# Graph potentials and mirrors of moduli of rank two bundles on curves.

Swarnava Mukhopadhyay
(joint work with Pieter Belmans and Sergey Galkin)

August 4, 2022

## Moduli space

Consider the smooth intersection of two quadrics $Q_1$ and $Q_2$

$$X_{2,2} \subset \mathbb{P}^5$$

This is a Fano three fold of Picard number one.

**Reinterpretation as moduli space**

- $C$ will denote a smooth projective curve of genus $g \geq 2$.
- $\mathbb{L}$ be a fixed line bundle on $C$.
- $M_C(\mathbb{L})$ will denote the moduli space of semi-stable rank two bundles with determinant $\mathbb{L}$.

## Properties

- For any $\mathbb{L}$ of odd degree (respectively even), the moduli spaces $M_C(\mathbb{L})$'s are isomorphic. We drop the $\mathbb{L}$ in the notation and simply denote $M_C^{\pm}$.

- If $C$ is hyperelliptic, then the moduli space has a more concrete description (Narasimhan-Ramanan, Newstead ($g = 2$), Desale-Ramanan).

$$M_C^- = \mathrm{OGr}_{q_1}(g-1, 2g+2) \cap \mathrm{OGr}_{q_2}(g-1, 2g+2).$$

- $M_C^-$ is smooth, Fano of dimension $3(g-1)$. Moreover (Drezet-Narasimhan)

$$\mathrm{Pic}(M_C^{\pm}) = \mathbb{Z}\Theta.$$

The canonical class $K_{M_C^-} = -2[\Theta]$, i.e. $M_C^-$ is of index two.

## Properties... continued

- Deformations of $M_C^{\pm}$ are controlled by deformations of $C$.
- The spaces $H^0(M_C^{\pm}, \Theta^{\otimes \ell})$ are known as **conformal blocks** and can be constructed as quotient of representations of $\widehat{\mathrm{SL}}_2(\mathbb{C}((t)))$. (Beauville-Laszlo, Faltings, Laszlo-Sorger, Kumar-Narasimhan-Ramananathan)
- As $C$ varies in $\overline{\mathcal{M}}_g$, the spaces $H^0(M_C^{\pm}, \Theta^{\otimes \ell})$ form a vector bundle (Tsuchiya-Ueno-Yamada/Wess-Zumino-Witten). denoted by $\mathbb{V}_{\pm}(\mathfrak{sl}(2), \ell)$ along with generalization to the parabolic bundles set-up.

# Mirror Symmetry for Fano $X$ and LG-models $(Y, w)$

**B-side**

- The bounded derived category $\mathbf{D}^b(X)$ and semi orthogonal decompositions.

- Matrix factorization category $\mathrm{MF}(Y, w)$ and their decomposition with respect to the critical values of $w$.

**A-side**

- Fukaya-Seidel category $\mathrm{FS}(Y, w)$ of a Landau-Ginzburg model.

- Fukaya Category $\mathrm{Fuk}(X)$, quantum cohomology ring $QH^*(X)$ and decomposition with respect to $c_1(X)\star_0$.

**Decompositions:** Eigen Values $(c_1(M)\star_0) =$ Critical Values $(w)$.

# Quantum periods $X$

Let $X_{0,k,m}$ denote the Kontsevich moduli space of stable maps $f$ from a rational curve with $k$ marked points and $\deg f^*(-K_X) = m$.

**Definition**

The $m \geq 2$-th descendent Gromov Witten number

$$p_m = \int_{X_{0,1,m}} \psi^{m-2}\, \mathrm{ev}_1^{-1}([pt]),$$

where $\psi$ is the *Psi* class on $X_{0,1,m}$ and $\mathrm{ev}_1 : X_{0,1,m} \to X$.

**Compute**

$$\widehat{G}_X(t) := \sum_{m \geq 0} m! p_m t^m \quad \text{for } p_0 = 1,\ p_1 = 0.$$

### Definition

Let $W : (\mathbb{C}^\times)^n \to \mathbb{C}$ be a Laurent polynomial. A classical period of $W$ is the following Laurent series.

$$\pi_W(t) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{1 - tW(x_1, \ldots, x_n)} \operatorname{dlog} \vec{x}$$

### Quantum=classical

Given $X$, can we find $W$ such that

$$\widehat{G}_X(t) = \pi_W(t)$$

## Example

- If $X = \mathbb{P}^3$, then $W = x + y + z + \frac{1}{xyz}$ and $G_X(t) = \sum_{d=0}^{\infty} \frac{t^{4d}}{(d!)^4}$
- If $X$ blow up of a line in $\mathbb{P}^3$, then $W = x + y + z + \frac{z}{x} + \frac{1}{yz}$.
- If $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$, then $W = x + y + z + \frac{z^2}{xy} + \frac{1}{z}$.

