Holomorphic Anomaly Equations and Crepant Resolution Correspondence for $[\mathbb{C}^n/\mathbb{Z}_n]$

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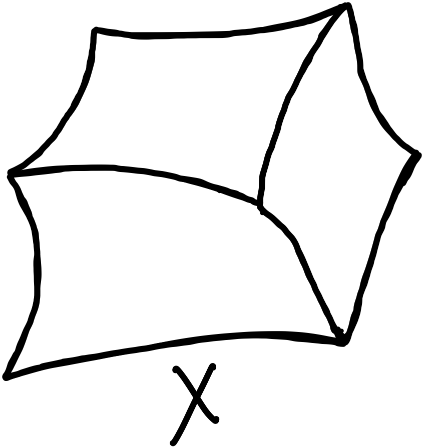
(The Ohio State University)

(Joint works with Hsian-Hua Tseng: arXiv:2301.08389, arXiv:2308.00780)

September 19, 2023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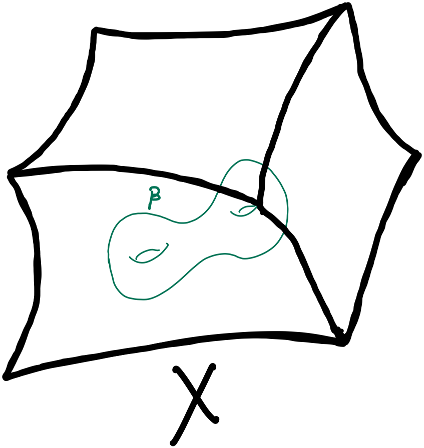
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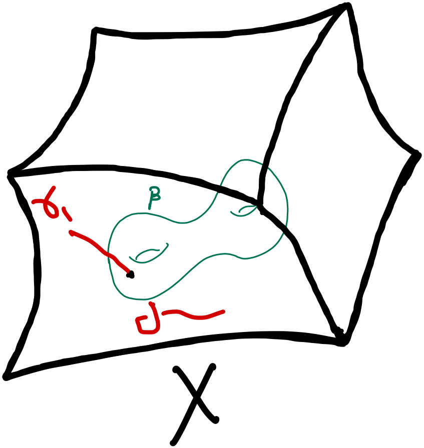
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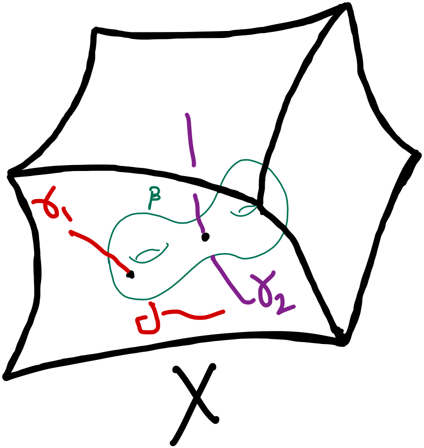
β
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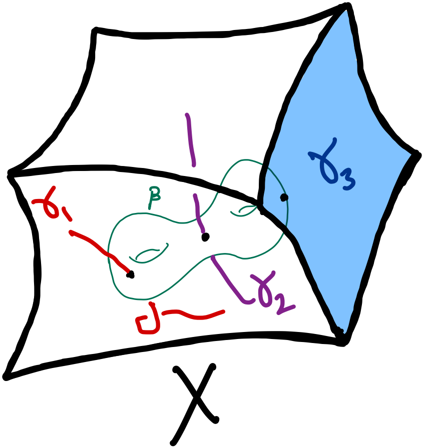
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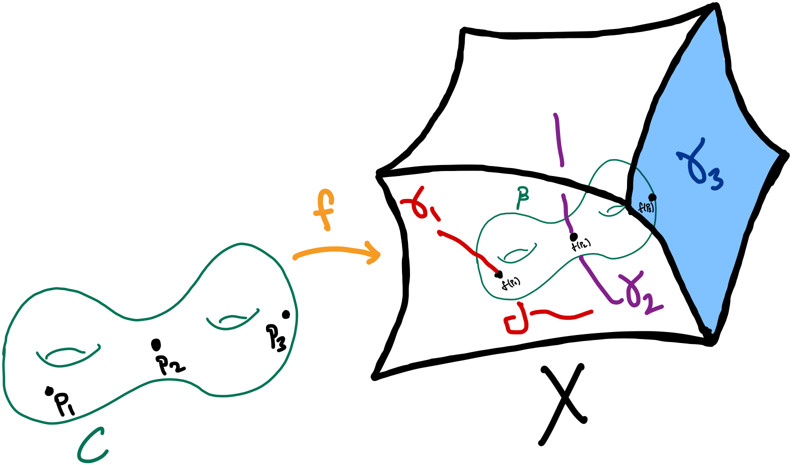
f
γ_1
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C

Let X be a smooth projective variety and $\beta \in H_2(X, \mathbb{Z})$. The moduli space $\overline{M}_{g,n}(X, \beta)$ is called the **moduli space of stable maps** and its points correspond to isomorphism classes of stable n-pointed maps $f : (C, p_1, ..., p_n) \to X$ satisfying $f_*([C]) = \beta$.

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The cotangent lines on the curves C at the i^{th} marked point patch together to form a line bundle $\mathbb{L}_i$ on $\overline{M}_{g,n}(X, \beta)$ and i^{th} **descendent class** is defined by

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For any $\gamma_1, ..., \gamma_n$ in $H^*(X, \mathbb{Q})$, the corresponding **Gromov-Witten invariant** is defined by:

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} \mathrm{ev}_i^* (\gamma_i) \, \psi_i^{m_i}$$

where $\mathrm{ev}_i : \overline{M}_{g,n}(X, \beta) \to X$ are defined by $\mathrm{ev}_i(f) = f(p_i)$ which are called the **evaluation maps**.

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When all $m_i = 0$, Gromov-Witten invariants are virtual counts of class β, genus g curves passing through Poincaré duals of the classes γ_i.

