Finding mirrors to Fano quiver flag zero loci.

Plan:
1) Motivation: why quiver flag zero loci? why mirrors?
2) Computing the A-side: the quantum period
3) Finding mirrors: the B-side. (Laurent polynomials) via a generalisation of the Gelfand-Cetlin toric degenerate

§ Motivation: Fano classification & mirror symmetry

Fano variety: smooth variety /$\mathbb{C}$ s.t. $-K_X$ is ample.

There are only finitely many n-dim Fano varieties up to deformation (Kollár-Miyaoka-Mori).

Classification: dim 1 & 2 classical

dim 3 105 (Iskovskikh, Mori-Mukai)

$\Rightarrow$ All dim ≤ 3 Fano varieties can be constructed as $Z(s) \subseteq \overset{\mathbb{C}\text{ v.s.}}{V} /\!/ G$, where $s \in \Gamma(\underbrace{E \times V^{ss}}_{\text{representation theoretic v.b.}} / G)$ and E is a representation of G.

In fact, they are either

1) toric complete intersection (V//G is a toric variety, G is abelian)
 - well understood

2) quiver flag zero loci (V//G is a quiver flag variety, G is non-Abelian)
 - topic for today

Theme: reduce type 2) to type 1) via

→ abelianisation

→ toric degenerations.

Quiver flag varieties:

- Q: acyclic quiver with a unique source $Q_0 = \{0, 1, \dots, \rho\}$
- $r = (1, r_1, \dots, r_\rho)$, $G = \prod_{i=1}^{\rho} Gl(r_i)$, $\theta = (1, 1, \dots, 1)$

$$V/\!/G = M_\theta(Q, r) = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}}) /\!/_\theta G \Big] \text{ quiver flag variety}$$

eg
$$\begin{tikzcd} 1 \arrow[r, "n"'] & r_1 \arrow[r] & r_2 \arrow[r] & \cdots \arrow[r] & r_\rho \end{tikzcd} \rightsquigarrow V/\!/G = Fl(n, r_1, \dots, r_\rho)$$

$M_\theta(Q,r)$ is a

- smooth projective variety
- fine moduli space

} Craw

- MDS

- $M_\theta(Q,r) = Z(s) \subseteq \prod_{i=1}^{\rho} Gr(\overset{\sim}{\underset{\text{\#paths } 0 \to i}{\underset{\shortparallel}{S_i}}}, r_i)$ $\quad S_i \searrow \swarrow Q_i$ $\quad s \in \Gamma(E)$ giving incidence conditions

Quiver flag are loci: $W_i := Q_i|_{M_\theta(Q,r)}$.

$Z(s)$, $s \in \Gamma(\bigoplus S^{\alpha_1^i} W_1 \otimes \cdots \otimes S^{\alpha_\rho^i} W_\rho)$

Dim 4: start by classifying Fano varieties of type 1 or 2.

Restrict ambient space $\dim V/\!/G \leq 8$. → can enumerate all dim 4 Fano subvarieties of type 1) (Coates-Kasprzyk-Prince) and

2) Coates – K – Kasprzyk

→ 141 new Fano fourfolds.

(CCGGKT...)

Program: Use mirror symmetry to classify Fano varieties.

Conjecturally (Kasprzyk – Tveiten):

n dim Fano varieties / deformation $\longleftrightarrow$ rigid maximally mutable Laurent polynomials $f \in \mathbb{C}[z_1^{\pm}, \dots, z_n^{\pm}]$ / mutation

X is mirror to f if

$$\underbrace{\hat{G}_X(t)}_{\text{regularised quantum period}} = \underbrace{\pi_f(t)}_{\text{classical period}}$$

$$G_X(t) = \sum_{i=0}^{\infty} \underbrace{a_i}_{} t^i$$

↳ genus 0 Gromov-Witten invariant

If X is a tci, Givental's mirror thm gives a closed form.

$$\pi_f(t) = \sum_{i=0}^{\infty} \mathrm{const}(f^i)\, t^i$$

Mutations $\quad f \xrightarrow[\text{mutation}]{} f'$

Compositions of

a) $(\mathbb{C}^*)^n \xrightarrow{\varphi_{A \in GL(n,\mathbb{Z})}} (\mathbb{C}^*)^n \qquad f' = \varphi_A^*(f)$

b) Let $h \in \mathbb{C}[z_1^{\pm}, \dots, z_n^{\pm}]$. Define

$$(\mathbb{C}^*)^n \overset{\varphi_h}{\dashrightarrow} (\mathbb{C}^*)^n \qquad \begin{array}{ll} z_i \to z_i & i<n \\ z_n \to h \cdot z_n & i=n \end{array}$$

$$f' = \varphi_h^*(f)$$

f is compatible with this mutation if the result, f', is a Laurent polynomial.

