Reductive quotients of klt singularities

jt with Daniel Greb / Kevin Langlois / Joaquín Moraga

(X normal alg var over $\mathbb{K}$ alg closed, char $\mathbb{K} = 0$)

Log terminal singularities

X has log terminal sing's, if for a resolution $f : Y \to X$, we can write

$$K_Y \sim f^*(K_X) + \sum a_i E_i$$

E_i excep div's

with $a_i > -1$.

⚠ K_X has to be $\mathbb{Q}$-Cartier (i.e. X $\mathbb{Q}$-Gorenstein)

Observation: in dim 2, lt sing's are precisely quotients of smooth pt's by finite subgroups* of $GL_2(\mathbb{K})$

Reductive groups

are those linear alg groups $G \subseteq GL_n(\mathbb{K})$, s.th. for all affine var Z with alg G-action, we can define

$$G \circlearrowright Z \to Z /\!/ G := \operatorname{Spec} \mathbb{K}[Z]^G$$

f.g.

Thm 1 (Schoutens, 2005)

$G \circlearrowright X$ log terminal, affine $\Rightarrow$ if $X/\!/G$ is $\mathbb{Q}$-Gorenstein, then it is log-terminal

⚠ Quotients tend to be non-$\mathbb{Q}$-Gorenstein (even in the toric case!)

Thm 2 (Boutot, 1987)

$G \circlearrowright X$ with rational sing $\Rightarrow$ $X/\!/G$ has rational sing's

⚠ lt $\subseteq$ rat sing, which are still well-behaved, but lack vanishing thm's, MMP, etc, ...

Thm 3 (Classical)

$G \circlearrowright X$ factorial $\Rightarrow$ $X/\!/G$ factorial

$\uparrow$ semisimple!

In birational geometry, often log pairs (X, Δ) are investigated to address
$0 \leq \Delta \leq 1$
adjunction, compare sing's of X and X, ...

We use Δ as a necessary evil to be able to define discrepancies even if K_X is <u>not</u> $\mathbb{Q}$-Cartier.

<u>Def</u>: We say (X, Δ) is klt if

$$K_Y + f_*^{-1}\Delta = f^*(K_X + \Delta) + \sum_i a_i E_i$$

with $a_i > -1$.

X is of klt type, if such Δ exists.

For such X, we have vanishing thm's, MMP, etc...

Thm 4 (Greb, Langlois, Moraga, '21)

$G \circlearrowright X$ of klt type, if a good quasiprojective quotient $X /\!\!/ G$ exists, it is of klt type.

Remarks

- Δ need ~~to~~ not be G-invariant
- How does $\Delta_{X/\!\!/G}$ on $X /\!\!/ G$ arise?
- Δ is a global object... is being klt local? Zariski, étale?
- alternative proof by Ziquan Zhuang Sep '22 via char p

Big picture (of the proof)

reductive $G \circlearrowright X$ klt type

Prop (BGLM)

étale loc klt $\Leftrightarrow$ Zariski loc klt $\overset{\text{if quasiproj!}}{\Leftarrow}$ klt

X affine

+ Luna étale slice thm

$X \ni x$ fixed point

quotient first by

- identity component of G

- derived subgroup
- torus
- group of components (finite!)

Ⓐ G finite

Standard: if Δ is not G-inv, take
$$\Delta^G := \frac{\sum g^*\Delta}{|G|}$$
Then [KM, 5.20] gives the result.

Ⓑ $G = \mathbb{T}$ torus

lift $\mathbb{T}$ to the local Cox space at $X \ni x$

X canonical Gorenstein

polyh. Div's & canonical bundle formula

✓

$$\begin{tikzcd}
\hat{X} \arrow[d, "/\!/\mathbb{T}\times\mathbb{T}'"'] \arrow[dr] & /\!/H_x = \mathbb{T}'\times A \\
 & X \arrow[d, "/\!/\mathbb{T}"] \\
Y \arrow[r, "/A"] & X/\!/\mathbb{T}
\end{tikzcd}$$

finite abelian

Y klt $\quad$ Ⓐ $\quad$ $X/\!/\mathbb{T}$ klt

Ⓒ G semisimple

lift G to the iterated local Cox space at $x \in X$ Ⓘⓘ

$x \in X$ factorial

Ⓘ

✓

Ⓘ follows from

<u>Prop</u> (BGLM '21)

[+ X klt $\Rightarrow$ X rational]

$x \in X$ fixed, factorial
$\circlearrowleft$
G semisimple

$\Rightarrow$ for $\pi : X \to X/\!/G$ and $\pi(x)$, $\exists$ open affine, locally factorial neighborhood $U \ni \pi(x)$

[Thm 2: U rational & K_U is $\mathbb{Q}$-Cartier $\Rightarrow$ canonical $\subseteq$ klt]

Ⓘⓘ Iteration of local Cox rings (Moraga, B '21)

