Generalization of the Murnaghan-Nakayama rule for K-k-Schur functions and k-Schur functions

Duc-Khanh Nguyen

University at Albany

Motivation

$f(x_1, x_2, \ldots)$ is symmetric if $f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$ $\forall \sigma \in S_\infty$

$\Lambda := \bigoplus_{n \geq 0} \Lambda^n$ graded ring of symmetric functions in variables $x_1, x_2, \ldots$ with coefficients in $\mathbb{Z}$

Λ^n has $\mathbb{Z}$-basis $\{ s_\lambda \mid \lambda \text{ is a partition of } n \}$.

Example polynomials in variables $x_1, \ldots, x_n$

$$e_r := \sum_{1 \leq i_1 < \ldots < i_r \leq n} x_{i_1} \cdots x_{i_r}$$ elementary symmetric polynomial

$$h_r := \sum_{1 \leq i_1 \leq \ldots \leq i_r \leq n} x_{i_1} \cdots x_{i_r}$$ complete homogeneous symmetric polynomial

$$p_r := x_1^r + \cdots + x_n^r$$ power-sum symmetric polynomial

$$s_\lambda := \frac{\det\left(x_i^{\lambda_j + n - j} \right)_{n \times n}}{\prod (x_i - x_j)_{1 \leq i < j \leq n}}$$ Schur function $\sim \lambda$

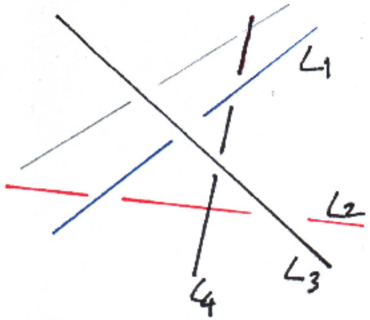
Fundamental rules: $\lambda \vdash n$, $0 \leq r \leq n$

Pieri rule: $e_r \cdot s_\lambda = \sum_{\mu \vdash n} * s_\mu$, $h_r \cdot s_\lambda = \sum_{\mu \vdash n} * s_\mu$

Murnaghan-Nakayama rule: $p_r \cdot s_\lambda = \sum_{\mu \vdash n} * s_\mu$

Littlewood-Richardson rule: $s_\lambda s_\mu = \sum_{\nu \vdash n} * s_\nu$

Counting problems in Algebraic Geometry



4 general line in $\mathbb{P}^3$

$\# \{ \text{lines meet all } L_1, \ldots, L_4 \} = ?$

(Co) homology theory

$$H^*(Gr(4,2)) = \bigoplus_{\lambda \subset \boxplus} \mathbb{Z}\sigma_\lambda$$

$$(\sigma_\square)^4 = 2\,\sigma_{\boxplus}$$

answer $= 2$

analogue

associate Schubert classes to symmetric functions

Representation Theory

$$V^\lambda \otimes V^\mu = \bigoplus c^\nu_{\lambda\mu} V^\nu$$

compute character table

characters as symmetric functions

Symmetric functions.

Fundamental rules:

Pieri rule $e_k, h_k \cdot s_\lambda$

Murnaghan Nakayama $p_r \cdot s_\lambda$

Littlewood-Richardson $s_\lambda \cdot s_\mu$

(quantum) K-theory

replace H^*, H_* by

QK^*, QK_*, K^*, K_*

associate Schubert classes to symmetric functions

We are here

$Gr = SL_{k+1}(\mathbb{C}((t))) / SL_{k+1}(\mathbb{C}[[t]])$ affine Grassmannian

$H_*(Gr) = \bigoplus_{\lambda_1 \le k} \mathbb{Z}\, \sigma_\lambda^{(k)}$ Schubert class $\sim \lambda$

$\sigma_\lambda^{(k)} \longleftrightarrow s_\lambda^{(k)}$ k-Schur function $\sim \lambda$

