A geometric proof of the classification of T-polygons

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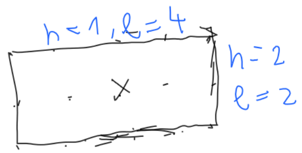
Def^n: A lattice polygon P is a Fano polygon if

- $0 \in \operatorname{int} P$
- the vertices of P are primitive lattice points.

P is a T-polygon if in addition $\forall$ edges E of P, the lattice length of E is divisible by the lattice height of E.

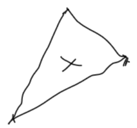
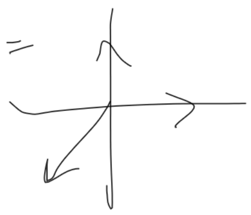
Ex:



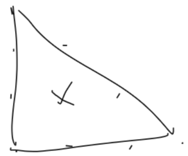
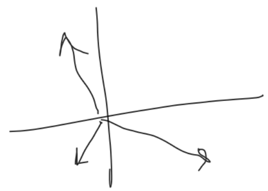
Mirror symmetry: Fano polygons are mirror to del Pezzo surfaces.

Given a Fano polygon P, consider the toric variety $X_{sp(P)}$ defined by the spanning fan of P.

The mirror of P is a generic qG-deformation of $X_{sp(P)}$

Ex: $P =$  $Sp(P) =$  $X_{Sp(P)} = \mathbb{P}^2 = X$

$f = x + y + \frac{1}{xy}$

$P =$  $Sp(P) =$  $X_{Sp(P)}$ = singular cubic surface

X = smooth cubic surface.

Conjecture: There is a 1-1 correspondence

$$\left\{ \begin{array}{l} \text{qG-families of del Pezzo} \\ \text{surfaces of class TG} \end{array} \right\} \longleftrightarrow \left\{ \text{Fano polygons} \right\} \Big/ \text{mutation}$$

T-polygons should be mirror to smooth del Pezzo surfaces.

$\exists$ 10 deformation families of smooth del Pezzos.

Kasprzyk-Nill-Prince: '15 $\exists$ 10 mutation equivalence classes of T-polygons.

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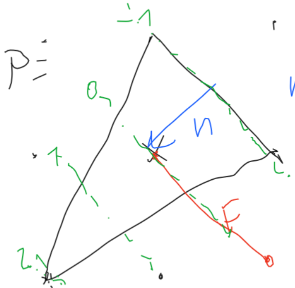
<u>Def^n</u>: Let P be a Fano polygon, n the inner normal vector to an edge of P.
& F a lattice segment s.t $F \subset n^{\perp}$

Define $P' = \mathrm{mut}_{(n,F)} P$ by

"removing $-k$ copies of F from P at height k for $k<0$" (height w.r.t n)

"adding k copies of F to P at height k for $k>0$"

$P =$



$n = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$F = \mathrm{Newt}(1 + \frac{x}{y})$.

$\downarrow$ mutation

$P' =$

If P & P' are mutation equivalent, they give rise to the same mirror del Pezzo.

If P is a T-polygon, then so is P'.

Fix a T-polygon P & consider a Laurent polynomial $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with $\mathrm{Newt} f = P$.

P defines a polarized toric variety (X_P, D_P) via the normal fan. $D_P = \sum_{E \subset P} h(E) D_E$.

Lattice points of P give a basis of sections for $\mathcal{O}(D_P)$.

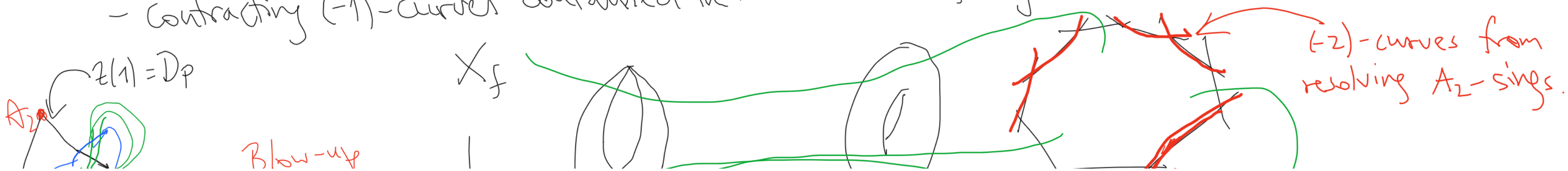
$\{\mathbb{1}, f\}$ give a pencil of sections of $\mathcal{O}(D_P)$ and define a rational map

$$\varphi : X_P \dashrightarrow \mathbb{P}^1$$
$$[1 : f]$$

$$\begin{tikzcd} & \tilde{X}_f \arrow[dl] \arrow[dr, "\pi"] & \\ \varphi : X_P \arrow[rr, dashed] & & \mathbb{P}^1 \end{tikzcd}$$

