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Another 80-dimensional extremal lattice

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Abstract. We show that the unimodular lattice associated to the rank 20 quaternionic matrix group $\mathbf{SL}_2(\mathbf{F}_{41}) \otimes \tilde{S}_3 \subset \mathbf{GL}_{80}(\mathbf{Z})$ is a fourth example of an 80-dimensional extremal lattice. Our method is to use the positivity of the Θ-series in conjunction with an enumeration of all the norm 10 vectors. The use of Aschbacher's theorem on subgroups of finite classical groups (reliant on the classification of finite simple groups) provides one proof that this lattice is distinct from the previous three, while computing the inner product distribution of the minimal vectors is an alternative method. We give details of both. The latter method also enables us to find the full automorphism group for each of the four lattices, and as already noted by Nebe, this fourth lattice has an additional 2-extension in its automorphism group.

Résumé. Nous montrons que le réseau unimodulaire associé au groupe de matrices quaternioniques $\mathbf{SL}_2(\mathbf{F}_{41}) \otimes \tilde{S}_3 \subset \mathbf{GL}_{80}(\mathbf{Z})$ de rang 20 donne un quatrième exemple d'un réseau extrémal en dimension 80. Notre méthode utilise la positivié de la série Θ ainsi que l'énumération des vecteurs de norme 10. L'utilisation du théorème d'Aschbacher sur les sous-groupes de groupes finis classiques (qui suit de la classification des groupes finis simples) permet de démontrer que ce réseau est différent des trois précédents. Une autre méthode est de calculer la distribution du produit scalaire des vecteurs minimaux. Nous donnons les deux preuves en détails. Cette dernière méthode nous permet également de déterminer complètement le groupe des automorphismes de ces quatre réseaux. Comme cela a déjà été noté par Nebe, ce quatrième réseau possède une 2-extension supplémentaire de son groupe d'automorphismes.

1. Introduction

Extremal unimodular lattices are of interest because they often have high packing densities and large kissing numbers. They can only exist in dimensions divisible by 8, the first example being E_8. Examples are known in each dimension up through 80, where three such lattices were known. In this dimension, being *extremal* means the smallest nonzero norm is 8.

The topic of constructing extremal lattices has seen a recent surge in interest, in part due to Nebe's demonstration [28] of such a lattice in dimension 72, answering a question that had been open for some time (and indeed, various experts seemed to favour the opinion that such a lattice could not exist [8, p. 129, Remark]). Bachoc and Nebe [3] had previously constructed two extremal lattices in dimension 80, proving these were extremal using coding theory; and more recently Stehlé and the author [38] used a more computationally intensive method to show that the lattice associated to the (binary) extended quadratic residue code of length 80 is a third example in this dimension. We use techniques similar to those exposed in [38] to prove the extremality of a fourth lattice in this dimension. This lattice corresponds to the rank 20 quaternionic matrix group $\mathbf{SL}_2(\mathbf{F}_{41}) \otimes \tilde{S}_3$ as constructed by Nebe in [25, §3], and in [25, Lemma 4.3(i)] it is noted that there is an additional 2-extension in the automorphism group.

1.1. Overview. Similar to the proof in [38], we show extremality by enumerating all the vectors of norm 10 and using the positivity of the Θ-series. We indicate various improvements over the methods used in [38], in particular those which allowed us to work with an automorphism group which does not have such a nice representation as with $\mathbf{SL}_2(\mathbf{F}_{79})$. We give two proofs that our lattice is not isometric to any of the previous lattices. The first uses Aschbacher's theorem on subgroups of finite classical groups [2], while the second involves computing the inner product distribution of the minimal vectors. We provide statistics about the inner product distribution of the minimal vectors for all known 80-dimensional extremal lattices. Finally, we comment on some examples we found in dimension 64, and our failure to find any new examples in dimension 48.

2. Our lattice

Our lattice L is given by Nebe [25, Remark 5.2] via a construction over the quaternion algebra $\mathcal{Q}_{\sqrt{41},\infty,\infty}$. There are two non-conjugate maximal orders, and the one of interest for us contains the maximal order of $\mathcal{Q}_{\infty,3}$,

Unwinding this notation, we find that it corresponds to writing one of the (complex-conjugate) 20-dimensional representations of $\mathbf{SL}_2(\mathbf{F}_{41})$ over a quaternion algebra, with the two possibilities being either the Hurwitz quaternions $\mathcal{Q}_{\infty,2}$, or our case of $\mathcal{Q}_{\infty,3}$. In either case we augment the automorphism group by the units of the quaternion ring, so $\tilde{S}_3$ for us.

