Real zeros of real odd Dirichlet L-functions

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Abstract

Let χ be a real, odd Dirichlet character of modulus d, and $L(s,\chi)$ be the associated Dirichlet L-function. As a consequence of the work of Low and Purdy, it is known that if $d \leq 800\,000$ and $d \neq 115\,147$, $357\,819$, $636\,184$, then $L(s,\chi)$ has no positive real zeros. By a simple extension of their ideas, and the advantage of thirty years of advances in computational power, we are able to prove that if $d \leq 300\,000\,000$, then $L(s,\chi)$ has no positive real zeros.

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1. Introduction.

Let χ be a primitive real Dirichlet character. We wish to determine whether the associated L-function $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ for $\Re s > 0$ has any positive real zeros. By classical arguments, we have $L(s,\chi) \neq 0$ for $s \geq 1$, and by the functional equation, we need only consider $1/2 \leq s < 1$. We shall restrict ourselves to odd characters, i.e. those such that $\chi(-1) = -1$, for reasons which will become clear. We remark that the best results for even χ have been obtained by Rosser, who, in unpublished work, showed that for moduli less than 986, $L(s,\chi)$ has no positive real zeros. The sharpest theoretical results have been obtained by Conrey and Soundararajan [2], who show that asymptotically at least 20% of the primitive real odd Dirichlet characters χ with 8 dividing the modulus have $L(s,\chi) > 0$ for $s > 0$.

By well-known arguments, the assumptions that χ is odd, primitive, and has modulus $d > 4$ implies that $\zeta(s)L(s,\chi)$ is the Dedekind zeta-function of the imaginary quadratic field of discriminant $-d$. In particular we have that $\chi(n) = (-d|n)$ where $(-d|n)$ is the Kronecker symbol. For $\Re s > 1$, we also have

$$\zeta(s)L(s,\chi) = \sum_{(a,b,c)\in Q} Z(s;a,b,c)$$

where

$$Q = \{(a,b,c) \in \mathbf{Z}^3 | -d = b^2 - 4ac,\ -a < b \leq a < c \quad \text{or} \quad 0 \leq b \leq a = c\}$$

is the set of reduced integral solutions of $-d = b^2 - 4ac$, and

$$Z(s;a,b,c) = \frac{1}{2} \sum\sum_{(m,n)\neq(0,0)} \frac{1}{(am^2 + bmn + cn^2)^s}.$$

Low [3] found three conditions which, if satisfied for a given modulus d, imply that $L(s,\chi)$ has no positive real zeros. Using an IBM 7094, he verified these conditions for moduli less than 593 000, finding one exception to his first condition. In his review of Low's work, Rieger [5] stated that an error analysis was lacking. To remedy the problem, Purdy [4] extended Low's work to moduli less than 800 000, and performed an accompanying error analysis. Purdy found three exceptions to Low's first condition: 115 147, 357 819, and 636 814. By a slight modification of Low's argument, we can overcome the fact that these three moduli fail his first condition. This is somewhat important, as many references to Low's work sloppily ignore this point. Moreover, due to thirty years of advances in computational power, we can extend these results up to moduli less than 300 000 000. We shall show the following:

Theorem. *Let χ be a real, odd Dirichlet character of modulus $d \leq 300\,000\,000$. Then $L(s,\chi)$ has no positive real zeros.*

2. Low's Three Conditions.

We let $-d$ be a fundamental discriminant with $d > 4$, and Q be the aforementioned set of reduced integral solutions of $-d = b^2 - 4ac$. We define

$$\alpha(a,d) = \log a + \log\bigl(8\pi e^{-\gamma}\bigr) - \frac{1}{2}\log d, \tag{1}$$

where γ is Euler's constant. We define b_n and β_n from the relations (for $|s-1| < 3$)

$$\log\bigl[(s-1)\zeta(s)\bigr] = \sum_{n=1}^{\infty}(-1)^n\beta_n(s-1)^n \qquad \text{and} \qquad b_n = (2^n-1)\frac{\zeta(n)}{n} + 2^n\beta_n.$$

