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Realizing Tropical Curves via Mirror Symmetry

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A – symplectic X^A

B - alg. geometry X^B

Q affine space.

Novikov Field

$$\Lambda := \left\{ \sum_i a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i \to \infty \right\}$$

Valuation

$$\mathrm{val}\left(\sum_i a T^{\lambda_i}\right) = \left(\min(\lambda_i) \text{ st. } a_i \neq 0 \right)$$

$$\mathrm{val}(0) = \infty$$

X^B: $X^B = (\Lambda^*)^n$

I'll give this coordinates $(z_1 \cdots z_n)$.

B-tropicalization

$$\mathrm{Trop}^B : X^B \to \mathbb{R}^n$$
$$(z_1, \cdots, z_n) \longrightarrow (\mathrm{val}(z_1) \cdots \mathrm{val}(z_n)).$$

Theorem (Gross-Bieri)

Given $Y^B \subset X^B$ a subvariety,

$\mathrm{Trop}(Y^B) := \{ \mathrm{Trop}(y) \mid y \in Y^B \}$ is a polyhedral complex of $\mathbb{R}^n$.

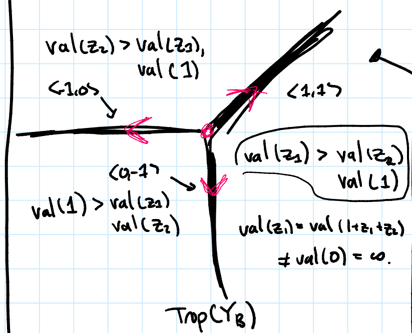
$\to$ A union of convex rational polytopes.

Example: $Y = (z, 0) \subset (\Lambda^*)^2$.

$\mathrm{Trop}(Y)$ = the line $(q, 0) \subset \mathbb{R}^2$.

Example

$Y^B := \{1 + z_1 + z_2 = 0\} \subset (\Lambda^*)^2$



Tropical Curve

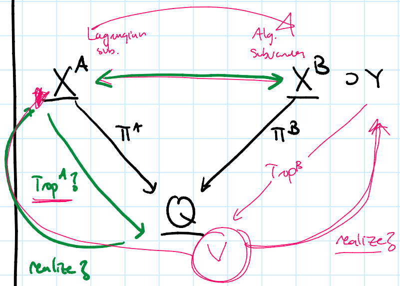
- A tropical curve is a piecewise linear graph in $\mathbb{R}^n$ st. at every edge we have a primitive $\mathbb{Z}^n$ vector describing the direction of the edge, and at each vertex $\sum_{e \ni v} \vec{v}_e = 0$.

B-realizability

Question: For given tropical curve $V \subset Q$, does there exist $Y_V \subset X^B$ st. $\mathrm{trop}^B(Y_V) = V$.

A: Not always. Q: when?

Main idea of talk



Näive A-tropicalization

1st Attempt.

$L \subset X^A$, look at $\pi^A(L)$.

(1) This is very rarely tropical.

(2) Given L, L' which are Hamiltonian isotopic (a usual equivalence preserving symp. geometry)

$$\pi_A(L) \neq \pi_A(L').$$

L
$$(\mathbb{C}^*)^n \xrightarrow{\pi^A} \mathbb{R}^n$$
$$(z_1^A \cdots z_n^A) \longrightarrow (\log|z_i^A|).$$

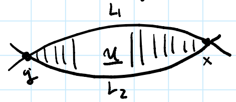
Floer Cohomology

Satisfying

Given Lagrangian submanifolds L_1, L_2, the Lagrangian intersection Floer cohomology is the chain complex

$$CF(L_1, L_2) := \bigoplus_{x \in L_1 \cap L_2} \Lambda\langle x \rangle.$$

$$\langle d(x), y \rangle = \sum_{u \in \mathcal{M}(x,y)} \pm T^{\omega(u)}$$



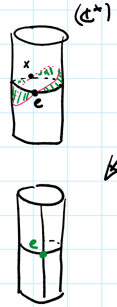
Example computations

- $CF(S^1, S^1) = C^\bullet(S^1)$

$$\Lambda\langle e \rangle \quad \downarrow\downarrow = 0 \quad \Lambda\langle x \rangle$$

- $CF(S^1, \mathbb{R})$

$$\Lambda\langle e \rangle$$



Upshots: $HF(L_1, L_2)$ is independent of Hamiltonian isotopy.

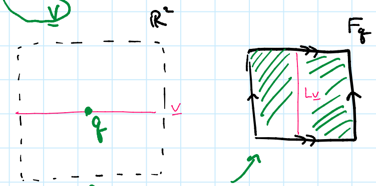
$$HF(L_1', L_2') = HF(L_1, L_2).$$

Example computations

$X^A = (\mathbb{C}^*)^2$, π, $\mathbb{R}^2$; $z_1^A z_2^A \mapsto \log|z_i^A|$.

