Moduli of boundary polarized Calabi-Yau pairs

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§ Motivation.

From classification of varieties, every variety is built from 3 fundamental types: (MMP).

- Fano varieties $\quad K_X < 0 \rightsquigarrow$ K-moduli theory (K-stability)
- Calabi-Yau varieties $\quad K_X = 0 \rightsquigarrow$ Many, ? (GIT, Hodge, mirror sym.)
- general type $\quad K_X > 0 \rightsquigarrow$ KSBA theory (generalize DM moduli)

Our approach : inspired by K-stability / KSBA,

more intrinsic

- Boundary polarized CY pairs.

(X, D) : $D^{\geq 0}$ ample, $\mathbb{Q}$-Cartier, $\mathbb{Q}$-divisor

$K_X + D \sim_{\mathbb{Q}} 0$. ($\Rightarrow$ X is Fano)

singularities are mild. (klt, lc, slc)

Ex. $(\mathbb{P}^2,$ cubic curve$)$ $\rightsquigarrow$ elliptic curve

$(\mathbb{P}^2, \frac{1}{2}C_6)$ $\overset{\text{double cover}}{\rightsquigarrow}$ K3 of deg 2.

$(\mathbb{P}^3, S_4)$ $\rightsquigarrow$ K3 of deg 4

$(\mathbb{P}^4, V_5)$ $\rightsquigarrow$ quintic 3-folds

More generally, if (V, L) is a polarized CY var.

then take $X = C_p(V, L)$ projective cone

$D =$ section at ∞ $\Rightarrow$ (X, D) is a $\underset{\text{boundary pol. CY pair.}}{\underline{\underline{\text{bpCY}}}}$.

Vague Q. Is there a compact moduli space for (X, D)

that is "canonical"?

(1) Natural singularity conditions $(\mathcal{X}, \mathcal{D}) \xrightarrow{\pi} B$.

(2) λ_{Hdg} is ample. $\pi^* \lambda_{Hdg} = K_{\mathcal{X}/B} + \mathcal{D}$.

Precise Q. Can we build a moduli space M^{CY}

that connects $M^{K} =$ K-moduli of $(X, (1-\varepsilon)D)$

and $M^{KSBA} =$ KSBA-moduli of $(X, (1+\varepsilon)D)$?

s.t. $M^{K} \to M^{CY} \leftarrow M^{KSBA}$.

In terms of stacks: $\mathcal{M}^{K} \cup \mathcal{M}^{KSBA}$
$\#$

$$\begin{tikzcd}
\mathcal{M}^{K} \arrow[r, hook, "\text{open}"] \arrow[d] & \mathcal{M}^{CY} \arrow[d] & \mathcal{M}^{KSBA} \arrow[l, hook', "\text{open}"'] \arrow[d] \\
M^{K} \arrow[r] & M^{CY} & M^{KSBA} \arrow[l]
\end{tikzcd}$$

good moduli space

• Moduli stack :

Objects : (X, D) bpCY pair s.t. (X, D) slc.

(In K-moduli , $(X, (1-\varepsilon)D)$ klt $\overset{\text{ACC}}{\Rightarrow}$ (X, D) is lc

In KSBA , $(X, (1+\varepsilon)D)$ slc $\Rightarrow$ (X, D) is slc)

Families : $f: (\mathcal{X}, \mathcal{D}) \longrightarrow B$

s.t. (1) f is a proj. flat , with slc fibers of pure dim.

(2) $\mathcal{D}$ is a relative $\mathbb{Q}$-Cartier $\mathbb{Q}$-div $/B$

(3) $K_{\mathcal{X}/B} + \mathcal{D} \sim_{\mathbb{Q}, B} 0$, $\mathcal{D}$ ample $/B$.

(4) Kollár conditions

Fix the volume V of $\mathcal{X}_b$ and degree d of $\mathcal{D}_b$ (coeff set).

$\hookrightarrow$ we get a moduli functor $\mathcal{M}^{CY}_{(n,V,d)}$

<u>Thm 1</u> (ABBDILW). $\mathcal{M}^{CY}$ is an Artin stack, <u>locally</u> of finite type with affine diagonal.

(Usually, $\mathcal{M}^{CY}$ is not bounded!).

Thm 2. (ABBDILW) $\mathcal{M}^{CY}$ satisfies S-completeness and Θ-reductivity.