$\sigma_\lambda^{(k)} \cdot \sigma_\mu^{(k)} = \sum_\nu * \sigma_\nu^{(k)}$

computed by $s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu * s_\nu^{(k)}$

$K_*(Gr) = \bigoplus_{\lambda_1 \le k} \mathbb{Z}\, \theta_\lambda^{(k)}$ (K) Schubert class $\sim \lambda$

$\theta_\lambda^{(k)} \longleftrightarrow g_\lambda^{(k)}$ K-k-Schur function $\sim \lambda$

$\theta_\lambda^{(k)} \cdot \theta_\mu^{(k)} = \sum_\nu * \theta_\nu^{(k)}$

computed by $g_\lambda^{(k)} g_\mu^{(k)} = \sum_\nu * g_\nu^{(k)}$

Definitions

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ $\lambda_i \in \mathbb{Z}_{\geq 0}$ partition

$$\sim \begin{array}{llll} \times & \times & \ldots & \times \quad (\lambda_1 \text{ boxes}) \\ \times & \times & \ldots \times & \quad (\lambda_2 \text{ boxes}) \\ \vdots & & & \quad \ddots \\ \times & & & \end{array}$$

Young diagram $\sim \lambda$

$$\lambda^t := \begin{array}{llll} \times & \times & \ldots & \times \\ \times & \times & \ldots & \times \\ \vdots & \uparrow & & \ddots \\ \times & (\lambda^t)_2 & & \\ \uparrow & & & \\ (\lambda^t)_1 & & & \end{array}$$

Conjugate partition of λ

For $\lambda \leq \mu$, $\mu/\lambda = \begin{array}{lllll} \otimes & \otimes & \times & \ldots & \times \\ \otimes & \otimes & \ldots & \times & \\ \otimes & \times & & & \\ \times & \times & & & \\ \times & & & & \end{array}$ is called a Ribbon if it does not contain $\begin{array}{ll} \times & \times \\ \times & \times \end{array}$

Remove λ from μ.

then $ht(\mu/\lambda) := \#$ vertical dominos $\begin{array}{l} \times \\ \times \end{array}$ in μ/λ

height

<u>Example</u>

$$\lambda = (4,2,1,1) \sim \begin{array}{lllll} \times & \times & \times & \times & \leftarrow 4 \\ \times & \times & \leftarrow 2 & & \\ \times & \leftarrow 1 & & & \\ \times & \leftarrow 1 & & & \end{array}$$

$$\lambda^t = (4,2,1,1) \text{ since } \begin{array}{llll} \times & \times & \times & \times \\ \times & \times & \uparrow & \uparrow \\ \times & \uparrow & 1 & 1 \\ \times & 2 & & \\ \uparrow & & & \\ 4 & & & \end{array}$$

$$\mu = (4,2,2,1,1,1) \sim \begin{array}{llll} \times & \times & \times & \times \\ \times & \times & & \\ \times & \times & & \\ \times & & & \\ \times & & & \\ \times & & & \end{array} \geq \lambda \text{ then } \mu/\lambda = \begin{array}{llll} \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & & \\ \otimes & \times & & \\ \otimes & & & \\ \times & & & \\ \times & & & \end{array}$$

is a ribbon, $ht(\mu/\lambda) = 1$

$\tilde{S}_{k+1}$ affine symmetric group generators : $s_0, \dots, s_k$

relations : $s_i^2 = id \quad \forall i \in \mathbb{Z}/(k+1)\mathbb{Z}$

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \forall i$ A

$s_i s_j = s_j s_i \quad \forall i-j \neq \pm 1$ B

S_{k+1} symmetric group generators : $s_1, \dots, s_k$.

