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Rank distribution in a family of cubic twists

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Abstract

In 1987, Zagier and Kramarz published a paper in which they presented evidence that a positive proportion of the even-signed cubic twists of the elliptic curve $X_0(27)$ should have positive rank. We extend their data, showing that it is more likely that the proportion goes to zero.

1.1 Introduction

Let E_m be the elliptic curve defined by the equation $x^3+y^3=m$, which is isomorphic to $y^2=x^3-432m^2$. The case of $m=1$ is the curve $X_0(27)$, and the cubefree positive m-values correspond to cubic twists.

These equations have a long history, dating back to Fermat. An early study was done by Sylvester [Syl] in 1879-80, and another voluminous study in 1951 by Selmer [Sel]. In between these two, Nagell [N, p.14] proved sundry results concerning non-solvability in many cases. In the late 1960s, Stephens [Ste1, Ste2] did numerical experiments with these curves with respect to the then-new Birch–Swinnerton-Dyer conjecture. Zagier and Kramarz [ZK] did a large numerical experiment in the mid 1980s, which led them to suggest that a positive proportion of the curves have rank 2 or greater. The best results in this regard appear to be due to Mai [M1, M2], who showed that, assuming the Parity Conjecture, for every $\epsilon>0$ at least $c_\epsilon T^{2/3-\epsilon}$ of the cubefree even twists up to T have rank 2. Elkies and Rogers [ER] have recently found that the curve

$$x^3+y^3=13293998056584952174157235$$

has rank at least 11. We shall mainly be concerned with rank 2 cubic twists, and in extending the numerical data of [ZK], showing that the purported positive proportion does not seem to persist. We also consider questions of the distribution of the size of the Tate–Shafarevitch groups attached to these curves, comment on effects stemming from the arithmetic of m, consider similar questions for quartic twists of $X_0(32)$, and discuss random matrix models for these.

We briefly review how to compute the central L-value of E_m. The first consideration is the sign of the functional equation, which was computed by Birch and Stephens [BS]. This is defined by $\epsilon = \prod_p \epsilon_p$ where for $p \neq 3$ we have that $\epsilon_p = \left(\frac{p}{3}\right)$ if $p|m$ and $\epsilon_p = +1$ if p does not divide m. For $p = 3$, we have that $\epsilon_3 = +1$ if $m \equiv \pm 1 \pmod 9$ or $3\|m$, and $\epsilon_3 = -1$ otherwise. Next, there is the conductor $N = \prod_p N_p$ where for $p \neq 3$ we have that $N_p = p^2$ if $p|m$ and $N_p = 1$ otherwise, while for $p = 3$ we have that $N_3 = 3^5$ if $3|m$, that $N_3 = 3^2$ if $m \equiv \pm 2 \pmod 9$, and $N_3 = 3^3$ otherwise. There are also Tamagawa numbers and considerations for the real period Ω; the effects of these are given in the last section (see also Table 1 of [ZK]).

When $\epsilon = +1$, the central L-value is given by

$$L(E_m, 1) = 2 \sum_n \frac{a_{m,n}}{n} e^{-2\pi n/\sqrt{N}},$$

where the conductor N is defined as above, and the $a_{m,n}$ can be computed as follows. For primes $p \not\equiv 1 \pmod 3$ and primes $p|3m$, we define $a_{m,p} = 0$. Given a prime $p \equiv 1 \pmod 3$, the set

$$A_p = \{a|\ a \equiv 2 \,(\mathrm{mod}\ 3),\ a^2 + 3b^2 = 4p \text{ for some } b \in \mathbf{Z}\}$$