Rigid max mutable Laurent polynomials

Let $P = \mathrm{Newt}(f)$. f is given by a choice of coefficients for the lattice points of P.

A Laurent polynomial is rigid maximally mutable if it is compatible with a maximal set of mutations, and the coefficients are uniquely determined by this property.

Evidence for the conjecture:

1) X tci $\xrightarrow[\substack{\text{Hori-Vafa} \\ \text{mirror}}]{}$ f } verified that it is rigid max mutable in search.

+ Przyjalkowski method

requires technical condition: existence of a convex partition

2) Conjectural mirrors found for 99/141 Fano quiver flag zero loci (K−)

②

$$\begin{tikzcd}
\underset{\text{quiver flag zero locus}}{\underline{X}} \subseteq M_\theta(Q,r) \arrow[r] & \underset{\text{"tci"}}{\underline{Z}} \subseteq \underbrace{M_\nu(L(\theta),(1,\dots,1))}_{\text{generalisation of Gelfand-Cetlin toric degeneration of flag varieties}} \arrow[dl, "\text{Przyjalkowski method (when convex partition exists)}"] \\
f',\ P=\mathrm{Newt}(f) \arrow[r, "\text{fix coefficients to be rigid max mutable (Kasprzyk code)}"] & f
\end{tikzcd}$$

WRONG

① Compute 20 terms of $G_X(t)$, $\pi_f(t)$ and compare

{ Computing $G_X(t)$: Abelianisation.

$$\begin{tikzcd}
 & E_G \arrow[d] & E_T \arrow[d] & \\
Z(s) \subseteq & V/\!/_\theta G \supseteq \underset{\text{max torus}}{T} & V/\!/_\theta T & \supseteq Z(\tilde{s})
\end{tikzcd}$$

Different dimensions → but turns out $V/\!/_\theta T$ is very useful

Results on: cohomology, quantum cohomology, Mori-wall-and-chamber structure, I functions

Even nicer when $V/\!/_\theta G = M_\theta(Q, r)$

Q

(Ciocan-Fontanine–Kim–Sabbah, K, Webb)

toric quiver flag variety

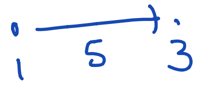
Q^{ab}

Thm: The quantum period of $M_\theta(Q, r)$ can be computed via the quantum period of $M_\theta(Q^{ab}, (1, \ldots, 1))$.

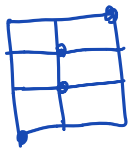
§ Laurent polynomial mirrors.

Hard to find a toric degeneration of $Z \subseteq M_\theta(Q, r)$ directly: instead try to generalise known constructions for flag varieties.

Ladder quivers



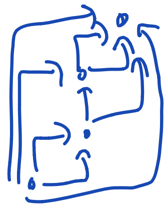
$Gr(5,3)$



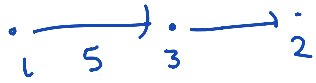
+ vertices

Orient

$\to \uparrow$



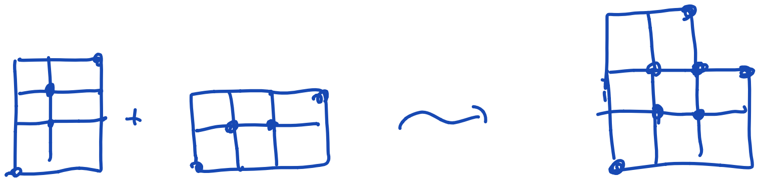
$L(Q)$

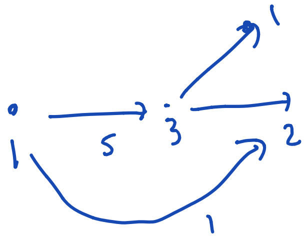


$Fl(5,3,2) \subseteq Gr(5,3) \times Gr(5,2)$

$(\mathbb{C}^5/V_1, \mathbb{C}^5/V_2)$

$V_2 \supseteq V_1$



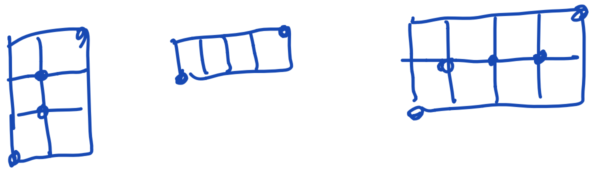


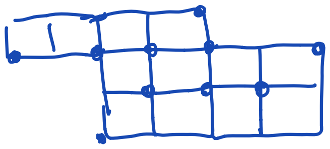
$M_6(Q, r) \subseteq Gr(5,3) \times Gr(5,1) \times Gr(6,2)$