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  In their papers, the following equations are described as holomorphic anomaly equations:

$$\partial_{\bar{j}}\partial_i F_1 = \mathrm{Tr}(-1)^F C_i \bar{C}_{\bar{j}} - \frac{1}{12} G_{i\bar{j}}\, \mathrm{Tr}(-1)^F,$$

$$\bar{\partial}_{\bar{i}} F_g = \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left(D_j D_k F_{g-1} + \frac{1}{2} \sum_{r=1}^{g-1} D_j F_r D_k F_{g-r} \right).$$

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- Oberdieck conjectured HAE for the Hilbert scheme of points of a $K3$ surface and proved some special cases for every $n \geqslant 1$. ('22).

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For $\phi_{c_1}, \dots, \phi_{c_m} \in H^*_{\mathrm{T},\mathrm{Orb}}([\mathbb{C}^n/\mathbb{Z}_n])$, consider the Gromov-Witten potential by

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after the following specializations of equivariant parameters:

$$\lambda_i = \begin{cases} e^{\frac{2\pi\sqrt{-1}i}{n}} e^{\frac{\pi\sqrt{-1}}{n}} & \text{if } n \text{ is even,} \\ e^{\frac{2\pi\sqrt{-1}i}{n}} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem (Genlik, Tseng ('23))

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3. *Let* $n \geqslant 4$ *be an even number with* $n = 2s$, *and* $g \geqslant 2$. *We have*

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These are the first holomorphic anomaly equations in arbitrary dimension ($n \geqslant 3$) and genera $g \geqslant 2$.

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after the following specialization of equivariant parameters

$$\chi_i = e^{\frac{2\pi\sqrt{-1}i}{n}}.$$

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2. *For* g *and* m *in the stable range* $2g-2+m>0$, *we have*

$$\mathcal{F}_{g,m}^{[\mathbb{C}^n/\mathbb{Z}_n]}(\phi_{c_1}, \dots, \phi_{c_m}) = (-1)^{1-g}\rho^{3g-3+m}\Upsilon\left(\mathcal{F}_{g,m}^{K\mathbb{P}^{n-1}}(H^{c_1}, \dots, H^{c_m})\right)$$

where $\Upsilon : \mathbb{F}_{K\mathbb{P}^{n-1}} \to \mathbb{F}_{[\mathbb{C}^n/\mathbb{Z}_n]}$ *is a ring isomorphism.*

A *stable graph* Γ is described by the following data:

1. V_Γ is the vertex set with a genus assignment $\mathrm{g} : \mathrm{V}_\Gamma \to \mathbb{Z}_{\geqslant 0}$,
2. E_Γ is the edge set,
3. L_Γ is the set of legs,
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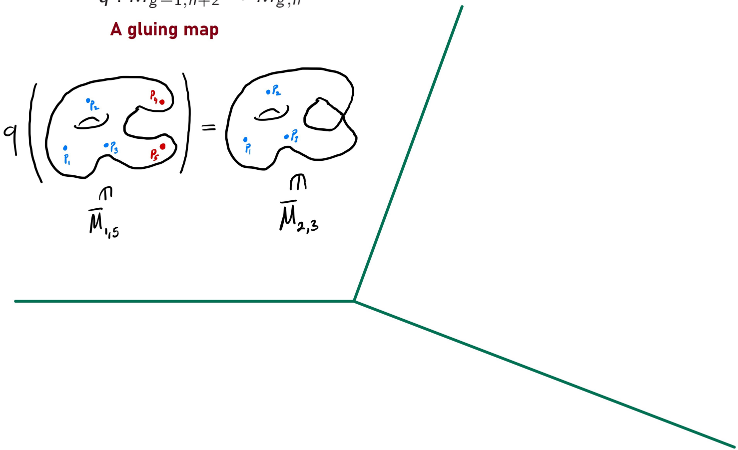
There is a canonical morphism

$$\iota_\Gamma : \prod_{\mathrm{V}_\Gamma} \overline{M}_{g(\mathfrak{v}),n(\mathfrak{v})} \to \overline{M}_{g,m}$$

with the image equal to the boundary stratum associated to the graph Γ.

$$q : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$$

A gluing map



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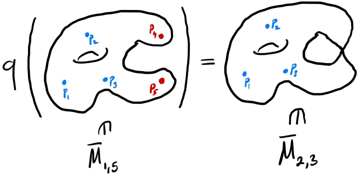

$$p : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$$

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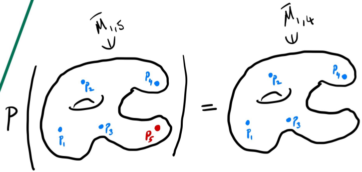

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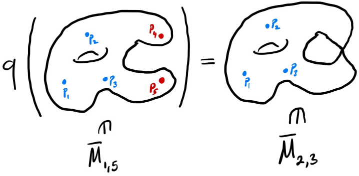

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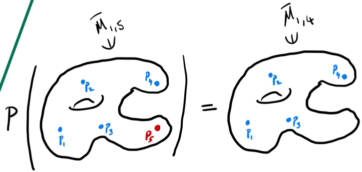

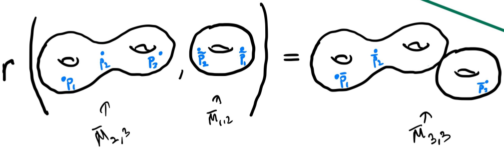

$r : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$ **A gluing map**

Let V be a finite dimesional $\mathbb{Q}$-vector space, η be symmetric non-degenerate bilinear form on V, and $\mathbb{1} \in V$ be a distinguished element.