has 3 elements. For such a prime we define $a_{1,p}$ to be the unique element in A_p for which $3|b$. We then define $a_{m,p}$ uniquely by the conditions $a_{m,p} \equiv m^{(p-1)/3} a_{1,p} \pmod p$ and $a_{m,p} \in A_p$ (this second condition is equivalent to $|a_{m,p}| < 2\sqrt{p}$ for $p > 13$ and not $p \geq 13$ as [ZK] claims). Having defined $a_{m,p}$ for all primes p, we can extend it to prime powers via the Hecke relations, and then to all positive integers via multiplicativity. In order to get a good approximation to the central value of the L-series, we need to use about $C\sqrt{N}$ coefficients for some constant C. When $\epsilon = -1$, there is a similar series for $L'(E_m, 1)$ with the exponential function replaced by an exponential integral — we did not deal with this case (Zagier and Kramarz looked at it for $m \leq 20000$) as it is much easier to compute the exponential homomorphism rapidly compared with the exponential integral — for the latter, a method involving local power series would likely be useful. One can note that Lieman [L] has shown that the values of $L(E_m, 1)$ are the coefficients of a metaplectic form as suggested in [ZK, 3.1], but this does not seem that useful for computational purposes.

1.2 Numerical data

Applying the above method for the cubefree $m \leq 10^7$ with $\epsilon = +1$, we find that about 17.7% of the twists have vanishing central L-value. This is to be compared to 23.3% for the $m \leq 70000$, and 20.5% for $m \leq 10^6$. If we take the best linear fit to a log-log regression, we find that the number of twists up to x with vanishing central L-value appears to grow like $x^{0.935}$. Heuristic models involving the expected size of Ш as in [ZK] imply that the growth should be more like $x^{5/6}$. Stronger models such as those in [CKRS] imply this should be more like $Bx^{5/6}(\log x)^C$ for some constants B and C; in the last section we make remarks about what random matrix theory implies about C.

There is also the question of arithmetic effects of m. Only 6.1% of the prime m in the above range have vanishing central L-value, while 11.3% of the m with two prime factors do, and 17.1% of the m with three prime factors. The number grows to 24.5% for four prime factors, and 35.3% for five prime factors, and is 51.4% for six or more prime factors. However, each of these percentages is about 20% lower than the comparative value when considering only the $m \leq 10^6$. So even if we restrict to prime m we expect that the proportion of twists with non-vanishing central L-value tends to zero. Note in this context that 3-descent can tell us much about the rank when we limit the number of prime factors of m (see [C]). For instance, when m is prime and E_m has even functional equation, we know that $m \equiv 1, 2, 5 \pmod 9$, and the rank is zero in the latter two cases. Thus the 6.1% of above might be re-interpreted as 18.3% of the cases where descent considerations do not force the rank to be zero. Using the results of [N], we could similarly derive such results when m has two prime factors. Also, one can recall that Elkies (see [E1]) has proven that the rank is exactly 1 for primes $m \equiv 4, 7 \pmod 9$; here in fact the conjecture is that the same is true for $m \equiv 8 \pmod 9$. We return to such considerations below when we discuss random matrix models.

We next make some comments about how often various Ш values occur. Zagier and Kramarz found that 26.3% of the even twists for $m \leq 70000$ have rank 0 and trivial Ш, while we find the percentage to be 18.8% for $m \leq 10^6$ and 14.1% for $m \leq 10^7$. Indeed, already in [ZK] this percentage was noted to be diminishing. More interesting might

Table 1.1. *Data for cubic twists*

	$r > 0$	$p = 2$	$p = 3$	$p = 5$	$p = 7$
$m \leq 10^5$	22.9	37.3	33.7	3.9	1.2
$m \leq 10^6$	20.5	42.1	40.1	5.9	2.4
$m \leq 10^7$	17.7	45.7	45.3	8.0	3.7
prime $m \leq 10^7$	6.1	53.5	5.8	14.5	8.2
Delaunay		58.3	36.1	20.7	14.5

be how often a given prime divides Ш, under the restriction to rank 0 twists. For instance, 32.4% of the even rank 0 twists with $m \leq 70000$ have 3 dividing Ш. This number increases to 40.1% for $m \leq 10^6$, and is 45.3% for $m \leq 10^7$. The heuristics of [De] imply a number more like 36.1%. There is a strong arithmetic impact from m, as for prime m the percentage for $m \leq 10^7$ is only 5.8%. However, this last datum is anomalous because of the special role that 3 plays in the cubic twists.

