A Néron-Ogg-Shafarevich criterion for K3 surfaces

(joint w/ B. Chiarellotto
C. Liedtke)

$\to$ $\underline{\mathcal{O}_K}$ = complete DVR
K = fraction field
k = residue field, perfect
$\ell \neq \mathrm{char}(k)$

K^s = sep. closure of K
$\bar{k}$ = res. field of K^s
$G_K = \mathrm{Gal}(K^s/K)$
$G_k = \mathrm{Gal}(\bar{k}/k)$

$$1 \to I_K \to G_K \to G_k \to 1$$

<u>Question</u> Given a smooth, proper variety X/K, does X have good reduction?

Classical Results

eg ① (Serre-Tate) $X = A$ abelian variety

A has good red^n $\iff$ $\underline{T_\ell A} := \varprojlim_n A[\ell^n](K^s) \circlearrowleft G_K$ is unramified

$\iff$ $\underline{H^1_{et}(A_{K^s}, \mathbb{Q}_\ell)} \circlearrowleft G_K$ is unramified

② (Oda) $X = C$ curve

C has good red^n $\iff$ $\pi_1^{et}(C_{K^s})_{\text{pro-}\ell} \circlearrowleft G_K$ outer is unramified

In both cases, good reduction (if it exists) is unique

- Néron models
- minimal models

K3 sufaces

X/K smooth, projective, geom. con. suface is called a <u>K3 surface</u> if

- $\omega_{X/K} := \Omega^2_{X/K} \cong \mathcal{O}_X$
- $\rightarrow$ $H^1(X, \mathcal{O}_X) = 0$

$\Rightarrow$ only interesting cohomology gp is $H^2_{et}(X_{K^s}, \mathbb{Q}_\ell)$

<u>Question</u> : Can we detect good redⁿ by looking at

$$G_K \curvearrowright H^2_{et}(X_{K^s}, \mathbb{Q}_\ell) \, ?$$

Complex case (Kulikov, 1970's)

$$\Delta := \{ z \in \mathbb{C} : |z| < 1 \}$$
$$\Delta^* := \Delta \setminus \{0\}$$

$\pi : \mathcal{X} \longrightarrow \Delta$ projective, flat map of complex manifolds
s.t. $\forall t \neq 0$ $\mathcal{X}_t := \pi^{-1}(t)$ is a K3 surface

Analogue of I_K-action:

$$\pi_1(\Delta^*, t) = \mathbb{Z} \circlearrowright H^2(\mathcal{X}_t, \mathbb{Q})$$

Kulikov (1970's)

If monodromy is trivial, then after a finite flat base change $\left(\Delta \xrightarrow{t \mapsto t^n} \Delta\right)$

$\exists$ modification $\mathcal{X}' \longrightarrow \mathcal{X}$ s.t. $\mathcal{X}' \to \Delta$ is a smooth family of K3 surfaces

"Proof": ① Use semistable red^n thm, can assume family is semistable

② Modify a semistable family to produce a "log K3 surface" $\mathcal{X}$ over $(\Delta, 0)$ i.e. $\Omega^2_{\mathcal{X}/\Delta}(\log \mathcal{X}_0) \cong \mathcal{O}_{\mathcal{X}}$

③ Classify possible shapes of the central fibre of log K3 surfaces & calculate monodromy in each case $\square$

Arithmetic and ve

Problem: Semistable red^n is not known in mixed char / equi char p

We'll assume our K3 surface X/K has potential semistable reduction ←

eg Known if
- $\mathrm{char}(k) = 0$
- $\mathrm{char}(k) = p$ & X admits a polarisation $\mathcal{L}$ of degree $\mathcal{L}^2 < p-4$ (Maulik)

Matsumoto (2014)

Arithmetic analogue of Kulikov's argument

- X/K K3 surface st $G_K \curvearrowright \underline{H^2_{et}(X_{K^s}, \mathbb{Q}_\ell)}$ unramified

  $\Rightarrow$ X has potential good red^n

  ($\exists\, L/K$ finite st X_L has good red^n)

(+ ε can assume that L/K is separable)

Liedtke - Matsumoto (2014)

Can actually take L/K to be unramified

"Proof:" Can assume L/K totally ramified (& Galois)

