SOME REMARKS ON HEEGNER POINT COMPUTATIONS

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1. Introduction

We make some remarks concerning Heegner point computations. One of our goals shall be to give an algorithm to find a non-torsion rational point on a given rank 1 elliptic curve. Much of this is taken from a section in Henri Cohen's latest book [6], and owes a great debt to Christophe Delaunay.

2. Definitions and Outline of Theory

Let τ be a quadratic surd in the upper-half-plane $\mathbf{H}$. Let $f_\tau = (A, B, C)$ be the associated integral primitive positive-definite binary quadratic form, so that $A\tau^2 + B\tau + C = 0$ with $A > 0$ and $\gcd(A, B, C) = 1$. The discriminant $\Delta(\tau)$ is $\Delta(f_\tau) = B^2 - 4AC$, which is negative. For simplicity, we take it to be fundamental.

Definition 2.1. A Heegner point of level N and discriminant D is a quadratic surd in the upper-half-plane with $\Delta(\tau) = D = \Delta(N\tau)$. We let $\mathcal{H}_N^D$ be the set of Heegner points of level N and discriminant D.

Proposition 2.2. *Let $\tau \in \mathbf{H}$ be a quadratic surd and $f_\tau = (A, B, C)$. Then $\tau \in \mathcal{H}_N^D$ iff $N|A$ and $\gcd(A/N, B, CN) = 1$.*

Proof. Note that $\tau = \frac{-B+\sqrt{D}}{2A}$ and $N\tau = \frac{-NB+N\sqrt{D}}{2A}$. For $\Delta(\tau) = \Delta(N\tau)$ we need $N\tau = \frac{-B'+\sqrt{D}}{2A'}$ and by equating imaginary and real parts we get $A = NA'$ and $B = B'$, so that $N|A$. Also note that $(A/N)(N\tau)^2 + B(N\tau) + (CN) = 0$, from which we get the rest of the lemma. □

Note that $\mathcal{H}_N^D$ will be empty unless $D = B^2 - 4N(A/N)C$ is a square modulo $4N$.

Lemma 2.3. *The set $\mathcal{H}_N^D$ is closed under $\Gamma_0(N)$-action.*

Proof. If $\gamma \in SL_2(\mathbf{Z}) \supseteq \Gamma_0(N)$ then $\Delta\big(\gamma(\tau)\big) = \Delta(\tau)$ since the discriminant is fixed. A computation shows that $\gamma \in \Gamma_0(N)$ and $\tau \in \mathcal{H}_N^D$ imply $\gamma(\tau) \in \mathcal{H}_N^D$. □

Lemma 2.4. *The set $\mathcal{H}_N^D$ is closed under the W_N-action that sends $\tau \to -1/N\tau$.*

Proof. Follows from the above proposition since $f_{(-1/N\tau)} = (CN, -B, A/N)$. □

Definition 2.5. We let $\mathcal{S}(D, N)$ be the set of square roots mod $2N$ of D mod $4N$.

Theorem 2.6. *There is a 1-1 map between $\mathcal{H}_N^D/\Gamma_0(N)$ and $\mathcal{S}(D, N) \times \mathcal{C}l\big(\mathbf{Q}(\sqrt{D})\big)$.*

Proof. This can be shown by chasing definitions. Essentially, $[\tau] \in \mathcal{H}_N^D/\Gamma_0(N)$ gets mapped to $(B \bmod 2N) \times [\mathbf{Z}+\tau\mathbf{Z}]$ where $f_\tau = (A, B, C)$, and in the other direction, when we are given $\beta \times l \in \mathcal{S}(D, N) \times \mathcal{C}l\big(\mathbf{Q}(\sqrt{D})\big)$ we take $(A, B, C) \in l$ with $N|A$ and $B \equiv \beta \pmod{2N}$, and then $\tau = \frac{-B+\sqrt{D}}{2A}$. □

From now on we let E be a global minimal model of a rational elliptic curve of conductor N, and take D to be a negative fundamental discriminant. We let $\mathbf{H}^\star$ be the union of $\mathbf{H}$ with the rationals and $i\infty$. We let $\mathcal{P}(z)$ be the function that sends $z \in \mathbf{C}/\Lambda$ to the point $\big(\wp(z), \wp'(z)\big)$ on E.

Theorem 2.7. *There is a surjective modular parametrisation map $\phi : X_0(N) \to E$, where $X_0(N) = \mathbf{H}^\star/\Gamma_0(N)$ and E can be viewed as $\mathbf{C}/\Lambda$ for some lattice Λ.*

Proof. This is due to Wiles and others [22, 21, 8, 7, 3]. Explicitly, we have that $\tau \in \mathbf{H}^\star$ gets mapped to the complex point $\phi(\tau) = 2\pi i \int_{i\infty}^{\tau} \psi_E = \sum_n (a_n/n) e^{2\pi i n\tau}$, where ψ_E is the modular form of weight 2 and level N associated to E. The lattice Λ is generated by the real and imaginary periods,[1] which we denote by Ω_{re} and Ω_{im}. We assume that the Manin constant is 1, which is conjectured always to be the case for curves of positive rank (see [20] and [18]). □

Theorem 2.8. *Let $\tau = \beta \times l \in \mathcal{H}_N^D$. Then $\mathcal{P}\big(\phi(\tau)\big)$ has its coordinates in the Hilbert class field of $\mathbf{Q}(\sqrt{D})$. Also we have*

(1) $\overline{\phi(\beta \times l)} = \phi\big(-\beta \times l^{-1}\big)$, *in* $\mathbf{C}/\Lambda$.
(2) $\phi\big(W_N(\beta \times l)\big) = \phi\big(-\beta \times ln^{-1}\big)$ *in* $\mathbf{C}/\Lambda$ *where* $n = \big[N\mathbf{Z} + \frac{\beta+\sqrt{D}}{2}\mathbf{Z}\big]$,
(3) $\mathcal{P}\big(\phi(\beta \times l)\big)^{\mathrm{Artin}(m)} = \mathcal{P}\big(\phi\big(\beta \times lm^{-1}\big)\big)$ *for all* $m \in \mathcal{C}l\big(\mathbf{Q}(\sqrt{D})\big)$.