$F_q = \pi^{-1}(q)$

$L_V = N^*V / N^*_{\mathbb{Z}} V$



$X^A \longrightarrow F_q$

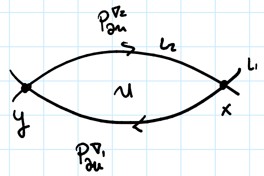
$z_1^A, z_2^A \mapsto \arg(z_i^A)$

$$CF(L_V, F_q) = CF(S^1, S^1) \otimes CF(S^1, \mathbb{R})$$
$$= C^\bullet(S^1) \otimes \Lambda$$

Generalization: Local System:

- Given Λ_Δ-local sytems on L_1, L_2, we have

$\underline{CF^\bullet((L_1,\nabla_1),(L_2,\nabla_2))} : \bigoplus_{x \in L_1 \cap L_2} \Lambda\langle x\rangle$



$$\langle d(x), y\rangle = \sum_{u \in \mathcal{M}(x,y)} P^{\nabla_2}_{\partial u} \circ (P^{\nabla_1}_{\partial u})^{-1} \cdot T^{\omega(u)}$$

Mirror Symmetry

Observation: Given $(\mathbb{C}^*)^n \xrightarrow{\pi} \mathbb{R}^n$ and $X^B \to \mathbb{R}^n$, we have a bijection

$$\left\{\begin{matrix}\text{pts on} \\ X^B\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Lagrangian torus} \\ \text{fibers of } \pi \\ (F_q, \nabla)\end{matrix}\right\}$$

$z \in X^B \longrightarrow \mathbb{R}^n \ni q \longleftarrow X^A \qquad (F_q, \nabla)$

Th^m (Abouzaid) Given a Lagrangian $L \subset X^A$ satisfying $\star$ then

$\exists$ a sheaf $\mathcal{F}(L)$ on X^B

s.t. $\hom(\mathcal{F}(L), \mathcal{O}_z) = HF^\bullet(L, (F_q, \nabla))$

$z \longleftrightarrow (F_q, \nabla)$

A-tropicalization

Given a Lagrangian Submanifold $L \subset X^A$ satisfying $\star$ then

$$\mathrm{Trop}^A(L) := \left\{ q \in \mathbb{R}^n \text{ st. } \exists \nabla \text{ a local system so that } HF(L, (F_q, \nabla)) \neq 0 \right\}.$$

"mirror to"

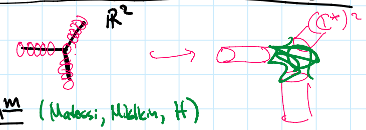
The set of points in Support$(\mathcal{F}(L))$ with $\mathrm{Trop}^B(z) = q$.

Easy $\mathrm{Trop}^A(L) \subset \pi^A(L)$

Proof: Suppose $q \notin \pi^A(L)$. Then $F_q := \pi^{-1}(q)$ has no intersection with L, so $CF(L, (F_q, \nabla)) = 0$.

Tropical Lagrangian Lift

Def^n A Lag. $L \subset X^A$ is a lift of tropical curve $V \subset Q$ if at every edge of V, L looks like $N^*_{\mathbb{R}}e / N^*_{\mathbb{Z}}e$.



Th^m (Matessi, Mikhalkin, H) Every V has a lift L_V.

Note: not all such Lagrangians satisfy $\star$

Roadmap to A-realizability $\star$

$$\begin{array}{ccc}
\text{check } \star & \text{Fuk}(X^A) \xrightarrow{\text{Abouzaid}} & D^bCoh(X^B) \\
L_V \subset X^A & & \downarrow \text{supp} \\
\text{always!} & B\text{-realizability} & X^B \\
 & Q \supset V & \xleftarrow{\mathrm{Trop}^B}
\end{array}$$

Th^m [H] whenever L_V satisfies condition $\star$ then the above diagram commutes or! $\star \Rightarrow$ B-realizability

what is $\star$ unobstructed?

- Exact
- $\omega(\pi_2(X^4, L))$.
- L is monotone Lagrangian submanifold.
- L is "unobstructed" $\star$

The curved A_∞ algebra $CF(L)$ constructed by Fukaya is homotopy eq. to a non-curved one.

Example Applications

Th^m: If $L \subset X^4$ and $\dim_{\mathbb{R}}(X^4) \leq 4$ then L is unobstructed.

$\Rightarrow$ All tropical curves in $\dim Q = 2$ are realizable.

$Q = \mathbb{R}^2$ Known (Mikhalkin).

$Q = T^2$ $\to$ algebraic realizability Nishinou 21

$\searrow$ analytic realizability New.

$\Rightarrow$ Th^m [H] All tropical hypersurfaces have L_V satisfying $\star$.

...If V is a tropical curve of genus 0, then L_V $\star$.

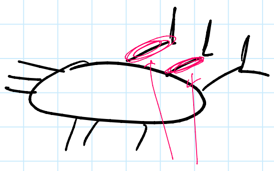
$\to$ Proven by Siebert-Nishinou on the A-side.

$L \longrightarrow \underline{H(L)}$ $\qquad$ $\dim(H(L))$

$\searrow \underline{HF(L)}$ $\qquad$ $\dim(HF(L))$

$\star \Longleftrightarrow$ The holomorphic disks w/ boundary on your Lagrangian cancel out in homology.

$\Rightarrow$ Minimum area disk must occur 2 times.



$\exists$ a holomorphic disk.