Hypergeometric motives over Q and their L-functions

We now give a different description of some of the above ansatz in the case of hypergeometric data defined over $\mathbf{Q}$. We then proceed to give a recipe for the L-functions attached to fibers...

1. Preliminaries

1.1. **Conventions on hypergeometric data.** Particularly for the purposes of L-functions it is useful to make various normalisations on the hypergeometric data. Firstly we assume that (α, β) is balanced of degree $d \geq 1$ and is non-resonant. Next, since the datum is defined over $\mathbf{Q}$, it can be defined by multisets of cyclotomic indices, denoted by $\mathcal{A}$ and $\mathcal{B}$. We can also write $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$ as the associated products of cyclotomic polynomials, with $\mathcal{P}_{\mathcal{A}} = \prod_{e\in\mathcal{A}} \Phi_e$.

Note that taking the fiber of $\mathcal{H}(\mathcal{A},\mathcal{B})$ at t gives the same motive as that for $\mathcal{H}(\mathcal{B},\mathcal{A})$ at $1/t$. Our convention for $(\mathcal{A},\mathcal{B})$ is as follows. If $1 \in \mathcal{A}\cup\mathcal{B}$, then we shall take $1 \in \mathcal{B}$ (this will have an effect on various formulas below). Else we shall take $\mathcal{B}$ so that it contains the largest element in $\mathcal{A}\cup\mathcal{B}$ (this choice is essentially arbitrary).

1.2. **Hodge functions.** Corresponding to the zig-zag diagram described in §**??**, for a given hypergeometric datum (α, β) write

$$\mathcal{D}(x) = \#\{\alpha_i : \alpha_i \in \alpha | \alpha_i \leq x\} - \#\{\beta_j : \beta_j \in \beta | 1 - \beta_j \leq x\},$$

where the collections of α's and β's are multisets. The *effective weight* w of the hypergeometric datum (α, β) is then given by $w+1 = \max \mathcal{D}(x) - \min \mathcal{D}(x)$. Letting $D = -\min \mathcal{D}(x)$, this effective weight is $2D$ more than the previously defined weight, and corresponds to Tate twisting by D. As an exercise, one can show that $w + 1 - 2D$ is equal to the multiplicity b_1 of 1 in $\mathcal{B}$. Also, whenever $1 \notin \mathcal{B}$ we have $D \geq 1$ (by our convention that β has the largest cyclotomic index).

The full Hodge polynomial is more complicated. One way to define it is via

$$H(T) = \sum_{\alpha_i \in \alpha} T^{\mathcal{D}(\alpha_i) - z_{\alpha_i}} \frac{T^{z_{\alpha_i}} - 1}{T - 1},$$

where z_{α_i} is the multiplicity of α_i (the sum itself includes each α_i only once). The degree of the numerator of $H(T)$ is $w + 1$, while the degree of denominator is the Tate-twisting parameter D. The Hodge polynomial $h(T)$ is in fact just the numerator of $H(T)$, that is (conjectured by Corti and Golyshev it seems), we have

$$h(T) = \sum_{p+q=w} h^{p,q} T^p.$$

For instance, for $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_6\Phi_{10}, \Phi_3\Phi_{12})$ we have $\mathcal{D}(1/10) = 0$, $\mathcal{D}(1/6) = 1$, $\mathcal{D}(3/10) = 2$, $\mathcal{D}(7/10) = -1$, $\mathcal{D}(5/6) = 0$, $\mathcal{D}(9/10) = 1$, so that the above Hodge function is $H(T) = (1/T + 1 + T + 1/T^2 + 1/T + 1) = (1 + 2T + 2T^2 + T^3)/T^2$. More complicated is the example $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_2^4\Phi_3^3\Phi_6^2, \Phi_1^2\Phi_4\Phi_5\Phi_9)$ where one gets $(1 + 3T + 3T^2 + 3T^3 + 3T^4 + T^5)/T^2$.

2. The hypergeometric trace

Intrinsic to our description of Euler factors will be the hypergeometric trace formula of Katz. The form of it that we present, specific to our case of hypergeometric data over $\mathbf{Q}$, was derived by Cohen and is more suitable for computation.

Let $q = p^f$ be a prime power, and define ω_p to be the (canonical) p-adic Teichmüller character given by $\omega_p(x) = x \cdot \exp\bigl(\frac{-1}{p-1}\log x^{p-1}\bigr)$ when $p > 2$, and let it be trivial for $p = 2$. Choosing π with $\pi^{p-1} = -p$, the Gauss sum is defined as

$$\mathfrak{g}_q(r) = \sum_{a \in \mathbf{F}_q^\star} \omega_p(a)^{-r} \zeta_\pi^{\mathrm{tr}(a)} \text{ where } \zeta_\pi^p = 1 \text{ with } \zeta_\pi \equiv 1 + \pi \pmod{\pi^2},$$

where there are various conventions with $\pm r$ and also a global minus sign.

Recall the γ_v defined by $\mathcal{P}_\mathcal{A}(T)/\mathcal{P}_\mathcal{B}(T) = \prod_v (T^v - 1)^{\gamma_v}$. The only restriction on them is that $\sum_v v\gamma_v = 0$. We also recall the scaling parameter $M = \prod_v v^{v\gamma_v}$

The canonical (p-adic) Gauss sum quotient for given hypergeometric data is then

$$G_q(r) = \prod_v \mathfrak{g}_q(rv)^{\gamma_v}.$$

Each nontrivial Gauss sum has size $\sqrt{q}$, so by using the Möbius-induced relation $\sum_v \gamma_v = -b_1$ (see §3.1), we find that $G_q(r)$ is generically of size $(1/\sqrt{q})^{b_1}$.

From this, for t with $v_p(Mt) = 0$ we then define the hypergeometric trace as

$$U_q(t) = \frac{1}{1-q}\left(\sum_{r=0}^{q-2} \omega_p(M/t)^r Q_q(r)\right) \tag{1}$$

where $Q_q(r) = (-1)^{m_0} q^{D+m_0-m_r} G_q(r)$ and m_r is the multiplicity of $\frac{-r}{q-1} \in \beta$ (these correspond to trivial characters in the previous language of Katz). We shall give a description of the derivation of this formula from the previous one below, but first make some comments.

- Firstly, the choice of r versus $-r$ reappears in the convention of t versus $1/t$. The two major implementations of L-functions of hypergeometric motives, those by Cohen in GP/PARI and Watkins in Magma, both use the latter convention. Due to this, in our description of L-functions in later sections, for consistency with calculations we shall refer to the parameter $\tilde{t} = 1/t$, and thus the Teichmüller character will be applied to $M\tilde{t}$.
- The Tate-twisting factor D is the degree of denominator of the Hodge function as in §1.2. Since $m_0 = b_1$ is the multiplicity of 1 in β while $\mathfrak{g}_q(0) = -1$, we find $G_q(0) = (-1)^{b_1} = (-1)^{m_0}$ and $Q_q(0) = q^D$.
- This formula gives a p-adic number, while we know that $U_p(t)$ is (after scaling) in fact an integer. By computing to sufficiently high p-adic precision (depending on the weight), we can recognise it as an integer.
- We can also give a brief heuristic as to why $U_q(t)$ might be expected to have size $q^{w/2}$ (which follows rigorously since the motive is pure, implying the Frobenius eigenvalues have this size). Indeed, generically we have $m_r = 0$ and the size of $G_q(r)$ is $(1/\sqrt{q})^{b_1}$, so that the typical summand in $U_q(t)$ has size $(\sqrt{q})^{2(D+b_1)-b_1}$. Upon using $w + 1 = b_1 + 2D$ this becomes $q^{(w+1)/2}$, and thus if the r-sum exhibits square-root cancellation, we expect $U_q(t)$ to have maximal size $(1/q) \cdot \sqrt{q} \cdot q^{(w+1)/2} = q^{w/2}$.
- Note that the above trace formula can be used whenever $v_p(M\tilde{t}) = 0$, even if p is wild (that is, it divides one of the cyclotomic indices in $\mathcal{A} \cup \mathcal{B}$). However, in some wild cases the Euler factor that is obtained from the trace formula is not that of the L-series, due to the inertia not being trivial. Also, the Katz rendition of the trace formula requires instead that $v_p(t) = 0$.

2.1. **Equivalence with previous form of trace formula.** We now explain how the above trace formula for $U_q(t)$ is equivalent to that given above for $H(\alpha,\beta|t)$ (except for the scaling by q^D). The first part is simply unwinding the definitions of p-adic Pochhammer symbols and the like, keeping track of the distinction between $\{\}_0$ and $\{\}_\infty$. Then, following some notes of Henri Cohen, we use the Gross-Koblitz formula to relate the products of p-adic Γ-functions over residue systems to products of Gauss sums.

The Katz/FRV trace formula says

$$H(\alpha,\beta|t) = \frac{1}{1-q}\sum_{\rho\in\frac{1}{q-1}\mathbf{Z}/\mathbf{Z}} (-p)^{\eta_f(\rho)}\frac{\prod_i(\alpha_i)_{\infty,\rho}}{\prod_i(\beta_i)_{0,\rho}}\omega_p(t)^{\rho(q-1)}.$$

where

$$(x)_{\nu,\rho} = \frac{\Gamma_{\nu,\rho}(x-\rho)}{\Gamma_{\nu,\rho}(x)} \quad\text{with}\quad \Gamma_{\nu,\rho}(x) = \prod_{i=0}^{f-1}\Gamma_p\big(\{p^i x\}_v\big).$$

Now swap ρ to $-\rho$ (inducing $t\to 1/t$), and put $\rho = r/(q-1)$ for $0\le r\le q-2$. Unraveling the p-adic Γ-functions we see that $H(\alpha,\beta|t)$ is

$$\frac{1}{1-q}\sum_{r=0}^{q-2}\omega(1/t)^r(-p)^{\eta_f(-r/(q-1))}\prod_j\prod_{i=0}^{f-1}\frac{\Gamma_p(\{p^i(\alpha_j+r/(q-1))\}_\infty)}{\Gamma_p(\{p^i(\beta_j+r/(q-1))\}_0)}\frac{\Gamma_p(\{p^i\beta_j\}_0)}{\Gamma_p(\{p^i\alpha_j\}_\infty)}.$$

Write

$$X_q(r) = \prod_{i=0}^{f-1}\frac{\prod_j\Gamma_p(\{p^i(\alpha_j+r/(q-1))\}_\infty)}{\prod_j\Gamma_p(\{p^i(\beta_j+r/(q-1))\}_\infty)},$$

and convert $\{\}_0$ into $\{\}_\infty = \{\}$. Since $\Gamma_p(0)=+1$ and $\Gamma_p(1)=-1$, this change gives a sign contribution. Writing $l(x)$ for the multiplicity of $-x$ in β, the above is

$$H(\alpha,\beta|t) = \frac{1}{1-q}\sum_{r=0}^{q-2}\omega(1/t)^r(-p)^{\eta_f(-r/(q-1))}\frac{(-1)^{l(r/(q-1))}X_q(r)}{(-1)^{l(0)}X_q(0)}.$$

Now we similarly unravel the exponent of $(-p)$. Recall that

$$\eta_f(x) = \sum_{i=0}^{f-1}\eta^{(i)}(p^i x) \quad\text{where}\quad \eta^{(i)} \text{ is the Landau function for } (p^i\alpha, p^i\beta).$$

Namely, the Landau function for (α,β) is

$$\eta(x) = \sum_i\big[(\alpha_i-x)_\infty-(\alpha_i)_\infty\big]-\big[(\beta_i-x)_0-(\beta_i)_0\big]$$

and so

$$\eta^{(i)}(p^i x) = \sum_j\big[\{p^i\alpha_j-p^i x\}_\infty-\{p^i\alpha_j\}_\infty\big]-\big[\{p^i\beta_j-p^i x\}_0-\{p^i\beta_j\}_0\big]$$

We write

$$S_f(x) = \sum_{i=0}^{f-1}\{p^i x\}_\infty \quad\text{and}\quad T_f(r) = \sum_j\big[S_f(\alpha_j+r/(q-1))-S_f(\beta_j+r/(q-1))\big]$$

so that

$$\eta_f\big(-r/(q-1)\big) = T_f(r)-T_f(0)-f\cdot\big[l\big(r/(q-1)\big)-l(0)\big],$$

the last term coming from the difference between $\{\}_0$ and $\{\}_\infty$.

Upon substituting for η_f and combining we then get

$$H(\alpha,\beta|t) = \frac{1}{1-q}\sum_{r=0}^{q-2}\omega(1/t)^r \frac{q^{m_0}}{q^{m_r}}\frac{X_q(r)}{X_q(0)}\frac{(-p)^{T_f(r)}}{(-p)^{T_f(0)}}, \tag{2}$$

as $p^f = q$ and the signs in $(-p)$ cancel out, and we wrote $m_r = l(r/(q-1))$.

Since our hypergeometric datum is over $\mathbf{Q}$, both $X_q(r)$ and the exponent $T_f(r)$ involve products or sums over reduced (coprime) residue systems modulo various $m \in \mathcal{A}\cup\mathcal{B}$. However, expanding products of p-adic Γ-function in a distributional manner is done over *complete* residue systems. Möbius inversion allows us to pass from one to the other, and also makes using the γ_v more natural than $\mathcal{A}$ and $\mathcal{B}$.

The transformations we sketch below appear in notes by Cohen. After making a Möbius inversion, we then use the Gross-Koblitz formula to replace the products of p-adic Γ-functions by Gauss sums, achieving (for some constant C)

$$\begin{aligned} X_q(r)(-p)^{T_f(r)} &= C\cdot\frac{\prod_{a\in\mathcal{A}}\prod_{d|a}(\omega_p(d)^{dr}\mathfrak{g}_q(dr))^{\mu(a/d)}}{\prod_{b\in\mathcal{B}}\prod_{e|b}(\omega_p(e)^{er}\mathfrak{g}_q(er))^{\mu(b/e)}} \\ &= C\cdot\omega_p(M)^r\cdot\prod_v \mathfrak{g}_q(rv)^{\gamma_v}. \end{aligned} \tag{3}$$

The second expression follows from the first by the definitions of M and γ_v.

Since we divide $X_q(r)(-p)^{T_f(r)}$ by $X_q(0)(-p)^{T_f(0)}$ in (2), the constant C cancels out. Furthermore, $\mathfrak{g}_q(0) = -1$ implies $X_q(0)(-p)^{T_f(0)} = C\cdot\prod_v(-1)^{\gamma_v} = C\cdot(-1)^{b_1}$. Upon accounting for the Tate-twisting factor q^D (ensuring the traces are integral) this then gives the trace formula (1) given above.

2.1.1. We now sketch the method of Cohen to show (3). We preliminarily record that for $p\nmid N$ we that for $x\in\mathbf{Q}$ we have

$$\sum_{i=0}^{f-1}\sum_{z=0}^{N-1}\{p^i(x+z/N)\} = \sum_{i=0}^{f-1}\{p^iNx\} + (N-1)\frac{f}{2}.$$

To show this, firstly note it is true for $f=1$ as both sides are invariant under the transformation $x\to x+1/N$ so that we can assume $0\le x<1/N$ when the result $\sum_{z\,(N)}\{x+z/N\} = \{Nx\}+(N-1)/2$ is immediate. For $f>1$, since $p\nmid N$ the map $z\to p^iz$ permutes $\mathbf{Z}/N\mathbf{Z}$, and we apply the $f=1$ case with x replaced by p^ix.

For integers $0\le r<q-1$, corresponding to the $T_f(r)$ we write

$$E_q(N,r) = (p-1)\sum_{z=0}^{N-1}\sum_{i=0}^{f-1}\left\{p^i\left(\frac{z}{N}+\frac{r}{q-1}\right)\right\} = s_q(Nr) + (p-1)(N-1)\frac{f}{2},$$

where

$$s_q(u) = (p-1)\sum_{i=0}^{f-1}\left\{p^i\frac{u}{q-1}\right\}$$

is the sum-of-digits function. We also define

$$B_q(N,r) = \pi^{E_q(N,r)}\prod_{z=0}^{N-1}\prod_{i=0}^{f-1}\Gamma_p\left(\left\{\frac{p^iz}{N}+\frac{p^ir}{q-1}\right\}\right)$$

and note that by Möbius inversion and the definitions of $X_q(r)$ and $T_f(r)$ we have

$$X_q(r)(-p)^{T_f(r)} = \frac{\prod_{a\in\mathcal{A}}\prod_{d|a} B_q(a,r)^{\mu(a/d)}}{\prod_{b\in\mathcal{B}}\prod_{e|b} B_q(b,r)^{\mu(b/e)}}.$$

To evaluate $B_q(N,r)$ we first recall the p-adic Γ-function distribution formula (see Theorem 11.6.14 of Cohen [**?**]), that for $x \in \mathbf{Q}$ we have

$$\prod_{z=0}^{N-1} \Gamma_p(x+z/N) = c_{p,N}\frac{\Gamma_p(Nx)}{N^{w_p(Nx-1)}},$$

where $c_{p,N}$ is a constant that can be given explicitly. Here $w_p(u) = [u - u\backslash p]$ with $s\backslash p$ defined as quotient without remainder (see Definition 11.6.1 of [**?**]) and $u^{[s-s\backslash p]}$ is well-defined as $\lim_{m\to s} u^{m-m\backslash p}$ (see Proposition 11.6.2 of [**?**]). Moreover, a version of this with fractional parts, namely

$$\prod_{z=0}^{N-1} \Gamma_p(\{x+z/N\}) = c_{p,N}\frac{\Gamma_p(\{Nx\})}{N^{w_p(\{Nx\}-1)}},$$

can be seen to be true, again as both sides are invariant under the transformation $x \to x + 1/N$, and when $0 < x < 1/N$ the distribution formula applies.

Then we apply the above with $x = p^i r/(q-1)$ and get

$$\prod_{z=0}^{N-1}\prod_{i=0}^{f-1} \Gamma_p\left(\left\{\frac{p^i z}{N} + \frac{p^i r}{q-1}\right\}\right) = c_{p,N}^f \omega_p(N)^{s_q(Nr)} \prod_{i=0}^{f-1} \Gamma_p\left(\left\{\frac{p^i N r}{q-1}\right\}\right),$$

Here the simplification of the $N^{\bullet}$-term to $\omega_p(N)^{s_q(Nr)}$ is obtained by a (tedious) calculation similar to Corollary 11.6.3 of Cohen; for brevity we omit the details.

Finally we note $s_q(Nr) \equiv Nr \pmod{p-1}$, so $\omega_p^{p-1} \equiv 1$ implies $\omega_p^{s_q(Nr)} = \omega_p^{Nr}$, while the Gross-Koblitz formula [**?**] says

$$\mathfrak{g}_q(r) = -\pi^{s_q(r)} \prod_{i=0}^{f-1} \Gamma_p\left(\left\{p^i \frac{r}{q-1}\right\}\right).$$

As the $(p-1)(N-1)\frac{f}{2}$ term in $E_q(N,r)$ is independent of r, its effect can be moved to the constant – similarly the minus sign in the Gross-Koblitz formula. Thus we obtain (3), with M arising as $\prod_{a\in\mathcal{A}}\prod_{d|a} d^{d\mu(a/d)} / \prod_{b\in\mathcal{B}}\prod_{e|b} e^{e\mu(b/e)} = \prod_v v^{v\gamma_v}$.

2.1.2. *Caveat.* Some modifications to the above calculations might need to made when $p = 2$. However, the final result remains the same. Note that $p = 2$ will never be "good" for the L-function, but we still may want to apply the trace formula.