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A **cohomological field theory (CohFT)** is a system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of S_n-equivariant tensors

$$\Omega_{g,n} \in H^* \left(\overline{M}_{g,n}, \mathbb{Q}\right) \otimes \left(V^*\right)^{\otimes n}$$

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$$q^\star \left(\Omega_{g,n}\left(v_1, \ldots, v_n\right)\right) = \sum_{j,k} \eta^{jk} \Omega_{g-1,n+2}\left(v_1, \ldots, v_n, e_j, e_k\right),$$

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$$\Omega_{g,n+1}\left(v_1, \ldots, v_n, \mathbb{1}\right) = p^\star \Omega_{g,n}\left(v_1, \ldots, v_n\right) \quad \text{and} \quad \Omega_{0,3}\left(v_1, v_2, \mathbb{1}\right) = \eta\left(v_1, v_2\right).$$

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A CohFT Ω defines a **quantum product** $\bullet$ on V by $\eta(v_1 \bullet v_2, v_3) = \Omega_{0,3}(v_1, v_2, v_3)$.

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A CohFT is **semisimple** if there exists a basis $\{e_i\}$ of idempotents,

$$e_i \bullet e_j = \delta_{ij} e_i .$$

Let $(\Omega, V, \eta, \mathbb{1})$ be a CohFT and

$$T(z) = T_2 z^2 + T_3 z^3 + \cdots \in V[[z]].$$

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Let R be a matrix series

$$R(z) = \sum_{k=0}^{\infty} R_k z^k \in \mathrm{Id} + z \cdot \mathrm{End}(V)[[z]]$$

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We define **a new CohFT** $R\Omega$:

$$(R\Omega)_{g,n} = \sum_{\Gamma \in \mathrm{G}_{g,n}} \frac{1}{|\mathrm{Aut}\ (\Gamma)|} \iota_{\Gamma\star} \left(\prod_{v \in V_\Gamma} \mathrm{Cont}(v) \prod_{e \in E_\Gamma} \mathrm{Cont}(e) \prod_{l \in L_\Gamma} \mathrm{Cont}(l) \right).$$

The topological part of Ω is given by

$$\omega = (\omega_{g,m} := \Omega_{g,m}|_{H^0(\overline{M}_{g,m})\otimes(V*)^{\otimes m}}),$$

and it is a CohFT as well. A topological CohFT can be obtained from $\omega_{0,3}$ by CohFT axioms.

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Theorem (Givental Teleman Classification)

For a semisimple CohFT Ω with unit, there exists a unique R-matrix which reconstructs Ω from its topological part ω,

$$\Omega = R(T(\omega)) \quad \textit{with} \quad T(z) = z((\mathrm{Id} - R(z)) \cdot 1) \in V[[z]],$$

as a CohFT.

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$\Omega_{g,m}|_{H^0(\overline{M}_{g,m})\otimes(V^*)^{\otimes m}}$ $\downarrow$ $\uparrow$ $\exists!\ R(z) \in \mathsf{Id} + z \cdot \mathrm{End}(V)[\![z]\!]$

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Flatness Equation

$$[R(z), \mathrm{d}\mathbf{u}] + z\Psi^{-1}\,\mathrm{d}(\Psi R(z)) = 0$$

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Quantum product $\bullet$

The J-function for $[\mathbb{C}^n/\mathbb{Z}_n]$ is defined by

$$J(\Theta, z) = \phi_0 + \frac{\Theta\phi_1}{z} + \sum_{i=0}^{n-1} \phi^i \left\langle\left\langle \frac{\phi_i}{z(z-\psi)} \right\rangle\right\rangle_{0,1}^{[\mathbb{C}^n/\mathbb{Z}_n]}.$$

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By methods of Coates-Corti-Iritani-Tseng, we define the I-function for $[\mathbb{C}^n/\mathbb{Z}_n]$:

$$I(x, z) = \sum_{k=0}^{\infty} \frac{x^k}{z^k k!} \prod_{\substack{b: 0 \leqslant b < \frac{k}{n} \\ \langle b \rangle = \langle \frac{k}{n} \rangle}} (1 + (-1)^n (bz)^n)\, \phi_k$$

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Theorem (Mirror Theorem)

We have $J(\Theta(x), z) = I(x, z)$ with the mirror transformation $\Theta(x) = I_1(x)$.

Define the following series in $\mathbb{C}[[x]]$:

$$L(x) = x\left(1 - (-1)^n \left(\frac{x}{n}\right)^n\right)^{-\frac{1}{n}}.$$

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Proposition

The I-function of $[\mathbb{C}^n/\mathbb{Z}_n]$ satisfies the following Picard-Fuchs equation

$$D^n I(x,z) + \frac{DL}{L}\sum_{k=1}^{n-1} s_{n,k} D^k I(x,z) = \frac{L^n}{z^n} I(x,z)$$

where $D = x\frac{d}{dx}$.

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We define the series $C_i \in \mathbb{C}[[x]]$ inductively as follows:

$$C_0 = I_0 = 1 \quad \text{and} \quad C_i = D\mathfrak{L}_{i-1}\ldots\mathfrak{L}_0 I_i \quad \text{where} \quad \mathfrak{L}_i = \frac{1}{C_i}D \quad \text{for} \quad i \geqslant 1,$$

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and $\mathfrak{L}_0$ is the identity.

For any $l \geqslant 0$, we further define

$$K_l = \prod_{i=0}^{l} C_i.$$

Proposition

For any $i, j \geqslant 0$, the quantum product is given by

$$\phi_i \bullet \phi_j = \frac{K_{i+j}}{K_i K_j} \phi_{i+j}.$$

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The proof relies on the following generation argument:

$$\phi_1 \bullet \phi_i = \frac{C_{i+1}}{C_1} \phi_{i+1},$$

and the following lemma was obtained by adapting methods of Zagier-Zinger for hypergeometric series.

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Lemma

We have the following identities for the series C_i and K_l

1. $C_{k+n} = C_k$ *for all* $k \geqslant 1$,
2. $\prod_{k=1}^{n} C_k = L^n$,
3. $C_k = C_{n+1-k}$ *for all* $1 \leqslant k \leqslant n$.
4. $K_{n+l} = L^n K_l$ *for all* $l \geqslant 0$, *in particular* $K_n = L^n$,
5. $K_l K_{n-l} = L^n$ *and* $K_l K_{\mathrm{Inv}(l)} = L^{l+\mathrm{Inv}(l)}$ *for all* $0 \leqslant l \leqslant n-1$.

