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The powers of logarithm for quadratic twists

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Abstract

We briefly describe how to get the power of logarithm in the asymptotic for the number of vanishings in the family of even quadratic twists of a given elliptic curve. There are four different possibilities, largely dependent on the rational 2-torsion structure of the curve we twist.

1.1 Introduction

Let E be a rational elliptic curve of conductor N and Δ its discriminant, with E_d its dth quadratic twist. The seminal paper [CKRS] modelled the value-distribution of $L(E_d, 1)$ via random matrix theory and applied a discretisation process to the coefficients of an associated modular form of weight 3/2. This led to the conjecture that asymptotically there are $c_E X^{3/4}(\log X)^{3/8-1}$ twists by *prime* $p < X$ with even functional equation and $L(E_p, 1) = 0$, where the 3/8 comes from random matrix theory, and the -1 comes from the prime number theorem.

We wish to determine a similar heuristic for the asymptotic for the number of twists by *all* fundamental discriminants $|d| < X$ such that $L(E_d, s)$ has even functional equation and $L(E_d, 1) = 0$. We find that the power of logarithm that we obtain depends on the growth rate of various local Tamagawa numbers of twists of E. Because of this, it is somewhat unfortunate that isogenous curves need not have the same local Tamagawa numbers. This is most particularly a problem when we have a curve with full rational 2-torsion and it is isogenous to one that only has one rational 2-torsion point; in this case, we should work with the curve with full 2-torsion. This makes the statement of our result a bit messy, but we have:

Heuristic 1.1.1 *Let E be a rational elliptic curve. Then the number of even quadratic twists E_d with $L(E_d, 1) = 0$ and $|d| < X$ is asymptotically $c'_E X^{3/4} (\log X)^{b_E + 3/8}$ where $c'_E > 0$ and*

- *$b_E = 1$ when E (or a curve isogenous to it) has full rational 2-torsion,*
- *$b_E = \sqrt{2}/2$ when E has one rational 2-torsion point (and no curve isogenous to E has full 2-torsion),*
- *$b_E = 1/3$ when E has no rational 2-torsion and Δ is square.*
- *$b_E = \sqrt{2}/2 - 1/3$ when E has no rational 2-torsion and Δ is not square.*

The $3/8$ in the exponent comes from random matrix theory, and so we only need concern ourselves with calculting b_E. Also, we do not consider the constant c'_E, as that would greatly complicate the discussion.

1.2 Discussion

The discretisation for the values of $L(E_d, 1)$ can be re-interpreted as saying that

$$L(E_d, 1) < \Omega(E_d) g(E_d) / T(E_d)^2 \implies L(E_d, 1) = 0$$

where Ω is the real period, g is the product of the Tamagawa factors, T is the order of the torsion subgroup. This comes from the Birch and Swinnerton-Dyer conjecture and the fact that the order of the Shafarevich-Tate group is an integer. From random matrix theory, we expect that there is some constant $c > 0$ such that the probability that $L(E_d, 1) \leq t$ tends to $ct^{1/2} (\log |d|)^{3/8}$ as $t \to 0$. Combining this distribution with the discretisation, we get that (as $|d| \to \infty$)

$$\mathrm{Prob}\big[L(E_d, 1) = 0\big] \sim c\sqrt{\Omega(E_d) g(E_d) / T(E_d)^2} (\log |d|)^{3/8}.$$

This becomes useful upon realising how these quantities vary in twist families. In particular, we have (up to a factor of 2 that we ignore) that $\Omega(E_d) = \Omega(E)/\sqrt{|d|}$ while $T(E_d)$ is constant for $|d|$ sufficiently large. This reduces our problem to a determination of how the Tamagawa product $g(E_d)$ varies; from the above we have that

$$\mathrm{Prob}\big[L(E_d, 1) = 0\big] \approx c' \sqrt{g(E_d)} (\log |d|)^{3/8} / |d|^{1/4},$$

and so the number of twists should be (here the d are fundamental)

$$N(X) \sim \sum_{\substack{|d| < X \\ E_d \text{ even}}} \mathrm{Prob}\big[L(E_d, 1) = 0\big] \approx \sum_{\substack{|d| < X \\ E_d \text{ even}}} c' \sqrt{g(E_d)} (\log |d|)^{3/8} / |d|^{1/4}.$$

and by partial summation we have that

$$N(X) \approx c'' X^{3/4} (\log X)^{3/8} \sum_{\substack{|d|<X \\ E_d \text{ even}}} \sqrt{g(E_d)},$$

We now compute the expected average value of $\sqrt{g(E_d)}$ via an analysis of the splitting behaviour of the cubic polynomial associated to E.

1.3 Tamagawa numbers

For simplicity, we now restrict to twisting by positive fundamental discriminants d with $\gcd(d, N) = 1$ and even sign in the functional equation.[1] We first isolate the contribution to the Tamagawa factor $g(E_d)$ coming from the primes that divide the discriminant of E, and call this $g(E)$. Writing $g_p(E_d)$ for the local Tamagawa number at p for the twist E_d, we have, up to a bounded factor B_d which includes $G(E)$ and other contributions from bad primes, that

$$g(E_d) = B_d \cdot \prod_{p|d} g_p(E_d).$$

We shall ignore B_d for the remainder of the discussion, as consideration of it does not change the power of logarithm. Again possibly ignoring a finite set of bad primes, when we twist by d, for primes $p|d$ the local Tamagawa number $g_p(E_d)$ at p for E_d is either 1, 2, or 4.[2] If we write E in the form $y^2 = f(x)$, these correspond to the cubic f having 0, 1, or 3 roots modulo p (provided that this model for E is good at p).