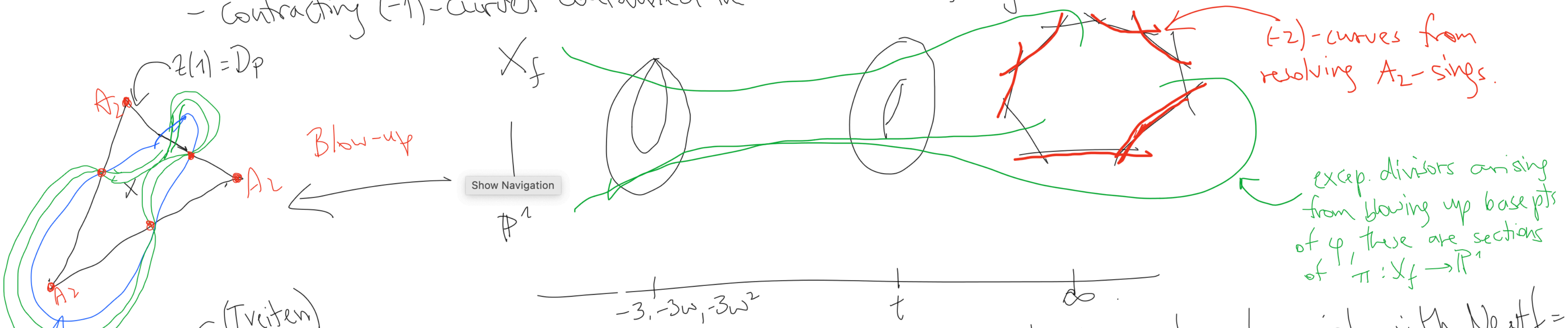
Define the surface X_f by

- resolving singularities of X_P to get a smooth toric surface $\tilde{X}_P$
- resolving the base locus of φ to get a smooth surface $\tilde{X}_f$
- contracting (-1)-curves contained in fibres of π to get a rel. minimal smooth surface X_f.



– resolving the base locus of φ to get …

– contracting (-1)-curves contained in fibres of π to get a rel. minimal smooth surface X_f.



<u>Prop</u>: (Tveiten) If P is a T-polygon & f is a <u>maximally mutable Laurent polynomial</u> with $\mathrm{Newt}\, f = P$, then X_f is an elliptic surface over $\mathbb{P}^1$.

↳ see below

<u>Def</u>: A Looijenga pair (X, D) is a smooth projective surface X with a nodal anticanonical divisor D.

$(X_P, \partial X_P)$ is a toric Looijenga pair.

anticanonical ...

$(\bar{X}_p, \partial \bar{X}_p)$ is a toric Looijenga pair.

$\Rightarrow (X_f, \pi^{-1}(\infty))$ is a Looijenga pair.

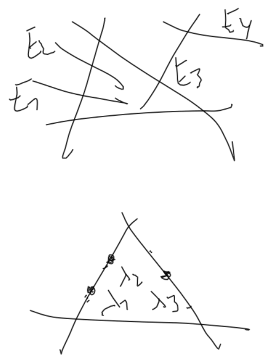
$\exists$ a Torelli thm (Gross-Hacking-Keel)

Given (X,D), define the lattice $\Lambda = \{ L \in \operatorname{Pic} X \mid L \cdot D_i = 0 \ \forall i \}$ $D = \sum D_i$.

Define the period point $\phi_X : \Lambda \to \operatorname{Pic}^0(D) = \mathbb{C}^\times$

$$L \mapsto L|_D$$

<u>Ex:</u>



$$\begin{array}{c} (X, D) \\ \downarrow \\ (\mathbb{P}^2, \partial \mathbb{P}^2) \end{array}$$

$E_1 - E_2, E_4 \in \Lambda$ $\qquad \phi_X(E_1 - E_2) = \mathcal{O}\left(\frac{\lambda_1}{\lambda_2}\right)$

$$\phi_X(E_4) = \mathcal{O}.$$

The period point roughly tells us the "location of the blowups"

The less generic the blowups, the larger the kernel of ϕ_X.