Proof. This is the theorem of complex multiplication of Shimura [16, 15]. □

Note that $\mathcal{P}\big(\overline{\phi(\tau)}\big) = \overline{\mathcal{P}\big(\phi(\tau)\big)}$, so that there is no danger of confusing complex conjugation in $\mathbf{C}/\Lambda$ with complex conjugation of the coordinates of the point on E. Using the third of these facts, we can take the trace of $\mathcal{P}\big(\phi(\tau)\big)$ and get a point that has coordinates in $\mathbf{Q}(\sqrt{D})$. Indeed, writing H for the Hilbert class field and K for $\mathbf{Q}(\sqrt{D})$ we get that

$$\begin{aligned} P = \mathrm{Trace}_{H/K}\big(\mathcal{P}\big(\phi(\tau)\big)\big) &= \sum_{\sigma\in \mathrm{Gal}(H/K)} \mathcal{P}\big(\phi(\tau)\big)^{\sigma} = \sum_{m\in\mathcal{C}l(K)} \mathcal{P}\big(\phi(\beta\times l)\big)^{\mathrm{Artin}(m)} \\ &= \sum_m \mathcal{P}\big(\phi\big(\beta\times lm^{-1}\big)\big) = \sum_{m\in\mathcal{C}l(K)} \mathcal{P}\big(\phi(\beta\times m)\big) \end{aligned}$$

has coordinates in $\mathbf{Q}(\sqrt{D})$. When E has odd functional equation, we can use the theorem to show that $P = \overline{P}$, and thus P has coordinates in $\mathbf{Q}$. In this case we have that $\psi_E = \psi_E \circ W_N$ which implies $\phi = \phi \circ W_N$, so that in $\mathbf{C}/\Lambda$ we have

$$\overline{\phi(\beta\times m)} = \overline{\phi\big(W_N(\beta\times m)\big)} = \overline{\phi\big(-\beta\times mn^{-1}\big)} = \phi\big(\beta\times m^{-1}n\big),$$

which gives us that

$$\overline{P} = \sum_m \mathcal{P}\big(\overline{\phi(\beta\times m)}\big) = \sum_m \mathcal{P}\big(\phi\big(\beta\times m^{-1}n\big)\big) = \sum_{m\in\mathcal{C}l(K)} \mathcal{P}\big(\phi(\beta\times m)\big) = P.$$

[1] Our convention is that the imaginary period is purely imaginary when the discriminant of E is positive, and in the negative discriminant case the real part of the imaginary period is $\Omega_{\mathrm{re}}/2$. The fundamental volume Ω_{vol} is the area of the period parallelogram.

We can rewrite this by introducing some new notation.

Definition 2.9. We write $\mathcal{H}_N^D(\beta)$ for subset of $\tau \in \mathcal{H}_N^D$ such that the associated form $f_\tau = (A, B, C)$ has $B \equiv \beta \pmod{2N}$. We write $\hat{\mathcal{H}}_N^D = \mathcal{H}_N^D/\Gamma_0(N)$, and noting that $\Gamma_0(N)$ acts on $\mathcal{H}_N^D(\beta)$, we write $\hat{\mathcal{H}}_N^D(\beta) = \mathcal{H}_N^D(\beta)/\Gamma_0(N)$.

Since $\hat{\mathcal{H}}_N^D(\beta)$ is in 1-1 correspondence with $\mathcal{C}l\big(\mathbf{Q}(\sqrt{D})\big)$, we get that

$$P = \sum_{m \in \mathcal{C}l(Q(\sqrt{D}))} \mathcal{P}\big(\phi(\beta \times m)\big) = \sum_{\tau \in \hat{\mathcal{H}}_N^D(\beta)} \mathcal{P}\big(\phi(\tau)\big).$$

3. The Gross-Zagier theorem and an algorithm

We now have a plan of how to find a non-torsion point on a curve of analytic rank 1. We select an auxiliary negative fundamental discriminant D such that D is a square modulo $4N$, choose $\beta \in \mathcal{S}(D, N)$, find τ-representatives for $\hat{\mathcal{H}}_N^D(\beta)$, compute $\phi(\tau)$ for each, sum these in $\mathbf{C}/\Lambda$, map the resulting point to E via the Weierstrass parametrization, and try to recognize the result as a rational point. One problem is that we might get a torsion point. Another problem is that we won't necessarily get a generator, and thus the point might have inflated height, which would increase our requirements on real-number precision. The Gross-Zagier Theorem tells us what height to expect, and combined with the Birch–Swinnerton-Dyer Conjecture, we get a prediction of what height a generator should have. Our heights will be the "larger" ones, and are thus twice those chosen by some other authors.

Theorem 3.1. *Suppose $D < -3$ is a fundamental discriminant with D a square modulo $4N$ and $\gcd(D, 2N) = 1$. Then*

$$h(P) = \frac{\sqrt{|D|}}{4\Omega_{\mathrm{vol}}} L'(E,1)L(E_D,1) \times 2^{\omega(\gcd(D,N))} \left(\frac{w(D)}{2}\right)^2.$$

Proof. This is due to Gross and Zagier [11]. Here E_D is the quadratic twist of E by D, while $w(D)$ is the number of units in $\mathbf{Q}(\sqrt{D})$ and $\omega(n)$ is the number of distinct prime factors of n. □

Calculations of Gross and Hayashi [12] indicate that this height formula is likely to be true for all negative fundamental discriminants D that are square mod $4N$.