2.1.3. *Modifications.* When $\mathcal{A}$ and $\mathcal{B}$ are not disjoint, most of the above still holds, with the $q^{D+m_0-m_r}$ values in (1) coming from the $\mathcal{B}$ of interest. However, the $(-1)^{m_0}$ in $Q_q(r)$ was actually derived as $\prod_\nu(-1)^{\gamma_\nu}$, and $\sum_\nu \gamma_\nu = -b_1$ need no longer hold. Thus this m_0 should correspond to the core datum, obtained by cancelling common cyclotomic indices in $\mathcal{A}$ and $\mathcal{B}$. Note that the resulting Euler factors for a resonant datum will have higher degree, and so it is typically superior to compute with the core datum and then post-multiply by a computable factor.

2.1.4. *Computing the hypergeometric trace.* To calculate the hypergeometric trace using (1), one first pre-computes $\mathfrak{g}_q(r)$ for all $0 \le r < q-1$. Here Cohen uses a method hinted at by Boyarsky [**?**], which (perhaps curiously) re-interprets the Gauss sums in terms of p-adic Γ-functions evaluated at multiples of $1/(q-1)$, and pre-computes $\Gamma_p\left(\frac{a}{q-1}\right)$ for all a with lowered amortised cost by an inductive formula.[1] This takes time $O(fq)$, with similar space requirements. Each term in $U_q(t)$ then takes essentially constant time (depending on the number of nonzero γ_v) to calculate, for a total time cost proportional to q. As noted previously, to recover $U_q(t) \in \mathbf{Z}$ one computes to sufficiently high p-adic precision, with the computational complexity linear in this precision. Of course, one can gain various constant factors by use of symmetry and duplication relations for the p-adic Γ-function.

[1] The FRV/Katz formula involves evaluating Γ_p at *shifts* of multiples of $1/(q-1)$.

Canonical schemes of hypergeometric motives over Q

In this section we describe a canonical scheme associated to a hypergeometric motive over $\mathbf{Q}$. At a good prime p, the hypergeometric trace should (somehow) be related to the number of points over $\mathbf{F}_p$.

3. Preliminaries

3.1. **Combinatorial transformations (and exercises).** Let $G^{\pm}$ be the multisets where v appears to multiplicity $|\gamma_v|$ where γ_v has the same sign as the $\pm$ symbol. For example with $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_2^2\Phi_6, \Phi_1^2\Phi_4)$ we have $\gamma = (-3, 2, -1, -1, 0, 1)$, so that $G^+ = \{2, 2, 6\}$ and $G^- = \{1, 1, 1, 3, 4\}$. Note that each member $e \in \mathcal{A} \cup \mathcal{B}$ contributes a count of $2^{\omega(e)}$ to the γ_v, distributed as a Möbius sum across the divisors. There can of course be cancellation across various e for the multisets $G^{\pm}$, but we see that $e = 1$ is the only case that gives an odd number of contributions, and similarly is the only case which does not give equidistributed contributions between $G^{\pm}$.

Letting b_1 be the multiplicity of $1 \in \mathcal{B}$, our conventions imply $|G^-| = |G^+| + b_1$.

3.2. **Relating Gauss sums to Jacobi sums.** Recall the Gauss sum defined in the previous section. Related to this is the p-adic Jacobi sum. For a vector $\vec{v} = (v_i)$ we define

$$J_q(\vec{v}) = \sum_{\substack{\vec{a} \in \mathbf{F}_q^\star \\ \sum_i a_i = 1}} \omega(a_i)^{v_i} \text{ with the relation } J_q(\vec{v}) = \frac{\prod_i \mathfrak{g}_q(v_i)}{\mathfrak{g}_q\left(\sum_i v_i\right)},$$

the latter holding when none of the v_i nor their sum is divisible by $(q-1)$.

We thus generically have

$$G_q(r) = \frac{\prod_i \mathfrak{g}_q(r g_i^+)}{\prod_j \mathfrak{g}_q(r g_j^-)} = \frac{J_q(r \cdot \vec{g}^+)}{J_q(r \cdot \vec{g}^-)},$$

where $\vec{g}^+ = (g_i^+)$ and $\vec{g}^- = (g_j^-)$ run over the multisets $G^{\pm}$ respectively.

4. The canonical associated scheme

Considering character orthogonality in the hypergeometric trace sum vis-â-vis the above Jacobi sum quotient, the canonical associated scheme is then defined (in the variables X_j, Y_i) by the equations $\sum_j X_j = \sum_i Y_i = 1$ and

$$\prod_{j=1}^{\#G^-} X_j^{g_j^-} = \frac{1}{M\tilde{t}} \prod_{i=1}^{\#G^+} Y_i^{g_i^+} \quad \text{where} \quad M = \prod_v v^{v\gamma_v} = \prod_{a \in \mathcal{A}} \prod_{d|a} d^{d\mu(a/d)} \Big/ \prod_{b \in \mathcal{B}} \prod_{e|b} e^{e\mu(b/e)}$$

and $\tilde{t}$ is the hypergeometric parameter. Note that this scheme (which is global) need not be in a very simplified form.

Let $r = \gcd(G^+ \cup G^-)$. The above canonical associated scheme splits into r reducible components over the algebra defined by $x^r - 1/M\tilde{t}$. When $r = 1$ the hypergeometric data is **primitive**.

4.1. The Belyi case. When $\sum_v |\gamma_v| = 3$ we are in the Belyi case, and can relate a 0-dimensional scheme to the hypergeometric data as $x^a(1-x)^b - 1/M\tilde{t}$, where $G^+ = \{a+b\}$ and $G^- = \{a,b\}$. The Jacobi sum here will be denoted by the shorthand $J(a,b)$, with it understood that one multiplies each argument by r when computing the sum over r in the hypergeometric trace. The cases here have weight 0, and will be described a bit more in §5.1.

4.2. Cases where a canonical curve occurs. There are three (or four) basic constructions to try to obtain a plane curve from the hypergeometric data. The first two occur when $\sum_v |\gamma_v| = 4$. When $|G^+| = |G^-| = 2$ we have a relation $a+b = c+d$ with $\{a,b\} = G^-$ and $\{c,d\} = G^+$. This then gives the quotient $J(a,b)/J(c,d)$, with the associated (possibly reducible) curve as $x^a(1-x)^b = y^c(1-y)^d/M\tilde{t}$. When we have $|G^+| = 1$, then $a = b+c+d$ with $\{a\} = G^+$ and $\{b,c,d\} = G^-$, so that the Jacobi sum is $J(b,c,d)$ and the curve is $x^b y^c (1-x-y)^d = 1/M\tilde{t}$.

The other construction(s) can occur when $\sum_v |\gamma_v| = 6$. Letting e be the largest index for which γ_e is nonzero, we need to have some a,b with $a+b=e$ with a,b in the opposite set of $G^\pm$ that e is in. When this occurs, the remaining three elements will similarly form a summation, say $c+d=f$. Thus we have both $J(a,b)$ and $J(c,d)$, and these form a product when $e,f \in G^+$, and else form a quotient. The product yields $x^a(1-x)^b y^c(1-y)^d = 1/M\tilde{t}$, and the quotient is as above.

5. Identifying weight 0 motives

The cases of weight 0 should correspond to Artin motives. In the Belyi case exhibiting the number field is rather easy. We also give a more difficult example (Feb 2013 Trieste), while David Roberts has done much more complicated examples.

5.1. Artin representations for Belyi motives. There is one case of weight 0 in degree 1, two twist families each in degree 2 and 3, and 7 twist families in degree 4. The cases of degree 1 and 2 have been computed by Cohen, though I prefer to handle one of his cases via twisting. Both of the degree 3 cases are also known, and 4 of the degree 4 cases. Some higher degree cases are also listed.

In all cases, the Artin representation is that for the algebra $x^a(1-x)^b - 1/M\tilde{t}$ minus that for a subalgebra $x^c - 1/M\tilde{t}$ where $c = \gcd(a,b)$. Here c is the imprimitivity index. The associated Jacobi sum can be seen to be $J(a,b)$ as we have $G^+ = \{a+b\}$ and $G^- = \{a,b\}$. A tabulation is in Table 1. In general, in degree d one should be able to handle any $a,b \le d$ with $a+b-c = d$ and $c|d$. In particular, here both $(a,b) = (d,d)$ and $(a,b) = (d,1)$ occur.

Note that in degree 1, the discriminant of $x(1-x) - 1/4\tilde{t}$ is $(\tilde{t}-1)/\tilde{t}$, giving the quadratic extension $\mathbf{Q}\big(\sqrt{\tilde{t}(\tilde{t}-1)}\big)$ as expected.

The twisting notation is interpreted as follows. For instance, to get the representation for $\mathcal{P}_\mathcal{A} = \Phi_6$ and $\mathcal{P}_\mathcal{B} = \Phi_1\Phi_2$ one sees $3t \cdot R_t$ in the twist column of the second line, which means to take the Artin representation R_t for the given t, and then twist it by the Kronecker character given by $3t$. In odd degree one has $R_{1/t}$ rather than R_t, and here one can note that α and β are (according to our convention) switched upon twisting.

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	M	algebra ($u = 1/M\tilde{t}$)	subalg	twist
Φ_2	Φ_1	2^2	$x(1-x)-u$	$\mathbf{Q}$	self
Φ_3	$\Phi_1\Phi_2$	$3^3/2^2$	$x^2(1-x)-u$	$\mathbf{Q}$	$3\tilde{t}\cdot R_{\tilde{t}}$
Φ_4	$\Phi_1\Phi_2$	2^4	$x^2(1-x)^2-u$	x^2-u	self
$\Phi_2\Phi_6$	$\Phi_1\Phi_3$	2^6	$x^3(1-x)^3-u$	x^3-u	self
$\Phi_2\Phi_4$	$\Phi_1\Phi_3$	$2^8/3^3$	$x^3(1-x)-u$	$\mathbf{Q}$	$-3\tilde{t}\cdot R_{1/\tilde{t}}$
Φ_5	$\Phi_1\Phi_2\Phi_3$	$5^5/2^23^3$	$x^3(1-x)^2-u$	$\mathbf{Q}$	$15\tilde{t}\cdot R_{\tilde{t}}$
Φ_5	$\Phi_1\Phi_2\Phi_4$	$5^5/2^8$	$x^4(1-x)-u$	$\mathbf{Q}$	$5\tilde{t}\cdot R_{\tilde{t}}$
$\Phi_3\Phi_6$	$\Phi_1\Phi_2\Phi_4$	$3^6/2^4$	$x^4(1-x)^2-u$	x^2-u	self
Φ_8	$\Phi_1\Phi_2\Phi_4$	2^8	$x^4(1-x)^4-u$	x^4-u	self
$\Phi_2\Phi_{10}$	$\Phi_1\Phi_5$	2^{10}	$x^5(1-x)^5-u$	x^5-u	self
$\Phi_2\Phi_3\Phi_6$	$\Phi_1\Phi_5$	$2^63^6/5^5$	$x^5(1-x)-u$	$\mathbf{Q}$	$-5\tilde{t}\cdot R_{1/\tilde{t}}$
$\Phi_4\Phi_{12}$	$\Phi_1\Phi_2\Phi_3\Phi_6$	2^{12}	$x^6(1-x)^6-u$	x^6-u	self
Φ_9	$\Phi_1\Phi_2\Phi_3\Phi_6$	$3^9/2^6$	$x^6(1-x)^3-u$	x^3-u	$3\tilde{t}\cdot R_{\tilde{t}}$
$\Phi_4\Phi_8$	$\Phi_1\Phi_2\Phi_3\Phi_6$	$2^{16}/3^6$	$x^6(1-x)^2-u$	x^2-u	self
Φ_7	$\Phi_1\Phi_2\Phi_3\Phi_6$	$7^7/2^63^6$	$x^6(1-x)-u$	$\mathbf{Q}$	$7\tilde{t}\cdot R_{\tilde{t}}$
Φ_7	$\Phi_1\Phi_2\Phi_5$	$7^7/2^25^5$	$x^5(1-x)^2-u$	$\mathbf{Q}$	$35\tilde{t}\cdot R_{\tilde{t}}$
Φ_7	$\Phi_1\Phi_2\Phi_3\Phi_4$	$7^7/2^83^3$	$x^4(1-x)^3-u$	$\mathbf{Q}$	$21\tilde{t}\cdot R_{\tilde{t}}$
$\Phi_2\Phi_{14}$	$\Phi_1\Phi_7$	2^{14}	$x^7(1-x)^7-u$	x^7-u	self
$\Phi_2\Phi_4\Phi_8$	$\Phi_1\Phi_7$	$2^{24}/7^7$	$x^7(1-x)-u$	$\mathbf{Q}$	$-7\tilde{t}\cdot R_{1/\tilde{t}}$
$\Phi_2\Phi_4\Phi_8$	$\Phi_1\Phi_3\Phi_5$	$2^{24}/3^35^5$	$x^5(1-x)^3-u$	$\mathbf{Q}$	$-15\tilde{t}\cdot R_{1/\tilde{t}}$
Φ_{16}	$\Phi_1\Phi_2\Phi_4\Phi_8$	2^{16}	$x^8(1-x)^8-u$	x^8-u	self
$\Phi_3\Phi_6\Phi_{12}$	$\Phi_1\Phi_2\Phi_4\Phi_8$	$3^{12}/2^8$	$x^8(1-x)^4-u$	x^4-u	self
$\Phi_5\Phi_{10}$	$\Phi_1\Phi_2\Phi_4\Phi_8$	$5^{10}/2^{16}$	$x^8(1-x)^2-u$	x^2-u	self
$\Phi_3\Phi_9$	$\Phi_1\Phi_2\Phi_4\Phi_8$	$3^{18}/2^{24}$	$x^8(1-x)-u$	$\mathbf{Q}$	$\tilde{t}\cdot R_{\tilde{t}}$
$\Phi_3\Phi_9$	$\Phi_1\Phi_2\Phi_7$	$3^{18}/2^27^7$	$x^7(1-x)^2-u$	$\mathbf{Q}$	$7\tilde{t}\cdot R_{\tilde{t}}$
$\Phi_5\Phi_{10}$	$\Phi_1\Phi_2\Phi_3\Phi_4\Phi_6$	$5^{10}/2^43^6$	$x^6(1-x)^4-u$	x^2-u	self
$\Phi_3\Phi_9$	$\Phi_1\Phi_2\Phi_4\Phi_5$	$3^{18}/2^85^5$	$x^5(1-x)^4-u$	$\mathbf{Q}$	$5\tilde{t}\cdot R_{\tilde{t}}$

TABLE 1. Some Belyi cases of weight 0 Artin representations

5.2. **A non-Belyi example.** We consider the weight 0 datum $\big(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}\big) = \big(\Phi_8,\Phi_1\Phi_2\Phi_3\big)$. We can compute a field of definition via considering the power series

$$f(z) = \sum_{n=0}^{\infty}\biggl(\prod_{k=1}^{\infty}(kn)!^{\gamma_k}\biggr)z^n,$$

and then linear algebra on (finite approximations to) $z^l f(z)^m$ yield a relation. In our case, the γ-list is $[1,-2,-3,-4,8]$, and we need only consider even m.

We obtain

$$\sum_{m=0}^{16} q_m(z)f(z)^{2m} = 0,$$

where the q_m are polynomials of degree not more than 12.

Via specialising z in $q_m(z)$, we then get a degree 32 field K (or algebra in general), which turns out to have $C_2 \wr S_4$ as its (generic) Galois group, and in fact the Galois closure is generated by the unique octic subfield of K. Interpolating the obtained fields suitably, we find that for $z = u/v$ the octic can be taken as

$$X^8 - v(2^{10}u + v)X^6 - 2^6uv^2(2^{12}u - 3^3v)X^4 + \\ + 2^{12}3^2u^2v^3(2^{14}u - 3^3v)X^2 + 2^{18}u^3v^3(2^{14}u - 3^3v)^2.$$

This then has a degree 4 Artin representation whose Euler factors for small p match that for the hypergeometric motive with parameter $\tilde{t} = 1/Mz$.The discriminant of this polynomial is a constant times $v^{21}u^{19}(2^{14}u - 27v)^8(2^{28}u^2 - 2^{15}11uv + 125v^2)^4$, but it seems that the latter factor disappears when computing the ring of integers.

Ideally the degree 4 representation from the hypergeometric data would already be the quotient of the octic Dedekind ζ-function by that for the quartic subfield. However, it turns out to need to be twisted by χ_z, that is, the quartic has 3 octic extensions inside the Galois closure, and we need a different one.

We thus took a compositum with $\mathbf{Q}(\sqrt{z})$ and computed the octic subfields, and interpolating four z-values was sufficient to yield the following form for the octic:

$$X^8 - 3v(2^{12}u + v)X^6 + v^2(2^{22}15u^2 + 2^{12}5uv + 3v^2)X^4 -$$

$$-v^3(2^{32}37u^3 + 2^{18}117u^2v + 2^8 21uv^2 + v^3)X^2 + 2^6(2^{10}u + 3v)^2(2^{14}u - 3v)^2uv^3.$$

This gives the desired Artin representation as the quotient of the octic by the obvious quartic subfield.

Note that the quartic field can be given by

$$X^4 + 8uvX^2 - uv^2X + 16u^2v^2, \text{ or } X^3(1 - X) - 64z.$$

In fact, the latter being in Belyi form, the octics are just given by

$$X^6(1 - X^2) - 64z, \text{ and } z^3X^6(1 - zX^2) - 64z,$$

the second (which is also $X^6(1 - zX^2) - 64/z^2$) giving the desired Artin representation vis-a-vis the quartic subfield. Indeed, we should probably be able to compute the octic as some sort of 3-point cover (which is how simplifications were made).

5.3. **Unidentified cases.** Two twist families of degree 4 and weight 0 are unidentified. They are $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_{12}, \Phi_1\Phi_2\Phi_e)$ for $e = 3, 4, 6$. For $e = 3$ attempting the analysis as in the previous subsection leads to a difficult but solvable linear algebra problem with $m = 48$ and degrees of $q(x)$ up to 44. However, the fields one obtains upon z-specialisation are too difficult to really work with.

In degree 5 there are two twist families still unidentified, namely $\mathcal{P}_A = \Phi_2\Phi_8$ and $\mathcal{P}_A = \Phi_2\Phi_4\Phi_6$, both with $\mathcal{P}_B = \Phi_1\Phi_5$. There are 7 unknown twist families in degree 6. Most likely we are running up against the fact that these are really *motives*, and not just varieties (number fields being the 0-dimensional case), so that a piece of the cohomology is desired, not the whole.

In terms of the Beukers-Heckmann classification, the $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_8, \Phi_1\Phi_2\Phi_3)$ case corresponds to a $4 = 3{+}1$ splitting in their Theorem 5.8, while the $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_{12}, \Phi_1\Phi_2\Phi_4)$ case is the $4 = 2{+}2$ imprimitive splitting, and $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_{12}, \Phi_1\Phi_2\Phi_3)$ is #37 in their table. The two twist families in degree 5 corresponds to $5 = 2 + 3$ and $5 = 1 + 4$ in their Theorem 5.8, while the degree 6 examples include both $6 = 5 + 1$ and $6 = 4 + 2$ and examples #45-49.

6. Curves for weight 1

Similarly, weight 1 looks to be given by curves, specifically elliptic curves (over $\mathbf{Q}$) in degree 2, while genus 2 curves and elliptic curves over a quadratic field (or algebra for special $\tilde{t}$) arise in degree 4. Many have been catalogued, with Cohen handling degree 2. One can reduce the problem in some cases via twisting (adding 1/2 to all the α and β). In all the examples I have done, one gets a sufficiently short Jacobi sums to reduce directly to a dimension 0 or 1 variety, which then can hopefully be identified.

6.1. **Elliptic curves in degree 2.**

Again Cohen has handled all 10 degree 2 cases (and again I prefer to simplify slightly via twisting).