The less generic the ...

<u>Torelli Thm</u>: In a family of Looijenga pairs, the period point determines the pair up to IM.
& any period point is realised by some pair (X, D).

X_f is an elliptic surface, so $D = \pi^{-1}(\infty)$ has strictly negative semi-definite intersection matrix (SNSD)
i.e. D is cycle of (-2)-curves or an irred. nodal curve. (Classification of singular fibres of elliptic surfaces)

<u>Thm</u>: (Friedman) There are 10 deformation families of Looijenga pairs (X, D) with D SNSD.

<u>Recap</u>: - Take T-polygon P
- Take f with $\mathrm{Newt} f = P$ $\rightsquigarrow (X_f, D)$ with D SNSD $\Rightarrow (X_f, D)$ is a member of 1 of these 10 families.

For fixed P, $\exists$ canonical choice of LP f with $\mathrm{Newt} f = P$, the unique normalized maximally mutable Laurent polynomial f. ($\longmapsto$ see Coates, Kasprzyk '21)

Ex: 

Thm: If f is norm. MM, then $\phi_{X_f} \equiv 1$. (MMLP = "as nongeneric as possible") polynomial f. ($\longmapsto$ see Coates, Kasprzyk '21)

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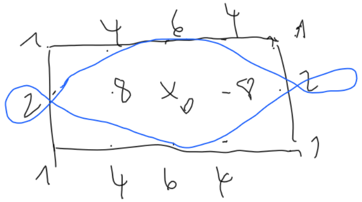
Ex: A norm MMLP



In each of the 10 families $\exists!$ pair (X_i, D_i) with $\phi_{X_i} \equiv 1$ by Torelli.

It follows that $(X_f, D) \cong (X_i, D_i)$ for some $1 \leq i \leq 10$.

Easy to find T-polygons P_i with norm. MMLP f_i s.t. $(X_{f_i}, D) \cong (X_i, D_i)$.

Obtain a diagram:

$$\begin{array}{ccc} & X_f = X_i & \\ \swarrow^{\text{bir}} & & \searrow^{\text{bir}} \\ X_P & & X_{P_i} \\ \cup & & \cup \\ (\mathbb{C}^\times)^2 & \overset{\varphi}{\dashrightarrow} & (\mathbb{C}^\times)^2 \end{array}$$

φ is volume-preserving (i.e. $\varphi^* \Omega = \Omega$ where $\Omega = \frac{dx}{x} \wedge \frac{dy}{y}$) since all blowups/blowdowns happen on the boundary.

Can arrange that $\varphi^* f_i = f$.

NTS: P is mutation equivalent to $Q := P_i$.

Let $g := f_i$.

Def^n: Fix a lattice N, $n \in N$, $F \in \mathbb{C}[n^{\perp}]$.

The automorphism $\mathbb{C}(N) \to \mathbb{C}(N)$

$$x^m \to x^m F^{\langle n, m \rangle}$$

The induced birational map $\mu_{(n,F)} : T_N \dashrightarrow T_N$ is called an alg. mutation.

A Laurent polynomial f is mutable wrt (n,F) if $\mu^*_{(n,F)} f$ is a Laurent poly.

Prop: If f is mutable wrt (n,F), then $\mathrm{Newt} f$ is mutable wrt $(n, \mathrm{Newt} F)$.

Thm: (L.) Let $\varphi : (\mathbb{C}^\times)^2 \dashrightarrow (\mathbb{C}^\times)^2$ be volume preserving & let f, g be Laurent polys s.t. $\varphi^* f = g$.

- φ factors as $\varphi = \varphi_n \circ \dots \circ \varphi_1$ where the φ_i are alg. mutations.
- $f_k := (\varphi_k \circ \dots \circ \varphi_1)^* f$ is a Laurent polynomial $\forall\ 1 \leq k \leq n$.

Proof: Uses modified version of Sarkisov algorithm for factoring birational maps.

Remark: The existence of the factorisation was proven by Blanc using different methods.

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Applying the Theorem, we obtain a sequence of mutations $f \mapsto f_1 \mapsto \cdots \mapsto f_n = g$

and an induced sequence of mutations $P = \mathrm{Newt} f \mapsto \cdots \mapsto \mathrm{Newt}(g) = Q$

$\Rightarrow$ P is mutation equivalent to Q. $\blacksquare$