We now write $P = lG + T$ where G is a generator and T is a torsion point, so that $h(P) = l^2 h(G)$. Then we replace $L'(E, 1)$ through use of the Birch–Swinnerton-Dyer conjecture [2] to get the following.[2]

Conjecture 3.2. *With notations as above we have that*

$$l^2 = \frac{\Omega_{\mathrm{re}}}{4\Omega_{\mathrm{vol}}} \left(\prod_p c_p \cdot \#\text{Ш}\right) \frac{\sqrt{|D|}}{\#E(\mathbf{Q})^2_{\mathrm{tors}}} L(E_D, 1) \cdot \left(\frac{w(D)}{2}\right)^2 2^{\omega(\gcd(D,N))}.$$

In particular, we note that we should use a quadratic twist E_D that has rank zero, so that $L(E_D, 1)$ does not vanish. The existence of such a twist is proven in [4]. Thus we have the following algorithm, which we shall work on improving.

[2] We use the convention that the Tamagawa number at infinity is equal to connected components of E over $\mathbf{R}$ — thus it is 1 for curves with negative discriminant and 2 for curves with positive discriminant.

Algorithm 3.3. *Given a rational elliptic curve E of conductor N of rank 1, find a non-torsion rational point.*

(1) *Compute $L'(E,1)$, which takes $\sqrt{N}$ time.*
(2) *Find a fundamental discriminant $D<0$ with D a square modulo $4N$ and $L(E_D,1)\neq 0$, so that the index l is nonzero.*
(3) *Choose $\beta\in\mathcal{S}(D,N)$ and compute (to sufficient precision) the complex number*
$$z=\sum_{\tau\in\hat{\mathcal{H}}_N^D(\beta)}\phi(\tau)=\sum_{\tau}\sum_{n}\frac{a_n}{n}e^{2\pi in\tau}.$$
(4) *Let m be the gcd of l and the exponent of the torsion group of E. If the discriminant of E is positive, check if*
$$\mathcal{P}\big((m\mathrm{Re}(z)+u\Omega_{\mathrm{re}})/ml\big)\quad\text{or}\quad\mathcal{P}\big((m\mathrm{Re}(z)+u\Omega_{\mathrm{re}})/ml+\Omega_{\mathrm{im}}/2\big)$$
is close to a rational point on E for $u=1,\ldots,lm$. If the discriminant of E is negative, let $o=\mathrm{Im}(z)/\mathrm{Im}(\Omega_{\mathrm{im}})$, and then check
$$\mathcal{P}\big((m\mathrm{Re}(z)+u\Omega_{\mathrm{re}})/ml+o\Omega_{\mathrm{re}}/2\big)$$
over the same u-range.

One can compute the index l in parallel with the $\phi(\tau)$, since both involve computing the a_n of the elliptic curve E. Of course, we should first take a small Euler product to ensure that the index is not going to be near zero.

3.1. **Step 3 of the Algorithm.** We now discuss how to do the third step efficiently. First note that we can sometimes pair $\phi(\tau)$ with its complex conjugate; by Theorem 2.8, in $\mathbf{C}/\Lambda$ we have
$$\overline{\phi(\beta\times l)}=\phi(-\beta\times l^{-1})=\phi\big(W_N(-\beta\times l^{-1})\big)=\phi\big(\beta\times(ln)^{-1}\big).$$
For $f=(A,B,C)\in\hat{\mathcal{H}}_N^D$ we write $\overline{f}=(A/N,-B,CN)$, so that when $g\sim\overline{f}$ we have $\phi(f)=\overline{\phi(g)}$ and thus $\phi(f)+\phi(g)=2\mathrm{Re}\phi(f)$ in $\mathbf{C}/\Lambda$. We refer to this as **pairing** the forms.

For $\sum_n(a_n/n)e^{2\pi in\tau}$ to converge rapidly, we wish for the imaginary part of our representative τ's to be large. It turns out the best we can do is essentially have the smallest imaginary part be about $1/N$ in size. We can achieve this via a trick of Delaunay, which introduces more Atkin-Lehner involutions.

Definition 3.4. Let $Q|N$ with $\gcd(Q,N/Q)=1$, and let $u,v\in\mathbf{Z}$ be such that $uQ^2-vN=Q$. The Atkin-Lehner involution W_Q sends τ to $\frac{uQ\tau+v}{N\tau+Q}$.

This defines W_Q up to transformations by elements in $\Gamma_0(N)$. One can check that $W_Q(W_Q(\tau))$ is in the $\Gamma_0(N)$-orbit of τ, and that the W_Q form an elementary abelian 2-group W of order $2^{\omega(N)}$. The important fact about the W_Q's that we shall use is that $\psi_E=\pm\psi_E\circ w_Q$, so that $\phi(\tau)=\pm\phi(W_Q(\tau))+\phi\big(W_Q(i\infty)\big)$. The sign can be computed as $\prod_{p|Q}\epsilon_p$ where ϵ_p is the local root number of E at p. Delaunay's idea is to maximise the imaginary part of τ over $\Gamma_0(N)$ and W rather than just $\Gamma_0(N)$ — the difficulty is that the action of W_Q need not preserve β. However, we still have that
$$P=\sum_{\tau\in\hat{\mathcal{H}}_N^D(\beta)}\phi(\tau)=\sum_{\tau\in\hat{\mathcal{H}}_N^D(\beta)}\epsilon_Q\phi\big(W_Q(\tau)\big)+(\text{torsion point}).$$

Explicitly, the analogue of the second part of Theorem 2.8 is that(?)

$$\phi\big(W_Q(\beta \times l)\big) = \epsilon_Q \phi(\tilde\beta \times l q_{\tilde\beta}^{-1}) + \phi\big(W_Q(i\infty)\big) \quad \text{where} \quad q_{\tilde\beta} = \left[Q\mathbf{Z} + \frac{-\tilde\beta + \sqrt{D}}{2}\mathbf{Z}\right].$$

Here $\tilde\beta$ is defined as follows. We make $\tilde\beta$ and β have opposite signs mod p^k for prime powers p^k with $p^k \| Q$ and $\gcd(p, D) = 1$, and else $\beta = \tilde\beta$. In particular, we have that $Q = \gcd(\beta - \tilde\beta, N)$ when N is odd.