To exemplify, consider $\big(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}\big) = \big(\Phi_3, \Phi_4\big)$. The Jacobi quotient is given by $J(4,1)/J(3,2)$, which yields the curve $x^4(1-x) = y^3(1-y)^2/M\tilde{t}$ with $M = 27/64$. Writing $u = 1/M\tilde{t}$, a model of a-invariants is $[0,0,-u,-u,0]$, or $[0,0,0,-12/\tilde{t},16/\tilde{t}^2]$.

This case is additionally interesting for it gives a sense as to what to expect when considering a degree 4 example with $\Phi_3\Phi_4$ as one of the components. The above elliptic curve has j-invariant $1728\frac{\tilde{t}}{\tilde{t}-1}$, and as noted (in essence) by FRV, by taking $\tilde{t} = t_0p^3$, the second model above reduces to $[0,0,0,-12/t_0p^3,16/t_0^2p^6]$, which can be scaled to $[0,0,0,-12p/t_0,16/t_0^2]$, so that the reduction modulo p is $y^2 = x^3 + 16/t_0^2$. Similarly, when $\tilde{t} = t_0/p^4$, one gets $[0,0,0,-12p^4/t_0,16p^8/t_0^2]$ which scales to $[0,0,0,-12/t_0,16p^2/t_0^2]$, and so modulo p is $y^2 = x^3 - (12/t_0)x$.

6.1.1. *Genus 2 quotient.*

The case $\big(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}\big) = \big(\Phi_3, \Phi_6\big)$ does not behave quite the same. This has a Jacobi quotient $J(1,1)/J(3,3)$ to give $x(1-x) = y^3(1-y)^3/Mt$ with $M = 1/16$. This is a genus 2 curve with V_4 as its automorphism group. One of the nonhyperelliptic involutions gives a quotient with j-invariant 0, while the other gives the elliptic curve we want.

$\mathcal{P}_A$	$\mathcal{P}_B$	Jacobi	M	curve ($u = 1/M\tilde{t}$)	twist
Φ_2^2	Φ_1^2	$J(1,1)^2$	2^4	$[1,u,u,0,0]$	self
Φ_3	Φ_6	$J(1,1)/J(3,3)$	$1/2^4$	$[-3u,4u+u^2/4]$ (quo)	self
Φ_6	Φ_1^2	$J(3,2,1)$	2^43^3	$[1,0,0,0,-u]$	$-3\tilde{t}\cdot E_{1/\tilde{t}}$
Φ_4	Φ_1^2	$J(2,1,1)$	2^6	$[1,0,0,u,0]$	$-\tilde{t}\cdot E_{1/\tilde{t}}$
Φ_3	Φ_1^2	$J(1,1,1)$	3^3	$[1,0,u,0,0]$	$-3\tilde{t}\cdot E_{1/\tilde{t}}$
Φ_3	Φ_4	$J(4,1)/J(3,2)$	$3^3/2^6$	$[0,0,-u,-u,0]$	$3\tilde{t}\cdot E_{1/\tilde{t}}$

Table 2. Degree 2 weight 1 elliptic curves

Note that $(0,0)$ is a 4-torsion point for the first case, while it is a 2-torsion point for the fourth case, a 3-torsion point for the fifth case, and has infinite order in the last case.

The model in the second case might be improved (and see the next section). The "guess" of $[0,0,u,0,u]$ (filling a_3 and a_6) is not correct, though presumably this is the other quotient (with j-invariant 0).

7. Imprimitivity: elliptic curves in higher degree

In the degree 2 case, we have 10 elliptic curves coming from the $\binom{5}{2}$ pairs of $\{\Phi_1^2, \Phi_2^2, \Phi_3, \Phi_4, \Phi_6\}$. In degree $2m$ we will have 10 elliptic curves defined over a degree m algebra coming from pairs from

$$\left\{\prod_{d|m}\Phi^2_{m/d},\ \prod_{\substack{d|m\\(d,2)=1}}\Phi^2_{2m/d},\ \prod_{\substack{d|m\\(d,3)=1}}\Phi_{3m/d},\ \prod_{\substack{d|m\\(d,4)=1}}\Phi_{4m/d},\ \prod_{\substack{d|m\\(d,6)=1}}\Phi_{6m/d}\right\}.$$

Call these $P_i(m)$ for $i = 1, 2, 3, 4, 6$. These correspond to the a-invariants. The twist classes depend on the parity of m. The (P_1, P_2) and (P_3, P_6) pairings always give self-twists, and the other 8 are also self-twists when m is even. When m is odd the twist can be computed by swapping the P_1 with P_2 and P_3 with P_6.

The results are listed in Table 3. The case with a_3 and a_6 has a quotient to get the given model (listed in short form). The case with a_1 and a_2 involves a_3 (maybe due to the squaring?), as else the curve would be singular. The case of a_2 with a_6 does not have a direct Jacobi decomposition (the notation $J_3(u)$ means $J(u, u, u)$ here), but we still can guess the corresponding elliptic curve by analogy (maybe there is a fibration?). Another model is $y^2 = x^3 - \frac{s}{4}x^2 + \frac{s}{2}x - \frac{s}{4} = x^3 - \frac{s}{4}(x-1)^2$.

$\mathcal{P}_A$	$\mathcal{P}_B$	Jacobi	M	$(s^m = 1/M\tilde{t})$
$P_6(m)$	$P_1(m)$	$J(3m, 2m, m)$	$2^{4m}3^{3m}$	$[1, 0, 0, 0, -s]$
$P_4(m)$	$P_1(m)$	$J(2m, m, m)$	2^{6m}	$[1, 0, 0, s, 0]$
$P_3(m)$	$P_1(m)$	$J(m, m, m)$	3^{3m}	$[1, 0, s, 0, 0]$
$P_2(m)$	$P_1(m)$	$J(m, m)^2$	2^{4m}	$[1, s, s, 0, 0]$
$P_4(m)$	$P_6(m)$	$J(6m, m)/J(4m, 3m)$	$2^{2m}/3^{3m}$	$[0, 0, 0, -s, s]$
$P_3(m)$	$P_6(m)$	$J(m, m)/J(3m, 3m)$	$1/2^{4m}$	$[-3s, 4s + s^2/4]$ quo
$P_2(m)$	$P_6(m)$	$J_3(m)/J_3(2m)$	$1/3^{3m}$	$[0, -s, 0, 8s, -16s]$
$P_3(m)$	$P_4(m)$	$J(4m, m)/J(3m, 2m)$	$3^{3m}/2^{6m}$	$[0, 0, -s, -s, 0]$
$P_2(m)$	$P_4(m)$	$J(m, m)/J(2m, 2m)$	$1/2^{2m}$	$[0, -s, 0, s, 0]$
$P_2(m)$	$P_3(m)$	$J(3m, m)/J(2m, 2m)$	$2^{4m}/3^{3m}$	$[0, -s, s, 0, 0]$

Table 3. Elliptic curve cases (over a degree m algebra) in degree $2m$

8. Genus 2 hyperelliptic curves in degree 4

Next we list the known cases of genus 2 curves in degree 4. First we give the 18 cases that come directly from Jacobi sums, and then those that come from quotients of higher genus curves.

To exemplify, take $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_6, \Phi_1^2\Phi_4)$. The Jacobi sum is given by $J(4, 1, 1)$, which yields $x^4y(1-x-y) = 1/M\tilde{t}$ with $M = 3^6/2^2$. Writing $u = 1/M\tilde{t}$, a model is $[1, 2, 1, 0, 0, 0, -4u]$, that is, the curve $y^2 = x^6 + 2x^5 + x^4 - 4/M\tilde{t}$.

In Table 4 we give the 18 cases. Perhaps some of the models could be improved to be more suggestive. Another item to note, for say the eighth case $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_4, \Phi_1^2\Phi_2^2)$, one can take $\tilde{t} = t_0p^{12}$ and study the Euler factor at p, where one should get $(1 + aT + pT^2)(1 + bT + pT^2)$ for some a, b.

8.1. Imprimitivity: a genus 2 curve over a quadratic algebra.

Consider the case of $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3^2\Phi_6^2, \Phi_4^2\Phi_8)$. The Jacobi quotient is $J(8, 4)/J(6, 6)$, and since $\gcd(8, 4, 6, 6) = 2$, we might expect a splitting over the quadratic algebra given by $x^2 - 1/M\tilde{t}$. Indeed, the canonical scheme splits as two curves of genus 2, and we obtain a hyperelliptic model $[1, 0, 0, 2s, -4s, 0, s^2]$ where $s = 1/\sqrt{M\tilde{t}}$ (with $M = 3^{12}/2^{20}$). This corresponds to the case of $A = \Phi_3^2$ and $\Phi_2^2\Phi_4$ in degree 4 (see the penultimate entry in Table 4).

$\mathcal{P}_A$	$\mathcal{P}_B$	Jacobi	M	curve ($u = 1/M\tilde{t}$)	twist
$\Phi_3\Phi_6$	$\Phi_1^2\Phi_4$	$J(4,1,1)$	$3^6/2^2$	$[1,2,1,0,0,0,-4u]$	$-\tilde{t}\cdot C_{1/\tilde{t}}$
Φ_5	$\Phi_1^2\Phi_4$	$J(4,1)J(1,1)$	$5^5/2^6$	$[1,2,1,0,0,-4u,-4u]$	$-5\tilde{t}\cdot C_{1/\tilde{t}}$
Φ_8	$\Phi_1^2\Phi_3$	$J(4,3,1)$	$2^{16}/3^3$	$[4,1,0,0,4u,0]$	$-3\tilde{t}\cdot C_{1/\tilde{t}}$
Φ_5	$\Phi_1^2\Phi_3$	$J(3,1,1)$	$5^5/3^3$	$[1,2,1,0,0,4u,0]$	$-15\tilde{t}\cdot C_{1/\tilde{t}}$
Φ_4^2	$\Phi_1^2\Phi_3$	$J(3,1)J(2,2)$	$2^{12}/3^3$	$[4,1,8u,0,4u^2,0]$	$-3\tilde{t}\cdot C_{\tilde{t}}$
$\Phi_2^2\Phi_4$	$\Phi_1^2\Phi_3$	$J(3,1)J(1,1)$	$2^{10}/3^3$	$[1,2,1,0,4u,4u,0]$	$3\tilde{t}\cdot C_{1/\tilde{t}}$
Φ_5	$\Phi_1^2\Phi_2^2$	$J(2,2,1)$	$5^5/2^4$	$[4,1,0,-2u,0,u^2]$	$5\tilde{t}\cdot C_{\tilde{t}}$
$\Phi_3\Phi_4$	$\Phi_1^2\Phi_2^2$	$J(2,2)J(2,1)$	$2^2 3^3$	$[4,1,-4u,-2u,0,u^2]$	$3\tilde{t}\cdot C_{\tilde{t}}$
Φ_5	Φ_8	$J(8,1)/J(5,4)$	$5^5/2^{16}$	$[1,4,0,0,0,-4u,0]$	$5\tilde{t}\cdot C_{1/\tilde{t}}$
Φ_5	$\Phi_3\Phi_6$	$J(6,1)/J(5,2)$	$5^5/2^4 3^6$	$[1,0,0,0,0,-4u,4u]$	$5\tilde{t}\cdot C_{1/\tilde{t}}$
Φ_5	$\Phi_2^2\Phi_6$	$J(6,2)/J(5,3)$	$5^5/2^8 3^3$	$[4,0,0,-4u,0,u^2]$	$-15\tilde{t}\cdot C_{\tilde{t}}$
Φ_3^2	$\Phi_2^2\Phi_6$	$J(2,1)/J(3,3)$	$3^3/2^8$	$[1,4,0,-2u,-4u,0,u^2]$	$-3\tilde{t}\cdot C_{\tilde{t}}$
Φ_4^2	Φ_5	$J(2,2)/J(4,1)$	$2^{12}/5^5$	$[4,0,-8u,0,4u^2,u^2]$	$5\tilde{t}\cdot C_{\tilde{t}}$
$\Phi_3\Phi_4$	Φ_5	$J(5,2)/J(4,3)$	$2^6 3^3/5^5$	$[4,0,-4u,0,0,u^2]$	$15\tilde{t}\cdot C_{\tilde{t}}$
$\Phi_2^2\Phi_4$	Φ_5	$J(5,1)/J(4,2)$	$2^{10}/5^5$	$[1,0,0,0,-4u,4u,0]$	$-5\tilde{t}\cdot C_{1/\tilde{t}}$
Φ_3^2	Φ_5	$J(5,1)/J(3,3)$	$3^6/5^5$	$[1,4,0,-2u,0,0,u^2]$	$5\tilde{t}\cdot C_{\tilde{t}}$
Φ_3^2	$\Phi_2^2\Phi_4$	$J(4,2)/J(3,3)$	$3^6/2^{10}$	$[1,0,0,2u,-4u,0,u^2]$	$-\tilde{t}\cdot C_{\tilde{t}}$
Φ_3^2	$\Phi_1^2\Phi_2^2$	$J(2,1)^2$	$3^6/2^4$	$[1,2,1,2u,-2u,0,u^2]$	$\tilde{t}\cdot C_{\tilde{t}}$

TABLE 4. Data for degree 4 weight 1 genus 2 curves

8.2. **Genus 2 hyperelliptic curves from quotients.** Here we list the cases of genus 2 curves that come from a quotient of the curve defined by the Jacobi sums.

$\mathcal{P}_A$	$\mathcal{P}_B$	Jacobi	M	curve ($u = 1/M\tilde{t}$)	twist
Φ_8	$\Phi_1^2\Phi_4$	$J(4,4)J(1,1)$	2^{10}	$[4,1,0,0,-16u,-4u]$	$-\tilde{t}\cdot C_{1/\tilde{t}}$
$\Phi_4\Phi_6$	$\Phi_1^2\Phi_3$	$J(3,3)J(2,2)$	2^{10}	$[4/u,0,-16,1,0,-4u]$	$-\tilde{t}\cdot C_{\tilde{t}}$
$\Phi_2^2\Phi_6$	$\Phi_1^2\Phi_3$	$J(3,3)J(1,1)$	2^8	$[4,1,0,16u,4u,0]$	self
Φ_5	Φ_{10}	$J(r,r)/J(5r,5r)$	$1/2^8$	$[1,2,0,0,0,u/8,u/4]$	self

TABLE 5. Data for degree 4 weight 1 genus 3 and 4 quotients

8.2.1. *First genus 3 case.* Here we have $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_8, \Phi_1^2\Phi_4)$. The Jacobi product $J(4,4)J(1,1)$ gives the genus 3 curve $x^4(1-x)^4y(1-y) = 1/M\tilde{t}$ with $M = 2^{10}$. Writing $u = 1/M\tilde{t}$, a hyperelliptic model is $[1,4,6,4,1,0,0,0,-4u]$. One of the nonhyperelliptic involutions gives a quotient of genus 2 (the other one gives an elliptic curve of j-invariant 1728), and this quotient has a model given by $[4,1,0,0,-16u,-4u]$. Twisting gives $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^2\Phi_4, \Phi_8)$, corresponding to $C_{1/\tilde{t}}$ twisted by $-\tilde{t}$.

8.2.2. *Second genus 3 case.* Here we have $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_4\Phi_6, \Phi_1^2\Phi_3)$. The Jacobi product $J(3,3)J(2,2)$ gives the genus 3 curve defined by $x^3(1-x)^3y^2(1-y)^2 = 1/M\tilde{t}$ with $M = 2^{10}$. Writing $u = 1/M\tilde{t}$, a model is $[4u^4, 0, 12u^3, u^2, 12u^2, 0, 4u, 0]$. One of the nonhyperelliptic involutions gives a quotient of genus 2 (the other one gives an elliptic curve of j-invariant 0), and this quotient has a hyperelliptic model given by $[4/u, 0, -16, 1, 0, -4u]$. Twisting gives $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_4, \Phi_2^2\Phi_6)$, corresponding to $C_{\tilde{t}}$ twisted by $-\tilde{t}$.

8.2.3. *Third genus 3 case.* Here we have $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^2\Phi_6, \Phi_1^2\Phi_3)$. The Jacobi product $J(3,3)J(1,1)$ gives the genus 3 curve $x^3(1-x)^3y(1-y) = 1/M\tilde{t}$ with $M = 2^8$. Writing $u = 1/M\tilde{t}$, a hyperelliptic model is $[1,4,6,4,1,0,4u,4u,0]$. One of the nonhyperelliptic involutions gives a quotient of genus 2 (the other one gives an elliptic curve of j-invariant 0), and this quotient has a hyperelliptic model given by $[4,1,0,16u,4u,0]$. This case is a self-twist.

8.2.4. *Genus 4 case.* Here we have $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_5, \Phi_{10})$. The Jacobi quotient $J(1,1)/J(5,5)$ gives the genus 4 curve $x(1-x) = y^5(1-y)^5/M\tilde{t}$ with $M = 1/2^8$. Writing $u = 1/M\tilde{t}$, a hyperelliptic model is $[1/4,0,0,0,0,u,5u,10u,10u,5u,u]$. Quotienting by either of the nonhyperelliptic involutions gives a genus 2 curve with a hyperelliptic model of $[1,2,0,0,0,u/8,u/4]$. This case is also a self-twist.

8.3. **Unresolved cases where a curve is known.** There are a few cases with a known curve, but no known coincidence with the hypergeometric traces as of yet.

8.3.1. *Genus 3 case 4.* Here we have $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_4, \Phi_8)$. The Jacobi quotient $J(2,1)/J(4,4)$ gives the genus 3 curve $x^2(1-x) = y^4(1-y)^4/M\tilde{t}$ with $M = 3^3/2^{10}$. However, this curve is not hyperelliptic. The automorphism group has a nontrivial element (note the symmetry in $y - 1/2$), and the quotient by this gives an elliptic curve with j-invariant 1728.

8.3.2. *Genus 3 case 5.* For $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^2\Phi_6, \Phi_{12})$ the quotient $J(3,1)/J(6,6)$ gives the genus 3 curve $x^3(1-x) = y^6(1-y)^6/M\tilde{t}$ with $M = 1/2^4 3^3$. The curve is not hyperelliptic, and there is a nontrivial element in the automorphism group.

8.4. **Unknown degree 4 weight 1 cases.** The above gives $10+2\cdot18+6+2\cdot2 = 56$ cases. This leaves 18, which fall in 9 twist classes. These are: Φ_{12} with any of $\{\Phi_1^2\Phi_4, \Phi_1^2\Phi_6, \Phi_5, \Phi_3\Phi_4, \Phi_3^2\}$; also Φ_8 with Φ_3^2 or $\Phi_2^2\Phi_3$; and Φ_5 with $\Phi_4\Phi_6$ or $\Phi_2^2\Phi_3$.

8.5. **Higher degree hyperelliptic curves.** A primitive weight 1 datum in degree $2g$ should be associated to a curve of genus g. Of course, once $g > 2$ we no longer expect such a curve to be necessarily hyperelliptic. However, there are a number of special families, and these tend involve a large percentage of the cases in small degree.

For instance in genus g we can take $G^+ = \{2, 2g+1\}$ and $G^- = \{1, 2g+2\}$, where the Jacobi quotient $J(2g+2,1)/J(2g+1,2)$ yields a projective model of $x^{2g+2}(z-x) = y^{2g+1}(z-y)^2/M\tilde{t}$, and one can complete the square in z to bring it into a hyperelliptic model. Similarly $G^+ = \{2g+2, 1, 1\}$ with $G^- = \{g+1, g+1, 2\}$ for $J(g+1,g+1)/J(1,1)$ yields $y(1-y) = x^{g+1}(1-x)^{g+1}/M\tilde{t}$ of genus g (where one can complete the square for y to see it as hyperelliptic). As the degree is $2\lceil g/2\rceil$, we should have a quotient of genus $\lceil g/2\rceil$.

g	#	hyp	tw
3	80	46	35
4	11	3	3
5	10	5	3
6	2	1	2
all	103	55	43

Table 6. Degree 6 statistics

g	#	hyp	tw
4	122	62	47
5	21	7	5
6	10	0	0
7	14	5	3
8	2	1	1
all	169	75	56

Table 7. Degree 8 statistics

8.5.1. *Degree 6 statistics.* There are 277 primitive degree 6 weight 1 hypergeometric selections, in 140 twist families. I compute that 103 of the selections give an irreducible curve via a Jacobi construction. In the end, there are 35 twist families of genus 3 hyperelliptic curves.