Similarly, 2 divides Ш 45.7% of the time for even rank 0 twists with $m \leq 10^7$, while only 42.1% of the time for $m \leq 10^6$ and 35.5% of the time for $m \leq 70000$. Here Delaunay predicts 58.1%. Here prime m are *more* likely to cause 2-divisibility of Ш, with the percentage here for $m \leq 10^7$ being 53.5%. As [ZK] notes, the expectation is that Ш should be of size $m^{1/3} \approx N^{1/6}$ for these cubic twists, larger than the expected $N^{1/12}$ in the general case. For 5-divisibility of Ш, the percentage increases from 3.6% to 5.9% to 8.0%. It seems unlikely that these percentages (for $p \neq 3$) will climb all the way to 100%, and without a better guess, one could posit that they are tending toward the number suggested by the Delaunay heuristic. In the table below, the $r > 0$ column counts percentages of curves for which the central L-value vanishes, while the other four columns denote how often a given prime divides the Ш-value of a nonvanishing twist.

It was pointed out to us by M. Rubinstein that quadratic twist data for Ш tend to the Delaunay number more readily upon including *all* even rank twists, instead of just the ones of rank 0. Indeed, as we except that the high rank twists should form an asymptotically negligible set, there is perhaps no reason not to include them in our data. Furthermore, additionally restricting to prime twists also tends to speed convergence toward the Delaunay heuristic. Upon implementing these two ideas, we get numbers of 56.3% for 2-divisibility, 19.7% for 5-divisibility, and 13.8% for 7-divisibility, which are fairly close to the percentages predicted by

Delaunay. For 3-divisibility we have only 11.6%, as the existence of 3-isogenies for our curves appears to have a definite impact (Rubinstein reports similar phenomena for quadratic twists).

One can do a similar experiment with quartic twists of $X_0(32)$, or even sextic twists of $X_0(27)$. We only looked at the former. For the computation of the sign of the functional equation in these cases, see [ST]. Note that [ZK] consider the **quadratic** twists of $X_0(32)$ given by $y^2 = x^3 - m^2x$ with $m \equiv 1 \pmod{16}$ for $m \leq 500000$, and they find that the percentage of vanishing twists is dropping fairly rapidly, it being 15.2% for $m \leq 50000$ and 10.6% for $m \leq 500000$. For the quartic twists of $X_0(32)$ we are looking at $y^2 = x^3 + mx$ where 4 does not divide m and m is free of fourth powers. Here we consider positive $m \leq 8000000$, of which 24.9% of the even twists have vanishing central value. This is less than the 27.4% for $m \leq 10^6$, and 29.8% for $m \leq 10^5$. Similar percentages occur for the negative m.

1.3 Computational techniques

The computations were carried out on a network of about 10 SPARC machines (mostly SPARC-V) over a 6-month period at the beginning of 2001. Our bound of $m \leq 10^7$ was chosen as we were mainly interested in the question of extra vanishing, and seemed sufficient to answer the question posed by [ZK] on whether the rate remained constant. With today's technology, extending the experiment to $m \leq 10^8$ should be feasible, as should a similar experiment looking at cubic twists with odd functional equation.

As stated in [ZK], the computation of the $a_{m,n}$ takes time $O(\log n)$ if n is prime and $O(1)$ time if not (using the multiplicativity relations, viewing the values for the primes dividing n as taking negligible time as they are already computed). We computed the values of $a_{1,p}$ for $p \leq 10^9$ once-and-for-all ahead of time, and then read these from disk as needed. Additionally, tricks such as fast modular exponentiation were used to speed up the computation of $m^{(p-1)/3} \bmod p$. Similarly, the computing of $e^{-2\pi n/\sqrt{N}}$ was faciliated by the fact that the exponential function is a homomorphism; for a given N, we computed various powers of $e^{-2\pi/\sqrt{N}}$ and then for each n multiplied these together as needed to get the desired value. For the computation of $L(E_m, 1)$, and the question of how far the infinite sum need be computed, we followed a method similar to that of [ZK], calculating the Ш-value $S_m = \frac{T^2}{c\Omega} L(E_m, 1)$ where T is the size of the torsion group, c is the Tamagawa number, and Ω is the real period (see pages 54–56 of [ZK] for these). We then stop the calculation when S_m is sufficiently close to an integer (possibly zero). As a check, we expect all the S_m values to be squares, which indeed does turn out to be the case.