Now, we define the series $A_i \in \mathbb{C}[[x]]$ for $0 \leqslant i \leqslant n$ by

$$A_i = \frac{1}{L}\left(i\frac{DL}{L} - \sum_{r=0}^{i}\frac{DC_r}{C_r}\right).$$

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After some change of variables:

$$R_{i,j}(z) = \sum_{k\geqslant 0} R_{i,j}^k z^k \rightsquigarrow P_{i,j}(z) = \sum_{k\geqslant 0} P_{i,j}^k z^k$$

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the flatness equation takes of the form

$$P^k_{\mathrm{Ion}(i)-1,j} = P^k_{i,j} + \frac{1}{L}DP^{k-1}_{i,j} + A_{n-i}P^{k-1}_{i,j}.$$

For example for $n = 6$, the equations look like

$$\begin{aligned}
P_{5,j}^{k} &= P_{0,j}^{k} + \frac{1}{L} DP_{0,j}^{k-1} \\
P_{4,j}^{k} &= P_{5,j}^{k} + \frac{1}{L} DP_{5,j}^{k-1} + A_1 P_{5,j}^{k-1} \\
P_{3,j}^{k} &= P_{4,j}^{k} + \frac{1}{L} DP_{4,j}^{k-1} + A_2 P_{4,j}^{k-1} \\
P_{2,j}^{k} &= P_{3,j}^{k} + \frac{1}{L} DP_{3,j}^{k-1} + A_3 P_{3,j}^{k-1} \\
P_{1,j}^{k} &= P_{2,j}^{k} + \frac{1}{L} DP_{2,j}^{k-1} + A_4 P_{2,j}^{k-1} \\
P_{0,j}^{k} &= P_{1,j}^{k} + \frac{1}{L} DP_{1,j}^{k-1} + A_5 P_{1,j}^{k-1}
\end{aligned}$$

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P^k_{2,j} &= P^k_{3,j} + \frac{1}{L} DP^{k-1}_{3,j} + A_3 P^{k-1}_{3,j} \\
P^k_{1,j} &= P^k_{2,j} + \frac{1}{L} DP^{k-1}_{2,j} + A_4 P^{k-1}_{2,j} \\
P^k_{0,j} &= P^k_{1,j} + \frac{1}{L} DP^{k-1}_{1,j} + A_5 P^{k-1}_{1,j}
\end{aligned}
$$

$$\mathbb{C}[L^{\pm 1}][\mathcal{DA}] := \mathbb{C}[L^{\pm 1}][A_1, ..., A_{n-1}, DA_1, ..., DA_{n-1}, D^2 A_1, ..., D^2 A_{n-1}, ...]$$

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$$

Lemma

We have $P^k_{0,j} \in \mathbb{C}[L]$. Hence, each $P^k_{i,j}$ lies in the differential ring $\mathbb{C}[L^{\pm 1}][\mathcal{D}\mathcal{A}]$

$DA =$

A_1	DA_1	D^2A_1	D^3A_1	D^4A_1	---
A_2	DA_2	D^2A_2	D^3A_2	D^4A_2	---
A_3	DA_3	D^2A_3	D^3A_3	D^4A_3	---
A_4	DA_4	D^2A_4	D^3A_4	D^4A_4	---
A_5	DA_5	D^2A_5	D^3A_5	D^4A_5	---

$$\mathfrak{A} := \{A_1, ..., D^{n-3}A_1\} \cup, \ldots \cup \{A_i, ..., D^{n-2-i}A_i\} \cup \ldots \cup \{A_{n-2}\}.$$

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Lemma (1st Simplification)

$\mathbb{C}[L^{\pm 1}][\mathcal{D}\mathcal{A}]$ *is a quotient of the ring* $\mathbb{C}[L^{\pm 1}][\mathfrak{A}]$.

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For the series A_i, *we have the following*

1. $A_i = -A_{n-i}$ *for all* $0 \leqslant i \leqslant n$,
2. $A_0 = A_n = 0$, *and* $A_{\frac{n}{2}} = 0$ *if* n *is even*,
3. $\sum_{i=0}^{n} A_i = 0$.

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Lemma (3rd Simplification)

For any $n \geqslant 3$, we have

$$2DA_{s-1} = \sum_{r=1}^{s-1} LA_r^2 - \sum_{r=1}^{s-2}(n-2r)DA_r - 2sf_{2s}(L) \quad \text{if} \quad n = 2s \geqslant 4,$$

$$DA_s = \sum_{r=1}^{s} LA_r^2 - \sum_{r=1}^{s-1}(n-2r)DA_r - (2s+1)f_{2s+1}(L) \quad \text{if} \quad n = 2s+1 \geqslant 3.$$

$DA =$

A_1	DA_1	D^2A_1	D^3A_1	D^4A_1	---
A_2	DA_2	D^2A_2	D^3A_2	D^4A_2	---
A_3	DA_3	D^2A_3	D^3A_3	D^4A_3	---
A_4	DA_4	D^2A_4	D^3A_4	D^4A_4	---
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A_1	DA_1	D^2A_1	D^3A_1	D^4A_1	---
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A_3	DA_3	D^2A_3	D^3A_3	D^4A_3	---
A_4	DA_4	D^2A_4	D^3A_4	D^4A_4	---
A_5	DA_5	D^2A_5	D^3A_5	D^4A_5	---