We list some properties of the b_n:

$$b_n < 2^{n+1} \tag{2}$$

$$\begin{gathered} 2.09 < b_2 < 2.10, \quad 2.66 < b_3 < 2.67, \quad 3.99 < b_4 < 4.0, \quad 6.39 < b_5 < 6.4 \\ b_6 < 10.67, \quad 18.28 < b_7 < 18.29, \quad 56.88 < b_9 < 56.89, \quad b_{11} < 186.19 \end{gathered} \tag{3}$$

$$b_{2n+1} > 594 \qquad \text{for} \quad n \geq 6 \tag{4}$$

A proof of (2) appears in Low's work, (3) can be verified directly, and (4) follows from Low's statement that $b_n > \frac{2^n}{n}\bigl(1 - 7.44(2/3)^{n-1}\bigr)$ for $n \geq 6$.

From [1], it is known that the analytic continuation of $Z(s;a,b,c)$ satisfies

$$\frac{1}{\sqrt{k}}\Bigl(\frac{ak}{\pi}\Bigr)^s\Gamma(s)Z(s;a,b,c) = f(s) + f(1-s) + H(s;a,b,c)$$

where $k = \sqrt{d}/2a$ and $f(s) = (k/\pi)^s\Gamma(s)\zeta(2s)/\sqrt{k}$ and $H(s;a,b,c)$ is an error term given by

$$H(s;a,b,c) = 4\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\cos\Bigl(\frac{\pi n b}{a}\Bigr)K_{s-1/2}(2\pi k n)$$

where $\sigma_l(n) = \sum_{m|n} m^l$ and $K_l(z)$ is the standard K-Bessel function given by $K_l(z) = \int_0^\infty e^{-z\cosh t}\cosh lt\,dt$.

The three conditions of Low are

$$\text{(i)} \quad S_0 = \sum_Q \frac{\alpha(a,d)}{\sqrt{a}} \geq 0$$

$$\text{(ii)} \quad U(s) = \sum_Q \frac{\alpha(a,d) - H(s;a,b,c)}{\sqrt{a}} > 0 \qquad \text{for} \quad 1/2 \leq s < 1$$

$$\text{(iii)} \quad S_n = \sum_Q \frac{1}{\sqrt{a}}\left(b_{2n+1} + \frac{\alpha^{2n+1}(a,d)}{(2n+1)!}\right) \geq 0 \qquad \text{for} \quad n \geq 1$$

It is proven in [3] that these imply $L(s,\chi) > 0$ for $0 < s < 1$. Note that (4) implies that (iii) trivially holds for sufficiently large n. In fact, the following lemma allows us to check condition (iii) quite efficiently.

Lemma. *Assume that $d \leq 10^{10}$. Then $S_n \geq 0$ for $n \geq 6$, and $S_1 \geq 0$ and $S_2 \geq 0$ imply that $S_n \geq 0$ for all $n \geq 1$.*

Since $1 \leq a \leq \sqrt{d}$, our assumption that $d \leq 10^{10}$ combined with (1) gives us that $-\sqrt{80} \leq \alpha(a,d) \leq 3$ for all (a,d) pairs. So for $n \geq 6$, we can use (4) to conclude that

$$b_{2n+1} + \frac{\alpha^{2n+1}(a,d)}{(2n+1)!} \geq 594 - \frac{(\sqrt{80})^{13}}{13!} \geq 200.$$