1.4 Random matrix models

In this section we make some comments about random matrix theory and the expected number of even cubic twists of $X_0(27)$ which have vanishing central L-value. We follow the ideas of [CKRS] and [DFK]. In our case of cubic twists, we expect, as do [CKRS], to have symmetry type O^+, that is, orthogonal with positive determinant. This is because the sign of our functional equation is always $+1$. Note that [DFK] have unitary symmetry in their type of cubic twist, due to the fact that the functional equation has an essentially arbitrary complex number (related to a Gauss sum) appearing in it.

We write $E = X_0(27)$ and E_d for the dth cubic twist of E. As given in equations (20), (22), and (16) of [CKRS], the assumption of orthogonal-plus symmetry implies that $P_E(N, x) = c_E N^{3/8}/\sqrt{x}$ should approximate (for small x) the probability density function for values of $L(E_d, 1)$, where $N \sim \log X$ and we integrate $\int_0^X P_E(N, x)\, dx$ to get an expected probability that $L(E_d, 1)$ is less than X. The idea is that we know that the actual values of $L(E_d, 1)$ are discretised (due to the Birch–Swinnerton-Dyer formula), and thus we declare (in a somewhat arbitrary manner) sufficiently small values of $L(E_d, 1)$ to indicate that in fact we have $L(E_d, 1) = 0$. We recall that BSD implies we have

$$\frac{L(E_d, 1)}{\Omega_d} = \prod_{p|3d} c_p \cdot \frac{|\text{Ш}_d|}{|T_d|^2}$$

where Ω_d is the real period of E_d, the c_p are Tamagawa numbers, Ш_d is the Shafarevitch–Tate group, and T_d is the torsion group of E_d. We are thus thinking of $|\text{Ш}_d|$ (which is a square) as our discretised variable, with everything else being computable. When $d > 2$ the torsion group is trivial. For cubefree d we have that $\Omega_d = \Omega_1/d^{1/3}$, except when $9|d$ in which case we have $\Omega_d = 3\Omega_1/d^{1/3}$. Note that in definition (8) of [CKRS], quadratic twists that are not relatively prime to the conductor are excluded; we will similarly exclude twists that are divisible by 3, though one could deal with them via making appropriate corrections. For the Tamagawa product we have that $c_3 = 3$ when $d \equiv \pm 1 \pmod 9$, $c_3 = 2$ when $d \equiv \pm 2 \pmod 9$, and $c_3 = 1$ otherwise, while $c_p = 3$ for primes $p \equiv 1 \pmod 3$ and $c_p = 1$ for primes $p \equiv 2 \pmod 3$. Given this divergent behaviour based upon prime divisibility, as in Conjecture 1 of [CKRS] we decide to restrict to prime twists, and additionally split the primes into congruence classes modulo 9. Indeed, it is calculable that the sign of the functional equation is odd when our prime twist d is congruent to $4, 7, 8 \pmod 9$, and by 3-descent we can verify that the rank is zero when d is 2 or 5 (mod 9). Moreover, again by 3-descent, we know that the rank is at most 2 (and the functional equation is even) when d is 1 mod 9. Computing as with equation (23) in [CKRS] we get the following:

Question 1.4.1 *Let V_T be the set of primes d less than T congruent to 1 modulo 9 with $L(E_d, 1) = 0$. Is there some constant $c \neq 0$ such that*

$$\sum_{d \in V_T} 1 \sim c T^{5/6} (\log T)^{-5/8} \qquad ?$$

Our data give a constant of approximately $c = 1/6$. The argument is similar for quartic twists of $X_0(32)$ or sextic twists of $X_0(27)$, and we can expect asymptotics for prime twists of order $T^{7/8}(\log T)^{-5/8}$ and $T^{11/12}(\log T)^{-5/8}$, and upon restricting to various congruence classes we should get appropriate constants in front of these. Via techniques from prime number theory and considerations from Tamagawa numbers, one should be able to argue as in [CKRS] to get an asymptotic for all cubefree twists.