Moreover, it satisfies $\exists$ part valuative crieterion for properness.

S-complete : $0 \in B$ pointed curve.

$$\begin{tikzcd} (\mathcal{X}, \mathcal{D}) \arrow[rr, dashed] \arrow[dr] & & (\mathcal{X}', \mathcal{D}') \arrow[dl] \\ & B & \end{tikzcd} \qquad (\mathcal{X}, \mathcal{D})\big|_{B \setminus \{0\}} \cong (\mathcal{X}', \mathcal{D}')\big|_{B \setminus \{0\}}$$

Then $\exists$ TCs

$$(\mathcal{X}_0, \mathcal{D}_0) \overset{TC}{\rightsquigarrow} \underset{\text{slc bpCY.}}{(Y, \Delta)} \overset{TC}{\leftsquigarrow} (\mathcal{X}'_0, \mathcal{D}'_0)$$

Def. (S-equivalence).

(X, D), (X', D') slc bpCY pairs.

We say they're S-equivalent if $\exists$ TCs

s.t. $(X, D) \overset{TC}{\rightsquigarrow} (Y, \Delta) \overset{TC}{\leftsquigarrow} (X', D')$

- Moduli spaces.

Consider $(\mathbb{P}^2, \frac{3}{d} C_d)$ bpCY.

$\mathcal{P}_d^K$ = K-moduli stack of $(\mathbb{P}^2, (\frac{3}{d} - \varepsilon) C_d)$

$\mathcal{P}_d^H$ = KSBA-moduli stack of $(\mathbb{P}^2, (\frac{3}{d} + \varepsilon) C_d)$

Both $\mathcal{P}_d^K$ & $\mathcal{P}_d^H$ are bounded, Artin stacks.

$\mathcal{P}_d^{CY}$ = closure of $\{(\mathbb{P}^2, \frac{3}{d} C_d)\}$ in bpCY moduli stack.

However, $\mathcal{P}_d^{CY}$ is unbounded if $3 \mid d$.

$\rightsquigarrow (\mathbb{P}^2, \cancel{*})$ S-equiv

type III deg.

$\underline{\text{Ex}}$. $(\mathbb{P}^2, C_t) \rightsquigarrow (\mathbb{P}(a^2, b^2, c^2), D_{\text{toric}})$ $\quad a^2+b^2+c^2 = 3abc$

lc bpCY. $\quad \infty$-many $\mathbb{Z}$-solutions.

family of smooth cubic curves.

$\pi : \mathcal{X} \longrightarrow B$ general fiber $\cong \mathbb{P}^2$

special fiber $\cong \mathbb{P}(a^2, b^2, c^2)$.

$\mathcal{P}_3^{CY}$ is unbounded.

Kawamata-Viehweg vanishing $\Rightarrow$ $\pi_* \omega_{\mathcal{X}/B}^{[-1]}$ flat.

$\Rightarrow H^0(\mathbb{P}^2, -K_{\mathbb{P}^2}) \rightsquigarrow H^0(\mathbb{P}(a^2, b^2, c^2), -K_{\mathbb{P}(a^2,b^2,c^2)})$ is flat.

Thm 3 (ABBDILW). $\exists$ a projective scheme P_d^{CY} param. S-equivalence classes of $\mathcal{P}_d^{CY}$.