Thus $S_n \geq 0$ for $n \geq 6$, as every term in the sum is positive. For the $n = 3$, $n = 4$, and $n = 5$ terms we can exploit the behavior of certain polynomials. We have that

$$\frac{x^7}{7!} - \frac{8}{3}\frac{x^5}{5!} + b_7 - \frac{8}{3}b_5 \geq 0$$

for $-\sqrt{80} \leq x \leq 3$. This follows from the fact that the critical points of the polynomial are at $x = 0$ and $x = \pm\sqrt{80}$. Evaluating at the critical points and using (3) gives the desired inequality. Letting x be $\alpha(a, d)$, multiplying by $1/\sqrt{a}$ and summing over Q, this implies that $S_3 \geq \frac{8}{3}S_2 \geq 0$. Similarly

$$\frac{x^9}{9!} - \frac{3}{2}\frac{x^7}{7!} + b_9 - \frac{3}{2}b_7 \geq 0$$

for $-\sqrt{80} \leq x \leq 3$, and so $S_4 \geq \frac{3}{2}S_3 \geq 0$. Finally,

$$\frac{x^{11}}{11!} - \frac{x^9}{9!} + b_{11} - b_9 \geq 0$$

for $-\sqrt{80} \leq x \leq 3$, and so $S_5 \geq S_4 \geq 0$. Thus if $S_1 \geq 0$ and $S_2 \geq 0$ and $d \leq 10^{10}$, we see that (iii) holds for all n, proving the Lemma.

In our range of moduli, the failures of Low's method come mostly from (i). A failure of (ii) would be very good evidence against GRH, while failures of (iii) might occur for the same spurious reasons that (i) fails for certain moduli. For moduli $d \leq 300\,000\,000$, we found that (ii) is always true, while (i) fails for 4507 moduli. One failure of (iii) was found; $-0.25 < S_2 < -0.23$ for $d = 290\,661\,067$. In the sequel, we shall have to deal with this modulus separately.

3. Dealing with failures to condition (i) in the range $d \leq 10^{10}$.

We follow the proof of [2, Theorem 2]. For $|r| < 1/2$, we have

$$\sum_Q \frac{1}{\sqrt{a}}[\alpha(a, d) + m(r)]h(r; a, d) \geq S_0 + \sum_{n=1}^{\infty} S_n r^{2n}, \tag{5}$$

where

$$m(r) = \sum_{n=1}^{\infty} b_{2n+1} r^{2n}, \qquad g(r) = r[\alpha(a, d) + m(r)], \qquad \text{and} \quad h(r; a, d) = \frac{\sinh[g(r)]}{g(r)} = \sum_{n=0}^{\infty} \frac{g^{2n}(r)}{(2n+1)!}.$$

We now show how to deal with failures to condition (i), and thus shall assume that $S_0 < 0$. We shall also assume that (ii) and (iii) hold. We want to multiply the left side of (5) by $l(r) = \exp\left(\sum_{n=1}^{\infty} b_{2n} r^{2n}\right) \geq 1$, and if $S_0 + S_1 r^2 \geq 0$, this will yield the valid inequality

$$\sum_Q \frac{l(r)}{\sqrt{a}}[\alpha(a, d) + m(r)]h(r; a, d) \geq S_1 r^2 + S_0 \geq S_0. \tag{6}$$

This is in contrast to Low, who always required $S_0 \geq 0$, which precluded him from being able to handle the cases when $S_0 < 0$. So for

$$|r| \geq \sqrt{-S_0/S_1}, \tag{7}$$

we see that (6) holds. As in Low's proof, (6) and (ii) imply that $L(s, \chi) > 0$ for $s = 0.5 + r$ for r in the range (7). So if $S_0 < 0$, we can conclude that $L(s, \chi) \neq 0$ for $s > 0.5 + \sqrt{-S_0/S_1}$. For our range of d, our computations show that (when $S_0 < 0$) we always have $\sqrt{-S_0/S_1} < 0.05$, and so we need only consider $|r| \leq 0.05$ in the remainder of this section. Because of numerical round-off errors which may occur, we shall actually consider all $S_0 \le 0.02$ instead of just negative S_0. This will not be a big issue; our main goal will be to show (6) for certain ranges of r.