The Q for which $\gcd(Q, D) > 1$ are different in that there is only one square root of D mod Q; thus β is preserved upon application of W_Q for the $Q|D$. With respect to complex conjugation, we can note that following. Suppose we have that $m \sim l^{-1} n_{\tilde\beta}$ so that $\overline{\phi(\beta \times l)} = \phi(\beta \times m)$. Then in $\mathbf{C}/\Lambda$ we have, up to torsion, that

$$\overline{\phi\big(W_Q(\beta \times l)\big)} = \epsilon_Q \phi(\tilde\beta \times l^{-1} q_{\tilde\beta} n_{\tilde\beta}) = \epsilon_Q \phi(\tilde\beta \times m q_{\tilde\beta}).$$

In the case that $q|\gcd(D, Q)$ we have that $q_{\tilde\beta} \sim q_{\tilde\beta}^{-1}$, and thus we get

$$\overline{\phi\big(W_Q(\beta \times l)\big)} = \epsilon_Q \phi(\tilde\beta \times m q_{\tilde\beta}^{-1}) = \epsilon_Q \phi\big(W_Q(\beta \times m)\big).$$

The point(?) of this is that if $(\beta \times l)$ can be paired iff $W_Q(\beta \times l)$ can be paired for $Q|D$. We now give the algorithm for finding good τ-representatives.

Algorithm 3.5. *Given D, N, find good τ-representatives.*

(1) *Choose $\beta \in \mathcal{S}(D, N)$. Set $U = \emptyset$ and $R = \emptyset$.*
(2) *While $\#R \neq \#\mathcal{C}l\big((Q\sqrt{D})\big)$ do:*
(3) *Loop over a from 1 to infinity and $b \in \mathcal{S}(D, N)$ [lift b from $\mathbf{Z}/2N$ to $\mathbf{Z}$]:*
(4) *Loop over all solutions s of $Ns^2 + bs + (b^2 - D)/4N \equiv 0$ modulo a:*
(5) *Let $f = \big(aN, b + 2Ns, ((b + 2Ns)^2 - D)/4aN\big)$.*
(6) *Loop over all positive divisors d of $\gcd(D, N)$ [which is squarefree]:*
(7) *Let $g = W_Q(f)/Q$ where Q is d times the product of the $p^k \| N$ with $b \not\equiv \beta$ mod p^k, so that $g \in \mathcal{H}_N^D(\beta)$.*
(8) *If the reductions of g and $\overline{g}$ are both not in R then append them to R, and append f to U with weight ϵ_Q when $g \sim \overline{g}$ and with weight $2\epsilon_Q$ when $g \not\sim \overline{g}$.*

We expect the maximal a to be of size $\#\mathcal{C}l(\mathbf{Q}(\sqrt{D}))/2\#W$. The above algorithm makes "parameter selection" fast compared to the computation of the $\phi(\tau)$.

3.2. **Step 4 of the Algorithm.** We now turn to the last step of our main algorithm, reconstructing a rational point on an elliptic curve from a real approximation. The most naïve method for this is simply to try to recognize the x-coordinate as a rational number. If our height calculation tells us to expect a point whose x-coordinate has a numerator and denominator of about H digits, the use of continued fractions will recognise it if we do all computations to about twice the precision, or $2H$ digits. We can note that by using a degree-n map to $\mathbf{P}^{n-1}$ and n-dimensional lattice reduction, this can be reduced to $nH/(n-1)$ digits for every $n \geq 3$ — we will discuss a similar idea later when we consider combining descent with our Heegner point computations. But in this case we can do better; we are able to recognise our rational point with only H digits of precision due to a trick of Cremona, coming from an idea in a paper of Silverman [17]. The idea is that we know the canonical height of our desired point, and this height decomposes into local heights; we have

$$h(P) = h_\infty(P) + \sum_{p|N} h_p(P) + \log \operatorname{denom}(x(P)).$$

The height at infinity $h_\infty(P)$ can be approximated from a real-number approximation to P, and there are finitely many possibilities for each $h_p(P)$ depending the reduction type of E at p. We compute the various heights to H digits of precision, and then can determine the denominator of $x(P)$ from this, our task being eased from the fact that it is square. Then from our real-approximation of P we can recover the x-coordinate, and from this[3] we get P. Note that we need to compute $L'(E,1)$ to a precision of H digits, but this takes only about $\sqrt{N}(\log H)^{O(1)}$ time. In practise, there can be many choices for the sums of local heights, and if additionally the index is large, then this step can be quite time-consuming. This can be curtailed by doing the calculations for the square root of the denominator of the x-coordinate to only about $H/2$ digits, and then not bothering with the elliptic exponential step unless the result is sufficiently close to an integer.