Finally we derive a version of Conjecture 2 of [CKRS] suitable for cubic, quartic, and sextic twists. For cubic twists, for a given prime $p \equiv 1 \pmod 3$ there are three solutions to $a^2 + 3b^2 = 4p$ with $a \equiv 2 \pmod 3$, and these correspond to the three possibilities for the Frobenius trace a_p. The argument given from (27)-(31) in [CKRS] does not differ (see below), and so we get the following:

Question 1.4.2 *Let $p \geq 5$ be prime, and for $1 \leq q \leq p-1$ let $F_p^q(T)$ be the set of cubefree positive integers $d \equiv q \pmod p$ that are less than T such that $x^3 + y^3 = d$ has even functional equation. Letting $a_{d,p}$ be the pth trace of Frobenius for $x^3 + y^3 = d$ (where d need not be cubefree), do we have*

$$\lim_{T \to \infty} \left(\sum_{d \in F_p^Y(T)} 1 \Big/ \sum_{d \in F_p^Z(T)} 1 \right) = \sqrt{\frac{p + 1 - a_{Y,p}}{p + 1 - a_{Z,p}}} \qquad ?$$

We can also make a similar calculation for quartic and sextic twists.

In Tables 1.2-1.5 below we list vanishing percentages in support of an affirmative answer to the above question; the c-column represents which congruence class is used. For $p=7$ the ratios should be $[\sqrt{3}:\sqrt{9}:\sqrt{12}]$, and for $p=13$ they should be $[\sqrt{9}:\sqrt{12}:\sqrt{21}]$.

We also have some data (see Tables 1.6-1.9) for the vanishing percentages for positive quartic twists of $X_0(32)$. For $p=5$ the ratios should be given by $\bigl[\sqrt{2}:\sqrt{4}:\sqrt{8}:\sqrt{10}\bigr]$; for $p=13$ they should be $\bigl[\sqrt{8}:\sqrt{10}:\sqrt{18}:\sqrt{20}\bigr]$.

The heuristic for Conjecture 2 in [CKRS] is based upon supposed cancellation from a quadratic character, whereas in our cubic twist case the source of cancellation is perhaps not so transparent. Therefore we go through the details. We have that

$$\sum_{d\in F_p^q(T)} L(E_d,1/2)^k = \sum_{d\in F_p^q(T)} \left(\sum_{n=1}^{\infty} \frac{a_{d,n}}{n}\right)^k = \sum_{d\in F_p^q(T)} \sum_{n=1}^{\infty} \frac{b_{d,n}}{n},$$

where $b_n = \sum_{n=n_1\cdots n_k} a_{n_1}\cdots a_{n_k}$ with the sum being over all ways of writing n as a product of k positive factors. If we invert the order of summation in this last expression, the sum over d should typically have much cancellation since the $b_{d,n}$ are essentially randomly distributed. This, however, is not the case for n that are a power of p, as here the value of a_{d,p^r} is fixed since d is fixed modulo p. Thus we should get a main contribution in the above by restricting to values of n that are powers of p (indeed, if we did this argument with no congruence restriction we would expect $n=1$ to give the main term). As in (31) of [CKRS] we thus get that

$$\sum_{d\in F_p^q(T)} L(E_d,1/2)^k \sim \sum_{d\in F_p^q(T)} \sum_{p^r} \frac{b_{d,p^r}}{p^r} = \sum_{d\in F_p^q(T)} \left(\sum_{p^r} \frac{a_{d,p^r}}{p^r}\right)^k =$$
$$= \left(\frac{p}{p+1-a_{d,p}}\right)^k \sum_{d\in F_p^q(T)} 1.$$

We complete our heuristic by first noting that the sets $F_p^q(T)$ have asymptotically equal sizes and then taking $k=-1/2$ as is suggested by the random matrix theory of [CKRS]. Note that a similar heuristic can be given for moments of higher derivatives, but the combinatorics become more difficult due to the presence of logarithms. In this context, the data of Elkies [E2] distinctively show a congruence-class phenomenon for rank 3 quadratic twists of $X_0(32)$.