We assume that we can use the Chebotarev density theorem to determine the frequency of each splitting behaviour of the cubic f. When E has full 2-torsion, the cubic f splits completely over the rationals, so we have $g_p(E_d) = 4$ for all $p|d$. When E has one rational 2-torsion point, the cubic f splits over $\mathbf{Q}$ as a quadratic factor times a linear factor, and the quadratic splits into two linear factors precisely when its discriminant is square mod p; thus asymptotically half the primes $p|d$ have $g_p(E_d) = 2$, and the other half yield $g_p(E_d) = 4$. Finally, when f is irreducible over the rationals, we have two cases, depending upon whether[3]

[1] A rigourous accounting would also separate the d into congruence classes modulo the discriminant (see [D]) but we omit this so as to focus on the main ideas. Indeed, the more pedantic analysis would only modify the constant c'_E and not the power of logarithm in the asymptotic.

[2] We can note that for $p > 2$ we have $g_p(E_d) = g_p(E_{p^\star})$ where $p^\star = p(-1)^{(p-1)/2}$, which essentially eliminates the dependence on d.

[3] The fact that the elliptic curve discriminant Δ and the discriminant of the cubic differ by a factor of 16 does not affect our analysis.

Δ is square: when it is square (such as with x^3-3x+1), asymptotically $1/3$ of the primes have $g_p(E_d)=4$ and the other $2/3$ have $g_p(E_d)=1$; when the discriminant is not square, the local Tamagawa factors are $g_p(E_d)=1,2,4$ in proportions $1/3$, $1/2$, and $1/6$.[4]

1.4 Analytic number theory

The problem of computing the average value of $\sqrt{g(E_d)}$ is now essentially one of analytic number theory; for simplicity,[5] we explain how to compute the average value at positive fundamental discriminants d of the multiplicative function $h(d)=\sqrt{g(E_d)}$, and so wish to compute an asymptotic for

$$F(X)=\sum_{d\le X}\mu^\star(d)^2h(d),$$

where the modified Möbius function $(\mu^\star)^2$ is the characteristic function of (positive) fundamental discriminants (this differs from μ^2 only at the prime 2). We analyse $F(X)$ via the behaviour of the logarithm of the Euler product $\prod_p\bigl(1+h(p)/p^s\bigr)$ as $s\to 1^+$. Explicitly, as $s\to 1^+$ we have that (ignoring the modification at the prime 2)

$$\log\prod_p\left(1+\frac{h(p)}{p^s}\right)\sim\sum_p\frac{h(p)}{p^s}\sim-(t_1+t_2\sqrt{2}+t_4\sqrt{4})\log(s-1),$$

where t_k is the probability that h takes on the value $\sqrt{k}$, and the last step is in analogy with the fact that $\sum_p 1/p^s\sim-\log(s-1)$. Via exponentiation we obtain $\prod_p\bigl(1+h(p)/p^s\bigr)\sim c/(s-1)^A$ for some constant $c\neq 0$, where $A=(t_1+t_2\sqrt{2}+t_4\sqrt{4})>0$. An application of the Tauberian theorem then gives us that $F(X)\sim c'X(\log X)^{A-1}$ for some $c'\neq 0$.

Finally, we conclude by computing the value of A in each of the four cases: $(t_1,t_2,t_4)=(0,0,1)$ and so $A=2$ for the case of full 2-torsion; $(t_1,t_2,t_4)=(0,1/2,1/2)$ and so $A=1+\sqrt{2}/2$ for the case of one rational 2-torsion point; $(t_1,t_2,t_4)=(2/3,0,1/3)$ and so $A=4/3$ when there is no 2-torsion and Δ is square; and $(t_1,t_2,t_4)=(1/3,1/2,1/6)$ and so $A=2/3+\sqrt{2}/2$ when there is no 2-torsion and Δ is non-square.

[4] Our use of the Chebotarev density theorem is not quite legitimate in general. We need to be more careful about our restriction to *even* twists (a condition that is given by congruences modulo N), which can give incompatibility conditions, especially in the case where f is irreducible and has non-square discriminant, as here the splitting condition cannot be given by congruence conditions modulo N.

[5] For computations regarding the restriction to d with $\gcd(d,N)=1$ and even sign, see [D, §6], especially Theorem 6.3 and Theorem 6.8 with $k=-1/2$; essentially Dirichlet characters mod N can be used to isolate the desired congruence classes.

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Bibliography

[CKRS] J. B. Conrey, J. P. Keating, M. O. Rubinstein, N. C. Snaith, *On the frequency of vanishing of quadratic twists of modular L-functions.* In *Number theory for the millennium, I* (Urbana, IL, 2000), edited by M. A. Bennett, B. C. Berndt, N. Boston, H. G. Diamond, A. J. Hildebrand and W. Philipp, A K Peters, Natick, MA (2002), 301–315. Available online at `arxiv.org/math.NT/0012043`

[D] C. Delaunay, *Moments of the Orders of Tate-Shafarevich Groups.* International Journal of Number Theory, **1** (2005), no. 2, 243–264.