We see that (6) holds trivially at $r = 0$, and we hope that by taking a power series expansion around $r = 0$, we will find that (6) holds in a region which overlaps (7). Since the b_i are nonnegative, to show (6), it suffices to show that

$$\sum_Q \frac{l(r)}{\sqrt{a}} \Big[\alpha(a, d) + b_3 r^2 + b_5 r^4\Big] h(r; a, d) \ge S_0. \tag{8}$$

Our computations show that $|S_0| \le 1/50$ and $S_1 \ge 1$ for all moduli d in our range for which $S_0 \le 1/50$, and so we will assume these inequalities for the remainder of the section. Using $0 \le b_n \le 2^{n+1}$, $|r| \le 1/20$, and (3), we get that

$$\sum_{n \ge 3} b_{2n} r^{2n} \le 12 r^6$$

and

$$1 + b_2 r^2 + (b_4 + b_2^2/2) r^4 \le l(r) \le \exp(b_2 r^2 + b_4 r^4 + 12 r^6) \le 1 + b_2 r^2 + (b_4 + b_2^2/2) r^4 + 23 r^6,$$

which gives $l(r) = 1 + b_2 r^2 + \big(b_4 + b_2^2/2\big) r^4 + 23\theta r^6$ for some $|\theta| \le 1$. We make the abbreviation $\alpha = \alpha(a, d)$, and note that

$$-\sqrt{80} \le \alpha \le \frac{g(r)}{r} = \alpha + \sum_{n=1}^{\infty} b_{2n+1} r^{2n} \le \alpha + \frac{4}{99} \le 3 + \frac{4}{99},$$

which gives that $g^2(r) \le 80 r^2 \le 1/5$, which implies

$$\sum_{n=3}^{\infty} \frac{g^{2n}(r)}{(2n+1)!} \le 80^3 r^6 \sum_{n=3}^{\infty} \frac{1}{5^{n-3}(2n+1)!} \le 102 r^6.$$

By a similar bounding of tails, we can obtain the results that

$$g^2(r) = \big(\alpha r + b_3 r^3 + 7\theta r^5\big)^2 \qquad \text{and} \qquad g^4(r) = \big(\alpha r + 3\theta r^3\big)^4$$

for some $|\theta| \le 1$, which need not be the same at every appearance. Thus we obtain that

$$h(r; a, d) = 1 + \sum_{n=1}^{\infty} \frac{g^{2n}(r)}{(2n+1)!} = 1 + \frac{\alpha^2}{6} r^2 + \Big(\frac{b_3 \alpha}{3} + \frac{\alpha^4}{120}\Big) r^4 + 200\theta r^6.$$

In the last step we have used $|\alpha| \le 9$. Expanding out (8), we get that the left side is

$$\sum_Q \frac{1}{\sqrt{a}} \Big(1 + b_2 r^2 + (b_4 + b_2^2/2) r^4 + 23\theta r^6\Big) \Big[\alpha(a, d) + b_3 r^2 + b_5 r^4\Big] \times$$

$$\times \left(1 + \frac{\alpha^2(a, d)}{6} r^2 + \Big(\frac{b_3 \alpha(a, d)}{3} + \frac{\alpha^4(a, d)}{120}\Big) r^4 + 200\theta r^6\right). \tag{9}$$

We wish to find a range for r such that (9) is greater than S_0. We multiply out (9) in order to find a lower bound for it. There will some obviously positive terms like $b_3 \alpha^2 r^4 / 2\sqrt{a}$ and $b_2 b_5 \alpha^2 r^8 / 6\sqrt{a}$ which we can ignore. Our multiplication will also yield multiples of S_1 and S_2 such as $S_1(b_4 + b_2^2/2) r^6$ and $S_2 b_2 r^6$. Since these are positive, we can ignore them also. We define $T = \sum_Q 1/\sqrt{a}$, and thus obtain a lower bound of

$$S_0 + (S_1 + b_2 S_0) r^2 + \big(S_2 + b_2 S_1 + [b_4 + b_2^2/2] S_0\big) r^4 - 2100 T r^6$$

for (9) by multiplying it out and using the bounds on the b_i and $|\alpha|$. Since $|S_0| \le 1/50$ and $S_1 \ge 1$ for all of our failures of (i), the r^4 term in the above is positive, and so we can ignore it. Thus we find that we have (8) for

$$|r| \le \min\left(0.05,\ \Big(\frac{S_1 + b_2 S_0}{2100 T}\Big)^{1/4}\right). \tag{10}$$