$\rightarrow \mathcal{Y}/\mathcal{O}_L$ smooth model for $\underline{X_L}$

Show - $H^2_{et}(X_{K^s}, \mathbb{Q}_\ell) \circlearrowleft G_K$ unramified

$\Rightarrow \mathrm{Gal}(L/K) \circlearrowleft X_L$ extends to $\mathcal{Y}$

- $\underline{\mathcal{Y}/\mathrm{Gal}(L/K)}$ is a smooth model for X $\square$

Counter examples

<u>Question</u> : Can we take L/K trivial ?

eg (Liedtke - Matsumoto) If $\underline{p \geq 5}$ then $\exists$ $\underline{X/\mathbb{Q}_p}$ K3 surface st.

① X does not have good red^n / $\mathbb{Q}_p$

② X does not have good red^n / $\mathbb{Q}_{p^2}$

$\Rightarrow$ no statement of the form

"X has good red^n $\Longleftrightarrow$ some invariant has unramified G_K-action"

Our results

Question: Can we "explain" this phenomenon "cohomologically"?

For the rest of the talk:

- X/K K3 surface
- L/K finite unramified Galois ext^n
- $\mathcal{Y}/\mathcal{O}_L$ smooth model for X_L
- k_L = res. field of L
- $G = \mathrm{Gal}(L/K) = \mathrm{Gal}(k_L/k)$

$G \circlearrowright X_L \longrightarrow G \dashrightarrow \mathcal{Y}$ rational action

Fact (Matsusaka–Mumford) This action is defined away from finitely many curves $C \subseteq \mathcal{Y}_{k_L} \rightsquigarrow G \dashrightarrow \mathcal{Y}_{k_L}$

$\Rightarrow G \circlearrowright \mathcal{Y}_{k_L}$ $\qquad Y := \mathcal{Y}_{k_L}/G \leftarrow$ K3 surface over k

If X has a smooth model $\mathcal{X}$ over $\mathcal{O}_K$, then

$Y = \mathcal{X}_k$ & [redacted] $H^2_{et}(X_{\bar{K}}, \mathbb{Q}_\ell) \overset{G\text{-equivariant}}{\cong} H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell)$

Theorem (CLL) If $H^2_{et}(X_{K^s}, \mathbb{Q}_\ell) \overset{G_K}{\cong} H^2_{et}(\underline{Y}_{\bar{k}}, \mathbb{Q}_\ell)$, then X has good redⁿ over K.

Proof proceeds by calculating the action of <u>Frobs</u> on cohomology

Flops

$\mathcal{Y}/\mathcal{O}_L$ = smooth model for X_L

$C \subseteq \mathcal{Y}_{k_L}$ -2 curve
- $C \cong \mathbb{P}^1_{k_L}$
- $C \cdot C = -2$

$$
\begin{tikzcd}
 & \mathcal{Y} \arrow[rrr, dashed, "f"] \arrow[d, "\text{blow down } C"'] & & & \mathcal{Y}^{+} \arrow[d, "\text{blow up}"] \\
\text{quadratic singularity } p \arrow[r] & \mathcal{Y}' & \hat{\mathcal{Y}}' \arrow[l, "\text{complete at } p"'] \arrow[r, "t", "\simeq"'] & \hat{\mathcal{Y}}' & \mathcal{Y}^{+\prime} \arrow[l, "\text{complete}"']
\end{tikzcd}
$$

Flops (ctd.)

$$\begin{array}{ccc} f : Y & \dashrightarrow & Y^+ \\ \cup\!\shortmid & & \cup\!\shortmid \\ C & & C^+ \end{array}$$

s.t. • $f : Y \setminus C \xrightarrow{\simeq} Y^+ \setminus C^+$

• f not regular

$$\begin{tikzcd} H^2_{et}(Y^+_{\bar{k}}, \mathbb{Q}_\ell) \arrow[r, "\simeq"] \arrow[d, "f^*_{\bar{k}}", "\wr"'] \arrow[dr, phantom, "\not\circlearrowleft"] & H^2_{et}(Y^+_{K^s}, \mathbb{Q}_\ell) \arrow[d, "f^*_L", "\wr"'] \\ H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell) \arrow[r, "\simeq"] & H^2_{et}(Y_{K^s}, \mathbb{Q}_\ell) \end{tikzcd}$$

