- FLAGS ON FANO 3-FOLD HYPERSURFACES - /$\mathbb{C}$

j w/ Takuzo Okada (in progress)

MOTIVATING QUESTION: when does a Fano variety admit a Kähler-Einstein metric? (KE)

↳ Ricci constant, i.e. $\mathrm{Ric}(\omega) = \lambda\omega$, ω Kähler form.

X algebraic variety

K_X ample and $K_X \equiv 0$ trivial] $\Longrightarrow \exists$ KE always

But: Fano varieties ($-K_X$ ample) have obstructions
↓
have to do with $\mathrm{Aut}(X)$

- Theorem [Matsushima '57]

X smooth Fano variety that admits a KE metric $\Longrightarrow$ $\mathrm{Aut}(X)$ is reductive

Q: notion that captures $\exists$ KE metrics? K-stability

History: • definition via TEST CONFIGURATIONS [Tian, Donaldson]
↓
Futaki invariant

K-stability of X is defined based on the positivity of

$$\boxed{\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0 \ \forall (\mathcal{X}, \mathcal{L})}$$

+ conditions.

Hard to check.

• [Liu, Xu] central fibre is klt

VALUATIVE CRITERION [Fujita, Li, Odaka...]

- def: α-invariant X Fano $\alpha(X) := \inf_{\substack{D \sim_{\mathbb{Q}} -K_X \\ \text{effective}}} \mathrm{lct}(X, D)$

↳ $\mathrm{lct} := \sup\{c \mid (X, cD) \text{ is log canonical}\}$

- Theorem: [Tian] [Fujita]

X $\mathbb{Q}$-Fano
if $\alpha(X) > \frac{\dim X}{\dim X + 1}$ ($\geq$) $\Longrightarrow X$ is K-stable (ss)

⟿ β-invariants & δ-invariants.

- **def:** X Fano, $\dim X = n$, $\underbrace{E \text{ prime divisor over } X}$
  ↳ $E \subset X$ or in some birational model of X
  $f : Y \to X$, $Y \supseteq E$

$$\boxed{A_X(E) := 1 + a_X(E) = 1 + \operatorname{ord}_E(K_Y - f^*K_X)}$$ log discrepancy

$$\boxed{S_X(E) := \frac{1}{(-K_X)^n} \int_0^{+\infty} \operatorname{Vol}(-f^*K_X - tE)\,dt}$$

↳ $\operatorname{Vol}(D) := \lim_{m \to \infty} \frac{h^0(mD)}{m^n/n!}$ leading term of global sections

- D nef $\Rightarrow \operatorname{Vol}(D) = D^n$
- D big $\Leftrightarrow \operatorname{Vol}(D) > 0$

finite integral $[0, \varepsilon(E)]$

where $\varepsilon(E) := \sup\{\lambda \in \mathbb{R}_{>0} \mid \operatorname{Vol}(-f^*K_X - tE) > 0\}$

PSEUDOEFF. THRESHOLD

- **def:** $\beta_X(E) := A_X(E) - S_X(E)$

$\delta_X(E) = \dfrac{A_X(E)}{S_X(E)}$ STABILITY THRESHOLD

- **Theorem:** [Fujita-Li] [Blum, Xu]

X Fano is K-stable (ss) $\Leftrightarrow$ $\forall E/X$ prime divisor $\beta_X(E) > 0$ (≥ 0) $\left(\delta_X(E) \overset{*}{>} 1 \ (\geq 1)\right)$

BIRATIONAL SUPERRIGIDITY & K-STABILITY

- **def:** a Mori fibre space $X \xrightarrow{\varphi} S$ is BIRATIONALLY RIGID if the only MFS in the birational class of $X \xrightarrow{\varphi} S$ is $X \xrightarrow{\varphi} S$ itself

BIRATIONALLY SUPERRIGID if also $\operatorname{Bir}(X) = \operatorname{Aut}(X)$

- **theorem:** [Stibitz, Zhuang '18]

X $\mathbb{Q}$-Fano, $\rho_X = 1$ birationally superrigid
if $\alpha(X) > \frac{1}{2}$ $\Rightarrow X$ is K-stable

- def: Fano 3-fold hypersurfaces $X = X_d \subseteq \mathbb{P}$ $\iota_X = 1$

$\mathbb{Q}$-factorial, terminal, quasi-smooth. (Reid's 95)

→ • All qsmooth members are bir rigid. (and some are superrigid)

[Corti, Pukhlikov, Reid] [Cheltsov, Park]