$s_{i_1 \dots i_k} := s_{i_1} \cdots s_{i_k}$

$\lambda = (\lambda_1, \lambda_2, \dots)$ s.t $\lambda_1 \leq k$.

$$\lambda \xmapsto{\phi} w_\lambda \in \tilde{S}^0_{k+1} := \{\text{minimum length coset representative of } \tilde{S}_{k+1}/S_{k+1}\} \xmapsto{5} K_\lambda, \qquad K_\lambda \xrightarrow{\mathfrak{P}} \lambda$$

Example $k = 4$

$$\lambda = (4,2,1,1) \longmapsto w_\lambda = s_{23043210} \longmapsto K_\lambda = (6,2,1,1)$$

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{array} \qquad \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{array}$$

- read

$\mathcal{A}_k :=$ associative algebra over $\mathbb{Z}$ generators: $A_0, \ldots, A_k$ relations : Ⓐ + Ⓑ

$$A_{i_1 \ldots i_R} := A_{i_1} \ldots A_{i_R}$$

For $0 \leq R \leq k$, $A \subseteq \mathbb{Z}/(k+1)\mathbb{Z}$, $|A| = R$

$$d_A := A_{i_1 \ldots i_R}, \quad i_A := A_{i_R \ldots i_1}$$

where $(i_1 \ldots i_R)$ is an rearrangement of A such that if $i, i+1 \in A$ then $i+1$ occurs before i

Example $k = 4$, $A = \{0,2,4\}$, $d_A = A_{042} = A_{024}$ $\quad i_A = A_{240} = A_{420}$

then $h_R := \sum_{A \in \binom{[0,k]}{R}} d_A$ noncommutative homogenous symmetric functions

$$e_R := \sum_{A \in \binom{[0,k]}{R}} i_A \text{ — elementary —}$$

$$s_{(R-i,1^i)} := \sum_{j=0}^{i} (-1)^j h_{R-i+j}\, e_{i-j} \text{ — hook Schur —}$$

$$p_R := \sum_{i=0}^{R-1} (-1)^i s_{(R-i,1^i)} \text{ — power sum —}$$

$\mathcal{A}_k \circlearrowright_\varphi \mathbb{C}[\tilde{S}_{k+1}]$ by $\alpha \underset{\varphi}{*} \omega$

$\Psi : \mathcal{A}_k \times \tilde{S}_{k+1} \longrightarrow \mathbb{R}$ is said to be φ-compatiable if $\Psi(\alpha\beta, \omega) = \Psi(\alpha, \beta \underset{\varphi}{*} \omega)\Psi(\beta, \omega)$

Fix φ, Ψ, we define $\{\mathcal{F}_\omega^{(k)}\}_{\omega \in \tilde{S}^\circ_{k+1}}$ to be a family symmetric functions

s.t

$$\mathcal{F}_{id}^{(k)} = 1$$

$$h_R \cdot \mathcal{F}_\omega^{(k)} = \sum_{\substack{A \in \binom{[0,k]}{R} \\ d_A \underset{\varphi}{*} \omega \in \tilde{S}^\circ_{k+1}}} \Psi(d_A, \omega) \mathcal{F}^{(k)}_{d_A \underset{\varphi}{*} \omega}$$

$$e_R \cdot \mathcal{F}_\omega^{(k)} = \sum_{\substack{B \in \binom{[0,k]}{R} \\ i_B \underset{\varphi}{*} \omega \in \tilde{S}^\circ_{k+1}}} \Psi(i_B, \omega) \mathcal{F}^{(k)}_{i_B \underset{\varphi}{*} \omega}$$

for $\omega_\lambda \in \tilde{S}^\circ_{k+1}$ $\mathcal{F}_\lambda^{(k)} := \mathcal{F}_{\omega_\lambda}^{(k)}$

