisms $A_n(I) \to A_n(J)$

Effective good divisibility of flag varieties

Theorem (Muñoz, __, Solá Conde, 2022)

The effective good divisibility of a flag variety $\mathrm{A}_n(I)$ is n, $\forall I \subset \Delta$.

For every $I \subseteq J \subseteq \Delta$ there are injections given by pullbacks

$$\mathrm{A}^\bullet(\mathrm{A}_n(I)) \to \mathrm{A}^\bullet(\mathrm{A}_n(J)) \to \mathrm{A}^\bullet(\mathrm{A}_n(\Delta))$$

hence

$$\mathrm{e.d.}(\mathrm{A}_n(\Delta)) \leq \mathrm{e.d.}(\mathrm{A}_n(J)) \leq \mathrm{e.d.}(\mathrm{A}_n(I))$$

Strategy: show that

- $\mathrm{e.d.}(\mathrm{A}_n(\Delta)) \geq n$
- $\mathrm{e.d.}(\mathrm{A}_n(I)) \leq n$ for every I of cardinality one

Upper bound

$$\boxed{\text{e.d.}(A_n(r)) \leq n}$$

- $\Sigma_p \subset \mathbb{G}(r,n)$, parametrizing linear spaces passing by p.
- $\Sigma_H \subset \mathbb{G}(r,n)$, parametrizing linear spaces contained in H.

$$\operatorname{codim}\Sigma_p + \operatorname{codim}\Sigma_H = n+1 \qquad [\Sigma_p]\cdot[\Sigma_H] = 0$$

$$\text{e.d.}(\mathrm{A}_n(r)) < \operatorname{codim}\Sigma_p + \operatorname{codim}\Sigma_H = n+1$$

Lower bound

$\boxed{\text{e.d.}(A_n(\Delta)) \geq n}$

We need a good understanding of the Chow ring of $X := \mathrm{A}_n(\Delta)$.

$\mathrm{A}^\bullet(X)$ can be described in terms of the Weyl group W of PGL_{n+1} which is the permutation group S_{n+1}.

Given $w \in W$ we consider the full flag F_w

$$\Lambda_1^w \subset \Lambda_2^w \subset \cdots \subset \Lambda_n^w$$

determined by the subspaces $\Lambda_j^w = \langle E_{w(1)}, \dots E_{w(j)} \rangle$ and the closure of its orbit under the (left) action of P_Δ:

$$X_w = \overline{P_\Delta F_w}$$

This is the Schubert variety associated with $w \in W$.

Weyl group

W is generated by the n transpositions $s_i = (i, i+1)$ with the relations

$$s_i^2 = 1 \qquad (s_i s_j)^2 = 1 \text{ if } |i-j| > 1 \qquad (s_i s_{i+1})^3 = 1$$

We have a well defined notion of length $\ell(w)$ of an element w, as the smallest k such that

$$w = s_{i_1} \dots s_{i_k}$$

An expression of w as product of $\ell(w)$ transpositions is called reduced.

The dimension of a Schubert variety is:

$$\dim X_w = \ell(w)$$

There exists a unique element w_0 of maximal length $\ell(w_0) = \dim X$, called the longest element, and

$$\operatorname{codim} X_w = \ell(w_0) - \ell(w) = \ell(w_0 w) := c(w)$$

Schubert cycles

The class of a Schubert variety in $A^\bullet(X)$ is called a Schubert cycle.

Properties of Schubert cycles

(1) The Schubert cycles form a $\mathbb{Z}$-basis of $A^\bullet(X)$.

(2) The cones of effective classes in $A^c(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ are polyhedral and generated by Schubert cycles of codim c, for every $c \geq 0$.

(3) The product of two Schubert cycles can be written as a sum of Schubert cycles with nonnegative coefficients.

Therefore the effective good divisibility can be computed considering only Schubert cycles. We need to prove that

$$[X_u] \cdot [X_v] \neq 0 \qquad \forall X_u, X_v \text{ with } \operatorname{codim} X_u + \operatorname{codim} X_v \leq n$$

Bruhat order

There exists a partial order on W, the Bruhat order, defined as $v \leq w$ iff some substring of a reduced expression of w is a reduced expression of v.

Intersection of Schubert varieties

$$[X_u] \cdot [X_{w_0 v}] \neq 0 \;\Leftrightarrow v \leq u$$

Our thesis

$$[X_u] \cdot [X_{w_0 v}] \neq 0 \qquad \forall X_u, X_{w_0 v} \text{ with } \operatorname{codim} X_u + \operatorname{codim} X_{w_0 v} \leq n$$

can be thus rephrased as

$$v \leq u \qquad \forall u, v \in W \text{ with } c(u) + c(w_0 v) \leq n$$

which can be rewritten as

$$v \leq u \qquad \forall u, v \in W \text{ with } c(u) + \ell(v) \leq n$$

Waving hands a bit

The problem is thus reduced to a problem on the Bruhat order in W.