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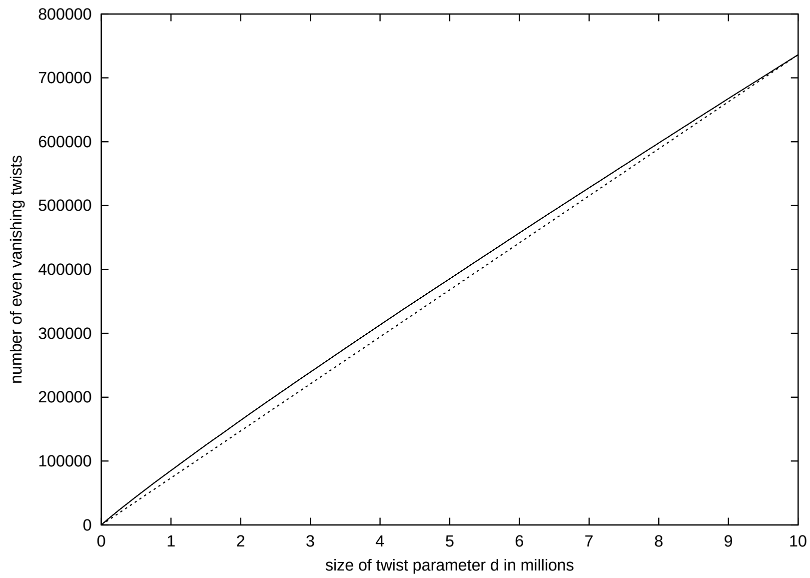

Fig. 1.1. Number of even vanishing cubic twists of $X_0(27)$ compared to a (dotted) straight line.

Table 1.2. $p = 5,\ X_0(27)$

c	$\#r > 0$	curves	
1	140463	838612	0.167
2	140549	838570	0.168
3	140613	838575	0.168
4	140750	838637	0.168

Table 1.3. $p = 7,\ X_0(27)$

c	$\#r > 0$	curves	
1	109569	595982	0.184
2	125728	595952	0.211
3	59440	595912	0.100
4	58759	595903	0.099
5	125714	595963	0.211
6	110125	595937	0.185

Table 1.4. $p = 11$, $X_0(27)$

c	$\#r > 0$	curves	
1	64989	378410	0.172
2	65211	378408	0.172
3	65001	378430	0.172
4	65008	378444	0.172
5	64956	378423	0.172
6	65208	378426	0.172
7	65054	378411	0.172
8	64773	378422	0.171
9	65164	378396	0.172
10	65338	378401	0.173

Table 1.5. $p = 13$, $X_0(27)$

c	$\#r > 0$	curves	
1	44504	320075	0.139
2	52214	320099	0.163
3	51754	320124	0.162
4	67352	320151	0.210
5	43064	320116	0.135
6	68325	320090	0.213
7	68702	320124	0.215
8	43215	320104	0.135
9	67584	320107	0.211
10	51465	320072	0.161
11	51827	320135	0.162
12	44858	320042	0.140

Table 1.6. $p = 5$, $X_0(32)$

c	$\#r > 0$	curves	
1	156097	749089	0.208
2	104136	749107	0.139
3	236861	749125	0.316
4	215944	749182	0.288

Table 1.7. $p = 7$, $X_0(32)$

c	$\#r > 0$	curves	
1	128846	538523	0.239
2	128491	538505	0.239
3	128553	538517	0.239
4	128597	538505	0.239
5	128053	538495	0.238
6	128335	538512	0.238

Table 1.8. $p = 11$, $X_0(32)$

c	$\#r > 0$	curves	
1	82653	341092	0.242
2	82782	341070	0.243
3	82581	341069	0.242
4	82392	341072	0.242
5	82806	341113	0.243
6	82448	341061	0.242
7	82661	341108	0.242
8	82388	341045	0.242
9	82720	341091	0.243
10	82948	341083	0.243

Table 1.9. $p = 13$, $X_0(32)$

c	$\#r > 0$	curves	
1	85079	287669	0.296
2	60843	287670	0.212
3	85408	287673	0.297
4	53551	287693	0.186
5	60788	287689	0.211
6	60926	287684	0.212
7	81716	287656	0.284
8	81500	287704	0.283
9	85480	287661	0.297
10	53852	287654	0.187
11	81525	287683	0.283
12	53688	287668	0.187