3.3. **Example.** We now give a complete example. Other explicit descriptions of computations with Heegner points appear in [1, 19, 13]. We take the curve given by $[1, -1, 0, -751055859, -7922219731979]$ for which the Heegner point has height 139.1747+. We select $D = -932$, for which the class number is 12 and the index is 4. We have $N = 11682$ and choose $\beta = 214$. Our first form is $(11682, 214, 1)$ to which we apply $W_1 = \mathrm{id}$. The reduction of this is $(1, 0, 233)$, and it pairs with itself under complex conjugation. Since we have $\gcd(D, N) = 2$, we can use W_2 without changing β; we get the form $(206717861394, 70769770, 6057)$ which reduces to the self-paired form $(2, 2, 117)$. Our next form is $(11682, 2338, 117)$ to which we apply W_{11} to get $(122225810454, 230158978, 108351)$ which reduces to $(11, 6, 22)$ and pairs with $(11, -6, 22)$. Applying W_{22} gives a form which reduces to $(11, -6, 22)$, so we ignore it. Next we have $(11682, 2810, 169)$ to which we apply W_9, getting a form that reduces to $(9, 2, 26)$ and pairs with $(9, -2, 26)$. Applying W_{18} gives a form that reduces to $(13, -2, 18)$ and pairs with $(13, 2, 18)$. Then we have $(11682, 4934, 521)$ to which we apply W_{99}, getting a form that reduces to $(3, -2, 78)$ and pairs with $(3, 2, 78)$. And finally applying W_{198} we get a form that reduces to $(6, -2, 39)$ and pairs with $(6, 2, 39)$, and so we have all of our τ-representatives. We note that W_{11}, W_9, and W_{18} switch the sign of the modular form, and thus the obtained forms get a weighting of -2. The self-paired forms get a weighting of $+1$, and the other two forms get a weighting of $+2$. For the non-self-paired forms we must remember to take the real part of the computed $\phi(\tau)$ when we double it.[4] The pairing turns is rather simple in this example, but need not be so perspicacious with respect to the class group. Note that we have only four distinct forms.

We need about 60 digits of precision if we use the Cremona-Silverman method to reconstruct the rational point, which means we must compute about 20000 terms of

[3]Elkies tells us that, given the height to precision H, the techniques of [10] (see Theorem 4 in particular) can reduce the precision to $o(H)$ as $H \to \infty$. The idea is that $h_\infty\big((x, y, z)\big) + 2\log z = C$ defines a *transcendental* arc, and thus the use of a sufficiently high degree Veronese embedding will reduce the needed precision substantially. The method might not be that practical.

[4]The self-paired forms f have $\phi(f) = \overline{\phi(f)}$ in $\mathbf{C}/\Lambda$ but not necessarily in $\mathbf{C}$ — the imaginary part cannot be ignored when the discriminant of E is negative and lm is odd.

the L-series. The curve E has negative discriminant and no rational torsion points. We compute a real-approximation to the Heegner point in $\mathbf{C}/\Lambda$ to be

$$z = 0.00680702983101357730368201485198918786991251635619740952608094.$$

We have $o = \mathrm{Im}(z)/\mathrm{Im}(\Omega_{\mathrm{im}}) = 0$, and with $u = 2$ we get that

$$\dot{z} = 0.00891152819280235244790996808333469812474933020620405901507952,$$

to which we apply the Cremona-Silverman method of recovery. The curve E is annoying for this method, in that we have many possibilities for $h_p(P)$. The height of the Heegner point is given by

$$h(P) = 139.174739524758127811521877478222781093487974225206369462318,$$

and the height at infinity[5] is given by

$$h_\infty(P) = 2.10306651755149369196435189022120441716979687181328497567075.$$

The reduction type at 2 is I_{25}, at 3 it is $I_{13}^\star$, at 11 it is I_1, and at 59 it is I_3. Thus we have $13 \times 3 \times 1 \times 2$ choices for the local heights. It turns out[6] that we have $h_p(P) = \frac{1}{6} v_p(\Delta) \log p$ for $p = 2, 11, 59$ while $h_3(P) = (13/6)\log 3$. Thus the denominator of the x-coordinate is the square of 12337088946900997614694947283, and the numerator is

$$5908330434812036124963415912002702659341205917464938175508715.$$

3.4. **Variants.** Next we mention the variant which has been described and investigated by Elkies [9]. Here we fix a rank zero curve, say the curve $E : y^2 = x^3 - x$ of conductor 32, and try to find points on rank one quadratic twists E_D with $D < 0$. It can be shown that E_D will have odd functional equation for $|D| \equiv 5, 6, 7 \pmod 8$. There is not necessarily a Gross-Zagier theorem in all these cases, and some involve mock Heegner points instead of Heegner points. However, we still have the prediction that $h(P) = c_D L(E, 1) L'(E_D, 1)$ for some $c_D > 0$. Elkies computes a point P in $\mathbf{C}/\Lambda$ via a method similar to the above — however, he generally[7] only attempts to determine if it is non-torsion, and thus need not worry as much about precision. There are about $\#\mathcal{C}l\big((\sqrt{D})\big) \sim \sqrt{|D|}$ conjugates of τ for which $\phi(\tau)$ needs to be computed; since we have an action of $\Gamma_0(32)$, computing each $\phi(\tau)$ takes essentially constant time, so we get an algorithm that takes about time $|D|^{1/2}$ to determine whether the computed point is non-torsion. Note that we don't know $L'(E_D, 1)$, which takes about $|D|$ time to compute. MacLeod [14] investigated a similar family of quadratic twists, those of a curve of conductor 128. The relevant curves are $y^2 = (x + p)(x^2 + p^2)$ with $p \equiv 7 \pmod 8$; with $p = 3167$ the height is 1022.64+.

4. Combination with descent

To find Heegner points of large height, say 500 or more, it is usually best first to do a descent on the elliptic curve, as this will tend to reduce the size of the rational point by a significant factor.[8] Upon doing a 2-descent, we need only $H/3$ digits of precision if we represent the covering curve as an intersection of quadrics in $\mathbf{P}^3$ and

[5]Note that Silverman [17] uses a different normalisation of height, and his choice of the parameter z when he computes the height at infinity corresponds to $\dot{z}/\Omega_{\mathrm{re}}$ for us. Also, his method is only linearly convergent, while that given in [5, 7.5.7] is quadratically convergent.

[6]This follows *a posterori* since P is nonsingular modulo these primes of multiplicative reduction,

[7]Elkies also computes a generator of height 239.6+ for $y^2 = x^3 - 1063^2 x$ in this manner.