$(\mathbb{C}^5/V_1, \mathbb{C}^5/V_2, \mathbb{C}^6/V_3)$

$V_1 \subseteq V_2$

$V_1 \oplus \mathbb{C} \subseteq V_3$





Works for any Y-shaped quiver

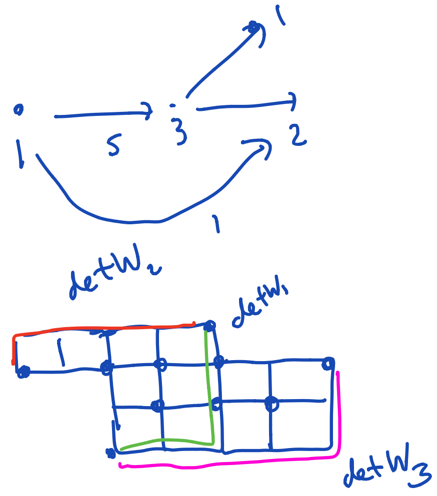
$Q \longrightarrow L(Q)$

Thm (K-) There is a toric degeneration of a Fano Y-shaped quiver to $M_\nu(L(Q),(1,\dots,1))$ where ν is the Fano stability condition.

Pf: via finding a SAGBI basis of $M_\theta(Q,r) \subseteq \prod Gr(\tilde{s}_i, r_i) \subseteq \prod \mathbb{P}^{a_i}$

i.e. using the embedding given by the $\det W_i$.

and comparing with $M_\nu(L(Q),(1,\dots,1)) \hookrightarrow \prod \mathbb{P}^{a_i}$.



path $\longrightarrow$ Weil divisor in $M_\nu(L(Q),(1,\dots,1))$.

Mirrors for subvarieties:

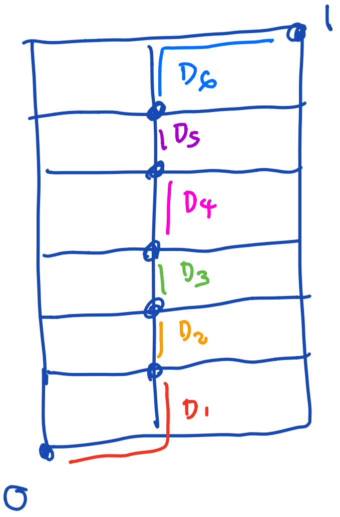
$Z \subseteq M_\theta(Q,r)$: if Z is a ci, then Z degenerates to a tci in $M_\nu(Q,r)$.

(used by Prince to find mirrors to ci in flag varieties)

General zero loci: use combinatorics of $L(Q)$.

$Gr(8,6) \subseteq \mathbb{P}^{\binom{8}{6}-1}$

Any path $0 \to 1 \rightsquigarrow \mathcal{O}(1) \Leftarrow \det W_1$



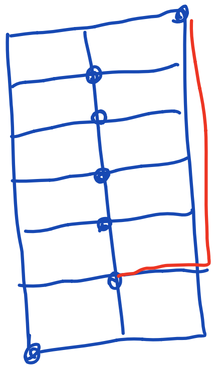
$$\mathcal{O}(D_1)\otimes\mathcal{O}(D_2)\otimes\mathcal{O}(D_3)\otimes\mathcal{O}(D_4)\otimes\mathcal{O}(D_5)\otimes\mathcal{O}(D_6)$$

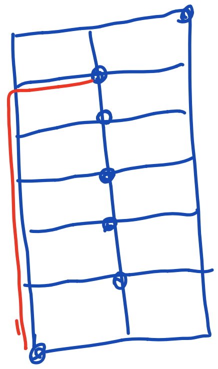
$$= \mathcal{O}(1)|_{M_\theta(L(Q),(1,\dots,1))} \to \det W_1$$