$\longrightarrow$ 1^{st} Simplification

A_1	DA_1	D^2A_1	D^3A_1
A_2	DA_2	D^2A_2	
A_3	DA_3		
A_4			

$DA =$

A_1	DA_1	D^2A_1	D^3A_1	D^4A_1	---
A_2	DA_2	D^2A_2	D^3A_2	D^4A_2	---
A_3	DA_3	D^2A_3	D^3A_3	D^4A_3	---
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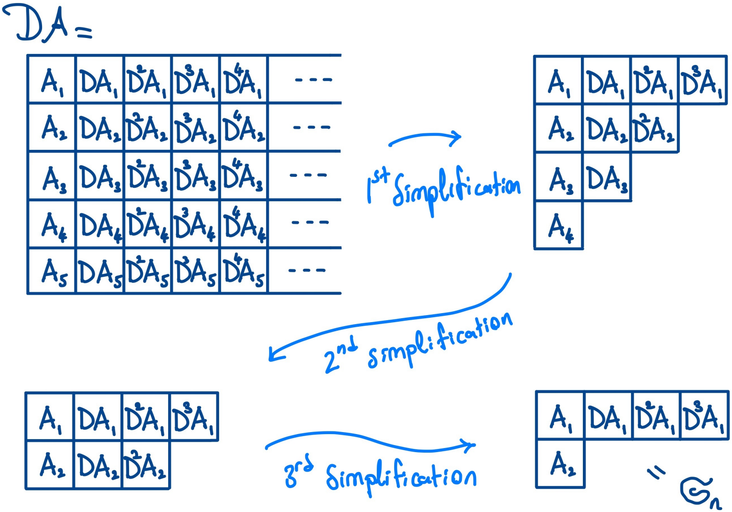
A_5	DA_5	D^2A_5	D^3A_5	D^4A_5	---

1st simplification

A_1	DA_1	D^2A_1	D^3A_1
A_2	DA_2	D^2A_2	
A_3	DA_3		
A_4			

2nd simplification

A_1	DA_1	D^2A_1	D^3A_1
A_2	DA_2	D^2A_2	


$DA =$
A_1 DA_1 D^2A_1 D^3A_1 D^4A_1 ---
A_2 DA_2 D^2A_2 D^3A_2 D^4A_2 ---
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1st simplification
A_1 DA_1 D^2A_1 D^3A_1
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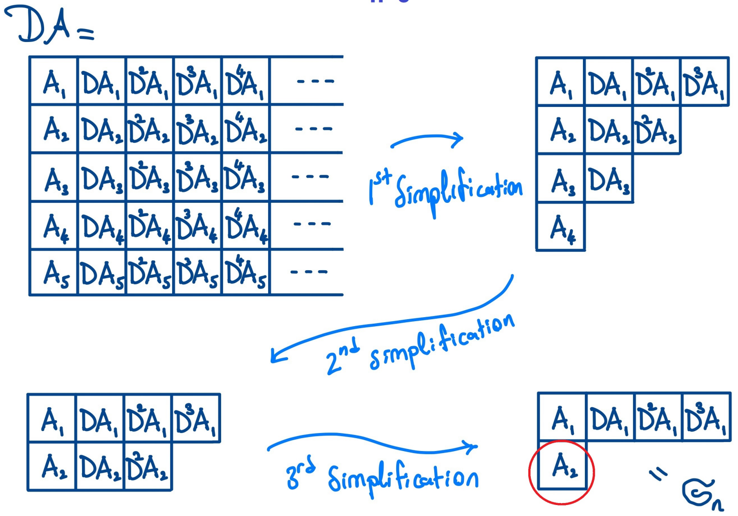
A_4
2nd simplification
A_1 DA_1 D^2A_1 D^3A_1
A_2 DA_2 D^2A_2
3rd simplification
A_1 DA_1 D^2A_1 D^3A_1
A_2
$= \mathfrak{S}_n$

n=6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Proposition

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We have a canonical lift of each $P_{i,j}^k$ to the free algebra $\mathbb{C}[L^{\pm 1}][\mathfrak{S}_n]$.

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$\mathbb{C}[L^{\pm 1}][\mathcal{D}\mathcal{A}]$ is a quotient of the ring $\mathbb{C}[L^{\pm 1}][\mathfrak{S}_n]$.

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Example:

$$P^k_{5,j} = P^k_{0,j} + \frac{1}{L} DP^{k-1}_{0,j} \in \mathbb{C}[L^{\pm 1}] \text{ since we have } DL, P^k_{0,j} \in \mathbb{C}[L],$$

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$$P^k_{4,j} = P^k_{5,j} + \frac{1}{L} D P^{k-1}_{5,j} + A_1 P^{k-1}_{5,j} \in \mathbb{C}[L^{\pm 1}][A_1],$$

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P_{3,j}^k &= P_{4,j}^k + \frac{1}{L} D P_{4,j}^{k-1} + A_2 P_{4,j}^{k-1} \in \mathbb{C}[L^{\pm 1}][A_1, DA_1, A_2],\\
P_{2,j}^k &= P_{3,j}^k + \frac{1}{L} D P_{3,j}^{k-1} \in \mathbb{C}[L^{\pm 1}][A_1, DA_1, D^2 A_1, A_2],
\end{aligned}
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$$P^k_{2,j} = P^k_{3,j} + \frac{1}{L} DP^{k-1}_{3,j} \in \mathbb{C}[L^{\pm 1}][A_1, DA_1, D^2A_1, A_2],$$
$$P^k_{1,j} = P^k_{2,j} + \frac{1}{L} DP^{k-1}_{2,j} - A_2 P^{k-1}_{2,j} \in \mathbb{C}[L^{\pm 1}][A_1, DA_1, D^2A_1, D^3A_1, A_2].$$

The following two lemmas are crucial in the proof of holomorphic anomaly equations.