If this overlaps the previous range (7) for r, we can conclude that $L(s, \chi)$ has no positive real zeros. Our programme found that these ranges do overlap for all failures of (i) for $d \le 300\,000\,000$. As an example, the smallest S_0 occurs for $d = 223\,191\,759$, for which $S_0 \approx -0.01638$. We have $S_1 \ge 314$, and $T \le 249$. Hence (7) implies (6) for $r \ge .0073$, while the range in (10) gives us (8) when $|r| \le \min(.05, .156)$.

4. The modulus 290 661 067.

Even though condition (iii) fails for $n = 2$ for this modulus, we can still use the ideas of the previous section to show that there are no positive real zeros of the associated L-function. Computations show that $S_0 \geq 0$, $S_1 \geq 3$, $S_2 \geq -1/4$, $S_3 \geq 250$, and $S_4 \geq 1200$, while the argument of Section 3 shows that $S_n \geq 0$ for $n \geq 5$. By (5) we have

$$\sum_Q \frac{1}{\sqrt{a}}[\alpha(a,d) + m(r)]h(r; a, d) \geq S_0 + S_1 r^2 + S_2 r^4.$$

In order to be able to obtain (6), we need both $S_0 + S_1 r^2 + S_2 r^4 \geq 0$ and $S_1 r^2 + S_2 r^4 \geq 0$. But for $|r| < 1/2$, both of these are true due to the inequalities $S_0 \geq 0$, $S_1 \geq 3$, and $S_2 \geq -1/4$. Thus we have (6), and so only need to verify (ii) to conclude that $L(s, \chi) > 0$ for real $s > 0$ for the quadratic character modulo 290 661 067. Such a verification was done as for the other moduli, as described in Section 5. We record the facts that this modulus is prime, and the class number of the associated imaginary quadratic field is 1101. Also, the lowest height zero of the associated L-function appears to be at approximately $s = 1/2 + 0.0477924709188i$.

5. How the programmes worked.

The first thing to do is generate members of Q, i.e. find solutions to $-d = b^2 - 4ac$ with certain bounds on the size of a, b, and c. The most expedient way to do this is to let b run from 0 to $\sqrt{d/3}$ and then factor $(b^2 + d)/4$ to find possible (a, c) pairs. Of course, factoring can be a time-consuming process. To speed up the factoring step, we create a ginormous array. In the qth position of the array, we store the smallest prime factor of q, storing 0 if q is prime. Such an array can be created very quickly by the use of sieve methods. The array needs to contain numbers up to $d/3$. Of course, we need not consider even numbers in the array, since it is so easy to divide by two. Since the smallest prime divisor of a composite number is less than its square root, and $d/3$ is no more than 10^8 for our range of moduli, we never have to store anything in our array greater than 10000, thus allowing the word size to be 16 bits. This allowed us to keep the memory requirements down, a great concern when dealing with an array with around 10^8 elements.

Having created this array which allows fast factorization, the programme next goes through the moduli d in the range it is given. It checks that d is in one the six congruence classes 3, 4, 7, 8, 11, or 15 modulo 16, and also that 9, 25, and 49 do not divide d. We just ignore the fact that higher squares may divide d, as it would be too time-consuming to check this. For each d, we let b run from 0 to $\sqrt{d/3}$, and find the factors of $(b^2 + d)/4$ by iteratively using our array to find the smallest prime factors. Note that unless $b = a$ or $a = c$ or $b = 0$, a triple (a, b, c) in Q has a distinct "conjugate" in Q, namely $(a, -b, c)$. When we speak of doing operations on members of Q in the sequel, it is understood that we only consider $b \geq 0$ and multiply by the necessary multiplicity. For (a, b, c) triples in Q, we make lower bound estimates for the contributions to S_0, S_1, S_2, and $U(s)$. We describe these more fully in the next paragraphs.