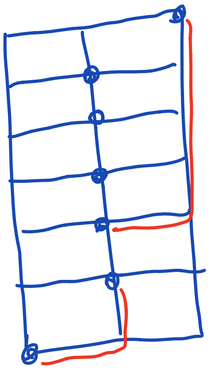
$$\mathcal{O}(D_1)\oplus\mathcal{O}(D_2)\oplus\mathcal{O}(D_3)\oplus\mathcal{O}(D_4)\oplus\mathcal{O}(D_5)\oplus\mathcal{O}(D_6)$$

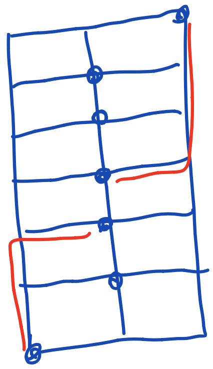
$$\sim W_1$$

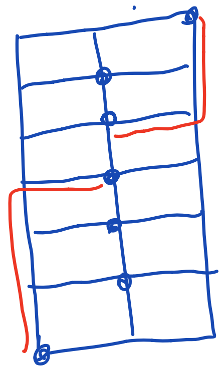
Can generalise to $S^\alpha W_1$ → eg. $\Lambda^5 W_1$ on $Gr(8,6)$

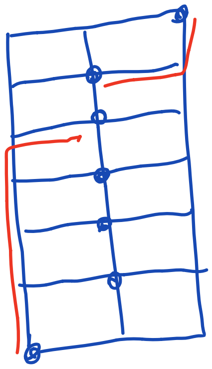








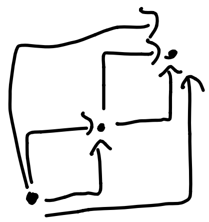




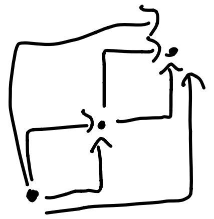
→ produce a Laurent polynomial mirror to fix $Z(s) \subseteq Gr(8,6)$, $s \in \Gamma(\det W_1 \oplus \det W_1 \oplus \Lambda^5 W_1)$, coefficients

This method produces only conjectural mirrors, unless $X = Gr(n,r)$, in which case $f = f_{EHX}$, the Eguchi-Hori-Xiang mirror.

eg. $Gr(4,2)$



L(Q)



· – ·

Thm (Marsh-Rietsh)

$$\Pi_{f_{EHX}}(t) = \tilde{G}_{Gr(n,r)}(t)$$

$\Bigg\}$ a corollary of their results on the Plücker coord mirror W_P

What is W_P?

Let me motivate it by telling you about the Gu-Sharpe mirror first.

Consider the abelianisation of

$$\overset{Q}{\downarrow} \qquad\qquad \overset{\mathcal{O}(1,0)\oplus\mathcal{O}(0,1)}{\downarrow}$$

$$Gr(4,2), \quad \mathbb{P}^3 \times \mathbb{P}^3 \quad \begin{bmatrix} 1&1&1&1&&&& \\ &&&&1&1&1&1 \end{bmatrix}$$

Mirror

$$\boxed{\begin{matrix} x_1 \\ + y_1 \end{matrix}} + \boxed{\begin{matrix} x_2 \\ y_2 \end{matrix}} + \boxed{\begin{matrix} x_3 \\ y_3 \end{matrix}} + \boxed{\begin{matrix} q_1/_{x_1x_2x_3} \\ q_2/_{y_1y_2y_3} \end{matrix}}$$

Each column corresponds to $c_1(\mathcal{O}(1,1))$

$\wr$

$c_1(Q) = s_\square$

Set $q_1 = q_2 = -q$

$$x_1 + x_2 + x_3 + {}^{-q}/_{x_1x_2x_3} + y_1 + y_2 + y_3 + {}^{-q}/_{y_1y_2y_3} = W_{GS}$$

Then the critical locus computes the quantum cohomology ring of $Gr(4,2)$, if one asserts $x \neq y$.

This mirror has the wrong number of variables...

Marsh-Rietsh: Use quantum cohomology to write $s_\square$ instead.