• a general qsmooth member is K-stable [Cheltsov]

SOFAR: [Kim, Okada, Won] computed α-invariants for superrigid Fano 3-fold hypersurfaces. (no generality assumption)

→ thm: $\boxed{\text{any qs Fano 3-fold hypersurface } \iota_X = 1 \text{ superrigid has } \alpha(X) \geq \tfrac{1}{2} \Rightarrow K\text{-stable}}$

$\delta_p(X) \geq \frac{4}{3} \alpha_p(X)$

[KOW] $\alpha_p(X) > \frac{3}{4} \Rightarrow \delta_p(X) > 1$

$\alpha(X) > \frac{1}{2}$ rigidity $\Rightarrow$?

Goal: prove K-stability for strictly birationally rigid Fano 3-fold hypersurfaces (without generality) (complete [KOW])

- Theorem: [C, Okada, '23]

$\boxed{\text{Any qsmooth Fano 3-fold hypersurface having } \iota_X = 1 \text{ that is strictly bir rigid} \Rightarrow K\text{-stable.}}$

→ USE δ-invariants.

- def: local δ-invariant (along Z)

$Z \subseteq X$ subvariety irreducible

$$\delta_Z(X) := \inf_{\substack{E/X \\ Z \subseteq C_X(E)}} \frac{A_X(E)}{S_X(E)}$$

[Abban-Zhuang] gives estimates for local δ-invariants

PERK: just looking at flags that realize the inequality ⊛

↓

ADMISSIBLE FLAGS $X \supset Y_1 \supset Y_2 \cdots \supset Y_n$

codim $Y_i = i$

bound $\delta(X)$ from below by computing local δ-invariants along each piece of a chosen admissible flag

3-folds $X \supset Y \supset Z \ni P$
irreducible

$$z := \max_{u \in \mathbb{Q}_{>0}} \{ -K_X - uY \text{ is psef} \}$$

Zariski decomp 1 (ZD) $\quad -K_X - uY = P(u) + N(u) \quad u \in [0, z]$

$$t(u) := \max_{v \in \mathbb{Q}_{>0}} \{ P(u)|_Y - vZ \text{ psef} \}$$

(ZD)2 $\quad P(u)|_Y - vZ = P(u,v) + N(u,v)$

-**Theorem** [From "Calabi Problem"]

ⓐ Δ_Z is the different: $K_Z + \Delta_Z = (K_Y + Z)|_Z$
ⓑ $A_Y(\tilde{Z})$ is the log discrepancy of $\tilde{Z}$ over arbitrary Y.

$\boxed{\begin{array}{l}\text{X is smooth Fano}\\ \text{For $P \in Z \subset Y \subset X$ we have the lower bound for the local stability threshold}\end{array}}$

$$\delta_P(X) \geq \min \left\{ \min_{p \in \tilde{Z}} \frac{1 - \mathrm{ord}_p(\Delta_{\tilde{Z}})}{S(W^{\tilde{Y}, \tilde{Z}}_{\bullet\bullet\bullet}; P)}, \frac{1 \quad A_Y(\tilde{Z})}{S(V^{\tilde{Y}}_{\bullet\bullet}; \tilde{Z})}, \frac{1}{S_X(Y)} \right\}$$

where:

$$S(W^{\tilde{Y}, \tilde{Z}}_{\bullet\bullet\bullet}; P) = \frac{3}{(-K_X)^3} \int_0^z \int_0^{\tilde{t}(u)} (\tilde{P}(u,v) \cdot \tilde{Z})^2 \, dv \, du + F_P(W^{\tilde{Y}, \tilde{Z}}_{\bullet\bullet\bullet})$$

for $F_P(W^{\tilde{Y}, \tilde{Z}}_{\bullet\bullet\bullet}) := \frac{6}{(-K_X)^3} \int_0^z \int_0^{\tilde{t}(u)} (\tilde{P}(u,v) \cdot \tilde{Z}) \cdot \mathrm{ord}_P(N_{\tilde{Y}}'(u)|_Z + \tilde{N}(u,v)|_{\tilde{Z}}) \, dv \, du$

$$S(V^{\tilde{Y}}_{\bullet\bullet}; \tilde{Z}) = \frac{3}{(-K_X)^3} \int_0^z h(u) \, du$$

for $h(u) := (\tilde{P}(u)^{m-1} \cdot \tilde{Y}) \cdot \mathrm{ord}_Z(\tilde{N}(u)|_{\tilde{Y}}) + \int_0^{+\infty} \mathrm{Vol}(\tilde{P}(u)|_{\tilde{Y}} - v\tilde{Z}) \, dv$

$$S_X(Y) = \frac{1}{(-K_X)^3} \int_0^z \mathrm{Vol}(\phi^*(-K_X) - xY) \, dx$$

X is singular $\rightsquigarrow$ $\phi: \tilde{X} \longrightarrow X$ at P.