Example

$\mathcal{A}_k \circlearrowright \mathbb{C}[\tilde{S}_{k+1}]$ def $A_i * \omega = \begin{cases} s_i\omega & \text{if } \ell(s_i\omega) > \ell(\omega), \\ \omega & \text{if } \ell(s_i\omega) < \ell(\omega). \end{cases}$ and $\Psi(\alpha, \omega) := (-1)^{\tilde{\ell}(\alpha) - \ell(\alpha * \omega) + \ell(\omega)}$

letters of α length function in $\tilde{S}_{k+1}$

then $\mathcal{F}_\omega^{(k)} = g_\omega^{(k)}$ K-k-Schur functions

$\mathcal{A}_k \circlearrowright \mathbb{C}[\tilde{S}_{k+1}]$ def $A_i \cdot \omega := \begin{cases} s_i\omega & \text{if } \ell(s_i\omega) > \ell(\omega), \\ 0 & \text{otherwise} \end{cases}$ and $\Psi(\alpha, \omega) := 1$

then $\mathcal{F}_\omega^{(k)} = s_\omega^{(k)}$ k-Schur functions

For $u \in \mathcal{A}_k$,

$S := \mathrm{supp}(u)$

$I_S :=$ canonical cyclic interval of S

1/ let a be the minimum in $[0, k]$ s.t $a \notin S$

2/ then I_S is : $a+1 < \dots < a-1$

Example $\mathcal{A}_4 \ni 0424 =: u$. $\mathrm{Supp}(u) = \{0, 2, 4\} =: S$

. $I_S = 0\ 1\ 2\ 3\ 4 = 2 < 3 < 4 < 0$

min not in S

8

u is called k-connected if S is an interval of I_S

weak hook word if it have a reduced word of form $\searrow\nearrow$ say hook type V

OR $\searrow\_\nearrow$ say hook type U

$u_i = u_{i+1}$ some i

$asc(u) := \#$ ascents of hook forms $\searrow\nearrow$, $\searrow\_\nearrow$ wrt to the order I_S

$\mathcal{E}_u = \{$ consecutive pairs $a < c$ st $\nexists\, a < b < c$ in hook form of $u \}$
$\neq c+1$

$c_{min} := \min\{c \mid (a<c) \text{ in } \mathcal{E}_u\}$ Fact $\# c_{min} \in \{0, 1, 2\}$.

Example

$S = 2 < \cancel{3} < 4 < 0 \rightsquigarrow$ not 4-connected

not an interval

$u = 0\,4\,2\,4 \rightsquigarrow$ hook type V

$\#$ ascent $= 1$

$0\;4\;2\;4 \rightarrow c_{min} = 4$, in left and right sides of V

no gap

pairs $a < c$ with gap between them

$4 > 2 \quad 2 < 4$

u → connected; u → disconnected.
disconnected. → $\# c_{min} = 2$ =: weak connected =: wc
disconnected. → $\# c_{min} = 1$ → c_{min} on left =: $\overline{wc}$, left
$\# c_{min} = 1$ → c_{min} on right =: $\overline{wc}$, right

Notations: $u \in V^{R}_{i,wc}$
- weak length (# letters) = R
- weak hook type $(u) = V$
- $asc(u) = i$
- weak connected

similar for $U^{R}_{i,wc}$, $U^{R}_{i,\overline{wc}\,\text{left}}$, $U^{R}_{i,\overline{wc}\,\text{right}}$

Example

$0424 \in V^{4}_{1,wc}$ $\quad$ $4224 \in U^{4}_{1,wc}$ $\quad$ $2240 \in U^{4}_{2,\overline{wc}\,\text{Right}}$

Main results

$f \doteq S$ means $f = \sum_{u \in S} u$.