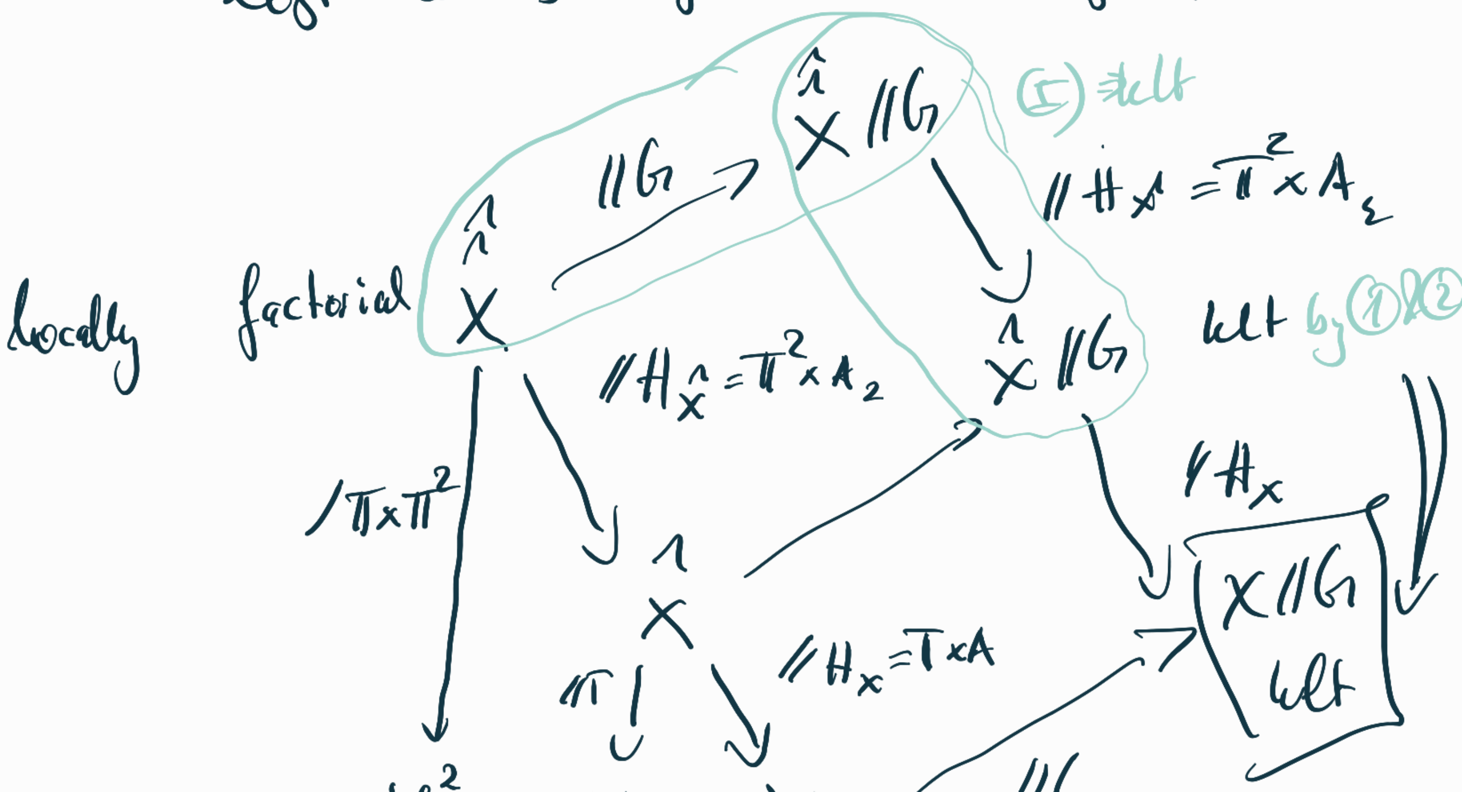
classical Cox ring, $Cl(X)$ f.g.

$$R_X := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D)$$

if R_X is f.g. then $\hat{X} := \mathrm{Spec}_X R_X \to X$ $/\!/H_X = \mathbb{T}\times A = \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$

- Fano type varieties have R_X
  - f.g. (BCHM)
  - log terminal (GOST, Brown)
- klt singularities X_x, have R_{X_x} (BM'21)
  - graded-local f.g over $\mathcal{O}_{X,x}$
  - canonical Gorenstein

BM'21: Iteration of Cox rings is possible & finite with factorial endpoints + can lift actions of connected groups



$$Y \xrightarrow{/\!/A_2} Y \xrightarrow{/\!/A} X \qquad /\!/G$$

$\square$ II

<u>Conseq</u>:

<u>Prop</u>: Projective GIT-quotients of Fano varieties are of Fano type.

[more generally: "quotients of MDS with klt Cox ring are again MDS with klt Cox ring"]