**Remark**

Observe that in all these cases $X$ is a toric variety and the Newton polytope of $W$ is the Fan polytope of the toric variety.

## Finding mirror potentials $W$

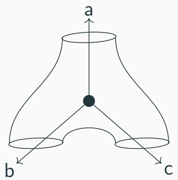
- (Hori-Vafa, Givental) If $X$ is a smooth toric Fano then we can take $W : (\mathbb{C}^{\times})^{\dim X} \to \mathbb{C}$ to the Newton polynomial of the Fan polytope. Similarly $W$ is known for Fano complete intersection in a toric variety.
- (Coates-Corti-Galkin-Kasprzyk) If $X$ is a smooth Fano three fold, then quantum periods are known.
- Many other results due to works of Batryrev-Ciocan-Fontanine-Kim-van-Straten, Bondal-Galkin, Coates, Przyalkowski,...

# Goal

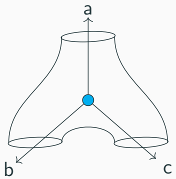
- Find a weak LG mirror $W$ for $M_C^-(2)$ ?
- Give an efficient way to compute periods of $W$.
- Compare the critical values and critcal sets to that of quantum cohomology of $M_C^-(2)$.
- Give evidence for natural decomposition of the derived category of $M_C^-(2)$.

# Graph potentials and trinion potentials

$$W_{\bullet} = abc + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}$$




$$W_{\color{cyan}\bullet} = \frac{1}{abc} + \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}$$




$$W_{\bullet}(a^{\pm}, b^{\pm}, c^{\pm}) = W_{\color{cyan}\bullet}(a^{\mp}, b^{\mp}, c^{\mp})$$

# Graph Potential

1. Trivalent graph correspond to decompositon of a surface into pair of pants.
2. Trivalent graphs also correspond to a strata in $\overline{M}_{g,n}$ of maximally degenerate curves.

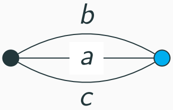
**Definiton**

Let $(\Gamma, c)$ be a colored trivalent graph and $c : V(\Gamma) \to \{\pm 1\}$, define

$$W_{\Gamma,c} := \sum_{v \in V(\Gamma)} W_{v,c(v)}.$$

# Examples: g=2

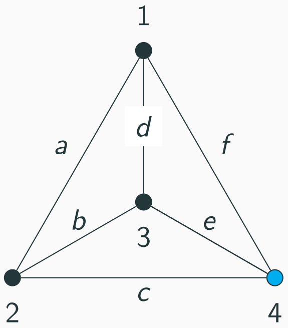

$$(b^2a + \frac{2}{a} + \frac{a}{b^2}) + (\frac{1}{ac^2} + 2a + \frac{c^2}{a})$$




$$(abc + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}) + (\frac{1}{abc} + \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c})$$

# Example $g = 3$

$$adf + \frac{f}{ad} + \frac{a}{df} + \frac{d}{af} +$$
$$bde + \frac{e}{bd} + \frac{b}{ed} + \frac{b}{de}$$
$$+abc + \frac{b}{ac} + \frac{c}{ab} + \frac{a}{bc}$$
$$+\left(\frac{1}{cef} + \frac{ef}{c} + \frac{cf}{e} + \frac{ce}{f}\right)$$

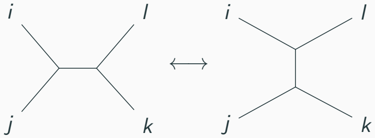


## Theorem (Belmans-Galkin-M)

Let $W \in \mathbb{C}[x_1^{\pm}, \ldots, x_e^{\pm}; y_1^{\pm} \ldots, y_\ell^{\pm}]$ be a Laurent polynomial, we will denote by $[W^m]$ the coeffiecient of $x_1^0 \ldots x_e^0$ in the $m$-th power of $W$.

We have the following result about graph potentials.

- The constant term $[(W_{\Gamma,c})^m]$ depends only on the genus $g$ of $\Gamma$ and total parity $\epsilon$ of the coloring $c$.



# Theorem on TQFT

Let $\Sigma_{g,n}$ be an oriented surface of genus $g$ with $n$ boundary components with the condition that $2g + n > 2$. To every pairs of pants decomposition of $\Sigma_{g,n}$, with dual graph $(\Gamma, n, c)$ the assignment defines a TQFT:

$$\mathcal{Z}_{\Sigma_{g,n}} := \bigotimes_{e \in E_{int}} \langle , \rangle_{a,b} \left( \bigotimes_{v \in V} \exp(tW_{\pm}(x_i, x_j, x_k)) \right), \in (\ell^2(\mathbb{Z}))^{\otimes n}$$

where $E_{int}$ are internal edges of $\Gamma$, $a, b$ are vertices adjacent to an edge $e \in E_{int}$, and $i, j, k$ are edges incident to a vertex $v$ of $\Gamma$.