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Lemma (Odd case)

Let $n \geqslant 3$ be an odd number with $n = 2s + 1$. We have the following identity

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Lemma (Even case)

Let $n \geqslant 4$ be an even number with $n = 2s$. We have the following identity

$$\frac{\partial P^k_{i,j}}{\partial A_{s-1}} = \delta_{i,s} P^{k-1}_{s+1,j} + \delta_{i,s-1} P^{k-1}_{s,j}.$$

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By Givental-Teleman classification of semisimple CohFTs, we have

$$\mathcal{F}^{[\mathbb{C}^n/\mathbb{Z}_n]}_{g,m}(\phi_{c_1}, \ldots, \phi_{c_m}) = \sum_{\Gamma \in \mathrm{G}^{\mathrm{Dec}}_{g,m}(n)} \mathrm{Cont}_\Gamma(\phi_{c_1}, \ldots, \phi_{c_m}).$$

Proposition (and Its Corollaries)

The contribution $\mathrm{Cont}_\Gamma(\phi_{c_1},\ldots,\phi_{c_m})$ *of a decorated stable graph* $\Gamma\in\mathrm{G}^{Dec}_{g,m}(n)$ *is*

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathrm{A}\in\mathbb{Z}_{\geqslant 0}^{\mathrm{F}(\Gamma)}}\prod_{\mathfrak{v}\in\mathrm{V}_\Gamma}\mathrm{Cont}^{\mathrm{A}}_\Gamma(\mathfrak{v})\prod_{\mathfrak{e}\in\mathrm{E}_\Gamma}\mathrm{Cont}^{\mathrm{A}}_\Gamma(\mathfrak{e})\prod_{\mathfrak{l}\in\mathrm{L}_\Gamma}\mathrm{Cont}^{\mathrm{A}}_\Gamma(\mathfrak{l})$$

where with $\mathrm{A}=(a_1,\ldots,a_m,b_1,\ldots,b_{|\mathrm{H}_\Gamma|})$ *where*

$$\mathrm{Cont}^{\mathrm{A}}_\Gamma(\mathfrak{v})=\sum_{k\geqslant 0}\frac{\eta(e_{\mathrm{p}(\mathfrak{v})},e_{\mathrm{p}(\mathfrak{v})})^{-\frac{2g-2+\mathrm{n}(\mathfrak{v})+k}{2}}}{k!}$$

$$\times\int_{\overline{M}_{\mathrm{g}(\mathfrak{v}),\mathrm{n}(\mathfrak{v})+k}}\psi_1^{a_{\mathfrak{v}1}}\cdots\psi_{\mathrm{l}(v)}^{a_{\mathfrak{v}\mathrm{l}(\mathfrak{v})}}\psi_{\mathrm{l}(\mathfrak{v})+1}^{b_{\mathfrak{v}1}}\cdots\psi_{\mathrm{n}(\mathfrak{v})}^{b_{\mathfrak{v}\mathrm{h}(\mathfrak{v})}}t_{\mathrm{p}(\mathfrak{v})}(\psi_{\mathrm{n}(\mathfrak{v})+1})\cdots t_{\mathrm{p}(\mathfrak{v})}(\psi_{\mathrm{n}(\mathfrak{v})+k})$$

where

$$t_{\mathrm{p}(\mathfrak{v})}(z)=\sum_{k\geqslant 2}\mathrm{T}_{\mathrm{p}(\mathfrak{v})k}z^k\quad\textit{with}\quad\mathrm{T}_{\mathrm{p}(\mathfrak{v})k}=\frac{(-1)^k}{n}P^k_{0,\mathrm{p}(\mathfrak{v})}\zeta^{-k\mathrm{p}(\mathfrak{v})}.$$

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where with $\mathrm{A} = (a_1, \ldots, a_m, b_1, \ldots, b_{|\mathrm{H}_\Gamma|})$ *where*

$$\mathrm{Cont}^{\mathrm{A}}_\Gamma(\mathfrak{v}) \text{ is a just polynomial in } P^k_{0,j}$$

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$$\mathrm{Cont}^{\mathrm{A}}_\Gamma(\mathfrak{v})\in\mathbb{C}[L],$$

$$\mathrm{Cont}^{\mathrm{A}}_\Gamma(\mathfrak{e})=\frac{(-1)^{b_{\mathfrak{e}1}+b_{\mathfrak{e}2}}}{n}\sum_{j=0}^{b_{\mathfrak{e}2}}(-1)^j\sum_{r=0}^{n-1}\frac{P^{b_{\mathfrak{e}1}+j+1}_{\mathrm{Inv}(r),\mathrm{p}(\mathfrak{v}_1)}P^{b_{\mathfrak{e}2}-j}_{r,\mathrm{p}(\mathfrak{v}_2)}}{\zeta^{(b_{\mathfrak{e}1}+j+1+\mathrm{Inv}(r))\mathrm{p}(\mathfrak{v}_1)}\zeta^{(b_{\mathfrak{e}2}-j+r)\mathrm{p}(\mathfrak{v}_2)}}$$

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\end{aligned}$$

Recall

$$\mathcal{F}^{[\mathbb{C}^n/\mathbb{Z}_n]}_{g,m}(\phi_{c_1},\ldots,\phi_{c_m})=\sum_{\Gamma\in\mathrm{G}^{\mathrm{Dec}}_{g,m}(n)}\mathsf{Cont}_\Gamma(\phi_{c_1},\ldots,\phi_{c_m}).$$

Theorem (Finite Generation Property)

We have $\mathcal{F}^{[\mathbb{C}^n/\mathbb{Z}_n]}_{g,m}(\phi_{c_1},\ldots,\phi_{c_m})\in\mathbb{F}_{[\mathbb{C}^n/\mathbb{Z}_n]}$.

Since $\mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{v}) \in \mathbb{C}[L]$ we have the following vanishing:

$$\frac{\partial \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{v})}{\partial A_{\lfloor \frac{n-1}{2} \rfloor}} = 0.$$

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Recall those two crucial lemmas:

Lemma (Odd case)

Let $n \geqslant 3$ be an odd number with $n = 2s + 1$. We have the following identity

$$\frac{\partial P^k_{i,j}}{\partial A_s} = \delta_{i,s} P^{k-1}_{s+1,j}.$$

Lemma (Even case)

Let $n \geqslant 4$ be an even number with $n = 2s$. We have the following identity

$$\frac{\partial P^k_{i,j}}{\partial A_{s-1}} = \delta_{i,s} P^{k-1}_{s+1,j} + \delta_{i,s-1} P^{k-1}_{s,j}.$$