New! <u>lemma</u> For $1 \le R \le k$, we have

$$P_R \doteq \sum_{i=0}^{R-1} (-1)^i V^R_{i,wc} + \sum_{i=1}^{R-1} (-1)^i (R-i)\, U^R_{i-1,wc} + \sum_{i=1}^{R-2} (-1)^i\, U^R_{i-1,\overline{wc},\text{left}}$$

New! <u>Theorem</u> (Murnaghan-Nakayama rule) if $\Psi(u,w)$ only depend on $\overset{R}{\overset{\shortparallel}{\tilde{\ell}(w)}}$, $\overset{w'}{\overset{\shortparallel}{u *_\varphi w}}$, w then we can write $\Psi(u,v) = \tilde{\Psi}(R,w',w)$

$$P_R *_\varphi \mathcal{F}^{(k)}_w = \sum_{w' \in \tilde{S}^{(0)}_{k+1}} \tilde{\Psi}(R,w',w) \left(\sum_{i=0}^{R-1} (-1)^i \left| V^{R,w'}_{i,wc} \right| + \sum_{i=1}^{R-1} (-1)^i (R-i) \left| U^{R,w'}_{i-1,wc} \right| + \sum_{i=0}^{R-2} (-1)^i \left| U^{R,w'}_{i-1,\overline{wc},\text{left}} \right| \right) \mathcal{F}^{(k)}_{w'}$$

means subset of words u s.t $u *_\varphi w = w'$

$\mathcal{A}_k \curvearrowright \mathbb{C}[\tilde{S}_{k+1}]$ def $A_i * w := \begin{cases} s_i w & \text{if } \ell(s_i w) > \ell(w), \\ w & \text{if } \ell(s_i w) < \ell(w). \end{cases}$ and $\Psi(\alpha, w) := (-1)^{\tilde{\ell}(\alpha) - \ell(\alpha * w) + \ell(w)}$

\# letters of α length function in $\tilde{S}_{k+1}$

then $\mathcal{F}_w^{(k)} = g_w^{(k)}$ K-k-Schur functions

New! Corollary 1 (Murnaghan – Nakayama rule for K-k-Schur functions)

★ $$p_r \cdot g_w^{(k)} = \sum_{w' \in \tilde{S}^{(0)}_{k+1}} (-1)^{R-\ell(w')+\ell(w)} \left[\sum_{i=0}^{R-1} (-1)^i \left| V^{R,w'}_{i,wc} \right| + \sum_{i=1}^{R-1} (-1)^i (R-i) \left| U^{R,w'}_{i-1,\,wc} \right| + \sum_{i=0}^{R-2} (-1)^i \left| U^{R,w'}_{i-1,\,\overline{wc}\,\text{left}} \right| \right] g_{w'}^{(k)}$$

$$= \sum_{\substack{\mu \in P_k \\ \text{s.t}}} (-1)^{R-\ell(w')+\ell(w)} \left[\sum_{i=0}^{R-1} (-1)^i \left| V^{R,\mu}_{i,wc} \right| + \sum_{i=1}^{R-1} (-1)^i (R-i) \left| U^{R,\mu}_{i-1,\,wc} \right| + \sum_{i=0}^{R-2} (-1)^i \left| U^{R,\mu}_{i-1,\,\overline{wc}\,\text{left}} \right| \right] g_{\mu}^{(k)}$$

(0) $\lambda \subseteq \mu$, $\lambda^{(k)} \subseteq \mu^{(k)}$

$\tilde{(1)}$ $|\mu/\lambda| \leq R$

(2) K_μ / K_λ is a Ribbon

$\tilde{(3)}$ K_μ / K_λ ~~is~~ k-connected or |support|<= r-1

$\tilde{(4)}$ $ht(\mu/\lambda) + ht(\mu^{(k)}/\lambda^{(k)}) \leq R-1$

$\mathcal{A}_k \curvearrowright \mathbb{C}[\widetilde{S}_{k+1}]$ def $A_i \cdot w := \begin{cases} s_i w & \text{if } \ell(s_i w) > \ell(w), \\ 0 & \text{otherwise} \end{cases}$ and $\Psi(\alpha, w) := 1$

then $\mathcal{F}_w^{(k)} = s_w^{(k)}$ k-Schur functions $A_i^2 \cdot w = 0$

Corollary 2 (Murnaghan - Nakayama rule for k-Schur functions) (A. Schilling - A. Zabrocki - J. Bandlow 2011)