# Explicit Formula

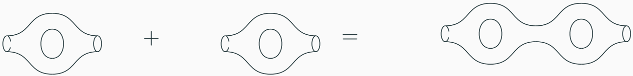
Let $Bes(z) := \sum_{m\geq 0} \frac{1}{m!^2} z^{2m}$ be the Bessel function.

**Theorem**

- For $\Gamma$ with no half edges (compact surfaces):

$$\sum_{m\geq 0} \frac{[(W_{\Gamma,c})^m]_{const}}{m!} t^m = \mathrm{Trace}(A^{g-1} S^{\epsilon+g}), \text{ where}$$

$$S(x^n) := x^{-n} \text{ and } A = Bes(t(x+y)) \cdot Bes(t(x^{-1}+y^{-1}))$$



**Example: g=2**

$$\sum_{n\geq 0} \frac{(2n!)^2}{n!^6} t^{2n}$$

# B side: Graph potentials and $M_C^{\pm}(2)$

**Theorem (Belmans-Galkin-M:20)**

- *The moduli space $M_C^{-}(2)$ (resp $M_C^{+}(2)$) has a natural toric $X_{\Gamma,c}$ degeneration associated to a trivalent graph $\Gamma$ whose Newton polynomial is the graph potential $W_{\Gamma,c}$.*

  **Remark:** *The degeneration (refining Manon:16) uses conformal blocks.*

- *If $\Gamma$ has no separating edges, then $X_{\Gamma,c}$ has terminal singularities and hence*
- *(Kiem-Li:04) $M_C^{+}(2)$ has terminal singularities for a generic curve.*

# A side: Graph potentials as weak LG models

**Theorem: Belmans-Galkin-M**

The $m$-th descendent Gromov-Witten invariant of $M_C^-(2)$ is $\frac{[(W_{\Gamma,c})^m]_{const}}{m!}$ for any closed graph $(\Gamma, c)$ of genus $g$ with odd parity. In particular

$$\widehat{G}_{M_C^-(2)}(t) = \pi_{W_{\Gamma,c}}(t)$$

**Remark:** Proposal of Eguchi-Hori-Xiong, for constructing mirror potential of Fano varieties. (Earlier: Abouzaid, Aroux, Coates-Corti-Galkin, FOOO, Givental, Konstevich, Katzarkov, Przylkowski, Nishinou-Nohara-Ueda, Orlov, Seidel).

# Conjectural semi-orthogonal decomposition

**Conjecture: Belmans-Galkin-M, Narasimhan**

Let $C$ be a smooth curve of genus $g$

$$\mathbf{D}^b(M_C^-(2)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathbf{D}^b(C), \cdots$$
$$\cdots, \mathbf{D}^b(\mathrm{Sym}^{g-2} C), \mathbf{D}^b(\mathrm{Sym}^{g\text{-}2} C), \mathbf{D}^b(\mathrm{Sym}^{g-1} C)\rangle.$$

**Theorem: Belmans-M:19**

$$\mathbf{D}^b(M_C^-(r)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathbf{D}^b(C), \mathcal{B}\rangle,$$

where $M_C^-(r)$ is the moduli space of rank $r$ bundles with fixed determinant of degree one.

**Remark:** Lee-Moon:22 has generalized BM:19 for any coprime degree.

## Theorem: Muñoz

The quantum multiplication $\star_0$ by $c_1(M_C^-)$ on quantum cohomology ring $QH^*(M_C)$ has the following eigen-space decomposition:

$$QH^*(M_C^-) = \bigoplus_{m=1-g}^{g-1} H_m,$$

- The eigen-values are

  $$8(1-g), 8(2-g)\sqrt{-1}, 8(3-g), \ldots, 8(g-3), 8(g-2)\sqrt{-1}, 8(g-1).$$

- $H_m$ are isomorphic as vector spaces to $H^*(\mathrm{Sym}^{g-1-|m|} C)$.

**Remark:** This decomposition is equivariant with respect to the natural $\mathrm{Sp}(2g)$ action on both sides.

**Theorem: Belmans-Galkin-M**

Let $\Gamma$ be the necklace graph with one colored vertex, then the set of critical values of $W_{\Gamma,c}$

$$\left\{ -8(g-1), -8\sqrt{-1}(g-2), \ldots, 0, \ldots, 8\sqrt{-1}(g-2), 8(g-1) \right\}$$

equals the eigen values (Muñoz) of quantum multiplication by $c_1(M_C^-(2))$.