2.1. A computation to realise L. Nebe gives one method for constructing L, namely by constructing a representation of a metacyclic group, and then solving norm equations in abelian fields (see [25, §3]). We chose to construct the lattice via a different, perhaps more circuitous route, via

the representation theory and G-module functionality in Magma [6]. Instead of starting from $\mathbf{SL}_2(\mathbf{F}_{41})$ as in [25], we worked directly with the group $G = \mathbf{SL}_2(\mathbf{F}_{41}) \otimes \tilde{S}_3$, and considered the rational 80-dimensional representations of it.[1] In particular, we wrote $\mathbf{SL}_2(\mathbf{F}_{41})$ and $\tilde{S}_3$ as permutation groups, and took their direct product. We then computed the rational characters of degree 80, and limited ourselves to those with a kernel given by identifying the central -1 elements in the two groups. This left two characters, and as is noted in [25, §5], the representation we seek is reducible over the reals, so we reject the remaining irreducible character. We computed a $\mathbf{Q}G$-module that affords this character using the `GModule` command[2] of Magma [6], and found that there is indeed a 2-dimensional space of symmetric forms fixed by this matrix group. Writing f, g for a basis of these, the determinant of $fx + gy$ is of the form $q(x, y)^{40}$ for some homogeneous quadratic polynomial q (depending on f, g), and so we solved the conic $q(x, y) = 1$. With minimal effort we found a solution (x, y) which made $fx + gy$ integral, and this gives a Gram matrix of our lattice L.

An alternative method to construct L would be: take the sum of the two 20-dimensional characters of $\mathbf{SL}_2(\mathbf{F}_{41})$, and write the resulting representation in degree 20 over $\mathcal{Q}_{3,\infty}$ (one can take the "tensor product" of this with the degree 1 quaternionic representation of the faithful irreducible degree 2 character of $\tilde{S}_3$, but this is just the action of the units). However, it seems that the best way to do this is to first write the G-module in dimension 80 over $\mathbf{Z}$, and then find i, j in the endomorphism ring with $i^2 = -1, j^2 = -3, ij = -ji$ to realise the module over $\mathcal{Q}_{3,\infty}$. In either case, we get not only the relevant Gram matrix but also the action of G on it.

3. Proving L is extremal

We use the general method outlined in [38], which was adapted from an idea in [1], and indeed is essentially already in [23]. We first note that an even lattice has a Θ-series $\Theta(L) = \sum_{\vec{v}} q^{\vec{v}\cdot\vec{v}/2}$ that is a modular form, and for a unimodular lattice L of dimension 80 this has weight 40 and level 1. The space of such modular forms has dimension 4, and a basis is given by

$$\begin{aligned}
f_0 &= 1 + 1250172000q^4 + 7541401190400q^5 + O(q^6),\\
f_1 &= q + 19291168q^4 + 37956369150q^5 + O(q^6),\\
f_2 &= q^2 + 156024q^4 + 57085952q^5 + O(q^6),\\
f_3 &= q^3 + 168q^4 - 12636q^5 + O(q^6).
\end{aligned}$$

[1]The additional 2-extension was not considered for two reasons: firstly, while the construction of G is relatively straightforward, it was not clear to me how the additional extension could be appended; and secondly, it was only belatedly that I found out about this 2-extension anyway.

[2]This took about 15 minutes, but this could vary as the underlying methods are in flux.

The Θ-series of L is then given by $\Theta(L) = f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3$ for some integers $a_i \geq 0$. A lattice is said to be *extremal* when all the a_i are zero. Indeed, in this case the minimal norm is as large as possible, and the Θ-series is given simply by the first element in the above basis. We show that $\Theta(L) = f_0$ by first showing that $a_1 = a_2 = 0$ via a brute-force search (relying on parallel enumeration code of Pujol [32]), which implies

$$\Theta(L) = f_0 + a_3 f_3 = 1 + a_3 q^3 + (\cdots) q^4 + (7541401190400 - 12636 a_3) q^5 + O(q^6).$$

We then proceed to search for 7541401190400 vectors of norm 10, and upon finding this amount, will have shown extremality because positivity then implies that $a_3 = 0$. The capacity to find this many vectors depends on a number of factors. We find one representative in each orbit under the known automorphisms, but this still leaves about 18.6 million orbits to be found in our case. By a coupon-collecting analysis, this implies that a bit over 300 million "random" vectors of norm 10 will need to be found, and below we give two methods that are able to achieve this. The reason why we find vectors of norm 10 rather than search for norm 6 vectors directly is that the latter would need to be exhaustive, and there is no apparent way to exploit the automorphisms in such a search.