$$p_R \cdot s_w^{(k)} = \sum_{w' \in \widetilde{S}^{(0)}_{k+1}} \left(\sum_{i=0}^{R-1} (-1)^i \left|V_{i,c}^{R,w'}\right| \right) s_{w'}^{(k)} = \sum_{\mu \in P_k} \left(\sum_{i=0}^{R-1} (-1)^i \left|V_{i,c}^{R,\mu}\right| \right) s_\mu^{(k)}$$

s.t.
(0) $\lambda \subseteq \mu$ and $\lambda^{(k)} \subseteq \mu^{(k)}$
(1) $|\mu/\lambda| = R$
(2) K_μ / K_λ is a Ribbon
(3) K_μ / K_λ is k-connected
(4) $ht(\mu/\lambda) + ht(\mu^{(k)}/\lambda^{(k)}) = R-1$

New! In Corollary 1 + 2 we give an effective algorithm to find the sets which contribute to the decompositions

Hence, we can compute coefficients by hand easily

Example (Murnaghan–Nakayama rule for K-k-Schur functions)

$$k=4,\quad R=4,\quad \lambda=(4,2,1,1) \longleftrightarrow \kappa_\lambda=(6,2,1,1)$$

$$\mu=(4,2,2,2,1) \longleftrightarrow \kappa_\mu=(7,3,2,2,1)$$

κ_λ

κ_μ

	1	2	3	4	5	6	7	8
1	0	1	2	3	4	0	1	2
2	4	0	1	2	3	4	0	1
3	3	4	0	1	2	3	4	0
4	2	3	4	0	1	2	3	4
5	1	2	3	4	0	1	2	3
6	0	1	2	3	4	0	1	2
7	4	0	1	2	3	4	0	1

an algorithm on skew tableau gives us hook words

$1034 \Updownarrow V_{1,wc}$ $\quad$ $3124 \Updownarrow V_{2,wc}$ $\quad$ $1340 \Updownarrow V_{2,wc}$ $\quad$ $3114 \Updownarrow U_{1,\overline{wc}\text{ left}}$ $\quad$ $\boxed{1134 \Updownarrow U_{2,\overline{wc}\text{ Right}}}$ $\quad$ $3134 \Updownarrow V_{2,wc}$

REMOVE

coeff of $g^{(4)}_\mu$ in $p_4 \cdot g^{(4)}_\lambda$ is:

$$(-1)^{4+|\mu|-|\lambda|}\left(\begin{aligned}&|V_{0,wc}|-|V_{1,wc}|+|V_{2,wc}|-|V_{3,wc}|\\ &-3|U_{0,wc}|+2|U_{1,wc}|-|U_{2,wc}|\\ &-|U_{0,\overline{wc}\text{ left}}|+|U_{1,\overline{wc}\text{ left}}|\end{aligned}\right) = (-1)^{4+11-8}\left(\begin{aligned}&0-1+3-0\\ &-3.0+2.0-1.0\\ &-0+1\end{aligned}\right)$$

$$=(-1)\,3=-3.$$

Future directions

MN rule for $\mathcal{F}_w^{(k)}$

$\cdots \quad g_\lambda^{(k)} \quad \cdots$ detailed in the future.

$g_\lambda^{(k)} \to s_\lambda^{(k)}$

dual k-Schur functions
affine Stable Grothendieck polynomials
closed k-Schur functions
closed K-k-Schur functions
etc

$H_*(Gr) \longleftrightarrow QH^*(Fl_{k+1})$ Peterson's isomorphism

$K_*(Gr) \longleftrightarrow QK^*(Fl_{k+1})$ K-Peterson's isomorphism

MN rule in $QK^*(Fl_{k+1})$ $QH^*(Fl_{k+1})$ .... (future)

Thank you for your attention!