Since the **C** library functions **log**, **exp**, **sqrt**, and **cos** are rather time-consuming to compute, and for the vast majority of d the conditions of Low will hold quite comfortably, it was decided that, as a first measure, crudely bounding these functions would be more efficacious in the long run, with sharper bounds then used only for the d which were cantankerous. The only thing of which we need to take the square root or natural logarithm many times is a, and we know that $a \leq \sqrt{d/3} \leq 10000$. Thus three arrays are created at the beginning of the programme, containing upper and lower bounds for $\sqrt{a}$, and lower bounds for $\log(a)$ for all integers $1 \leq a \leq 10000$. To check condition (ii), we also need to be able to upper bound $H(s; a, b, c)$ easily, and by the results of [1] and [2, Theorem 4] we have

$$H(s; a, b, c) \leq \frac{2\cos(\pi b/a)}{k^{1/2} e^{2\pi k}} + \frac{1}{25}\frac{|2\cos(\pi b/a)|}{k^{1/2} e^{2\pi k}},$$

where $k = \sqrt{d}/2a$. To expedite calculation of the terms involving k, two arrays are created which contain upper and lower bounds for the values of $k^{-1/2} e^{-2\pi k}$ for $0 \leq k \leq 2$ at k-intervals of 1/25. The upper bounds contain a factor of $2 + \frac{1}{25} = 2.08$, while the lower bounds carry a factor of 1.92, as these quantities would appear in the calculations in any case. The programme then simply rounds k either up or down to the nearest demarcation and looks up the appropriate bound in the tables. The cosine is bounded by fourth order Taylor polynomials. If $2b < a$, then upper bounds for $\cos(\pi b/a)$ and the expression involving k are used, while if $2b > a$ the corresponding lower bounds are taken.

For a given d, each member of Q has its contribution to the four sums in question bounded in this manner. In order to calculate $\alpha(a, d)$, the value of $\log(d)$ is computed once with the **C** library function. Similarly, $\sqrt{d}$ is computed once with the **C** library function so that $k = \sqrt{d}/2a$ can be determined. The computations for a given d can be ended early if all four sums are positive and it is ascertained that additional values of b can only add nonnegative values to the four sums. The values of b_3 and b_5 that occur in S_1 and S_2 were pre-computed, and lower bounds for these are used in the programme. Similarly, the value of $\log(8\pi e^{-\gamma})$ was computed once and a lower bound is used in the programme.

Note that the repetitive computations of programme involved with individual members of Q use nothing but the four basic operations, while the values in tables at the beginning can easily be checked against more accurate computations to bound their error. It is therefore a straightforward but tedious task to undertake an error analysis as in [4]. In practice, any d with computed bounds for S_0, S_1, S_2 or $U(s)$ which are less than 0.001 are written to an output file, which is sufficient to compensate for any accumulated round-off error. As mentioned above, for all moduli we considered we had $S_1 \geq 0$, and S_2 was positive for all d save 290 661 067. Note that any modulus d which is not written to the output file satisfies all of Low's conditions, and thus we can conclude that the corresponding L-function has no positive real zeros.

The moduli which remain are then handled by a more robust programme, using a multi-precision package. To put the level of precision in perspective, we remark the smallest lower bound for $U(s)$ in our range of d is 2.073×10^{-6} for the modulus $d = 175\,990\,483$. This secondary programme computes the sum $T = \sum_Q 1/\sqrt{a}$ which appeared in Section 3, and also checks to see if d has a nontrivial odd square factor (recall that previously we had only checked this for 9, 25, and 49; for instance, $d = 13\,932\,787$ will fail the first programme, but is divisible by 121). There were 4507 moduli with $S_0 < 0$ which got passed to this secondary programme, of which 64 were eliminated by having a nontrivial odd square factor. For each of the 4443 remaining moduli, it is relatively easy to verify that the ranges (7) and (10) of Section 3 overlap, but condition (ii) can often be difficult to verify, since the L-function is so small near its central point in many of these cases. As a first measure, we do not make a time-consuming effort to get an impressive upper bound on $H(s; a, b, c)$, but we are less crude than previously. In fact, often the crude bound of the first programme is enough to conclude that $U(s)$ is positive (throughout the desired range), but the secondary programme still verified this anyway. If our lower bound for $U(s)$ is still negative, then a heavy-duty tertiary programme with even sharper bounds and more precision is used (these was necessary for a total of nine moduli).