None of the genus 3 curves (hyperelliptic or otherwise) is a self-twist, the same holds for genus 4, but one of the genus 5 and both of the genus 6 curves are self-twists. In fact, none of the nonhyperelliptic genus 3 or 4 curves has a twist that yields a curve via a Jacobi construction (one pair of nonhyperelliptic genus 5 curves has this property).

8.5.2. *Degree 8 statistics.* There are 927 primitive degree 8 weight 1 hypergeometric selections, in 466 twist families. I compute that 169 of the selections give an irreducible curve via a Jacobi construction. In the end, there are 47 twist families of genus 4 hyperelliptic curves.

None of the genus 4 curves (hyperelliptic or otherwise) is a self-twist, similarly for genus 5 and 6, but one of the genus 7 and both of the genus 8 curves are self-twists.

9. Jacobi motives

Jacobi motives were perhaps at one time an attempt to understand tame primes of hypergeometric motives, but by now have a life of their own.

9.1. The formalism (following Anderson) is given in Schappacher in terms of the relationship to Grössencharacters, and sundry results appear elsewhere in the literature. With $n_j \in \mathbf{Z}$ and $x_j \in \mathbf{Q}/\mathbf{Z}$, let $\theta = \sum_j n_j \langle x_j \rangle$ be an element of the free group on $\mathbf{Q}/\mathbf{Z}$ with $\sum_j n_j x_j \in \mathbf{Z}$.

9.1.1. *Field of definition.* Letting m be the least common multiple of the denominators of the x_j, the natural field of definition K_θ is a subfield of $\mathbf{Q}(\zeta_m)$, corresponding by class field theory to quotienting out $(\mathbf{Z}/m\mathbf{Z})^\star$ by any elements which leave θ fixed when scaling by them. When scaling by -1 fixes θ the field K_θ is totally real, and otherwise it is a CM field. We write $u \circ \theta = \sum_j n_j \langle u x_j \rangle$ for the scaling of θ by u.

9.1.2. *Weight and Hodge structure.* The weight of the motive is defined as $\sum_j n_j$, with the local Hodge weight associated to a coprime residue class u mod m given by $\sum_j n_j \{u x_j\}$. In the former case the individual weights at each u must all be the same and the action of complex conjugation on $h^{p,p}$ is $(p, p, (-1)^p)$ at each infinite place, while in the latter case the Hodge structure can be obtained by pairing u and $-u$, with $(p, p, +)$ and $(p, p, -)$ occurring equally when applicable.

9.1.3. *Good Euler factors.* The Euler factor for primes $p \nmid m$ can be given as follows. Determine the smallest positive f with $m|(p^f - 1)$, and consider the splitting by scaled orbit sets as

$$\Big\{ \{ap^i \circ \theta\}_{i=0}^{f-1} : 1 \leq a \leq m-1, \gcd(a, m) = 1 \Big\}.$$

For each orbit, sum a representative as $\sum_i ap^i \circ \theta = \sum_i \sum_j n_j \langle ap^i x_j \rangle$ and compute

$$R(a) = (-p)^v \prod_{i=0}^{f-1} \prod_j \Gamma_p\big(\{ap^i x_j\}\big)^{n_j} \quad \text{where} \quad v = \sum_{i=0}^{f-1} \sum_j n_j \{ap^i x_j\},$$

with each Γ_p-evaluation being invertible in $\mathbf{Z}_p$. The Euler factor is then given by the product $\prod_a (1 - R(a) T^e)^{-1}$, where e is the degree of K_θ divided by the number of a-orbits. As usual, one computes $R(a)$ by p-adic methods to sufficient precision to be able to recognise its $\mathbf{Z}_p$-invertible part as a unique integer in a Weil interval (scaled by the known valuation). In particular, in the case of one orbit when e equals $\deg(K)$, then $R(a)$ will correspond to a trivial product of Gauss sums.

9.1.4. *Conductors.* The conductor of Jacobi sum motives is known to differing levels of explicitness in various cases, and Weil gave the general upper bound that it divides m^2 in $\mathcal{O}_K$.

9.1.5. *Root numbers.* (empty section)

9.2. Small examples. There are various special cases of Jacobi sum motives, particularly when small numbers occur, and we recall a few of these. The first case is $2\langle 1/2\rangle$, when the weight is 2, the Hodge type is $(1,1,-)$, the field of definition is the totally real subfield of $\mathbf{Q}(\zeta_4)$ (namely $\mathbf{Q}$), and computation gives us that we have a Tate twist of the Kronecker character χ_{-4}. Similarly $\langle 1/3\rangle + \langle 2/3\rangle$ gives a Tate twist of χ_{-3}, and $\langle 1/4\rangle + \langle 3/4\rangle$ gives a Tate twist of χ_{-8}, while $\langle 1/5\rangle + \langle 4/5\rangle$ gives a Tate twist of the nontrivial Hecke character of modulus $\mathfrak{p}_5\infty_1\infty_2$ for $\mathbf{Q}(\sqrt{5})$.

Schappacher notes $2\langle 2/3\rangle - \langle 1/3\rangle$ corresponds to a canonical Grössencharacter of modulus $\mathfrak{p}_3^2$ over $\mathbf{Q}(\zeta_3)$ which is also the same as an elliptic curve of conductor 27, and similarly $\langle 1/2\rangle + \langle 3/4\rangle - \langle 1/4\rangle$ for the Grössencharacter of modulus $\mathfrak{p}_2^3$ over $\mathbf{Q}(\zeta_4)$ or elliptic curve of conductor 32. With $\langle 5/8\rangle + \langle 7/8\rangle - \langle 1/2\rangle$ there is stability upon scaling by 3, and the field is thus $\mathbf{Q}(\sqrt{-2})$, with the Grössencharacter having modulus $\mathfrak{p}_2^5$ corresponding to an elliptic curve with conductor 256.

9.2.1. *Relations.* Note that there can be relations between various of these Jacobi motives. In particular, scaling by an invertible element modulo m does not change the motive (at least over $\mathbf{Q}$, though it can conjugate it over K), but there can also be other relations that depend on occurrence of unital motives. For instance the Jacobi sum motive for $Z_5 = \langle 1/5\rangle + \langle 4/5\rangle - \langle 2/5\rangle - \langle 3/5\rangle$ corresponds to the ζ-function of $\mathbf{Q}(\sqrt{5})$ by the above formalism, and one can induce this to the unital motive on $\mathbf{Q}(\zeta_5)$. The $\mathbf{Q}$-motives for $\theta_1 = 2\langle 1/5\rangle - \langle 2/5\rangle$ and $\theta_2 = \langle 1/5\rangle + \langle 2/5\rangle - \langle 3/5\rangle$ can then be seen to be equal since $\theta_1 = 3\circ\theta_2 + Z_5$. Indeed as rank 1 motives on $\mathbf{Q}(\zeta_5)$ we have[2]

$$\mathbf{J}(\theta_1) = \mathbf{J}(3\circ\theta_2) \underset{\mathbf{Q}(\zeta_5)}{\otimes} \mathbf{J}(Z_5),$$

and $\mathbf{J}(Z_5)$ is unital. Similarly, $Z_7 = \langle 1/7\rangle + \langle 6/7\rangle - \langle 2/7\rangle - \langle 5/7\rangle$ corresponds to the ζ-function for the real cubic subfield of $\mathbf{Q}(\zeta_7)$, so that $\kappa_1 = 2\langle 1/7\rangle - \langle 2/7\rangle$ and $\kappa_3 = \langle 1/7\rangle + \langle 3/7\rangle - \langle 4/7\rangle$ can then be seen to yield the same $\mathbf{Q}$-motive since $\kappa_1 - 5\circ\kappa_3 = Z_7$. However, $\kappa_2 = \langle 1/7\rangle + \langle 2/7\rangle - \langle 3/7\rangle$ corresponds to a different motive, as there is no scaling u mod 7 such that $\kappa_1 - u\circ\kappa_2$ has effective weight 0.

Another example can be seen with $\theta_1 = 2\langle 3/5\rangle + \langle 4/5\rangle$ and $\theta_2 = 2\langle 3/5\rangle - \langle 1/5\rangle$. Here these differ by $Z = \langle 1/5\rangle + \langle 4/5\rangle$, and we can compute that

$$\mathbf{I} = \mathrm{Ind}_{\mathbf{Q}(\sqrt{5})}^{\mathbf{Q}(\zeta_5)} \mathbf{J}(Z)$$

is in fact a Tate twist of the unital motive.[3] Indeed, $\mathbf{J}(Z)$ over $\mathbf{Q}(\sqrt{5})$ has an Euler factor given by $(1\mp 2pT + p^2T^2)$ at primes $p \equiv \pm 1 \pmod 5$ and $(1+p^2T^2)$ at primes $p \equiv \pm 2 \pmod 5$. Alternatively we can say that primes $p \equiv \pm 1 \pmod 5$ each have two eigenvalues $\pm p$, while primes $p \equiv \pm 2 \pmod 5$ have one eigenvalue $-p^2$. This implies that over $\mathbf{Q}(\zeta_5)$ for $p \equiv 1 \pmod 5$ we get four eigenvalues of p, while for $p \equiv 4 \pmod 5$ we get two eigenvalues of $(-p)^2$ as these primes do not split further, and similarly with a single eigenvalue of $(-p^2)^2$ with $p \equiv 2, 3 \pmod 5$. Thus in all cases we find that there are f eigenvalues of p^f, where f is the smallest positive integer with $5|(p^f-1)$. This implies that $\mathbf{I}$ is a Tate twist of the unital motive over $\mathbf{Q}(\zeta_5)$, and since $\mathbf{J}(\theta_1) = \mathbf{J}(\theta_2)\otimes\mathbf{I}$ over this field, we find that $\mathbf{J}(\theta_1)$ is a Tate

[2]Upon rewriting the Γ_p-expressions by the Gross-Koblitz formula, at the level of Gauss sums for a prime $p \equiv 1 \pmod 5$ we are asserting that $\mathfrak{g}_p(1/5)^2/\mathfrak{g}_p(2/5) = \mathfrak{g}_p(3/5)\mathfrak{g}_p(6/5)/\mathfrak{g}_p(9/5)$, as can be verified since $\mathfrak{g}_p(1/5)\mathfrak{g}_p(4/5) = \mathfrak{g}_p(2/5)g_p(3/5) = p$.

[3]This is the same argument that would show, e.g., that a quadratic Dirichlet character motive over $\mathbf{Q}$ becomes unital when induced to the quadratic extension.

twist of $\mathbf{J}(\theta_2)$ (indeed, it is a Tate twist over $\mathbf{Q}(\zeta_5)$ as we have written it, though in general we could scale one of the θ's and end up with the analogous result in terms of $\mathbf{Q}$-motives).

9.2.2. *An example.* We take $\theta = \langle 1/12\rangle + \langle 5/12\rangle + \langle 1/2\rangle$ which has a scaling symmetry by 5, so that the field of definition is $\mathbf{Q}(\zeta_4)$. The motive is of weight 3 with $h^{1,2} = h^{2,1} = 1$. The conductor is $\mathfrak{p}_2^4 \cdot 3 = 12$, and the Euler factor at primes which are 3 mod 4 is $(1 + p^3T^2)^{-1}$.

Writing $p = a^2 + 4b^2$, for $p \equiv 5 \pmod{12}$ the Euler factor is $(1 \pm 4bpT + p^3T^2)^{-1}$ with the plus sign when both or neither $a \equiv \pm 1 \pmod{12}$ and $b \equiv 1 \pmod 3$ holds, while for $p \equiv 1 \pmod{12}$ the Euler factor is given by $(1 - 2apT + p^3T^2)^{-1}$ where $(a \bmod 12) \in \{1, 3, 5\}$.

9.2.3. *An Euler factor calculation with Gauss sums.* We consider the example of the Jacobi sum motive given by $\theta = 2\langle 1/8\rangle + \langle 3/8\rangle + \langle 7/8\rangle - \langle 1/2\rangle$. Here there are no scalings that fix θ, so that $K_\theta = \mathbf{Q}(\zeta_8)$, while the weight is 3 with Hodge type $h^{1,2} = h^{2,1} = 2$. Consider a prime $p \equiv 5 \pmod 8$, so that $f = 2$ and the orbit sets are $\{\theta, 5 \circ \theta\}$ and $\{3 \circ \theta, 7 \circ \theta\}$. The value of v is 3 for either orbit, so we get that $R(1)$ and $R(5)$ are both

$$(-p)^3\big[\Gamma_p(1/8)^2\Gamma_p(3/8)\Gamma_p(7/8)/\Gamma_p(1/2)\big]\big[\Gamma_p(5/8)^2\Gamma_p(7/8)\Gamma_p(3/8)/\Gamma_p(1/2)\big],$$

while $R(3)$ and $R(7)$ are just a rearrangement of this, namely

$$(-p)^3\big[\Gamma_p(3/8)^2\Gamma_p(1/8)\Gamma_p(5/8)/\Gamma_p(1/2)\big]\big[\Gamma_p(7/8)^2\Gamma_p(5/8)\Gamma_p(1/8)/\Gamma_p(1/2)\big].$$

Using the distribution formula for Γ_p, namely Prop 11.6.14 of Cohen with $s = 1/8$ and $N = 4$, we have

$$\Gamma_p(1/8)\Gamma_p(3/8)\Gamma_p(5/8)\Gamma_p(7/8) = -\Big(\frac{-2}{p}\Big)\Gamma_p(1/2)^2/4^{[(-1/2)-(-1/2)\backslash p]},$$

and we can simplify the denominator by Corollary 11.6.3 of Cohen with $s = 1/2$, so $a = (p-1)/2$ and $a\backslash p = 0$, giving $\omega_p(4)^{-(p-1)/2} = 1/\omega_p(2)^{p-1} = 1$ as its evaluation. Upon squaring, we are left with $R(1) = (-p)^3\Gamma_p(1/2)^{4-2} = p^3$ and find that the Euler factor is $(1 - 2 \cdot p^3T^2 + p^6T^4)^{-1}$.

9.3. Kummer twists. We can also consider twists of Jacobi motives. Schematically, upon taking parameters $t \in \mathbf{Q}^\star$ and $\rho \in \mathbf{Q}/\mathbf{Z}$, for the Euler factors at good primes p we multiply $R(a)$ by $\omega_p(t)$ raised to the $\rho(p^f - 1)\sum_i ap^i$ power. Note that we need $v_p(t) = 0$ to be able to compute the Teichmüller character ω_p at t, and the definition of f (from m) might need modification as explained below.

Indeed, we can consider the systemology of above with respect to a triple (θ, t, ρ), or alternatively to a pair (θ, t^ρ) where with the latter we would consider the action of roots of unity. With the former notation, we would rather want to define an effective ρ, namely if t is an kth power we consider $\rho_t = k\rho$, and the orbits defining K and the Euler factors will then be with respect to the pair (θ, ρ_t), with m now being the least common multiple of the denominators in θ and that of ρ_t. The Hodge structure stays the same, except that the central $h^{p,p}$ signs are switched when $\rho_t = 1/2$ and t is negative. Note that when $t = 1$ we are in the plain Jacobi case, and similarly when ρ_t is integral.

9.3.1. *Twists of the Fermat cubic, and congruent number curve.* Recall from above that $\theta = 2\langle 2/3\rangle - \langle 1/3\rangle$ corresponds to an elliptic curve of conductor 27. We can take various twists of this by choosing t and ρ appropriately. In particular, with $\rho = 1/2$ we will obtain quadratic twists (note that these do not enlarge the field K), while $\rho = \pm 1/3$ will give cubic twists, with $\rho = \pm 1/6$ for sextic twists. Similarly, for $\theta = \langle 1/2\rangle + \langle 3/4\rangle - \langle 1/4\rangle$ we can take quartic twists of the associated congruent number curve (of conductor 32) via $\rho = \pm 1/4$.

9.3.2. *Various examples.* One can already Kummer twist $\theta = 0$ and obtain nontrivial results. For instance, twisting by $t^{1/2}$ gives the Kronecker character for t, and $2^{1/3}$ will yield the nontrivial 2-dimensional Artin representation of $\mathbf{Q}(2^{1/3})$. Similarly, with $\theta = \langle 1/5\rangle + \langle 4/5\rangle$ taking the Kummer twist by $5^{1/5}$ results in a Tate twist of the irreducible 4-dimensional Artin representation for $\mathbf{Q}(5^{1/5})$, with the conductor of the L-function being 5^9. This same conductor appears with the Kummer twist by $5^{1/5}$ of $\theta = 2\langle 3/5\rangle - \langle 1/5\rangle$. Moreover, the four conjugate Kummer twists here all have congruent Euler factors mod 5.

L-functions of hypergeometric motives over $\mathbf{Q}$

In this chapter we try to give as explicitly as possible a recipe for computing the L-function data for a hypergeometric motive defined over $\mathbf{Q}$. We shall not prove many of our statements, and indeed most of them are currently conjectural.

Following Serre [**?**](?), an L-function can be specified by defining the degree d, the weight w, the conductor N, the Γ-factors whose product is $\Gamma_{\mathbf{M}}(s)$, and an inverse Euler factor E_p at each prime (which is a polynomial of degree at most d). In fact the L-function proper only depends on said Euler factors, being defined by

$$L_{\mathbf{M}}(s) = \prod_p E_p(p^{-s})^{-1},$$

and for the completed L-function

$$\Lambda_{\mathbf{M}}(s) = N^{s/2} L_{\mathbf{M}}(s)\Gamma_{\mathbf{M}}(s)$$

there is expected to be a sign $\epsilon_{\mathbf{M}} = \pm 1$ (explicit in principle) such that

$$\Lambda_{\mathbf{M}}(s) = \epsilon_{\mathbf{M}}\Lambda_{\mathbf{M}}(w+1-s)$$

The degree d is the same as the degree of the hypergeometric datum, while the weight w needs to be adjusted as explained above with the Hodge function. This ensures that the the pth coefficient of the L-series will be integral and generically not divisible by p.

Our hypergeometric parameter shall again be $\tilde{t}$, the reciprocal of the parameter appearing originally in the trace formula a la Katz. As previously noted, this is the choice made in the GP/PARI and Magma implementations, so that the reader can test the examples if desired.

10. Γ-factors

As explained below, the above Hodge polynomial determines the Γ-factors in the presumed functional equation, except for the action of complex conjugation on the $h^{p,p}$ pieces (these can occur in even weight).

There are three intervals, $\tilde{t} < 0$, $0 < \tilde{t} < 1$, and $1 < \tilde{t}$. In odd degree we have that trace of complex conjugation is $h^+ - h^- = \pm 1$, while in even degree it can be 0 or ± 2. Taking the u-value from the Table 8, we have $h^+ - h^- = u \cdot (-1)^D$.

degree	$\tilde{t} < 0$	$0 < \tilde{t} < 1$	$1 < \tilde{t}$
odd	$+1$	-1	$+1$
even	0	0	$+2$

Table 8. Central Γ-factor u-values

Note that this breaks the $t \to 1/t$ symmetry for positive t. However, in even weight we have $1 \in \mathcal{B}$ (with odd multiplicity), and so this fixes our normalisation.