[8]During the mid 1990s, Cremona and Siksek worked out a few examples using 2-descent.

use 4-dimensional lattice reduction, and if we do a 4-descent we need only $H/12$ digits. We first explain how these lattice reduction methods work, and then show how to use them in our application. It might also be prudent to point out that if E has nontrivial rational isogenies, then one should work with the isogenous curve for which the height of the generator will be the smallest.[9]

4.1. **Lattice Reduction.** Most of theory here is due to Elkies [10]. We first describe a p-adic method — this is not immediately relevant to us as we do not know how to approximate the Heegner point in such a manner, but it helps to understand the idea. Let $F(W, X, Y) = 0$ be a curve in $\mathbf{P}^2$. We wish to find rational points on F. Let $(1 : x_s : y_s)$ be a (nonsingular) point modulo some prime p, and lift this to a solution $(1 : x_0 : y_0)$ modulo p^2. Then determine d such that any linear combination of $(1 : x_0 : y_0)$ and $(0 : p : dp)$ will be a solution mod p^2 (computing d essentially involves taking a derivative). Then perform lattice reduction on the rows of the matrix

$$\begin{pmatrix} 1 & x_0 & y_0 \\ 0 & p & dp \\ 0 & 0 & p^2 \end{pmatrix}.$$

Finally search for global solutions to F by taking small linear combinations of the rows of the lattice-reduced matrix. If we choose p to be around B for some height bound B, upon looping through all local solutions modulo p we should find all global points whose coordinates are of size B; in general we take p of size $B^{2/n}$ in projective n-space. This can be used, for instance, to search for points on a cubic model of an elliptic curve.[10]

Over the real numbers the description is more complicated. Here we deal with the transformation matrix of the lattice reduction. If we wanted to do 2-dimensional reduction, that is, continued fractions, on a real number x_0, we would perform lattice reduction on the rows of the matrix

$$M_2 = \begin{pmatrix} 1 & -x_0B \\ 0 & B \end{pmatrix}$$

to get good rational approximations to x_0. We can note that $\begin{pmatrix} 1 & x_0 \end{pmatrix} M_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, and that the transformation matrix T for which TM_2 is lattice-reduced has the property that $\begin{pmatrix} 1 & 0 \end{pmatrix} T$ is approximately proportional to $\begin{pmatrix} 1 & x_0 \end{pmatrix}$. In four dimensions we take a point $(1 : x_0 : y_0 : z_0)$ on some curve, assuming that derivatives of y and z with respect to x are defined at this point. The matrix we use here is[11]

$$M_4 = \begin{pmatrix} 1 & -x_0B & (y'x_0 - y_0)B^2 & \bigl(-e(y'x_0 - y_0) + z'x_0 - z_0\bigr)B^3 \\ 0 & B & y'B^2 & (ey' - z')B^3 \\ 0 & 0 & B^2 & -eB^3 \\ 0 & 0 & 0 & B^3 \end{pmatrix}.$$

Here $e = z''/y''$ and all the derivatives are with respect to x and are to be evaluated at $(1 : x_0 : y_0 : z_0)$. Note that if we have computed $(1 : x_0 : y_0 : z_0)$ to H digits of precision, we must "lift" it to precision $3H$ to use this. Similar to the

[9]Because Ш might have different size for the various isogenous curves, we cannot always tell beforehand which curve(s) will have a generator of smallest height.

[10]This description of the Elkies method in the p-adic case was noted to the author by Womack (see Section 2.9 of [23]), and might be due to Heath-Brown, if not already known to Elkies.

[11]The 3-dimensional version is just the upper-left corner.

2-dimensional case of above we have that $\begin{pmatrix}1 & x_0 & y_0 & z_0\end{pmatrix} M_4 = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}$, and $\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix} T$ is approximately proportional to $\begin{pmatrix}1 & x_0 & y_0 & z_0\end{pmatrix}$.

4.2. **Results.** We now combine descent with the Heegner point method. We assume that we have a descendent $C \to E$, and for each point $\mathcal{P}(\dot{z})$ given by the above algorithm we compute its real pre-images on C. For a 2-descendent quartic, the x-coordinate has size $H/4$, but the y-coordinate on the quartic will be of size $H/2$. Either continued fractions on the x-coordinate or 3-dimensional lattice reduction on both coordinates and the curve requires a precision of $H/2$ digits — however if our 2-descendent is given as an intersection of quadrics in $\mathbf{P}^3$, then we only need a precision of $(H/2)(2/3)$ since the Elkies method does better in higher dimension. For a 4-descendent represented as an intersection of quadrics in $\mathbf{P}^3$, the coordinates are of size $H/8$, and so we need a precision of $(H/8)(2/3)$ to recover our point.

We now give two examples of Heegner points of large height.[12] First we consider E given by $[0, 1, 1, -4912150272, -132513750628709]$, for which $N = 421859$. Here the Heegner point is of height $3239.048+$. We refer to reader to [23] for how to do a 4-descent. The intersection of quadrics that gives the 4-descendent is given by the matrices

$$\begin{pmatrix} 1 & 3 & 14 & 4 \\ 3 & 7 & 9 & 8 \\ 14 & 9 & -8 & 19 \\ 4 & 8 & 19 & 13 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 16 & -10 & 5 & -5 \\ -10 & 29 & -3 & -5 \\ 5 & -3 & -1 & 8 \\ -5 & -5 & 8 & 13 \end{pmatrix}.$$

We used $D = -795$ for which the index is 4. The class group has size 4; upon using the pairing from complex conjugation we need only 2 forms, which we can take to be $(421859, 234525, 32595)$ and $(421859, 384997, 87839)$. We need to use about $3239/12 \log(10) \approx 120$ digits of precision and take around 1.3 million terms of the L-series. For our approximation to a generator on $\mathbf{C}/\Lambda$ we get

$$\dot{z} = 0.0082583151840681431245098564622255839109520795462317571566289712763512600656062689191498313057421234300078042601843027605 5.$$