We describe the estimates used in the generic secondary programme. We write $v = s - 1/2$, and from [3], we have that

$$H(s; a, b, c) \leq 4\cos\Big(\frac{\pi b}{a}\Big)K_v(2\pi k) + 4\big[2^v + 2^{-v}\big]\cos\Big(\frac{2\pi b}{a}\Big)K_v(4\pi k) + \frac{7}{\sqrt{k}e^{6\pi k}}. \tag{11}$$

We need only consider v in the range $0 \leq v < 1/2$, and in this range from [1] we have (for all positive integers n)

$$1 - \frac{1 - 4v^2}{16\pi nk} \leq 2(kn)^{1/2}e^{2\pi kn}K_v(2\pi kn) \leq 1. \tag{12}$$

Thus if $\cos(\pi b/a)$ is positive, then the first term in (11) is no greater than $2\cos(\pi b/a)/\sqrt{k}e^{2\pi k}$, while if $\cos(\pi b/a)$ is negative, the corresponding bound is

$$\frac{2\cos(\pi b/a)}{\sqrt{k}e^{2\pi k}}\left[1 - \frac{1}{16\pi k}\right].$$

Similarly, if $\cos(2\pi b/a)$ is positive, then the second term of (11) is no more than $3\cos(2\pi b/a)/\sqrt{k}e^{4\pi k}$, and if $\cos(2\pi b/a)$ is negative, the bound is

$$\frac{3\cos(2\pi b/a)}{\sqrt{k}e^{4\pi k}}\left[1 - \frac{1}{32\pi k}\right].$$

Plugging these into (11), we get the bounds that are used in the secondary programme; this gave us a positive lower bound on $U(s)$ for all but nine of the moduli, the outliers being 119 299 192, 124 414 899, 157 619 067, 172 672 063, 174 523 771, 175 990 483, 177 351 128, 192 214 136, and 246 616 980.

Equation (12) is the tip of the iceberg; in fact, we have a whole family of inequalities of this type. The relevant one for us comes from [1], and approximates $K_v(2\pi kn)$ more sharply in some cases:

$$1-\frac{1-4v^2}{16\pi nk}+\frac{(1-4v^2)(9-4v^2)}{2!\,(16\pi nk)^2}-\frac{(1-4v^2)(9-4v^2)(25-4v^2)}{3!\,(16\pi nk)^3}$$
$$\le 2(kn)^{1/2}e^{2\pi kn}K_v(2\pi kn)\le 1-\frac{1-4v^2}{16\pi nk}+\frac{(1-4v^2)(9-4v^2)}{2!\,(16\pi nk)^2}. \quad (13)$$

For the nine remaining moduli, we divided up the v-range into ten equal parts, and uses the better of the above bounds (12) and (13) to bound the Bessel functions in each of these ranges. We also note the bounds

$$n^{s-1/2}\sigma_{1-2s}(n)\le n^{1/2}\sigma_{-1}(n) \qquad \text{and} \qquad \sigma_{-1}(n)\le\frac{3}{2}(n-1) \qquad \text{for} \quad n\ge 2,$$

both of which are proven in [1]. These are used in conjunction with the above. For each v-range, $H(s)$ is estimated using these. In theory, if the lower bound were not positive, the v-range could be further divided, or expansions with more terms could be used in place of (13), but for the nine moduli we considered at this stage, the above was sufficient to show condition (ii).

6. Results.

The programmes were run during off-peak hours on a SPARC over a 3-month period in 1997. As indicated above, we were able to show that (ii) always holds for the d in our range, while failures of (i) and (iii) could be handled. In all cases, the programmes were able to show that $L(s,\chi)$ had no positive real zeros, in accordance with the well-believed conjectures.

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