One can compare the above with Table 5.3 of Deligne, where on $H^{p,p}$ the complex conjugation action is $F_\infty = (-1)^{p+\epsilon}$, yielding a Γ-factor of $\Gamma_{\mathbf{R}}(s+\epsilon-p)$. That is, $h^+ = h^{p,p,0}$ when p is even and $h^+ = h^{p,p,1}$ when p is odd. For instance, for $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_1^3\Phi_3, \Phi_2^3\Phi_6)$ at $t = 3/2$, we have $w = 2$ and $D = 0$ so that $h^+ - h^- = 1$, so that the Hodge structure is $2h^{0,2} + h^{1,1,\epsilon}$ with $\epsilon = 1$. Similarly, for $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_3^2, \Phi_1\Phi_2\Phi_6)$ at $t = 3/2$, we have $w = 2$ and $D = 1$ so that $h^+ - h^- = -2$, so that the Hodge structure is $h^{0,2} + 2h^{1,1,\epsilon}$ with $\epsilon = 0$.

The full Γ-factor $\Gamma_{\mathbf{M}}(s)$ is then given by

$$\Gamma_{\mathbf{M}}(s) = \prod_{p<q} \Gamma_{\mathbf{C}}(s-p)^{h_{p,q}} \cdot \Gamma_{\mathbf{R}}(s-p)^{h^{p,p,0}} \Gamma_{\mathbf{R}}(s+1-p)^{h^{p,p,1}},$$

where $\Gamma_{\mathbf{R}}(s) = \Gamma(s/2)/\pi^{s/2}$ and $\Gamma_{\mathbf{C}}(s) = 2 \cdot \Gamma(s)/(2\pi)^s = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$.

11. Euler factors and local conductors

We next describe how to compute the Euler factor and local conductor for a given prime. There are four types of primes. First are the (potentially) wild primes, those which divide an element of $\mathcal{A} \cup \mathcal{B}$. Next are the tame primes, which are nonwild primes with $v_p(\tilde{t}) \neq 0$. Then come the "multiplicative" primes, which are primes that are neither wild nor tame, and have $v_p(\tilde{t}-1) \geq 0$. Finally are the good primes, which are primes that are none of the above. In other words, they do not divide a cyclotomic index, and have $v_p\big(\tilde{t}(\tilde{t}-1)\big) = 0$.

Note that any of the above types of primes could be "good" in the classical sense, namely that p does not divide the conductor and the Euler factor has full degree.

11.1. Good primes. The local conductor at a good prime p is trivial, while the Euler factor is given by

$$E(T) = \exp\big(-\sum_n U_{p^n}(\tilde{t})T^n/n\big).$$

With hypergeometric data of degree d, one need only compute the traces for p^f with $f \leq \lfloor d/2 \rfloor$, as the rest of the (inverse) Euler factor via use of the local functional equation, namely that $E(T) = \pm p^w T^d E(1/p^w T)$. To be able to use this, one needs to know the sign, which we give below. When d is even and the sign is -1, one can also omit the computation of the central term. We first note that when the weight is odd, the sign is always $+1$.

11.1.1. *Local sign, even weight.* There are two cases for even weight. When the degree is odd, note that 1 has odd multiplicity in $\mathcal{B}$, and 2 has odd multiplicity in $\mathcal{A}$ (this is just because these are the only cyclotomic polynomials of odd degree). The β have an integral sum, and the character χ_u is the Kronecker character for

$$u = (-\tilde{t})(1-\tilde{t}) \prod_{b \in \mathcal{B}} \Delta(\Phi_b),$$

where $\Delta(\Phi_b)$ is the discriminant of the bth cyclotomic polynomial.

When the degree is even, the assumption of even weight implies (by a symmetry argument) again that 1 has odd multiplicity in $\mathcal{B}$, and so does 2 in this case. Thus the α have an integral sum this time, and the character χ_u is the Kronecker character for

$$u = (1-\tilde{t}) \prod_{a \in \mathcal{A}} \Delta(\Phi_a).$$

In all cases, for $0 \leq j \leq \lfloor d/2 \rfloor$, the $(d-j)$th coefficient in the Euler factor is the jth coefficient multiplied by $(-1)^d \chi_u(p) p^{w(d-2j)/2}$.

11.1.2. *Examples.* For $\big(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}\big)=\big(\Phi_4^2,\Phi_1^2\Phi_6\big)$ of weight 3, at $\tilde t=5$ the Euler factor at $p=101$ is given by

$$1-1044T+1500658T^2-1044\cdot 101^3T^3+101^6T^4.$$

For $\big(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}\big)=\big(\Phi_3^4,\Phi_1^2\Phi_4^3\big)$ of weight 7, at $\tilde t=5$ the Euler factor at $p=19$ is

$$1-32656T+12495435311T^2+11328313645088T^3+837256931112824684T^4+$$
$$19^7\cdot 11328313645088T^5+19^{14}\cdot 12495435311T^6-32656\cdot 19^{21}T^7+19^{28}T^8.$$

Note that 19^6 divides the T^4 term here (whereas $101\|1500658$ above).

For $\big(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}\big)=\big(\Phi_4\Phi_5,\Phi_1\Phi_2\Phi_3^2\big)$ of weight 2, the character is χ_u for $u=5(\tilde t-1)$. Thus for $\tilde t=7$ and $p=41$ the functional equation is odd, with the Euler factor given by

$$1+23T-123T^2+41^2\cdot 123T^4-41^4\cdot 23T^5-41^6T^6.$$

For $\big(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}\big)=\big(\Phi_2\Phi_3\Phi_5,\Phi_1\Phi_4^3\big)$ of weight 4, the character χ_u for $u=-\tilde t(\tilde t-1)$. Thus for $\tilde t=7$ and $p=31$ the functional equation is odd (recall that one negates the character value in the odd degree case), with the Euler factor given by

$$1+685T-1246448T^2-1088920632T^3-31^2c_3T^4-31^6c_2T^5-31^{10}\cdot 685T^6-31^{14}T^7.$$

Here $31^3|1088920632$, which (as David points out) follows since Newton polygons lie above Hodge polygons.

11.2. Multiplicative primes. These are primes p with $v_p(\tilde t-1)>0$ which do not divide any of the cyclotomic data. They are "multiplicative" primes, in that p exactly divides the conductor N. From the standpoint of hypergeometric differential equations, there are $(d-1)$ independent holomorphic solutions about $\tilde t=1$, and thus this many independent eigenvectors.

The method of p-adic Γ-functions can again be used to compute the traces (even at $p=2$), which can then be lumped into the Euler factor. However, the computation of the local sign is different.

Furthermore, when the weight is even and $v_p(\tilde t-1)$ is also even, then this supposed multiplicative prime is actually good (for instance, $\big(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}\big)=\big(\Phi_2,\Phi_1\big)$ with $\tilde t=-8$ at $p=3$). In this case, one computes part of the Euler factor as below from the hypergeometric trace, and then multiplies it by $\big(1-\chi_u(p)p^{w/2}T\big)$ to get the full degree d polynomial. Here $u=Kt_0$ where $\tilde t-1=t_0p^v$ with $\gcd(t_0,p)=1$, and $K=(-1)^{(b_1-1)/2}\cdot 2\prod_v v^{\gamma_v}$.

11.3. Local sign, odd weight (even degree). Here the sign can be determined from χ_u, which is the Kronecker character of $u=(-1)^{b_1/2}\prod_v v^{\gamma_v}$ where b_1 is the multiplicity of 1 in $\mathcal{B}$ (necessarily even). By §**??** we also have $u=\prod_v(\zeta_4 v)^{\gamma_v}$.

Note that this sign is for the degree 1 piece. The complete Euler factor is given by $\big(1-\chi_u(p)p^{(w-1)/2}T\big)\tilde E(T)$, where $\tilde E$ is of degree $(d-2)$ and satisfies a functional equation with sign $+1$ (and weight w).

11.3.1. *Local sign, even weight, even degree.* Here χ_u is the Kronecker character for $u=-(-1)^{d/2}(-1)^{(b_1-1)/2}\cdot 2\prod_v v^{\tilde\gamma_v}$, where the $\tilde\gamma$ come from $[\alpha+1/2,\beta]$. That is, shift all the α's by $1/2$, cancel any β intersection, and compute the $\tilde\gamma$ from that. Alternatively, $\mathcal{P}_\mathcal{A}(-T)/\mathcal{P}_\mathcal{B}(T)=\prod_v(T^v-1)^{\tilde\gamma_v}$. As above, this sign is for the degree 1 piece, given by $(1-\chi_u(p)p^{w/2}T)$.

11.3.2. *Local sign, odd degree (even weight).* Here χ_u is the Kronecker character of $u = (-1)^{(d-1)/2}\prod_v v^{\hat{\gamma}_v}$, where the $\hat{\gamma}$ come from $[\alpha, \beta + 1/2]$. That is, shift all the β's by 1/2, cancel any α intersection, and compute the $\hat{\gamma}$ from that. Alternatively, $\mathcal{P}_\mathcal{A}(T)/\mathcal{P}_\mathcal{B}(-T) = -\prod_v (T^v - 1)^{\hat{\gamma}_v}$. Here the sign is for the entire Euler factor.

11.3.3. *Examples.* Take $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_{12}, \Phi_1^2\Phi_6^2)$ of weight 3. The character here is χ_u with $u = -3$. With $\tilde{t} = 8$ and $p = 7$, we get the Euler factor as

$$(1 - 7T)(1 + 8T + 105T^2 + 7^3 \cdot 8T^3 + 7^6T^4).$$

For $p = \tilde{t} - 1 = 11$, the Euler factor is $(1 + 11T)(1 - 14T + 22T^2 - 11^3 \cdot 14T^3 + 11^6T^4)$.

Take $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_4^3, \Phi_1^3\Phi_2\Phi_6)$ of weight 4. This has character χ_u for $u = -8$. With $\tilde{t} = 8$ and $p = 7$, we get the Euler factor as

$$(1 + 49T)(1 - 36T + 2758T^2 - 7^4 \cdot 36T^3 + 7^8T^4).$$

For $p = \tilde{t} - 1 = 11$ we get $(1 - 121T)(1 - 36T + 12694T^2 - 11^4 \cdot 36T^3 + 11^8T^4)$.

Take $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2\Phi_4\Phi_{10}, \Phi_1\Phi_{18})$ of weight 2. The character is χ_u for $u = -6$. With $\tilde{t} = 14$ and $p = 13$ the Euler factor is

$$1 - 4T - 13T^2 + 13^2 \cdot 13T^4 + 13^4 \cdot 4T^5 - 13^6T^6.$$

11.4. Tame primes. These are primes that do not divide the cyclotomic data, and for which $v_p(\tilde{t}) > 0$ or $v_p(1/\tilde{t}) > 0$. These correspond respectively[4] to monodromy about ∞ or 0. From the standpoint of hypergeometric differential equations, one has solutions like $z^{1-\beta_i}{}_nF_{n-1}(z)$, and when $v_p(\tilde{t})\beta_i$ is an integer, this becomes holomorphic about 0. Of course, if there are repeated β's, then one gets log-factors, which are not holomorphic.

Given a polynomial $\Phi_m^r \| \mathcal{P}_\mathcal{A}$, there will be Jordan blocks of size r with each of the mth primitive roots of unity as eigenvalues. When $m \nmid v_p(\tilde{t})$, then from these blocks one simply has a trivial Euler factor with conductor exponent $r\phi(m)$.

11.4.1. *Nontrivial Euler factors.* When $\tilde{t}$ is chosen with $m|v_p(\tilde{t})$, then there will be an Euler factor of degree $\phi(m)$ from these blocks, with the conductor exponent from them being $(r-1)\phi(m)$. The weight of such an Euler factor (Corti and Golyshev?) should be $w + 1 - r$.

Consider $\Phi_m \in \mathcal{A}$ at $\tilde{t} = t_0 \cdot p^{vm}$ (where $v > 0$). Let $q = p^f$ be the smallest power of p with $m|(q-1)$. We have a sequence of p-adic numbers

$$\left[\omega_p(Mt_0)^{j(q-1)/m} Q_q\Big(\frac{j(q-1)}{m}\Big) : 0 \le j < m \;\Big|\; \gcd(j, m) = 1\right].$$

Denoting the above sequence elements by η, we then have that $\prod_\eta (1 - \eta T^f)$ is an fth power, and so let $E(T)$ be its fth root (thus E has degree $\phi(m)$). The weight w_m of this Euler factor should be $w + 1 - a_m$, where a_m is the multiplicity of $m \in \mathcal{A}$.

It is exactly analogous for $m \in \mathcal{B}$ at $\tilde{t} = t_0/p^{vm}$. Note that when $m = 1$ we simply have $E(T) = (1 - p^DT)$ since $w_1 = w + 1 - b_1 = 2D$.

When computing the Euler factor, we could again use a local functional equation, but it is not that much worse simply to compute the above quantities to a sufficiently high precision. A more worrisome difficulty is when $q = p^f$ is large, for then we need to do (at least) q computations. Here it would be useful to know (say) when

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[4]This is due to our choice of $\tilde{t}$. With the opposite choice, then $v_p > 0$ corresponds to 0 and $\mathcal{B}$.

we are simply going to get a result like $(1 \pm p^{w_m\phi(m)/2}T^{\phi(m)})$, which should be determinable via cyclotomy and the local sign.

11.4.2. *Examples and relation to Grössencharacters.* An alternative method to compute the Euler factors could be via Grössencharacters. However, I have never been able to pin down the right normalisation, and it is usually easier to figure out the correspondence in retrospect.

As an example, take $\bigl(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}\bigr) = \bigl(\Phi_5, \Phi_1^4\bigr)$ for $\tilde{t} = 11^5$. The Euler factor at 11 is given by a canonical Grössencharacter of $\mathbf{Q}(\zeta_5)$ of ∞-type $[(3,0),(1,2)]$ and ideal of norm 5^2. Namely, the Euler factor is $(1 - 89T + 3861T^2 - 89 \cdot 11^3T^3 + 11^6T^4)$.

Another example (showing the weight drop) is $\bigl(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}\bigr) = \bigl(\Phi_3^2, \Phi_1^4\bigr)$ with $\tilde{t} = 7^3$, where the Euler factor at $p = 7$ is $(1 + 13T + 49T^2)$. This matches the $[(2,0)]$ Grössencharacter of $\mathbf{Q}(\zeta_3)$ whose modulus has norm 3^2.

11.5. Wild primes... ...

12. Sign of the (local/global) functional equation

...

13. Examples

...

14. Degeneration to $t = 1$

One can also consider the L-series obtained from $\tilde{t} = 1$. Doing this results in an L-function of smaller degree, with only the wild primes being bad. The L-factor will be as with multiplicative primes, except in the odd weight case the degree 1 piece is dropped (and so the total degree is two less than the original). The Γ-factors are easily described except for the central piece in even weight. In the case where the initial degree is odd we have $h^+ = h^-$. When the initial degree is even, we have that $h^+ - h^- = \tau \cdot (-1)^{d/2}$ where $\tau = +1$ when $u < 0$ in §11.3.1, while $\tau = -1$ when $u > 0$. Alternatively, $h^+ - h^- = (-1)^{w/2+D}$, upon unwinding all this.

14.1. Examples. ...

Statement about checking L-functions numerically.

Various experiments and data

15. The 14 unipotent families in degree 4

Consider the 14 cases of degree 4 hypergeometric data with $\mathcal{P}_{\mathcal{B}} = \Phi_1^4$. For each case, we verified the functional equation for $\tilde{t} = -1$ to six or more digits, with the conductor as given in Table 9. The case of $\mathcal{P}_{\mathcal{A}} = \Phi_3\Phi_4$ has a quadratic Euler factor of $1 + 8T^2$ at $p = 2$. All the other wild Euler factors are trivial.

$\mathcal{P}_{\mathcal{A}}$	Φ_2^4	$\Phi_2^2\Phi_3$	Φ_3^2	$\Phi_2^2\Phi_4$	$\Phi_3\Phi_4$	Φ_4^2	Φ_5
cond	2^9	$2^7 3^5$	3^6	2^{10}	$2^7 3^5$	2^{13}	5^5
$\mathcal{P}_{\mathcal{A}}$	$\Phi_2^2\Phi_6$	$\Phi_3\Phi_6$	$\Phi_4\Phi_6$	Φ_6^2	Φ_8	Φ_{10}	Φ_{12}
cond	$2^9 3^5$	$2^7 3^6$	$2^{10} 3^5$	$2^9 3^6$	2^{17}	$2^9 5^5$	$2^{13} 3^6$

Table 9. Fourteen unipotent cases in degree 4

Comparatively, in degree 1 one has the conductor 2^3, while in degree 2 one has conductors 2^5, $2 \cdot 3^3$, 2^7, and $2^5 3^3$. In degree 3, one can multiply any of the $\mathcal{P}_A$ from degree 2 by Φ_2, and the resulting conductors are 2^7, $2^5 \cdot 3^4$, 2^9, and $2^7 3^4$ (in all cases $h^- = 1$). One can try the same in degree 5, but the conductors tend to be a bit too large to test numerically.

A related idea is to consider $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_2^d, \Phi_1^d)$ at $\tilde{t} = -1$. Checking the first 9 cases numerically, the conductor is 2^{2d+1}. Similarly, one can consider a family like $(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}) = (\Phi_4^d, \Phi_1^d\Phi_2^d)$ at $\tilde{t} = -1$. Here for the first 4 cases the conductor as 2^{1+6d}.

16. L-function (im)primitivity and generalised Sato-Tate

Some cases have an imprimitive L-function (that is, it factors into smaller degree L-functions – do not confuse this with tensor product of L-functions!). For instance, when an Artin representation exists. (?!) There are also higher weight examples.

One way to test for imprimitivity is via the measure $\frac{1}{\pi(X)} \sum_p^X |c_p|^2/p^w$. Assuming conjectures for the Selberg class of L-functions, the limit as $X \to \infty$ will be an integer, and is 1 precisely when the L-function is primitive. [This assumes one has tested for poles via $\sum_p^X c_p/p^{w/2} \sim r\pi(X)$ and divided out by $\zeta(s)^r$ if necessary]. In general, a primitive product $\prod_i L_i(s)^{u_i}$ gives a imprimitivity measure of $\sum_i |u_i|^2$. This can be compared to factorisations of Euler factors.

From this, we can fairly easily execute an experimental test for primitivity; of course, the convergence of the above sum depends on the conductor, but rather weakly (say a logarithm). In practise, one will usually be within 10% of an integer with the primes up to 1000 or 2000.

16.1. Sato-Tate analysis. In a similar manner, given a general Sato-Tate conjecture, we expect that the eigenvalue distribution to generically be either the full symplectic or (special)[5] orthogonal group. Using Larsen's alternative, applicable most specifically for positive weight and degree at least 3, this is equivalent to the fourth moment of the trace of the eigenvalues (linear coefficient of the Euler factor) being 3. This gives a useful test, though using higher moments can also aid in guessing the distribution. Of course, when the hypergeometric data itself is already imprimitive, there will be a different eigenvalue distribution.

[5]In deg 4 wt 2, the SO case has 4th moment 4, while O has 3. Else both have 4th moment of 3.

We ran through all hypergeometric data (up to twist) up through degree 7 and averaged across about fifty $\tilde{t}$-values for primes up to 1000. We found no positive weight non-primitive hypergeometric data whose moments differed to any great extent from that predicted by symplectic or orthogonal groups. Similarly, in the data below, other than the $\tilde{t} = -1$ symplectic cases noted in the next subsubsection, we found no examples of $\tilde{t}$-values that gave a *primitive* L-function which did not appear to have fourth trace moment equal to 3. However, we did find various exceptional t-values where the L-function was imprimitive.

16.1.1. *Special $\tilde{t}$-values.* Given any $\big(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}\big) = \big(\Phi_2^k\Phi_3^l, \Phi_1^k\Phi_6^l\big)$ the specialisation to $\tilde{t} = -1$ gives an alternative distribution. In the symplectic case (even degree) the fourth moment appears to be 6, while in the orthogonal case, the second moment is 2 (and thus already imprimitive), except for $(k,l) = (5,1)$ when it is 3. For the case of $(k,l) = (4,0)$ the $\tilde{t} = -1$ specialisation is related to the Siegel modular form of level $(2,4)$ of van Geemen and van Straten [**?**], and comes from a tensor product.