We find the real pre-images of this on the 4-descendent and then via 4-dimensional lattice reduction we find the rational point

(905858492223506210113393024249323265421924744748543313130312163382040538806700779447013024918525728237312026342662199441467025094898245325290448879879478594723551249394712957 29,
582078484694685672492501009047456045174915846216541010656493376894960360413180680196461593316523862645798797467270954347431710750081346713999645139248706071573407853276610717 57,
-526601834730048318750844106429182505324580075229565235152794642481747302402870184241487015871230002692215306004659982891124430459579653775304125009412020595687436569112817668 81,
120566343955994724443268162651063750286159426913472902595469949397024821835521392662156637479278825638673289873881696613629128450397637909394604102581491799075589326751709650806),

which can then be mapped back to E. Even though we only used 120 digits of precision, we are still able to find a point with approximately 3/2 as many digits. Note that if we did not use descent, but recovered the point on the original curve using the Cremona-Silverman method, we would need 12 times the precision and 12 times as many terms in the L-series — this could be a total time factor of as much as 12^3, depending upon the efficiency of our high-precision arithmetic. This computation takes no more than about 5-10 minutes.

Finally we give a more extreme example — this is the largest example which we have computed. The curve is from the database of Stein and Watkins [18]. Let E be given by $[0, 0, 1, -5115523309, -140826120488927]$, for which $N = 66157667$

[12] These examples exemplify the experimental and heuristic correlation between large heights and large cancellation in $c_4^3 - c_6^2 = 1728\Delta$.

and the Heegner point is of height 12557+. The intersection of quadrics that gives the 4-descendent is

$$\begin{pmatrix} 0 & 1 & 3 & 3 \\ 1 & 5 & -1 & -6 \\ 3 & -1 & 8 & -2 \\ 3 & -6 & -2 & 16 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 12 & -21 & -10 & -68 \\ -21 & -13 & -7 & -27 \\ -10 & -7 & 3 & -7 \\ -68 & -27 & -7 & 15 \end{pmatrix}.$$

We select $D = -1435$ for which our 2 forms are $(66157667, 2599591, 25537)$ and $(66157667, 37610323, 5345323)$ and the index is 2. We need 460 digits of precision and 600 million terms of the L-series. This takes less than a day. We list the x-coordinate of the point on the original elliptic curve. It has numerator

36777053718667750661400564234182717008793226949228558472621877006165354634927101580536513437032674306114130646450005288670465199839976647884079191530786174150
72739338026281573250924797082687602171017553858718167805487654785022844156276828471927526818990949626599378706300367603592935770218062374839710749312284163465
07852381696883227650072039964481597215995993299744934117106289850389364006552497835877740257534533113775202882210048356163645919345794812074571029660897173224
37033770105616573500859064029709029870912150626669726646199320182539736999955086814229431275632217741073053282806475960497536924235099356803072693704991160726
41097827468479512837941192989412144907943309029865829912295694015235199387427463761071907702040105138183490127866378892547110594555551738109049119276198990318
55149292325338589831979737026402711049742594116000380601480839982975557506035851728035645241044229165029649347049289119188596869401159325131363345962579503132
33984727542244009455382470518922565367745951286311791172183855293430912450813449336643740809392436203974991190741697350414232211175705858420072502263211616472
01649986417295226774605259994990779421258204288795260637356926859910185168629387960475973239865371541712483169437963732171919939969937146546295368843960579247
90938647656663281596178145722116098216500930333824321806726937018190136190556573208870483553355670787931266569286578590367793505932745987173797308807240343 01
86773944374984180945671588419372032890146155265988262840584220975675716781666213994508186464210853359598997571625925924015283405094065447961714768592250085694
44498220453860921224090969785448172188478976405134778065983291776042463808123777390491844755507773416209859765703930378802827649670195524084007307548226764414
81715385344001979832232652414888335865567377214360456003296961668177481944809066257442596772347829664126972931904101685281128944780074646796760942430959617022
25747987408940356496503888537981786692004892981452026849367750705907376590267163808736648884967028363262685745931232451074203488781017631238933476570202755912 4
88242478005942708620520821859733932900091898676772594580806760650987034535395255769756395437005076256407298723407894063143944684005844559206833619762001218344
30751233901473228497490561998078486251074993528871318797403348087370426900997556442577081254910572185107856605139877331015042842121106080690743578173268489400
49905689831262195394796701235841454775289708109709179574420369768404606625566320124229276012675987126600451637743296191727204021714708356339998761242059527579
20338556769918233682548621595584500438080514815332972700352873822470382792932239463850701180823069589872686033969240544031038574440588486055874154005176700326
31121206127732481340391088277796488544415738156553014768406246154666005139690428085145098272500791416214774673484501826722500527091164944262537169595848931680
75409677471286049057274622409403118704320452610723920107960346829752289510659856743701508334879787536416279769396881980413954888575128268715223707826035870523
02844262030644936842506142828799181077337962070672500038239594129356776240932360470386373655773263995890088045077860119731559277310730347065365574614438066227
07622411087809371872157210456836892493613836792026761820382217165481998924123604782787923229739171920575447007099501678380795077013113325989801385729993920818
30165442425133956460687682012192837224621339985921328279251116804395344383979390113997419447930029756609766453919938465190843618873242881837330238304638885942
79378938418880142666851776166056447837041357949318307502656863359340665652409440494482130055919971289855607602603992142786359126343515867623548693540215307461
89992899582554597632108309638569296964800046983072736238483149014714600896056552029642747991419063454749142059564274298254654925893866404955146903330024475746
16354371499624965242017117105423172633649354158697143177894405148105963373839941141857432381177094972972684361267292500063135565983416420055444131545100343345
24662047071238116636236628372968629480617587599286317636619851856158018862057707210320063041448677873470583163922956715800916558720872094859132869301288586404
42589125454268580397484571921012318872311624898317615607628176460097441336323549031828235965636277950827328087547939511112374216436584203379248450122647406094
03517113074066372354767593988595936388113589303510201838944421274614625032834824261067352402237899497839202009881472197450206269281573668922975906582209394279
53187053452755989894263352359355056053114113015603211922694308617337435444029085864973053536009094312149332025225287171092144929593300160658102876231441792884
66664888540622702346704213752456372574449563979215782406566937885352945871994541770838871930542220307771671498466518108722622109421676741544945695403509866953
16727762828023246483921500347404889696803754466002975574006558127013908324990321257223041794224979546710070039394431032500967717918210997094334680733501444683
96122825088243240736795841228512083604591663154848919522994493400258965092989359393577217235439331087432419973874470183959253201676376403284079570698454395013
81234605867495003402016724626400855369636521155009147176245904149069225438646928549072337653348704931901764847439772432025275648964681387210234070849306330191
79038041239611544624083258348136637213230084906083526213683231531105290336750385743792050893130528314337942393060136915457253067727886206663888425022179164712
3563828956462530983567929499493346622977494903591722345188975062941907415400740881