Moreover the dimensions of the critical set with absolute critical value $8(g-1-k)$ is $k$.

## Recent updates

- Theorem: Bondal-Orlov:95
  If $g = 2$, then $\mathbf{D}^b(M_C^-(2)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathbf{D}^b(pt) \rangle$.
- Theorem: Narasimhan:15, Kuznetsov-Fonarev:18
  $\mathbf{D}^b(M_C^-(2)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathcal{C} \rangle$.
- Theorem: Lee-Narasimhan
  If $C$ is not hyperelliptic, then
  $\mathbf{D}^b(M_C^-(2)) = \langle \mathbf{D}^b \operatorname{Sym}^2(C), \mathcal{C}' \rangle$.
- Theorem: Tevelev-Torres
  $\mathbf{D}^b(M_C^-(2)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(pt), \cdots, \mathbf{D}^b(\mathrm{Sym}^{\text{g-1}}\mathrm{C}), \mathcal{A} \rangle$.
- Theorem: Xu-Yau
  $\mathbf{D}^b(M_C^-(2)) = \langle \{\Theta^\ell \otimes \mathbf{D}^b(\operatorname{Sym}^i(C))\}_{0 \le \ell < 2, i < g - \ell}, \mathcal{A}' \rangle$ with some generalizations for principal bundles.

# Outline of the general machinery: Step I

Let $\mathcal{X} \to B$ be a degeneration of a smooth Fano $X$ such that the degeneration preserves second Betti numbers and $X_0$ is toric.

- Consider the moment map $\mu : X_0 \twoheadrightarrow P \subset \mathbb{R}^{\dim X_0}$ and construct a monotone Lagrangian torus $L = \mu^{-1}(u)$ in $X_0$.

- Using the toric degeneration and symplectic parallel transport, we construct a monotone Lagrangian torus in $X$ (Nishinou-Nohara-Ueda, Harada-Kaveh).

# cont..

**Theorem: Belmans-Galkin-M**

The Newton polytope of the Floer potential $m_0(L)$ counting Maslov index two disc in $X$ with boundary in $L$ equals that of the fan polytope of $X_0$.
In particular if the fan polytope has no non-vertex lattice points, then we can compute $m_0(L)$
(Galkin-Mikhalkin, generalizing Nishinou-Nohara-Ueda).

**Quantum periods v/s Floer potential**

It is known that (Tonkonog, Bondal-Galkin, Mikhalkin) that $G_{M_C^-(2)}(t)$ can be computed via periods of $m_0(L)$.

## Step II: Construct a toric degeneration of $M_C^-(2)$

Let $(\Gamma, c)$ be a trivalent graph with one (zero) colored vertex of genus $g$. The moduli spaces $M_C^-(2)$ ($M_C^+(2)$-even degree determinant) degenerates to a toric variety $X_{\Gamma,c}$. whose moment polytope in $\mathbb{R}^{|E|}$ is given by:

If $c(v) = (-1)^\epsilon$,

- $(-1)^\epsilon(x + y + z) \geq -1$.
- $(-1)^\epsilon(x - y - z) \geq -1$.
- $(-1)^\epsilon(-x - y + z) \geq -1$.
- $(-1)^\epsilon(-x + y - z) \geq -1$.

with respect to a lattice $L_\Gamma$ in $\mathbb{Z}^{|E|}$ of index $2^g$.

## Steps...

- Consider the section ring $\oplus_{\ell>0} H^0(M_C^{\pm}, \Theta^{\ell})$ and using the identification with conformal blocks $\mathbb{V}_{\pm}(\mathfrak{sl}(2), \ell)_{|C}$, we get a sheaf of algebras over $\overline{M}_g$.
- The factorization theorem relates $\mathbb{V}_{\pm}(\mathfrak{sl}(2), \ell)_{|C}$ to a conformal block on its normalization.
- Hence as curve degenerates, the section ring degerates to product of the fusion ring for $\mathfrak{sl}(2)$ and which has a very explicit description in terms of the quantum Clebsch-Gordan equations.
- This gives the toric degeneration.

## The case $X = M_C^-(2)$

- If $\Gamma$ has no separating edges $X_{\Gamma,c}$ has terminal singularities.
- Let $P_{\Gamma,c}$ be the moment polytope, then $L = \mu^{-1}(\vec{0})$ is monotone, Lagrangian.
- Hence $m_0(L) = W_{\Gamma,c}$, when $\Gamma$ has no separating edges.
- The case of general $\Gamma$ follows from the TQFT results since periods of $W_{\Gamma,c}$ only depend on parity of $c$ and the genus of $\Gamma$.