16.2. Artin representations. In the Belyi cases of Artin representations, we see imprimitivity in a quite noticeable way. Indeed, taking $\big(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}\big) = \big(\Phi_3, \Phi_1\Phi_2\big)$ with the algebra $x^2(1-x) - 1/M\tilde{t}$ as an example, one can simply solve $\tilde{t} = 1/Mx^2(1-x)$ and plug in x-values. The resulting $\tilde{t}$-values will all give imprimitive L-functions (indeed, they will all have $\zeta(s)$ as a factor). Furthermore, the cubic polynomial has discriminant $16(\tilde{t}-1)/27\tilde{t}^2$, and so when $3(\tilde{t}-1)$ is square the Galois group will be $\mathrm{Alt}(3)$. There can also be (in general) splittings of a more complicated sort.

In general, we shall be most interested in the cases where the weight is as large as possible, namely one (maybe two) less than the degree.

16.3. Degree 2 weight 1. For completeness we list the cases of degree 2. Here each of the 10 elliptic curve families yields a parametrised family F of j-invariants say $j_F(\tilde{t})$, and we can solve $j_F(\tilde{t}) = j$ for the 13 CM invariants j of class number 1. In Table 10 we list these up to twist; for $\big(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}\big) = \big(\Phi_3, \Phi_4\big)$ (and twist), where $j_F(\tilde{t})$ has degree 1, we give this parametrisation (there are 11 nondegenerate rational $\tilde{t}$-solutions, see the $\big(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}\big) = \big(\Phi_2^3, \Phi_1\Phi_3\big)$ case of Table 12, and take $\frac{1}{1-\tilde{t}}$).

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	$\tilde{t}$	j
Φ_3	Φ_1^2	-8	$-3\cdot 160^3$
Φ_3	Φ_1^2	$8/9$	0
Φ_3	Φ_1^2	2	$2\cdot 30^3$
Φ_3	Φ_6	$(125/128)^\pm$	$-3\cdot 160^3$
Φ_3	Φ_6	-1	12^3
Φ_2^2	Φ_1^2	$1/2$	12^3
Φ_2^2	Φ_1^2	$-1, 2$	66^3
Φ_2^2	Φ_3	$1/2$	12^3

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	$\tilde{t}$	j
Φ_4	Φ_1^2	$63/64$	-15^3
Φ_4	Φ_1^2	$3/4$	0
Φ_4	Φ_1^2	$9/8$	12^3
Φ_4	Φ_1^2	2	20^3
Φ_4	Φ_1^2	-3	$2\cdot 30^3$
Φ_4	Φ_1^2	9	66^3
Φ_4	Φ_1^2	-63	255^3
Φ_3	Φ_4		$12^3\frac{t}{\tilde{t}-1}$

TABLE 10. Degree 2 weight 1 CM elliptic curves

16.4. Degree 3 weight 2. The results (possibly incomplete?) for degree 3 and weight 2 are in Tables 12 and 13, where we don't list twists. Of necessity, the imprimitive L-functions must split into degree 1 and 2, with the degree 1 part from a Dirichlet character (Tate-twisted to have weight 2), while the degree 2 part is from a $[(2,0)]$ Grössencharacter of an associated imaginary quadratic field.

The notation for a Grössencharacter Ψ lists the ∞-type in the superscript and the discriminant of the imaginary quadratic field in the subscript. Inside the parenthesis is an ideal, and in the case where K has class number 1 an norm-induced Hecke character (possibly trivial) on it. Namely we have that $\psi_m(\mathfrak{a}) = \chi_m(\mathrm{N}\mathfrak{a})$ for $\mathfrak{a}$ that are coprime to m. When K does not have class number 1, then there is still ambiguity of a Hilbert character (or choice of embedding) in any event, and here we specify via values at distinguishing primes. Similarly, for example in the $\tilde{t} = -25920$ case with $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2\Phi_4, \Phi_1^3)$, we conflate the above notation to allow $m = \zeta_5$ and indicate the trace at 13 to distinguish.

The $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^3, \Phi_1\Phi_3)$ contains the various class number 1 j-invariants, and indeed this can somehow be realised as the "symmetric square" (up to twist) of the $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3, \Phi_4)$ case above. Note that for $p = 7, 13, 19, 43, 67, 163$ the conductors here are merely p, rather than p^2 as in the elliptic curve case. The $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^3, \Phi_1\Phi_6)$ case similarly has 11 exceptional $\tilde{t}$-values, with prime factors of $(p+1)/4$ appearing. The $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2\Phi_4, \Phi_1\Phi_3)$ case has 11 exceptional $\tilde{t}$-values, but their (partial) classification eludes me.

The $\tilde{t}$-values for $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^3, \Phi_1^3)$ are (up to $\tilde{t} \to 1/\tilde{t}$, and noting also that they use $-\tilde{t}$ in place of $\tilde{t}$) those appearing in Theorem 1.2 of Ahlgren, Ono, and Penniston's work on *Zeta Functions of an Infinite Family of K3 Surfaces.* [**?**] They denote these as modular, showing in their §5 that these are the only ones which are associated to a weight 3 modular form. Note, however, here we are only considering *rational* $\tilde{t}$-values that induce an alternative distribution. For higher degree cases, one might even expect that the number of expectional $\tilde{t}$-values over $\bar{\mathbf{Q}}$ is finite.

Our $\tilde{t}$-values here correspond to the 7 exceptional values for $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_4, \Phi_1^2)$, upon taking $\tilde{t} \to 1 - \tilde{t}$ from Table 10. Indeed, we have (see AOP, up to notation) that $\mathrm{Sym}^2\mathcal{H}(\{4\}, \{1,1\}|\tilde{t}) = \mathcal{H}(\{2,2,2\}, \{1,1,1\}|1-\tilde{t}) \otimes \chi_{\tilde{t}(\tilde{t}-1)}$.

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	#
Φ_2^3	Φ_1^3	7
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	15
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	12
Φ_2^3	$\Phi_1\Phi_3$	11
$\Phi_2\Phi_4$	Φ_1^3	14
Φ_2^3	$\Phi_1\Phi_6$	11

TABLE 11. Counts of exceptional t in degree 3 and weight 2

16.5. Degree 4. One can list Artin/Belyi splittings as before for weight 0. In weight 1, I suspect that there could be families of genus 2 curves whose Jacobian splits as a product of two elliptic curves. I generated some data for weight 2, but it seems hard to catalogue, as the the splittings involve a 3-dimensional piece that is not easy to name. None of them appeared to split into two 2-dimensional parts.[6]

[6] In this dimension, it seems that SO has a different fourth moment than O, namely 4 instead of 3, generating much "exceptional" data.

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	$\tilde{t}$	decomposition	N
Φ_2^3	Φ_1^3	-1	$\chi_{-4} \oplus \Psi^{2,0}_{-8}(\mathfrak{p}_2^2, \psi_{-4})$	2^7
Φ_2^3	Φ_1^3	$-1/8$	$\chi_{-4} \oplus \Psi^{2,0}_{-4}(\mathfrak{p}_2^2, \psi_1)$	2^6
Φ_2^3	Φ_1^3	4	$\zeta \oplus \Psi^{2,0}_{-3}(2\,\mathcal{O}_K, \psi_1)$	$2^2 3$
Φ_2^3	Φ_1^3	$1/64$	$\zeta \oplus \Psi^{2,0}_{-7}(\mathcal{O}_K, \psi_1)$	7
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	-1	$\zeta \oplus \Psi^{2,0}_{-24}(\mathcal{O}_K, \mathfrak{p}_2 \to 2)$	$2^3 3$
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	$-1/27$	$\zeta \oplus \Psi^{2,0}_{-84}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to 3)$	$2^2 \cdot 3 \cdot 7$
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	$125/27$	$\zeta \oplus \Psi^{2,0}_{-120}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to 3)$	$2^3 \cdot 3 \cdot 5$
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	$256/135$	$\zeta \oplus \Psi^{2,0}_{-3}(5\,\mathcal{O}_K, \psi_5)$	$3 \cdot 5^2$
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	$-2^8 3^3 7/5^6$	$\chi_{21} \oplus \Psi^{2,0}_{-3}\big(7\,\mathcal{O}_K, \psi_{\zeta_7}, \mathfrak{p}_{19} \to \frac{11\pm 21\sqrt{-3}}{2}\big)$	$3 \cdot 7^3$
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	$-15^6/17^3$	$\chi_{17} \oplus \Psi^{2,0}_{-102}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to -3)$	$2 \cdot 3 \cdot 17^2$
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	$3^6/5^6$	$\chi_{-4} \oplus \Psi^{2,0}_{-228}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to -3)$	$2^4 \cdot 3 \cdot 19$
$\Phi_2\Phi_3$	$\Phi_1\Phi_6$	$-11^6/3^3 5^6$	$\zeta \oplus \Psi^{2,0}_{-372}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to 3)$	$2^2 \cdot 3 \cdot 31$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$-2^{10}/3^7$	$\zeta \oplus \Psi^{2,0}_{-19}(\mathcal{O}_K, \psi_1)$	19
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$-5^3/3^7$	$\zeta \oplus \Psi^{2,0}_{-40}(\mathcal{O}_K, \mathfrak{p}_2 \to 2)$	$2^3 5$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$2^{10}/7^4$	$\chi_{-3} \oplus \Psi^{2,0}_{-51}(\mathcal{O}_K, \mathfrak{p}_3 \to -3)$	$3^2 17$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$27/196$	$\chi_{-7} \oplus \Psi^{2,0}_{-84}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to -3)$	$2^2 \cdot 3 \cdot 7^2$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$5^6/7^2 11^4$	$\chi_{-7} \oplus \Psi^{2,0}_{-168}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to -3)$	$2^3 \cdot 3 \cdot 7^2$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$-15^3/7^4$	$\chi_8 \oplus \Psi^{2,0}_{-30}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to -3)$	$2^4 \cdot 3 \cdot 5$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$5^6/2^2 3^7$	$\zeta \oplus \Psi^{2,0}_{-52}(\mathcal{O}_K, \mathfrak{p}_2 \to 2)$	$2^2 13$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$-2^{10} 3^3 5/11^4$	$\chi_5 \oplus \Psi^{2,0}_{-3}(5\,\mathcal{O}_K, \psi_1)$	$3 \cdot 5^3$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$5^6/2^4 11^2$	$\chi_{33} \oplus \Psi^{2,0}_{-132}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to -3)$	$2^2 3^2 11^2$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$-15^6/2^6 7^4 19^2$	$\chi_{57} \oplus \Psi^{2,0}_{-228}(\mathcal{O}_K, \mathfrak{p}_2 \to 2, \mathfrak{p}_3 \to -3)$	$2^2 3^2 19^2$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$5^6 11^3/3^7 7^4$	$\zeta \oplus \Psi^{2,0}_{-88}(\mathcal{O}_K, \mathfrak{p}_2 \to 2)$	$2^3 11$
$\Phi_2\Phi_3$	$\Phi_1\Phi_4$	$-2^{10} 5^6/3^7 7^4$	$\zeta \oplus \Psi^{2,0}_{-43}(\mathcal{O}_K, \psi_1)$	43
Φ_2^3	$\Phi_1\Phi_3$	$-64/125$	$\zeta \oplus \Psi^{2,0}_{-7}(\mathcal{O}_K, \psi_1)$	7
Φ_2^3	$\Phi_1\Phi_3$	$-3^3/2^9$	$\zeta \oplus \Psi^{2,0}_{-11}(\mathcal{O}_K, \psi_1)$	11
Φ_2^3	$\Phi_1\Phi_3$	$-1/2^9$	$\zeta \oplus \Psi^{2,0}_{-19}(\mathcal{O}_K, \psi_1)$	19
Φ_2^3	$\Phi_1\Phi_3$	$-9/64000$	$\zeta \oplus \Psi^{2,0}_{-3}(\mathfrak{p}_3^2, \psi_1)$	3^3
Φ_2^3	$\Phi_1\Phi_3$	$-1/80^3$	$\zeta \oplus \Psi^{2,0}_{-43}(\mathcal{O}_K, \psi_1)$	43
Φ_2^3	$\Phi_1\Phi_3$	$-1/440^3$	$\zeta \oplus \Psi^{2,0}_{-67}(\mathcal{O}_K, \psi_1)$	67
Φ_2^3	$\Phi_1\Phi_3$	$-1/53360^3$	$\zeta \oplus \Psi^{2,0}_{-163}(\mathcal{O}_K, \psi_1)$	163
Φ_2^3	$\Phi_1\Phi_3$	$27/125$	$\chi_{-4} \oplus \Psi^{2,0}_{-8}(\mathfrak{p}_2^2, \psi_{-4})$	2^7
Φ_2^3	$\Phi_1\Phi_3$	$4/125$	$\chi_{-4} \oplus \Psi^{2,0}_{-3}(4\,\mathcal{O}_K, \psi_{-4})$	$2^6 3$
Φ_2^3	$\Phi_1\Phi_3$	$8/11^3$	$\chi_{-8} \oplus \Psi^{2,0}_{-4}(4\,\mathcal{O}_K, \psi_1)$	2^9
Φ_2^3	$\Phi_1\Phi_3$	$64/85^3$	$\chi_{-4} \oplus \Psi^{2,0}_{-7}(4\,\mathcal{O}_K, \psi_{-4})$	$2^6 7$

TABLE 12. Deg 3 wt 2 imprimitive L-functions (up to twist), part 1

An imprimitive weight 3 example would necessarily split into two 2-dimensional parts: one with Hodge structure $(2,1)$, presumably a Tate twist of an elliptic curve; and the other with Hodge structure $(3,0)$, presumably from a weight 4 newform.

I found a few examples that split here, given in Table 14. Here f_{180} is the weight 4 newform $\sum_n c_n q^n$ of level 180 with $c_{11} = 30$, while f_{54} has level 54 and $c_{11} = -57$. The Grössencharacter in the first case corresponds to the elliptic curve of conductor 36. In the second case, $p = 2$ is partially good in both factors. Both of these can also be twisted, of course.

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	$\tilde{t}$	decomposition	N
$\Phi_2\Phi_4$	Φ_1^3	3^2	$\chi_{-3} \oplus \Psi^{2,0}_{-24}(\mathcal{O}_K, \mathfrak{p}_2 \to -2)$	$2^3 3^2$
$\Phi_2\Phi_4$	Φ_1^3	$3^4/2^8$	$\zeta \oplus \Psi^{2,0}_{-7}(\mathcal{O}_K, \psi_1)$	7
$\Phi_2\Phi_4$	Φ_1^3	$3^4/2^5$	$\chi_{-4} \oplus \Psi^{2,0}_{-4}(\mathfrak{p}_2^2, \psi_1)$	2^6
$\Phi_2\Phi_4$	Φ_1^3	3^4	$\chi_{-8} \oplus \Psi^{2,0}_{-40}(\mathcal{O}_K, \mathfrak{p}_2 \to -2)$	$2^6 5$
$\Phi_2\Phi_4$	Φ_1^3	-2^2	$\chi_{-4} \oplus \Psi^{2,0}_{-20}(\mathcal{O}_K, \mathfrak{p}_2 \to -2)$	$2^4 5$
$\Phi_2\Phi_4$	Φ_1^3	$-2^4 3$	$\chi_{-3} \oplus \Psi^{2,0}_{-4}(3\,\mathcal{O}_K, \psi_1)$	$2^2 3^3$
$\Phi_2\Phi_4$	Φ_1^3	$-3^2/2^4$	$\chi_{-3} \oplus \Psi^{2,0}_{-3}(2\,\mathcal{O}_K, \psi_1)$	$2^2 3^2$
$\Phi_2\Phi_4$	Φ_1^3	$-2^2 3^4$	$\chi_{-4} \oplus \Psi^{2,0}_{-52}(\mathcal{O}_K, \mathfrak{p}_2 \to -2)$	$2^4 13$
$\Phi_2\Phi_4$	Φ_1^3	$-2^6 3^4 5$	$\chi_{-20} \oplus \Psi^{2,0}_{-4}(5\,\mathcal{O}_K, \psi_{\zeta_5}, \mathfrak{p}_{13} \to 12 \pm 5i)$	$2^4 5^3$
$\Phi_2\Phi_4$	Φ_1^3	$-63^2/2^8$	$\chi_{-7} \oplus \Psi^{2,0}_{-7}(\mathcal{O}_K, \psi_1)$	7^2
$\Phi_2\Phi_4$	Φ_1^3	7^4	$\chi_{-3} \oplus \Psi^{2,0}_{-8}(\mathcal{O}_K, \psi_1)$	$2^3 3$
$\Phi_2\Phi_4$	Φ_1^3	$3^4 11^2$	$\chi_{-11} \oplus \Psi^{2,0}_{-88}(\mathcal{O}_K, \mathfrak{p}_2 \to -2)$	$2^3 11^2$
$\Phi_2\Phi_4$	Φ_1^3	$3^8 11^4$	$\chi_{-8} \oplus \Psi^{2,0}_{-232}(\mathcal{O}_K, \mathfrak{p}_2 \to -2)$	$2^6 29$
$\Phi_2\Phi_4$	Φ_1^3	$-2^2 3^4 7^4$	$\chi_{-4} \oplus \Psi^{2,0}_{-148}(\mathcal{O}_K, \mathfrak{p}_2 \to -2)$	$2^4 37$
Φ_2^3	$\Phi_1\Phi_6$	$-3^2/2^4$	$\zeta \oplus \Psi^{2,0}_{-3}(\mathfrak{p}_3^2, \psi_1)$	3^3
Φ_2^3	$\Phi_1\Phi_6$	-2^2	$\zeta \oplus \Psi^{2,0}_{-15}(\mathcal{O}_K, \mathfrak{p}_3 \to 3)$	$3 \cdot 5$
Φ_2^3	$\Phi_1\Phi_6$	$-1/2^4 3^3 7$	$\zeta \oplus \Psi^{2,0}_{-3}\big(7\,\mathcal{O}_K, \psi_{\zeta_7}, \mathfrak{p}_{19} \to -\frac{11 \pm 21\sqrt{-3}}{2}\big)$	$3 \cdot 7^2$
Φ_2^3	$\Phi_1\Phi_6$	$3^3/2^4$	$\zeta \oplus \Psi^{2,0}_{-11}(\mathcal{O}_K, \psi_1)$	11
Φ_2^3	$\Phi_1\Phi_6$	$-1/2^4$	$\zeta \oplus \Psi^{2,0}_{-51}(\mathcal{O}_K, \mathfrak{p}_3 \to 3)$	$3 \cdot 17$
Φ_2^3	$\Phi_1\Phi_6$	$-1/2^{10}$	$\zeta \oplus \Psi^{2,0}_{-123}(\mathcal{O}_K, \mathfrak{p}_3 \to 3)$	$3 \cdot 41$
Φ_2^3	$\Phi_1\Phi_6$	$2/3^3$	$\chi_{-8} \oplus \Psi^{2,0}_{-3}(8\,\mathcal{O}_K, \psi_8)$	$2^9 3$
Φ_2^3	$\Phi_1\Phi_6$	$1/2$	$\chi_{-8} \oplus \Psi^{2,0}_{-24}(\mathfrak{p}_2^2, \psi_{-4}, \mathfrak{p}_3 \to 3)$	$2^8 3$
Φ_2^3	$\Phi_1\Phi_6$	$3^3/2$	$\chi_8 \oplus \Psi^{2,0}_{-8}(\mathfrak{p}_2^2, \psi_{-4})$	2^7
Φ_2^3	$\Phi_1\Phi_6$	$-1/80$	$\zeta \oplus \Psi^{2,0}_{-3}(5\,\mathcal{O}_K, \psi_5)$	$3 \cdot 5^2$
Φ_2^3	$\Phi_1\Phi_6$	$4/125$	$\chi_{-20} \oplus \Psi^{2,0}_{-15}(4\,\mathcal{O}_K, \mathfrak{p}_3 \to 3)$	$2^6 \cdot 3 \cdot 5^2$

TABLE 13. Degree 3 weight 2 imprimitive L-functions, part 2

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	$\tilde{t}$	decomposition	N
$\Phi_3\Phi_4$	Φ_1^4	$-1/4$	$\Psi^{2,1}_{-3}(2\mathfrak{p}_3, \psi_0) \oplus f_{180}$	$2^4 3^4 5$
Φ_3^2	Φ_1^4	-8	$E_{54a} \oplus f_{54}$	$2^2 3^6$

TABLE 14. Degree 4 weight 3 imprimitive L-functions

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	$\tilde{t}$	N	Euler factor
Φ_2^5	Φ_1^5	-4	$2^4 5$	1
Φ_2^5	Φ_1^5	-1024	41	$1 + 5T + 10T^2 + 80T^3 + 256T^4$
$\Phi_2\Phi_3\Phi_4$	Φ_1^5	$-16/27$	$2^2 43$	$(1-4T)(1-16T^2)$ $1 + 14T + 102T^2 + 3^4 14T^3 + 3^8 T^4$
$\Phi_2^3\Phi_4$	Φ_1^5	16	$2^2 15$	$(1-4T)(1+6T+16T^2)$
$\Phi_2\Phi_3^2$	$\Phi_1^3\Phi_6$	$4/27$	69	$(1-4T)\cdot(1+4T)(1+2T+16T^2)$ $(1-9T)\cdot(1+15T+81T^2)$
$\Phi_2^3\Phi_3$	Φ_1^5	$-1/27$	$2^4 7$	$1-16T^2$ $1 + 8T + 30T^2 + 3^4 8T^3 + 3^8 T^4$

TABLE 15. Degree 5 weight 4 imprimitive L-functions

16.6. Higher degrees. David Roberts notes the example $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^5, \Phi_1^5)$ of degree 5 and weight 4, where $\tilde{t} = -4, -2^{10}$ both exhibit imprimitivity. In each case there is a factor of $\zeta(s-2)$ in the L-function. In the first case the conductor is 80, while in the second case the conductor is 41 with an inverse Euler factor at 2 of $(1 + 5T + 10T^2 + 5 \cdot 2^4 T^3 + 2^8 T^4)$. We have found four additional examples, again all having a factor of $\zeta(s-2)$ in the L-function. In various cases deriving the good Euler factor from the hypergeometric trace formula with $v_p(M\tilde{t}) = 0$ was useful, either directly, or giving a good guess in the correct direction.