$Y \supset Z \ni P \rightsquigarrow \tilde{Y} \supset \tilde{Z} \ni p \rightsquigarrow p \in \tilde{Z}$

$\tilde{Z} = E \cap \tilde{Y}$, $E = \mathrm{Exc}(\phi)$

BIRATIONAL RIGIDITY

Untwisting

$\exists$ Sarkisov link initiated by ϕ

$$\iota_p : X \dashrightarrow X$$
$$\Downarrow$$

QI or EI

Excluding

Σ by ϕ breaks.

non-BI

$(F=0)$ $\qquad -K_X \sim A$

$$X = X_d \subset \mathbb{P}^4(a_0 x_0, a_1 x_1, a_2 x_2, a_3 x_3, a_4 x_4)$$

$$P \sim \frac{1}{a_k}(a_{i_1}, a_{i_2}, a_{i_3}) \qquad a_{i_1} \le a_{i_2} \le a_{i_3}$$

$$F = x_k^2 x_j + x_k f + g \qquad x_j$$

$\boxed{QI}$ • <u>degenerate case</u> $\quad x_j \mid f \implies$ no link

E

$\phi^* A$

$\phi^* A - \frac{1}{a_k} E$

$a_j \, \phi^* A - \frac{d}{a_k} E$

$\mathrm{Eff}(\tilde{X})$

$\mathrm{Mov}(\tilde{X})$

• <u>non-degenerate case</u>: $\quad x_j \nmid f \implies \exists$ link

E

$\phi^* A$

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "i"] \arrow[d] & \tilde{X}' \arrow[d] \\ X & X \end{tikzcd}$$

• Exceptional case $\implies$ no link

- Proof: [C, Okada, '23]

For $P \in X$ QI $\qquad Y \supset Z \ni P \qquad Y := (x_{i_3} = 0)|_X \sim a_{i_3} A$

$$Z := Y \cap H_{x_{i_1}}$$

$\implies \varepsilon = \frac{1}{a_{i_3}} \qquad$ 2D $\qquad -K_X \sim Y \qquad P(u) = (1 - a_{i_3} u) A$

$$N(u) = 0$$

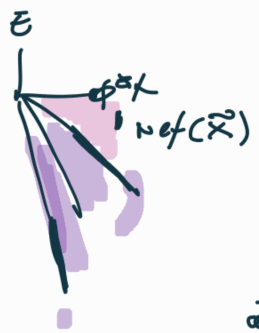
- Prop: [C, Okuda]

- **$P \in X$ smooth** $\qquad t(u) = \dfrac{1 - Mu}{a_{i_1}} \qquad M$ highest weight in wP^4.

ZD $\quad P(u)|_Y - vZ \qquad \begin{cases} P(u,v) = (1 - Mu - a_{i_1} v) A|_Y \\ N(u,v) = 0 \end{cases}$

$v \in [0, t(u)]$

- **$P \in X$ QF** : • non degenerate case ($\exists$ lin K)



$0 \leq v \leq \dfrac{1}{a_k}(1 - a_{i_3} u) \qquad \tilde{P}(u)|_{\tilde{Y}} - v\tilde{Z}$

$$\begin{cases} \tilde{P}(u,v) = (1 - a_{i_3} u)\, \phi^* A|_{\tilde{Y}} - v\tilde{Z} \\ \tilde{N}(u,v) = 0 \end{cases}$$

$$\frac{1}{a_k}(1 - a_{i_3} u) \leq v \leq \frac{d}{(d - a_k) a_k}(1 - a_{i_3} u) =: \tilde{t}(u)$$

$$\tilde{P}(u,v) = \lambda\left(\phi^* A|_{\tilde{Y}} - \frac{1}{a_k}\tilde{Z}\right)$$

$$\tilde{N}(u,v) = \mu\left(\phi^* A|_{\tilde{Y}} - \frac{d - a_k}{a_j a_k}\tilde{Z}\right)$$

λ, μ solve $\qquad \begin{cases} \lambda + \mu = 1 - a_{i_3} u \\ \dfrac{1}{a_k}\lambda + \dfrac{d - a_k}{a_j a_k}\mu = v \end{cases}$