Since $\mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{v}) \in \mathbb{C}[L]$ we have the following vanishing:

$$\frac{\partial \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{v})}{\partial A_{\lfloor \frac{n-1}{2} \rfloor}} = 0.$$

Those two crucial lemmas result in the following two crucial lemmas :)

Lemma

Let $n \geqslant 3$ be an odd number with $n = 2s + 1$, then we have

$$\frac{\partial}{\partial A_s} \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{e}) = \frac{(-1)^{b_{\mathfrak{e}1}+b_{\mathfrak{e}2}}}{2s+1} \frac{P^{b_{\mathfrak{e}1}}_{s+1,\mathrm{p}(\mathfrak{v}_1)} P^{b_{\mathfrak{e}2}}_{s+1,\mathrm{p}(\mathfrak{v}_2)}}{\zeta^{(b_{\mathfrak{e}1}+s+1)\mathrm{p}(\mathfrak{e}_1)} \zeta^{(b_{\mathfrak{e}2}+s+1)\mathrm{p}(\mathfrak{v}_2)}}.$$

Lemma

Let $n \geqslant 4$ be an even number with $n = 2s$, then we have

$$\frac{\partial}{\partial A_{s-1}} \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{e}) = \frac{(-1)^{b_{\mathfrak{e}1}+b_{\mathfrak{e}2}}}{2s} \left(\frac{P^{b_{\mathfrak{e}1}}_{s+1,\mathrm{p}(\mathfrak{v}_1)} P^{b_{\mathfrak{e}2}}_{s,\mathrm{p}(\mathfrak{v}_2)}}{\zeta^{(b_{\mathfrak{e}1}+s+1)\mathrm{p}(\mathfrak{v}_1)} \zeta^{(b_{\mathfrak{e}2}+s)\mathrm{p}(\mathfrak{v}_2)}} + \frac{P^{b_{\mathfrak{e}1}}_{s,\mathrm{p}(\mathfrak{v}_1)} P^{b_{\mathfrak{e}2}}_{s+1,\mathrm{p}(\mathfrak{v}_2)}}{\zeta^{(b_{\mathfrak{e}1}+s)\mathrm{p}(\mathfrak{v}_1)} \zeta^{(b_{\mathfrak{e}2}+s+1)\mathrm{p}(\mathfrak{v}_2)}} \right).$$

For $\mathcal{F}_g^{[\mathbb{C}^n/\mathbb{Z}_n]}$, the graph contributions are like this:

$$\mathrm{Cont}_\Gamma = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathrm{A}\in\mathbb{Z}_{\geqslant 0}^{\mathrm{F}(\Gamma)}} \prod_{\mathfrak{v}\in\mathrm{V}_\Gamma} \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{v}) \prod_{\mathfrak{e}\in\mathrm{E}_\Gamma} \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{e})$$

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For $n = 2s + 1$ (the odd case), we see

$$\begin{aligned} \frac{\partial \mathrm{Cont}_\Gamma}{\partial A_s} &= \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathrm{A} \in \mathbb{Z}_{\geqslant 0}^{\mathrm{F}(\Gamma)}} \prod_{\mathfrak{v} \in \mathrm{V}_\Gamma} \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{v}) \frac{\partial}{\partial A_s} \left(\prod_{\mathfrak{e} \in \mathrm{E}_\Gamma} \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{e}) \right) \\ &= \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathrm{A} \in \mathbb{Z}_{\geqslant 0}^{\mathrm{F}(\Gamma)}} \prod_{\mathfrak{v} \in \mathrm{V}_\Gamma} \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{v}) \prod_{\substack{\mathfrak{e} \in \mathrm{E}_\Gamma \\ \mathfrak{e} \neq \tilde{\mathfrak{e}}}} \mathrm{Cont}_\Gamma^{\mathrm{A}}(\mathfrak{e}) \frac{\partial \mathrm{Cont}_\Gamma^{\mathrm{A}}(\tilde{\mathfrak{e}})}{\partial A_s} \end{aligned}$$

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Idea: Taking derivative wrt A_s equivalent to breaking an edge into two legs.

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$$\mathrm{Cont}_{\Gamma^0_{\tilde{\mathfrak{e}}}}(\phi_s, \phi_s) \quad \text{or} \quad \mathrm{Cont}_{\Gamma^1_{\tilde{\mathfrak{e}}}}(\phi_s)\,\mathrm{Cont}_{\Gamma^2_{\tilde{\mathfrak{e}}}}(\phi_s)$$

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Idea: Taking derivative wrt A_s equivalent to breaking an edge into two legs.

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$$\frac{C_{s+1}}{(2s+1)L} \frac{\partial}{\partial A_s} \mathcal{F}_g^{[\mathbb{C}^n/\mathbb{Z}_n]} = \frac{1}{2} \mathcal{F}_{g-1,2}^{[\mathbb{C}^n/\mathbb{Z}_n]}(\phi_s, \phi_s) + \frac{1}{2} \sum_{i=1}^{g-1} \mathcal{F}_{g-i,1}^{[\mathbb{C}^n/\mathbb{Z}_n]}(\phi_s)\, \mathcal{F}_{i,1}^{[\mathbb{C}^n/\mathbb{Z}_n]}(\phi_s) .$$

The I-function of $K\mathbb{P}^{n-1}$ is

$$I^{K\mathbb{P}^{n-1}}(q,z) = \sum_{d\geqslant 0} q^d (-1)^{nd} \frac{\prod_{k=0}^{nd-1}(nH+kz)}{\prod_{k=1}^{d}((H+kz)^n - H^n)}.$$

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Theorem

The mirror theorem implies the equality

$$e^{H\log Q/z} J^{K\mathbb{P}^{n-1}}(Q,z) = e^{H\log q/z} I^{K\mathbb{P}^{n-1}}(q,z),$$

subject to the change of variables (mirror map)

$$\log Q = \log q + n \sum_{d\geqslant 1} q^d (-1)^{nd} \frac{(nd-1)!}{(d!)^n}.$$

Define

$$L^{K\mathbb{P}^{n-1}} = (1 - (-n)^n q)^{-1/n} \in 1 + q\mathbb{C}[[q]].$$

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With an analogous approach to $[\mathbb{C}^n/\mathbb{Z}_n]$, we also introduce the series $C_i^{K\mathbb{P}^{n-1}}$, $K_i^{K\mathbb{P}^{n-1}}$, $A_i^{K\mathbb{P}^{n-1}}$ lying in $\mathbb{C}[\![q]\!]$.