17. Symmetric powers

The above imprimitivity note was made after I failed to be able to numerically compute $L(\mathrm{Sym}^2 \mathcal{H}_{-1}, s)$ for the data $\mathcal{H} = (\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^3, \Phi_1^3)$. In fact, the Sym^2 L-function splits as $\zeta(s-2)^2 \cdot L(\Psi_{-8}^{2,0}(\mathcal{O}_K, \psi_1), s-1) \cdot L(\Psi_{-8}^{4,0}(\mathcal{O}_K, \psi_1), s)$.

An example with a primitive L-function is $L(\mathrm{Sym}^2 \mathcal{H}_2, s) = \zeta(s-2) L(U_5, s)$. Here the Sym^2-conductor is 2^{10}, and the Euler factor of U_5 at 2 is trivial.

One can easily go higher in the imprimitive cases. For instance,

$$L(\mathrm{Sym}^3 \mathcal{H}_{-1}, s) = L(\chi_{-4}, s-3)^2 \cdot L(\Psi_{-8}^{2,0}, s-2)^2 \cdot L(\Psi_{-8}^{4,0}, s-1) \cdot L(\Psi_{-8}^{6,0}, s),$$

where all Grössencharacters Ψ are for $(\mathfrak{p}_2^2, \psi_{-4})$. Note that the total conductor is 2^{24}, which gives a sense that primitive cases will be hard to test numerically.

Here is a way to acknowledge that Sym^2 of a degree 3 L-function should have at least one factor of $\zeta(s)$ (or the trivial representation). By Newton the generic Sym^2 Euler factor looks like $(1 - b_p T + \cdots)$ where $b_p = c_p^2 - c_{p^2}$ with the original Euler factor as $(1 - c_p T + c_{p^2} T^2 + \cdots)$. Now the assumption of degree 3 implies $c_{p^2} = -p^{w/2} \chi(p) c_p$ by duality, and so we find $b_p = c_p(\chi(p) p^{w/2} + c_p)$. Upon noting $\chi(p) c_p p^{w/2}$ has mean 0, one is left with c_p^2, and so we surmise that the pole-order of Sym^2 is the imprimitivity measure of the original L-function (for degree 3).

Another way to conclude the same is probably to use representation theory of orthogonal groups. Indeed, the mth symmetric power of a degree 3 orthogonal L-function has the $(m-2)$nd as a factor (upon suitable translation), and one gets a "new" constituent of degree $2m+1$ at each step.

17.1. Assorted examples. We catalogue some examples that appear in the L-series chapter of the Magma Handbook [**?**]. In Table 16, we list a hypergeometric datum and a $\tilde{t}$-value, with its degree, weight, and conductor. Then we list a symmetrization to apply, and the resulting degree, weight and conductor, and the Euler factor at 2 if nontrivial. The Euler factors at other primes can be computed from the action of the symmetrization on Frobenius eigenvalues.

Note that we take the "new" part of the various symmetrizations of the orthogonal or symplectic data (see [**?**], or Magma handbook). For instance, the symmetric powers of a degree 3 orthogonal representation have degrees $6, 10, 15, \ldots$, but these split up as $(1+5), (3+7), (1+5+9), \ldots$, since the irreducible representations of $SO(3)$ are all of odd degree.[7] Similarly, the [2]-symmetrization (symmetric square)

[7] There can be various twisting when transferring this fact to L-functions. For instance, the $[2,1]$-symmetrization of a degree 3 orthogonal L-function has degree 8, splitting as the original L-function multiplied by the $[2]_O$-symmetrization twisted by the determinant (which can be viewed a degree 1 L-function corresponding to the $[1,1,1]$-symmetrization) Or a more simple example, the $[1,1]_O$-symmetrization is the original L-function twisted by its determinant.

$\mathcal{P}_\mathcal{A}$	$\mathcal{P}_\mathcal{B}$	$\tilde{t}$	d	w	N	Sym	d	w	N	Euler factor
Φ_2^3	Φ_1^3	1/2	3	2	2^8	$[3]_O$	7	6	2^{18}	
Φ_2^3	Φ_1^3	2	3	2	2^9	$[4]_O$	9	8	2^{18}	$(1-2^4T)$
Φ_2^5	$\Phi_1^3\Phi_4$	1	4	4	2^7	$[2]_O$	9	8	2^{15}	$(1-2^4T)$
Φ_2^5	$\Phi_1^3\Phi_4$	1	4	4	2^7	$[2,2]_O$	10	16	2^{21}	
Φ_2^5	Φ_1^5	1	4	4	2^8	$[2]_O$	9	8	2^{12}	$(1+2^8T^2)(1+2^4T)$
Φ_2^5	Φ_1^5	1	4	4	2^8	$[2,2]_O$	10	16	2^{12}	$(1-2^8T)^2(1+2^8T)^2$
Φ_4^2	$\Phi_1^2\Phi_2^2$	2	4	1	2^{16}	$[1,1]_{Sp}$	5	2	2^{16}	
$\Phi_4\Phi_2^4$	Φ_1^6	1	4	5	2^8	$[1,1]_{Sp}$	5	10	2^8	
Φ_4^3	$\Phi_1\Phi_2^5$	1	5	4	2^{11}	$[1,1]$	10	8	2^{24}	

TABLE 16. Various symmetric powers

of an orthogonal degree 4 representation has degree 10 with a trivial constituent, thus yielding a new part of degree 9, while the $[2,2]$-symmetrization is itself of degree 20, with both a trivial constituent and one from the $[2]$-symmetrization.

For the symplectic case, we list two examples of degree 4 and their alternating squares ($[1,1]$-symmetrizations), which is naturally a degree 6 representation but has a trivial constituent. Conversely, for an orthogonal degree 5 L-function the $[1,1]$-symmetrization does not split, and the "new" part has the full degree of 10.

Some partial information about conductors can sometimes be computed using Swan slopes. For instance, with $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_2^5,\Phi_1^5)$ the Swan conductor (at 2) for the $[2]_O$-symmetrization is $15-9=6$ (though computing this is nontrivial), which when added to the tame conductor $9-3=6$ gives the listed result of 12. Similarly, the $[2,2]_O$-symmetrization here has Swan conductor is $16-10=6$, and again when adding the tame conductor $10-4=6$ we get 12 as listed. Incidentally, this $[2,2]_O$-symmetrization has a $\zeta(s-8)$ factor dividing its L-function (and thus a pole), even though this is not forced by general group theoretic considerations.

18. DATA AT $\tilde{t}=1$

One can also gather data for the $t=1$ degeneration of a hypergeometric datum. The degree of the L-function drops by 1 in even weight and by 2 in odd weight. The Euler factors at non-wild primes can be computed by treating them as multiplicative primes (§11.2),

18.1. Hodge structure. In odd weight the Hodge structure loses one of the central $H^{p,q}$ pieces with $q-p=1$, while in even weight it loses one of the central $H^{p,p,\epsilon}$ pieces. When the hypergeometric datum has odd degree (so that the degree of the L-function is even), then the eigenvalues of complex conjugation are equalised so that $h^+=h^-$. When the hypergeometric datum has even degree, the subtracted piece is $H^{p,p,\epsilon}$ with $\epsilon=(-1)^{(b_1-1)/2}$.

18.2. Twisting character. The $\tilde{t}=1$ degeneration of a hypergeometric datum and its twist are related by twisting the resulting L-functions by the Kronecker character corresponding to $(-1)^{w+1}\prod_v \mathrm{Disc}(\Phi_v)$ where w is the weight and v runs over all cyclotomic indices in $\mathcal{A}\cup\mathcal{B}$.

In particular, when the twisting factor is 1 the twists give the same HGM (and L-function). But there are additional examples of differing data giving the same

HGM at $\tilde{t}=1$, for instance $\left(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}}\right)=\left(\Phi_4,\Phi_3\Phi_6\right)$ and $\left(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}}\right)=\left(\Phi_2^2\Phi_4,\Phi_{12}\right)$ are both associated to the elliptic curve isogeny class 864a.

18.3. Poles and imprimitivity. Looking at the Belyi examples (§5.1) one finds that except for the (highly imprimitive) examples with $a=b$, that the $\tilde{t}=1$ degeneration will have the Riemann ζ-function as a constituent of its L-function, and thus have a pole. There are other weight 0 examples, and indeed some with higher weight that we mention below.

As before, one might expect there to be various splitting of weight 1 data from split Jacobians, and in general higher weight should show fewer splittings.

18.3.1. *Degree 6 symplectic examples.* The examples of odd weight in degree 6 will have L-functions of degree 4, which could then split into two pieces of degree 2. In Tables 17-19 we list these examples, pairing them by twists when appropriate. Note that having an imprimitive hypergeometric datum is not overly correlated with having a split L-function for the $t=1$ degeneration (up to twist there are six weight 1 data that are 3-imprimitive in degree 6, of which two appear in the table).

$\mathcal{P}_{\mathcal{A}}$	$\mathcal{P}_{\mathcal{B}}$	decomp	notes
$\Phi_5\Phi_6$	$\Phi_1^2\Phi_3\Phi_4$	50B,300A	
$\Phi_2^2\Phi_4\Phi_6$	$\Phi_3\Phi_{10}$	400C,1200L	
$\Phi_5\Phi_6$	$\Phi_1^2\Phi_2^2\Phi_3$	50B,75C	
$\Phi_3\Phi_{10}$	$\Phi_1^2\Phi_2^2\Phi_6$	50A,75A	
Φ_9	Φ_{18}	648B,648D	3-imprimitive, twists by -3 of each other
Φ_7	Φ_{14}	392C,392F	
$\Phi_5\Phi_6$	$\Phi_3\Phi_{10}$	20A,300D	
$\Phi_2^2\Phi_{10}$	$\Phi_1^2\Phi_5$	40A,200B	
$\Phi_2^2\Phi_6^2$	$\Phi_1^2\Phi_3^2$	24A,72A	3-imprimitive, twists by -3 of each other

TABLE 17. Splittings for $\tilde{t}=1$ data in degree 6 weight 1

$\mathcal{P}_{\mathcal{A}}$	$\mathcal{P}_{\mathcal{B}}$	w	N	a_p	w	N	a_p
$\Phi_3\Phi_{10}$	$\Phi_1^4\Phi_6$	4	100	$(26,45,44)$	4	300	$(-7,-54,-55)$
$\Phi_2^4\Phi_3$	$\Phi_5\Phi_6$	4	100	$(-26,45,-44)$	4	300	$(7,-54,55)$
$\Phi_3\Phi_5$	$\Phi_6\Phi_{10}$	4	40	$(-34,16,58)$	4	600	$(4,-28,-16)$
$\Phi_2^2\Phi_5$	$\Phi_1^2\Phi_{10}$	4	10	$(-8,-4,12)$	4	50	$(2,26,-28)$
$\Phi_2^4\Phi_6$	$\Phi_1^4\Phi_3$	4	8	$(-2,24,-44)$	4	24	$(14,-24,-28)$

TABLE 18. Splittings for $\tilde{t}=1$ data in degree 6 weight 3

$\mathcal{P}_{\mathcal{A}}$	$\mathcal{P}_{\mathcal{B}}$	w	N	a_p	w	N	a_p
Φ_3^3	Φ_6^3	4	72	$(-16,-12,-64)$	6	72	$(-16,12,-448)$
$\Phi_2^2\Phi_3^2$	$\Phi_1^2\Phi_6^2$	4	18	$(-6,-16,-12)$	6	6	$(-66,176,-60)$
$\Phi_2^4\Phi_3$	$\Phi_1^4\Phi_6$	4	36	$(18,8,-36)$	6	4	$(54,-88,540)$
Φ_2^6	Φ_1^6	4	8	$(-2,24,-44)$	6	8	$(-74,-24,124)$

TABLE 19. Splittings for $\tilde{t}=1$ data in degree 6 weight 5

$\mathcal{P}_{\mathcal{A}}$	$\mathcal{P}_{\mathcal{B}}$	w	N	a_p
Φ_3^4	Φ_6^4	6	36	$(-54,-88,-540)$
$\Phi_2^2\Phi_3^3$	$\Phi_1^2\Phi_6^3$	6	18	$(96,-148,-384)$
$\Phi_2^4\Phi_3^2$	$\Phi_1^4\Phi_6^2$	6	576	$(6,-40,-564)$
$\Phi_2^6\Phi_3$	$\Phi_1^6\Phi_6$	6	4	$(54,-88,540)$
Φ_2^8	Φ_1^8	6	4	$(54,-88,540)$

TABLE 20. Degree 2 pieces of $\tilde{t}=1$ splittings in degree 8 weight 7

18.3.2. *A maximal weight imprimitive family.* Similarly to the data with $\tilde{t}=-1$, there is a symplectic family in maximal weight which (experimentally) always splits, namely $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_2^k\Phi_3^l,\Phi_1^k\Phi_6^l)$ with k even.

For degree 10 hypergeometric data at $\tilde{t}=1$, thus of an L-function of degree 8, the L-functions for these families split into two pieces of degree 4.

18.3.3. *Sundry.* There is a four-fold splitting for the 6-imprimitive Belyi example given by $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_4\Phi_{12},\Phi_1\Phi_2\Phi_3\Phi_6)$, where the $\tilde{t}=1$ degeneration splits as $\chi_{-4}\oplus\chi_8\oplus\chi_{12}\oplus\theta_{12^2}$ where θ_{12^2} is a 2-dimensional Artin representation of conductor 144 over the octic field $y^8-8y^6+12y^4-8y^2+4$

There is one weight 4 splitting example from primitive degree 10 hypergeometric data, namely $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_3^3\Phi_8,\Phi_1\Phi_2^3\Phi_6^3)$ which has a pole, with its twist also being imprimitive.

There are (up to twist) three weight 4 examples of splittings (all primitive) in degree 11: $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_2\Phi_3^3\Phi_{12},\Phi_1\Phi_8\Phi_9)$, $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_2\Phi_3^2\Phi_4\Phi_6^2,\Phi_1\Phi_{12}\Phi_{18})$ and $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_2^5\Phi_6^3,\Phi_1^3\Phi_3^3\Phi_4)$. Each of these has a ζ-constituent (the same is true for the twist of the third), and third splits further into pieces of dimension 2 and 7, the smaller piece being associated to (a Tate twist) of the $(2,0)$ Grössencharacter for $\mathbf{Q}(\sqrt{-3})$ with modulus 2 (namely $\Psi_{-3}^{2,0}(2\,\mathcal{O}_K,\psi_1)$ in previous notation). Here (and in other places), we inferred the splitting from what the factorization of Euler factors at small primes might suggest.

18.4. Analytic rank beyond the sign. David Roberts noted the first example of a $t=1$ degeneration (in odd weight) having extra vanishing[8] for its central value, and we give a complete catalogue of known examples below.

We should first note that when the sign of the functional equation is even, then it seems that observed analytic rank 2 is quite common when Φ_1 is not part of the hypergeometric data, and conversely is quite rare when it is.

18.4.1. *Degree 4, weight 1.* Here 32 of the 74 examples have even parity, and 28 of the 32 have Φ_1 as part of the hypergeometric data, with all 28 having rank 0. On the other hand, 3 of the remaining 4 cases have observed analytic rank 2, namely $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_3^2,\Phi_8)$ and its twist which each give the elliptic curve 3456l, and also $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_{10},\Phi_{12})$ which gives 21600ct. The datum $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_3\Phi_4,\Phi_2^2\Phi_6)$ has rank 0.

All 42 of the odd parity examples have analytic rank 1. It is also the case that only one of these 42 has Φ_1^2 involved, namely $(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}})=(\Phi_8,\Phi_1^2\Phi_2^2)$.

[8]Here "extra vanishing" means any vanishing not forced by the functional equation, *including* when breaking down into primitive constituents of the L-function.

18.4.2. *Degree 4, weight 3.* Here 22 of the 47 examples have even parity, of which 18 have Φ_1 involved. One of these 18 has observed analytic rank 2, namely $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_{12},\Phi_1^4)$ which is associated to a modular form of level 864. All 4 of those which exclude Φ_1 have rank 2, including $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_2^4,\Phi_{12})$ (the twist of the above), $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_3^2,\Phi_4^2)$ and its twist (both again associated to this same modular form of level 864), and $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_3^2,\Phi_{10})$ which is associated to a modular form of level 5400.

Again all 25 odd parity examples are analytic rank 1, and 7 of these involve Φ_1^4.

18.4.3. *Degree 6, weight 1.* Here we have 287 hypergeometric data, of which 168 have even parity and 119 have odd parity. Of the latter, two of them have observed analytic rank 3, namely $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_6\Phi_8,\Phi_9)$ and its twist.

With even parity, there are 69 of analytic rank 0 and 99 of observed analytic rank 2 (caveat, I did not check for higher ranks). The splitting is almost completely based upon whether Φ_1^2 is part of the hypergeometric data. The only example of rank 2 with Φ_1^2 extant is $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_6\Phi_8,\Phi_1^2\Phi_5)$, and the only example of rank 0 without it is $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_2^2\Phi_4\Phi_6,\Phi_3\Phi_8)$.