$\Rightarrow$ failure to commute $=$ $s_C : H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell) \to H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell)$

$\alpha \longmapsto \alpha + (\alpha \cup [C])[C]$

The Weyl group

Take $\underline{\mathcal{L}}$ a polarisation on $\underline{X}$ s.t. $\mathcal{L}_{k_L}$ big & nef on Y_{k_L}

$\underline{S} = \{ C \subseteq Y_{\bar{k}} : -2$ curves C s.t. $C \cdot \mathcal{L}_{k_L} = 0 \}$ ←

assume that all elements of $\underline{S}$ are defined over $\underline{k_L}$

Def The Weyl gp $\underline{W} = W(X, \mathcal{L}) \subseteq GL(H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell))$ is the subgroup generated by reflections in $[C]$ $\forall C \in S$

Regularity of G-action

$\sigma \in G \qquad \underline{f_\sigma} : Y \dashrightarrow Y^\sigma$

$$\begin{tikzcd}
H^2_{et}(Y^\sigma_{\bar{k}}, \mathbb{Q}_\ell) \arrow[r, "\simeq"] \arrow[d, dashed, "\wr"] & H^2_{et}(Y^\sigma_{k^s}, \mathbb{Q}_\ell) \arrow[d, dashed, "\wr"] \\
H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell) \arrow[r, "\simeq"] & H^2_{et}(Y_{k^s}, \mathbb{Q}_\ell)
\end{tikzcd}$$

$S_\sigma :=$ failure of square to commute $\in \underline{\underline{GL(H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell))}}$

Regularity of G-action (ctd)

Proposition ① $s_\sigma \in W \subseteq GL(H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell))$

② $s_\sigma \equiv 1 \iff$ G-action is regular ←

③ $\sigma \mapsto s_\sigma$ is a 1-cocycle for $G \curvearrowright \underline{W}$

④ $\mathcal{Y}'/\mathcal{O}_L$ a different model

$\Rightarrow$ cocycles s_σ, s'_σ are cohomologous

$\therefore$ get a well-defined element $[s] \in H^1(G, W)$

(depends on $X, \mathcal{L}$)

Regularity of G-action (ctd)

Corollary X has good red^n over K $\iff$ $\exists$ model $\mathcal{Y}$ for which

$\iff$ G-action is regular

$\iff$ $[s] = 1$ in $H^1(G, W)$

Back to cohomology

$$W \hookrightarrow GL\left(H^2_{et}(Y_{\bar k}, \mathbb{Q}_\ell)\right)$$

$$\rightsquigarrow [s]_\ell \in H^1\left(G_k, GL\left(H^2_{et}(Y_{\bar k}, \mathbb{Q}_\ell)\right)\right)$$

By construction $H^2_{et}(Y_{\bar k}, \mathbb{Q}_\ell) \overset{G_k}{\cong} H^2_{et}(X_{k^s}, \mathbb{Q}_\ell)$

$\iff [s]_\ell$ is trivial

ADE classification

Main Theorem now follows from:

Proposition The map $H^1(G, \underline{W}) \to H^1(G_k, \underline{GL(H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell))})$ has trivial kernel.

"Proof":
- Replace $H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell)$ by $\mathbb{Q}_\ell$-span of $[C] \in \underline{S}$
- Quadratic space (V, u) classified by a Dynkin diagram
- Explicit calculation in each case $\square$

Final remarks

- $\exists$ p-adic version of main result
  - if $\mathrm{char}(K) = 0$ $\quad H^2_{et}$ unramified $\longrightarrow$ H^2_{et} crystalline

$$H^2_{et}(X_{K^s}, \mathbb{Q}_\ell) \overset{G_K}{\cong} H^2_{et}(Y_{\bar{k}}, \mathbb{Q}_\ell) \longrightarrow D_{cris}(H^2_{et}(X_K)) \cong H^2_{cris}(Y/K)$$

  - if $\mathrm{char}(K) = p$ bit more involved $\quad ((\varphi, \nabla)$-modules / Robba ring $\mathcal{R}_K)$