Fact: The cohomology of $Gr(n,r)$ is generated by s_λ $\underbrace{\lambda \subseteq r \times n-r}$

$\hookrightarrow$ such partitions also index $I \subseteq \{1,\dots,n\}$, $|I| = r$

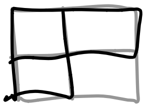
eg $Gr(4,2)$ $\quad \underset{1}{\overset{\shortparallel}{s_\emptyset}}, \; s_{\ydiagram{1}}, \; s_{\ydiagram{1,1}}, \; s_{\ydiagram{2}}, \; s_{\ydiagram{2,1}}, \; s_{\ydiagram{2,2}}$

$$s_\emptyset * s_{\ydiagram{1}} = s_{\ydiagram{1}}$$

$$s_{\ydiagram{1}} * s_{\ydiagram{1,1}} = s_{\ydiagram{2,1}}$$

$$s_{\ydiagram{1}} * s_{\ydiagram{2}} = s_{\ydiagram{2,1}}$$

$$s_{\ydiagram{1}} * s_{\ydiagram{2,2}} = q\, s_{\ydiagram{1}}$$



vertical steps $\rightarrow \{1,3\}$

$$\frac{s_{\ydiagram{1}}}{s_\emptyset} + \frac{s_{\ydiagram{2,1}}}{s_{\ydiagram{1,1}}} + \frac{s_{\ydiagram{2,1}}}{s_{\ydiagram{2}}} + \frac{q s_{\ydiagram{1}}}{s_{\ydiagram{2,1}}}$$

$$W_p = \frac{P_{24}}{P_{34}} + \underset{\substack{\wr \\ \text{splits} \\ \text{into} \\ 2}}{\frac{P_{13}}{P_{23}}} + \underset{\substack{\wr \\ \text{splits} \\ \text{into} \\ 2}}{\frac{P_{13}}{P_{14}}} + \frac{q P_{24}}{P_{12}} \Bigg\} \text{Plücker coordinates}$$

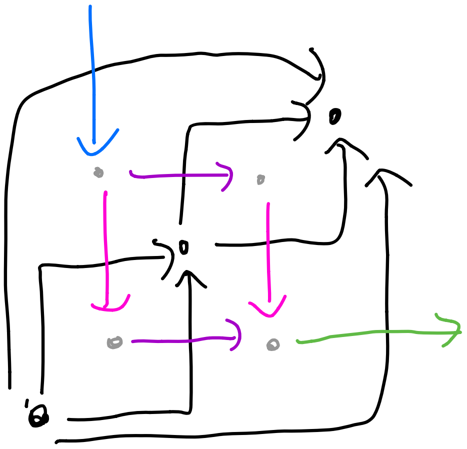
$s_{\ydiagram{1}} = D_1 + D_2$

(in general, choices given by cluster structure: one gives EHX mirror)

$$P_{13} = \frac{P_{12}P_{34} + P_{14}P_{23}}{P_{24}}$$

$$\overset{\cong}{\rightsquigarrow} f_{EHX}$$

$$W_p = \underline{\frac{p_{24}}{p_{34}}} + \underline{\frac{p_{13}}{p_{23}}} + \underline{\frac{p_{13}}{p_{14}}} + \underline{\frac{q\,p_{24}}{p_{12}}}$$



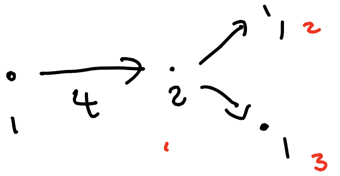
$$\begin{tikzcd} W_p \arrow[d] & \text{“}S_\square\text{” splits into } r \text{ or } n-r \text{ or } 1 \text{ monomials} & = & Gr(4,2) \arrow[d] & Q,\ S^* \arrow[d] \\ f_{EHX} & & & \text{GC toric degeneration} & \text{splitting via paths} \end{tikzcd}$$

WIP with Weigu: Write down a "Plücker coordinate mirror" for

- $\to \mathrm{Fl}(n, r_1, \cdots, r_\rho)$ ✓
- other quiver flag varieties ?

In examples, can use the proposed Plücker coordinate mirror to find Laurent polynomial varieties to quiver flag zero loci.

eg



Q

$$E_1 = \det W_1 \otimes W_2$$

$$E_2 = \det W_1 \otimes W_2 \otimes W_3$$

$$E_3 = \det W_1 \otimes W_2 \otimes W_3$$

$$E_4 = W_2 \otimes W_3$$

4 of 99/141

$$\begin{tikzcd} & Q, E_i \arrow[dl, "\text{GC degeneration}"'] \arrow[dr, "\text{Plücker coordinate}"] & \\ g_i \arrow[rr, leftrightarrow, bend right, "\text{mutation equivalent}"'] & & f_i \end{tikzcd}$$