Note that three of rank 2 examples are imprimitive (see Table 17), namely $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_9,\Phi_{18})$, $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_7,\Phi_{14})$, and $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_2^2\Phi_4\Phi_6,\Phi_3\Phi_{10})$. In each case we have a sum of two elliptic curves each of rank 1.

18.4.4. *Degree 6, weight 3.* There are 238 odd parity hypergeometric data here, of which 5 have observed analytic rank 3 (see Table 21). The third and fourth are the same motive.

$\mathcal{P}_\mathcal{A}$	$\Phi_2^2\Phi_{12}$	$\Phi_2^2\Phi_{12}$	$\Phi_4^2\Phi_6$	$\Phi_3\Phi_4^2$	$\Phi_4^2\Phi_6$
$\mathcal{P}_\mathcal{B}$	Φ_{14}	Φ_7	Φ_{18}	Φ_9	Φ_{10}

TABLE 21. Degree 6 weight 3 cases of rank 3

Of the 249 even parity data, 80 of them have observed analytic rank 2. The only rank 0 case where Φ_1 does not appear is $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_4^2,\Phi_2^4\Phi_6)$, while there are 8 cases where a datum with Φ_1 has rank 2 (Table 22). The second and seventh examples are the same motive.

$\mathcal{P}_\mathcal{A}$	Φ_{18}	Φ_{18}	Φ_{18}	Φ_{18}	$\Phi_2^2\Phi_{12}$	$\Phi_2^2\Phi_{10}$	Φ_9	Φ_7
$\mathcal{P}_\mathcal{B}$	$\Phi_1^4\Phi_6$	$\Phi_1^4\Phi_3$	$\Phi_1^4\Phi_2^2$	$\Phi_1^2\Phi_4^2$	$\Phi_1^2\Phi_{10}$	$\Phi_1^2\Phi_{12}$	$\Phi_1^4\Phi_3$	$\Phi_1^2\Phi_{12}$

TABLE 22. Degree 6 weight 3 cases of rank 2 with Φ_1

Note that $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_5,\Phi_6\Phi_{10})$ and $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_2^4\Phi_3,\Phi_5\Phi_6)$ are imprimitive, and in both cases the constituents each have rank 1.

18.4.5. *Degree 6, weight 5.* There are 65 odd parity data here, and all have rank 1. Of the 77 with even parity, 42 have Φ_1 involved and all these have rank 0. The 35 others all have observed rank 2, with $(\mathcal{P}_\mathcal{A},\mathcal{P}_\mathcal{B}) = (\Phi_6^3,\Phi_3^3)$ being the product of two rank 1 constituents.

18.4.6. *Higher degree.* The natural(?) conjecture might be that even parity data that do not involve Φ_1 have rank 2 (or higher?) except when Φ_3 and Φ_6 appear on opposing sides and the rest of the data is products of various Φ_{2^v}. The three known rank 0 cases are $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_4, \Phi_2^2\Phi_6)$, $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^2\Phi_4\Phi_6, \Phi_3\Phi_8)$, and $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3\Phi_4^2, \Phi_2^4\Phi_6)$.

David Roberts has also computed the rank 2 example $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3^4, \Phi_1^8)$, which is degree 6 and weight 7 with conductor 3^9.

18.5. Root numbers. It looks difficult to determine the root numbers. For instance $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3^2, \Phi_8)$ has even parity while $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3^2, \Phi_2^2\Phi_4)$ has odd parity, which might lead one to think that the local root numbers at 2 for Φ_8 and $\Phi_2^2\Phi_4$ are opposites. However one finds that $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_2^2\Phi_4, \Phi_1^2\Phi_3)$ and $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_8, \Phi_1^2\Phi_3)$ are both even parity, which seems to indicate that there is something additional occurring.

18.6. Wild Euler factors. For all 1257 hypergeometric data of degree 6 (and 336 examples of lesser degree) we were able to (numerically) compute the conductor and Euler factors at wild primes for the $\tilde{t} = 1$ degeneration. Now included with Magma.

Expand this ...

Two worked examples with p-adic monodromy

19. Worked example

This is my attempt to understand the relation of inertia and Euler factors to hypergeometric differential equations, in the simplest case of $\bigl(\mathcal{P}_{\mathcal{A}},\mathcal{P}_{\mathcal{B}}\bigr)=\bigl(\Phi_2,\Phi_1\bigr)$. Maybe it extends to other Artin cases (where again one has algebraic functions).

Here we have (maybe switching α and β) the differential equation

$$z\theta F(z)=\Bigl(\theta-\frac{1}{2}\Bigr)F(z)\quad\text{or}\quad z^2F'(z)=zF'(z)-\frac{1}{2}F(z),$$

where $\theta=z\frac{d}{dz}$ and the solution is $F(z)=\sqrt{\frac{z}{z-1}}$. Of course, this is only up to a constant, but I think we want to choose the global normalisation so that $F(\infty)=1$. At any rate, this appears to work.

We consider what happens to $F(z)$ as a $\mathbf{C}_p$-solution about $z=t$ in the previous parametrisation. We go through the various cases, at first considering $p>2$.

19.1. Tame primes. When $v_p(t)>0$, this valuation v can either be even or odd. When it is odd, writing $t=t_0p^v$ the solution about $z=t$ is

$$\frac{\sqrt{t_0}\sqrt{p}\cdot p^{(v-1)/2}}{\sqrt{p^v-1}}+F'(t)(z-t)+\cdots,$$

and this is defined over a ramified extension of $\mathbf{Q}_p$, implying that the inertia group is nontrivial (if I remember correctly). Thus the Euler factor is trivial, as expected. The local conductor is p^1, as the inertia is C_2, hence tame for odd primes.

When v is even, writing $t=t_0p^v$ the solution is

$$\frac{\sqrt{-t_0}\,p^{v/2}}{\sqrt{1-t_0p^v}}+F'(t)(z-t)+\cdots.$$

The constant term is defined over $\mathbf{Q}_p$ precisely when $-t_0$ is square modulo p, and else over an unramified extension. Thus inertia is trivial, while the action of Frobenius fixes this solution when $-t_0$ is square, and else negates it. So the Euler factor is $(1-\chi_p(-t_0)T)$.

This agrees with the calculation given in the tame prime section (§11.4). That is, there we get $\omega_p(4t_0)^{(p-1)/2}\cdot(-1)\cdot p\cdot\mathfrak{g}_p(0)/\mathfrak{g}_p\bigl(\frac{p-1}{2}\bigr)^2$. Now the trivial Gauss sum is $\mathfrak{g}_p(0)=-1$ while the evaluation of quadratic Gauss sums gives $\mathfrak{g}_p\bigl(\frac{p-1}{2}\bigr)^2=\bigl(\frac{-1}{p}\bigr)p$, and the ω_p-term is just $\chi_p(t_0)$, so that we do indeed get $\chi_p(-t_0)$.

The situation is analogous for $v_p(t)<0$. We then invert $z\to 1/z$ and get the solution has constant term $\sqrt{1/(1-p^v/t_0)}$, which is in $\mathbf{Q}_p$. Thus inertia is trivial and Frobenius fixes the solution, giving $(1-T)$ as the Euler factor.

19.2. Multiplicative primes. With $v_p(t-1)>0$ there are again two cases depending on whether the valuation is odd or even. Writing $t-1=p^vt_0$, the constant term of the power-series solution about $z=t$ is given by $\sqrt{\frac{1+p^vt_0}{p^vt_0}}$, and this is in a ramified extension when v is odd. When v is even, it is in $\mathbf{Q}_p$ precisely when t_0 is square. Thus we get $\bigl(1-\chi_p(t_0)T\bigr)$ as the Euler factor.

Again this agrees with the formula given in §11.2 above, where $K=1$ for this data. Note that the differential equation seems to give the Euler factor, though the hypergeometric trace does not in this case.

19.3. Good primes. Here we plainly have $\sqrt{\frac{t}{t-1}}$ as the constant term of the series expansion about $z = t$, and thus Frobenius fixes the solution when $t(t-1)$ is a square modulo p. This agrees with the identification of the L-series as being associated to the quadratic algebra $\mathbf{Q}\big(\sqrt{t(t-1)}\big)$.

19.4. The wild prime 2. Finally we get to the wild prime. In some sense this is the most interesting (if also most tedious) case, as it should tell us how the 2-adic solution of the differential equation relates to the conductor and Euler factor.

When $v_2(t) > 0$ we write $t = t_0 2^v$ and have $\sqrt{\frac{-t_0 2^v}{1-2^v t_0}}$ as the constant term of the power-series expansion. When $v \geq 3$ the denominator has a square root in $\mathbf{Q}_2$, and we are left to analyse the extension $\mathbf{Q}_2\big(\sqrt{-t_0 2^v}\big)$. This has trivial inertia when v is even and $t_0 \equiv 3 \pmod 4$, with trivial Frobenius action when $t_0 \equiv 7 \pmod 8$. When v is odd the local conductor is 2^3; for even v and $t_0 \equiv 1 \pmod 4$ it is 2^2.

When $v = 2$, one proceeds similarly with the extension $\mathbf{Q}_2\big(\sqrt{\frac{-t_0}{1+4t_0}}\big)$, where the inertia is as before, but the Frobenius action is now trivial (that is, the extension is trivial) when $t_0 \equiv 3 \pmod 8$. When $v = 1$ one again has local conductor 2^3.

The above agrees with the L-series of the quadratic algebra, but this is kind of obvious, seeing as how we took everything from $F(z) = \sqrt{\frac{z}{z-1}}$.

The rest is tedious (and I think I convinced myself by now). In short, one is able, at least in this case, to read off the local information at p from considering inertia and Frobenius act on the $\overline{\mathbf{Q}}_p[[z]]$-solution space around $z = t$.

20. Another worked example

Here I try to work out $(\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}) = (\Phi_3, \Phi_1\Phi_2)$. I don't know the general method to derive the algebraic functions. I managed to get them via evaluating the power series at enough points and interpolating/guessing. I get

$$F_-(z) = 2^{-1/3} z^{2/3}\left(\frac{-2}{z(1-z)} + \frac{2}{z(1-z)^{3/2}}\right)^{1/3} = \frac{z^{1/3}}{(1-z)^{1/2}}\big(1-\sqrt{1-z}\big)^{1/3},$$

$$F_+(z) = 2^{1/3} z^{1/3}\left(\frac{1/2}{(1-z)} + \frac{1/2}{(1-z)^{3/2}}\right)^{1/3} = \frac{z^{1/3}}{(1-z)^{1/2}}\big(1+\sqrt{1-z}\big)^{1/3}.$$

The above expressions also show that $F_-(z) - F_+(z)$ is holomorphic about $z = 1$, Around $z = \infty$, we write $u = 1/z$ and get

$$F_\pm(u) = \frac{1}{\sqrt{u-1}}\big(\sqrt{u} \pm \sqrt{u-1}\big)^{1/3} = \frac{1}{(u-1)^{1/3}}\left(\frac{\sqrt{u}}{\sqrt{u-1}} \pm 1\right)^{1/3},$$

so that their difference will be holomorphic at $u = 0$, and their sum will be $\sqrt{u}$ times something holomorphic.

From symbolic algebra, I get that

$$F'_\pm(z) = \left[\frac{\sqrt{1-z}}{2z(1-z)^2} \mp \frac{1/6}{z(1-z)}\right] z^{1/3}\big(1 \pm \sqrt{1-z}\big)^{1/3}.$$

About $z = t$ we have $F_\pm(z) = F_\pm(t) + F'_\pm(t)(z-t) + O\big((z-t)^2\big)$, and the question is how inertia and Frobenius act on these solutions.

It seems that a/the "natural" basis/scaling for the solution space is $\{S(z), D(z)\}$ where $D(z) = F_-(z) - F_+(z)$ and $S(z) = \sqrt{-3} \cdot \big[F_+(z) + F_-(z)\big]$ (up to $\mathbf{Q}$-factors).

I am not sure from where this global normalisation is derived. The point is that $S(z)$ will incur an extra minus sign when Frobenius acts for primes $p \equiv 2 \pmod 3$.

20.1. Tame primes. The first case has $v_p(t) > 0$. Writing $t = t_0 p^v$ we get (where each term in the power series expansions is correct to its leading p-adic digit)

$$F_-(z) \sim p^{2v/3} t_0^{1/3} (t_0/2)^{1/3} + \frac{2}{3p^v t_0} t_0^{1/3} p^{v/3} (p^v t_0/2)^{1/3} (z-t) + O\big((z-t)^2\big),$$

$$F_+(z) \sim p^{v/3} t_0^{1/3} 2^{1/3} + \frac{1}{3p^v t_0} t_0^{1/3} p^{v/3} 2^{1/3} (z-t) + O\big((z-t)^2\big),$$

In particular, we find that there is nontrivial inertia when $3 \nmid v$. When $3|v$ there are naturally two cases. When $p \equiv 1 \pmod 3$ we have $-3 \in \mathbf{Q}_p^2$ and so Frobenius fixes the solutions when $2t_0 \in \mathbf{Q}_p^3$, with the action otherwise being $\mathrm{diag}(\zeta_3, \zeta_3^{-1})$; thus the Euler factors are $(1-2T+T^2)$ and $(1+T+T^2)$ respectively. In the case where $p \equiv 2 \pmod 3$ we have $-3 \notin \mathbf{Q}_p^2$ and $2t_0 \in \mathbf{Q}_p^3$, and so $D(z)$ is fixed by Frobenius while $S(z)$ is negated due to the $\sqrt{-3}$, giving us an Euler factor of $(1-T^2)$.

The second case has $v_p(1/t) > 0$. Using the above u-expansion about ∞ and writing $t = t_0/p^v$, when v is odd we see Frobenius fixes $F_-(z) - F_+(z) \in \mathbf{Q}_p[[z]]$, while inertia is nontrivial on the sum. Thus the Euler factor is $(1-T)$. When v is even, the difference-solution $D(z)$ is as before, while the leading term of $S(z)$ is p-adically close to $\sqrt{-3} \cdot \frac{(2/\sqrt{t_0})p^{v/2}}{3\sqrt{-1}}$. Thus $S(z)$ is fixed by Frobenius when $3t_0 \in \mathbf{Q}_p^2$. The overall Euler factor is $(1-T)(1 \pm T)$, with the minus sign when $3t_0 \in \mathbf{Q}_p^2$.

20.2. Multiplicative primes. Here $v_p(t-1) > 0$. Writing $t - 1 = p^v t_0$ we have

$$F_\pm(z) \sim \frac{1}{\sqrt{-p^v t_0}} (1 \pm \sqrt{-p^v t_0})^{1/3} + O(z-t).$$

The difference $D(z)$ is thus defined over $\mathbf{Q}_p$. The sum $S(z)$ has leading term close to $2\sqrt{-3}/\sqrt{-p^v t_0}$, with nontrivial inertia when v is odd. When v is even, Frobenius again acts trivially when $3t_0 \in \mathbf{Q}_p^2$. Thus the Euler factor is $(1-T)$ when v is odd, and is $(1-T)(1 \pm T)$ when v is even, with the minus sign when $3t_0 \in \mathbf{Q}_p^2$.

20.3. Good primes. First we consider the cases where $1 - t \in \mathbf{Q}_p^2$, when we have $F_\pm(t)/F'_\pm(t) \in \mathbf{Q}_p$. Indeed, when $p \equiv 2 \pmod 3$ we have $F_\pm(t) \in \mathbf{Q}_p$ also, and so $F_\pm(z)$ are fixed by Frobenius. Thus $D(z)$ is fixed and $S(z)$ is negated, so the Euler factor is $(1-T)(1+T)$. When $p \equiv 1 \pmod 3$, we must consider if $t(1 \pm \sqrt{1-t}) \in \mathbf{Q}_p^3$. If so, then Frobenius fixes $F_\pm(t)$, and else it multiplies them by $\zeta_3^\pm$ [note that $F_-(z)F_+(z) = \frac{z}{1-z}$, fixing the product of cube roots]. In the former case we get $(1-T)^2$ as the Euler factor, and else $(1-\zeta_3 T)(1-\zeta_3^{-1}T) = (1+T+T^2)$.

For the second case we have that $1 - t \notin \mathbf{Q}_p^2$. We write $K = \mathbf{Q}_p(\sqrt{1-t})$ and need to determine whether $t(1-\sqrt{1-t}) \in K^3$, For primes $p \equiv 1 \pmod 3$ it seems this is always the case (reciprocity law?). Thus Frobenius acts by negating $\sqrt{1-t}$, sending $F_\pm(z)$ to $-F_\mp(z)$, with Euler factor $(1-T^2)$. Similarly, for $p \equiv 2 \pmod 3$, when $t(1-\sqrt{1-t}) \in K^3$ we again have $F_\pm(z) \to -F_\mp(z)$ so that $D(z)$ is fixed by Frobenius, and indeed so is $S(z)$ via the $\sqrt{-3}$ factor [which Frobenius negates], for an Euler factor of $(1-T)^2$. Else the action sends $F_\pm(z) \to -\zeta_3^\pm F_\mp(z)$, and we get $D(z) \to \big(S(z) - D(z)\big)/2$ and $S(z) \to -\big(S(z) + 3D(z)\big)/2$ [the leading minus sign from negating $\sqrt{-3}$], yielding a trace of -1 for an Euler factor of $(1+T+T^2)$.

20.4. A special case. Note that when $t = \frac{343}{100}$ the L-function is $\zeta(s)^2$. Indeed, for good primes we have that $(1-t)$ is square-equivalent to (-3). Thus for primes $p \equiv 1 \pmod 3$ we are interested in whether $\frac{7^3}{(1+9)^2}\left(1 \pm \frac{3^2}{5}\sqrt{-3}\right)$ is a cube mod p, which I guess follows from some sort of cubic reciprocity law. With $p \equiv 2 \pmod 3$ the relevant expression is again always in K^3. This shows that the Euler factor is $(1-T)^2$ at all good primes, so by general nonsense it equals $\zeta(s)^2$ at all primes.

Indeed, the L-function is $\zeta(s)^2$ for any $t = 1/Mx^2(1-x)$ for which $-3x^2+2x+1$ is square (our example being the orbit $x = 5/21, 20/21, -4/21$).

20.5. Wild primes. Well, I guess one gets an algorithm to compute here, but I don't see it as any easier than doing the same with the Artin representations.

TODO:

- Expand wild primes section (with data, FRV/David guesses)
- Wild primes for $t = 1$ (David's slope stuff)
- Other non-Belyi cases (section 5) (maybe from Beukers/Heckmann?)
- Beukers/Mellitt on canonical schemes

21. Twisting

The term twisting refers to transforming hypergeometric data via adding 1/2 to all elements in α and β. Alternatively (in the case over $\mathbf{Q}$), all elements of $\mathcal{A}$ and $\mathcal{B}$ with 2-valuation less than 2 are either doubled (valuation 0) or halved (valuation 1). Our $(\mathcal{A}, \mathcal{B})$-convention can cause them to be switched when this twisting is done.

The hypergeometric traces behave in a specified way upon twisting. Namely, writing $U_q^{\mathcal{H}}(t)$ for the qth trace for parameter t with data $\mathcal{H}$, (for good primes) the twist $\mathcal{T}$ will have (here χ_u is the Kronecker character $\left(\frac{u}{\cdot}\right)$, and Δ is the discriminant)

$$U_q^{\mathcal{T}}(\tilde{t}) = U_q^{\mathcal{H}}(\tilde{t}^{\pm})\chi_u(q) \text{ where } u = (-\tilde{t})(-1)^w \prod_{e \in \mathcal{A} \cup \mathcal{B}} \Delta(\Phi_e),$$

with the $\pm$ sign in the $\tilde{t}$-exponent as -1 when $\mathcal{A}$ and $\mathcal{B}$ are switched.