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Lemma

For all $i, j \geqslant 0$, the quantum product is given by

$$H^i \bullet H^j = \frac{K_{i+j}^{K\mathbb{P}^{n-1}}}{K_i^{K\mathbb{P}^{n-1}} K_j^{K\mathbb{P}^{n-1}}} H^{i+j}.$$

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The flatness equations for $K\mathbb{P}^{n-1}$ reads as

$$P^{k,K\mathbb{P}^{n-1}}_{\mathrm{Ion}(i)-1,j} = P^{k,K\mathbb{P}^{n-1}}_{i,j} + \frac{1}{L^{K\mathbb{P}^{n-1}}} \mathrm{D}_{K\mathbb{P}^{n-1}} P^{k-1,K\mathbb{P}^{n-1}}_{i,j} + A^{K\mathbb{P}^{n-1}}_{n-i} P^{k-1,K\mathbb{P}^{n-1}}_{i,j}.$$

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$$\mathrm{D}_{K\mathbb{P}^{n-1}} = q\frac{d}{dq} = -\frac{1}{n}x\frac{d}{dx} = -\frac{1}{n}\mathrm{D}_{[\mathbb{C}^n/\mathbb{Z}_n]}.$$

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Lemma

The series $-\sqrt{-1}\mathrm{P}_{0,j}^{[\mathbb{C}^n/\mathbb{Z}_n]}(z)$ *and* $\mathrm{P}_{0,j}^{K\mathbb{P}^{n-1}}(\rho z)$ *match after identification.*

In addition, we formally identify the following:

$$C_i^{K\mathbb{P}^{n-1}} \mapsto -\frac{\rho}{n} C_i^{[\mathbb{C}^n/\mathbb{Z}_n]},$$
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The matrix series $-\sqrt{-1}\mathrm{P}^{[\mathbb{C}^n/\mathbb{Z}_n]}(z)$ *and* $\mathrm{P}^{K\mathbb{P}^{n-1}}(\rho z)$ *match after identifications.*

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The identifications above define a ring isomorphism:

$$\Upsilon : \mathbb{F}_{K\mathbb{P}^{n-1}} \to \mathbb{F}_{[\mathbb{C}^n/\mathbb{Z}_n]}.$$

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By the Givental-Teleman classification the Gromov-Witten potential of $K\mathbb{P}^{n-1}$ is given by

$$\mathcal{F}_{g,m}^{K\mathbb{P}^{n-1}}(H^{c_1}, \ldots, H^{c_m}) = \sum_{\Gamma \in \mathrm{G}_{g,m}^{\mathrm{Dec}}(n)} \mathrm{Cont}_\Gamma^{K\mathbb{P}^{n-1}}(H^{c_1}, \ldots, H^{c_m}).$$

Proposition

For each graph $\Gamma \in \mathrm{G}^{Dec}_{g,m}(n)$*, the contribution* $\mathrm{Cont}^{K\mathbb{P}^{n-1}}_{\Gamma}(H^{c_1}, \dots, H^{c_m})$ *is given by*

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathrm{A} \in \mathbb{Z}^{\mathrm{F}(\Gamma)}_{\geqslant 0}} \prod_{\mathfrak{v} \in \mathrm{V}_\Gamma} \mathrm{Cont}^{\mathrm{A}}_{\Gamma}(\mathfrak{v}) \prod_{\mathfrak{e} \in \mathrm{E}_\Gamma} \mathrm{Cont}^{\mathrm{A}}_{\Gamma}(\mathfrak{e}) \prod_{\mathfrak{l} \in \mathrm{L}_\Gamma} \mathrm{Cont}^{\mathrm{A}}_{\Gamma}(\mathfrak{l}).$$

$$\mathrm{Cont}^{\mathrm{A}}_{\Gamma}(\mathfrak{v}) \in \mathbb{C}[(L^{K\mathbb{P}^{n-1}})^{\pm 1}],$$

$$\mathrm{Cont}^{\mathrm{A}}_{\Gamma}(\mathfrak{e}) \in \mathbb{C}[(L^{K\mathbb{P}^{n-1}})^{\pm 1}][\mathfrak{S}^{K\mathbb{P}^{n-1}}_n],$$

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Theorem (Finite generation property for $K\mathbb{P}^{n-1}$)

$$\mathcal{F}^{K\mathbb{P}^{n-1}}_{g,m}(H^{c_1},\dots,H^{c_m})\in\mathbb{C}[(L^{K\mathbb{P}^{n-1}})^{\pm 1}][\mathfrak{S}^{K\mathbb{P}^{n-1}}_n][\mathfrak{C}^{K\mathbb{P}^{n-1}}_n]=\mathbb{F}_{K\mathbb{P}^{n-1}}$$

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For each graph $\Gamma \in \mathrm{G}^{Dec}_{g,m}(n)$, *the contribution* $\mathrm{Cont}^{K\mathbb{P}^{n-1}}_\Gamma(H^{c_1}, \dots, H^{c_m})$ *is given by*

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$$\begin{aligned}
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Theorem (Crepant Resolution Correspondence)

For g and m in the stable range $2g-2+m>0$, the ring isomorphism Υ yields

$$\mathcal{F}^{[\mathbb{C}^n/\mathbb{Z}_n]}_{g,m}(\phi_{c_1}, \dots, \phi_{c_m}) = (-1)^{1-g}\rho^{3g-3+m}\Upsilon\left(\mathcal{F}^{K\mathbb{P}^{n-1}}_{g,m}(H^{c_1}, \dots, H^{c_m})\right).$$