and denominator (which is square)

42550442729747398883181472438884631491027966762628698441290370019390690825369783206694892509489159845392479060050865239326960543711865223880415469150998005445
00632766713011953565768977996873028383821265775526344346357123541112589188755147040471997532333827695072210398747506143870850088480631610375575105629402969923
37990669688363242672664077782923842468765734764552501489904789995181590663825882826884437614702086601072912460205046718987111919759741772924082107906614768188
25928778559738389606781437786937703940490529701550198752240522047885413980183556155907460046868073063343753429797282371426595980125377117083031923094847501187
81539765280348857873798901748823243463804440411715478920014959236515352089221657498291013202443206177483221913021678256235579893414436481218449503035888524081
15740619547664383566612706438064220965934420574805698187965300040168887347070624862281490478227347630906482652560940554823360522517964582349779334663133777009
32389603015388462267191386514299244523629967196928070338170422172904291954458588735665717672690912582485100989972418433868929605913487922691271558721181323613
70012052090648575376001600422257728323075912369868818631162455043825487235531941123620115037512323584244014680824673299882453235229917216799756372612183353210
23686195578854195273414555800682053521914530044820400116481027789228810525723047625080568425427769667815771926774952742389468234144023574739674218399717724794
64490794909903308233044977729321377913661153832551845135705059041770761773002335204714530513728518042273513609336859761902134080074259874956489916483283558415
18790233089971640826391421865446416159176533305374315501526687536310287865258225341831409297024404559487986223470061428509852910119956281813557240650373612303
80612421682647619158275971206216911075102110620022250292103973758280786545321293000346117555930108145915118154377991059233962745297863017035414522428825832481
10185696903019214515170137956322244850592956107013529355675348962971488951184208519602258834025155300581173742601529286315229193789791848640470464974617150056
62259748200507982418704000034189585130679648460399502892795419432308411176961777683169393789440445404586765622160449971094877274592261820143359041089932670055
51298147417813688141747600315791793741837036656460760259084286414504016916928653373373749351673138873697355678073568445462500597124999569428768348290694470131
39695758671845345053150692524956190311186650121760981782607840168999098928254294766906595754090551526543788821211073870016226804688310128261364457467501381181
99895452299627549452761944871112397182275542812517516928333071853473496479471177474588594469094399278566301150327219929702835948186371102871019478295556041049
94569790530090215161267032801442076978425884340578290684944705663842203873434946895313693410268205549042641332729214043530781277783400932521461065846050516266
06369309829333522153142406896441472229879201250507247335030253750414919268086142811991181135615102063450273874178955396056033628717223631646597545299321971795
72083803798469762841343018748973041067954703588990548236248494723651739045289521882710298950201026345472967707812057733059495444795253020394607206147343184720
12891382027947113716845979329657679391121796342422193808825527319590678236462421470515380610223708802325911166125101655807805544810736511736112015725788207983
04346277884275334856562425654767711062664505338976464447359505881649914260980555651233453080870411547161166201727321046199497573313805291689786242134493722493
65401328137208190879603847948651527305255792412098085899417962189727321732456666040832323196614881188508752637063922635721727888588861006697067295082662856621
42789555098127098307594733822740958886080483268977873098575694622813361281769384832759774254005221673936338256702877699726892622342424638164907882484814569895
78820841326437902307074825587297298227228464352952356348415245051280414970934534815003363838566586486090914244124413024752619262170944161605020754084937204942
16372326712489077723971624589353187301664632745253191939259669507578605710768387023661969449344902879751178806889386035847044662250758843856626244205371560027
87345086683247957409590801131490445553949472979089821198999392392205812011786324196553409374966862790769096287105922007091926687147181741236537543028535996580
45947455276513698860733871942939615110361783065380776837197220049720205456646520373809780940384932684877526623946791294274354580233914301360048309031821084423
72425264218315757261837807638185861100198403815128178367563386992036547922694358498639362771798041618046013662235878191246173024597377861825966407677879013966
44826035500413324947183241852334621567713018309711089099896390758911615128829646468875441800478905861321031022715201049054765075321382013759982613256651365903
08677310205945594774669134858545209082084155986593443637589125910770359979110742764732875552438147802286623083638506353523937829379144982925749510228726599819
64923226850485187083410037527178126285256712196330538305579411947408618370648081044119314858092966789985880558618838749736465935723939056195292855739824197002
85747166387929856818643218648078765165327952717275940366217333921605018810173490426678520993395921957402807842113084883601771499771585515616148588419257652541
10965702365800067847782401155042150831554731143366545320353097125730540763006480842903826240579636224118544527127667643337427417482571764186299386690494751101
4583148762951888762273247975157547523726572052209999592135037352690220398771 6.

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