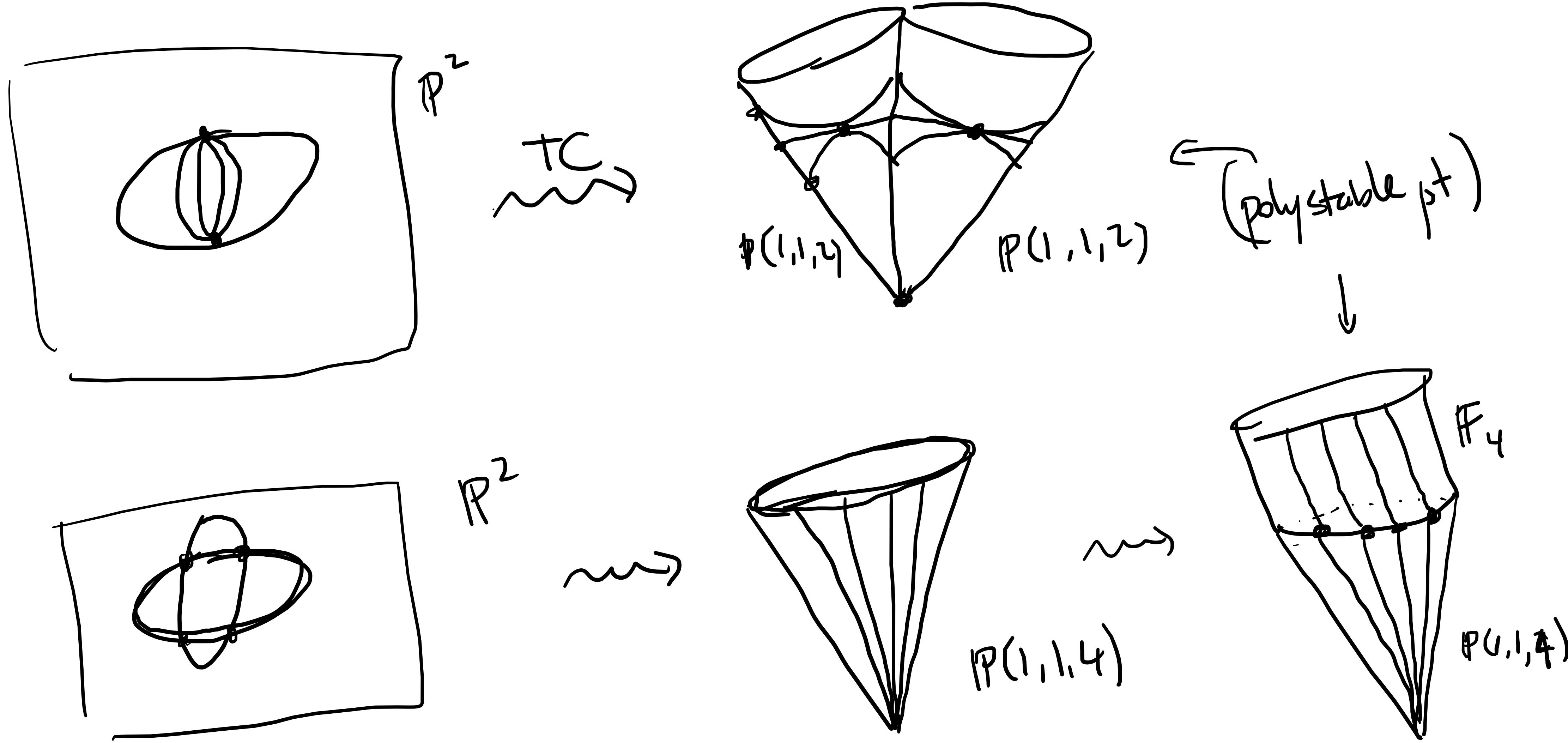
Moreover, we have bir. morphism

$$P_d^K \longrightarrow P_d^{CY} \longleftarrow P_d^H.$$

And λ_{Hdg} is ample on P_d^{CY}. ($\Rightarrow \lambda_{Hdg}$ is semiample on P_d^K & P_d^H)

(Baily-Borel type compactification).

Ex. $\underline{\underline{d=6}}$ type II degenerations (min lc center has dim $= 1$).



Idea of proof on thm 3.

$\forall m \in \mathbb{N}$, $\mathfrak{P}_d^m := \{ (X, D) \in \mathfrak{P}_d^{CY} \mid \operatorname{ind}_x K_X \leq m \ \forall x \in X \}$.

We show $\mathfrak{P}_d^m$ is bounded.

Challenge: Show $\mathfrak{P}_d^m$ admits a good moduli space!

Using complements, translate this question to the case $d = 3$.

We use twisted elliptic curves & deformation theory (Olsson).

Then we show P_d^m (gms of $\mathcal{P}_d^m$) stabilizes

as $m \longrightarrow \infty$.

Extra argument to show λ_{Hdg} is ample.

(Kollár, Ambro, Fujino-Gongyo).

Everything is based on explicit geometry of degenerations of $\mathbb{P}^2$.

For general del Pezzo surface

$$(X_t, D_t) \rightsquigarrow (X_0, D_0)$$

smooth dP surface

slc.

For $\mathbb{P}^2$: $\exists$ 1-complement on X_0

i.e. $\exists\, \Gamma_0 \in |-K_{X_0}|$ s.t. (X_0, Γ_0) is slc.

It may fail for other del Pezzo surfaces.

We can find 2-complement.