This can be solved by induction, considering

- $W_J \subset W$ the subgroup generated by $s_2, \dots, s_n$
- $W^J = \{w \in W \mid \ell(ws_i) > \ell(w) \; \forall i = 2, \dots, n\}$, which is the set of left cosets of W_J

Geometrically we are considering the projection $X \to \mathbb{P}^n$.

- The fibers are complete flag manifolds of type A_{n-1}, with Weyl group W_J.
- Elements of W^J, which are right substrings of $s_n s_{n-1} \dots s_2 s_1$ corresponds to Schubert cells of $\mathbb{P}^n$.

Waving hands a lot

Any element $w \in W$ has a unique decomposition

$$w = w^J w_J \qquad w^J \in W^J, w_J \in W_J$$

We show that, in our assumptions we can write

$$v = v_1 v_2 \qquad \text{with} \qquad v_1 \leq u^J \text{ and } v_2 \leq u_J$$

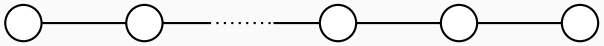
so that $v = v_1 v_2 \leq u^J u_J = u$.

RH spaces of classical type

RH spaces of classical type

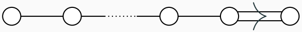
Any RH space of classical type is a flag of linear spaces, determined by a subset of the corresponding Dynkin diagram

A_n



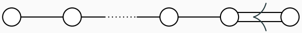
Flags of linear subspaces of $\mathbb{P}^n$

B_n



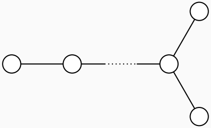
Flags of linear subspaces of $\mathbb{Q}^{2n-1}$

C_n



Flags of linear subspaces of $\mathbb{P}^{2n-1}$
isotropic w.r.t. a contact form

D_n



Flags of linear subspaces of $\mathbb{Q}^{2n-2}$

Results

Theorem (Muñoz, __ , Solá Conde, 2022)

The effective good divisibility of a complete flag manifold of classical type $\mathcal{D}$ is

$\mathcal{D}$	A_n	B_n	C_n	D_n
$\mathrm{e.d.}(\mathcal{D}(\Delta)$	n	$2n-1$	$2n-1$	$2n-3$

Theorem (Muñoz, __ , Solá Conde, 2022)

For any rational homogeneous variety of classical type $\mathcal{D}(I)$ we have

$$\mathrm{e.d.}(\mathcal{D}(I)) = \mathrm{e.d.}(\mathcal{D}(\Delta)$$

unless $\mathcal{D} = \mathrm{D}_n$ *and* $I \cap \{1, n-1, n\} = \emptyset$, *where* $\mathrm{e.d.}(\mathrm{D}_n(I)) = 2n-2$.

Ideas

For complete flag manifolds, again one studies Schubert varieties and reduces to a problem on the Bruhat order on Weyl groups.

The Weyl groups are different, but the the idea of the proof is the same.

For Picard number one varieties the effective bound is again given by the pair Σ_p and Σ_H except for $\mathrm{D}_n(1), \mathrm{D}_n(n-1), \mathrm{D}_n(n)$, due to the presence of different maximal disjoint pairs in quadrics and spinor varieties.

Applications

Morphisms to RH varieties

Theorem (Muñoz, __, Solá Conde, 2022)

$\mathcal{D}$ Dynkin diagram of classical type with set of nodes Δ, $\emptyset \neq I \subset \Delta$. M complex projective variety s.t. $\text{e.d.}(M) > \text{e.d.}(\mathcal{D}(I))$. *Then every morphism from M to $\mathcal{D}(I)$ is constant.*

The variety $\mathcal{D}(I)$ can be identified with a subvariety of $\prod_{i \in I} \mathcal{D}(i)$.

A morphism $f : M \to \mathcal{D}(I)$ is constant if and only the induced morphism $f : M \to \mathcal{D}(i)$ is constant for every i.

We may assume that I consists of a single element i.

We have already shown the statement in the case A_n. The other cases are an adaptation of the alternative proof of Tango's Theorem.

Morphisms between RH varieties

Corollary

Let $\mathcal{D}'$ be a Dynkin diagram of classical type, whose set of nodes is Δ' and $\mathcal{D} \subsetneq \mathcal{D}'$ be a proper subdiagram, with set of nodes Δ. Then, for any $I \subset \Delta, I' \subset \Delta'$, any morphism $\phi : \mathcal{D}'(I') \to \mathcal{D}(I)$ is constant.

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