3.1. No vectors of norm 2 or 4. Unlike the case of [38, §5.1], we are not able to relate our lattice to a coding theory construction so as to eliminate the possibility of vectors of norm 2 or 4. However, after finding a sufficiently good basis for the lattice using block Korkine-Zolotareff (BKZ) reduction [35] (with a dimension parameter of about 30 — this is usually all that is useful, and takes less than 10 minutes), it only takes only a couple of cpu-months to do an exhaustive search, and parallel code for this is now available from Pujol [32] (described in [10], and see also [9]). Using 12 cpus and Pujol's code, it took about 4 days to show that our lattice has no vectors of norm 2 or 4. We had to make a slight modification to the code to allow an integral Gram matrix (rather than a basis) as the input.[3]

3.2. Vectors of norm 10 with nontrivial stabiliser. First we find all the vectors of norm 10 that have a nontrivial stabiliser. To do this, we compute the conjugacy classes of G, and then for each nontrivial conjugacy class, take a representative g of it and search for vectors in the sublattice fixed by g. We find that there are 140 nontrivial conjugacy classes, and the largest sublattice fixed by any of these is of dimension 40. We are able to find all vectors of norm up to 10 in such sublattices in about 15 minutes. For the orbits of vectors of norm 10, we find:

- 34342 orbits with stabiliser of size 2,
- 260 orbits with stabiliser of size 3,

[3]This induces minor changes in the error analysis [33] of the floating point computations.

- 56 orbits with stabiliser of size 6,
- 10 orbits with stabiliser of size 10.

See Section 3.4 for how to recognise orbits. Assuming the lattice is extremal, this leaves 18230412 free orbits to be found.

3.3. Finding vectors via pruning. As described in [38], the idea of pruning (perhaps first noted in [36, p. 195]) is to follow the standard enumeration technique of Kannan [18] or Fincke-Pohst [12], but to limit the search region to areas which are considered more likely to possess short vectors. Explicitly, using the standard notation for lattices (as found in e.g. [38, §6]), rather than solve the series of inequalities

$$\sum_{i=j}^{80} y_i^2 \|\vec{b}_i^\star\|^2 \leq 10 \quad \text{for all } 1 \leq j \leq 80,$$

we introduce a pruning array $P_j = 1 - \frac{(j-1)}{80}$ and solve

$$\sum_{i=j}^{80} y_i^2 \|\vec{b}_i^\star\|^2 \leq 10 \cdot P_j \quad \text{for all } 1 \leq j \leq 80.$$

To describe this loosely, this ensures that any initial segment of the coordinates does not take up more than its "fair share" of the available norm.

The first step in any lattice-searching method is to obtain a good basis. Here LLL [21] by itself is not completely satisfactory, but after applying BKZ [35] with a dimension parameter of 30, we have a reasonable basis. We would run the pruned-enumeration code for 100 seconds on a given basis, before making a perturbation of it as in [38]. With the above choice of P_j we obtained about 400 norm 10 vectors per cpu-second using the Magma implementation of Stehlé.

Recent work appearing in [14, Appendix D] has improved the tree traversal process; while we do not have exact timings, a guess is that it would be 30-40% faster at the cost of increasing the memory usage slightly.

3.3.1. *An alternative method to find vectors of norm 10.* As noted in [1], an alternative method to try to find vectors of norm 10 is to take random pairs of (known) norm 8 vectors, hoping that their inner product is of size 3.

We do not have a complete analysis of this method, but can note that the primary step will be the computation of an inner product. Done in the most obvious manner, this would take about 80^2 multiply-and-adds; but by (say) first diagonalising the Gram matrix over the reals, we are able to reduce the calculation to 80 such operations. The distribution given in Section 4 indicates that a random inner product between two vectors of norm 8 will have about a 1-in-380 chance of having size 3. We can thus achieve about 10000 norm 10 vectors per cpu-second, which is notably

faster than the pruning techniques. However, see Section 7.2 below for some difficulties with this method.

3.4. Recognising orbits. One difficulty in mimicking the strategy of [38] for our lattice L is that it is not so clear how to find orbit representatives as easily as with $\mathbf{SL}_2(\mathbf{F}_{79})$ (for which there is a doubly transitive action of signed permutations on the coordinates). We overcome many of the difficulties by noting that an orbit can be recognised via a baby-steps giant-steps technique involving subgroups (or even subsets). Indeed, suppose we have two vectors $\vec{v}$ and $\vec{w}$ in the same orbit, so that $\vec{v}g = \vec{w}$ for some element $g \in G$. We assume that we have G written as BA, where in practise this decomposition will be exact with A a subgroup and B just representatives of the cosets of A. Then we have $\vec{v}ba = \vec{w}$ for some elements $a \in A$, $b \in B$, and so by comparing $\vec{v}B$ with $\vec{w}A$ we will detect whether $\vec{v}$ and $\vec{w}$ are in the same orbit.

In our case, we take A to be a subgroup of size 820 in G, and further mod out by $-1 \in A$. This means that the set B is of size 504. For every vector $\vec{v}$ we find, we compute $\vec{v}b$ for each $b \in B$ and use a hash table to detect if it is the same as any $\vec{w}a$ that was seen previously. If so, then we have already counted this orbit. If not, we compute $\vec{v}A$ and store these vectors (we can also compute the stabiliser of $\vec{v}$ at this step from $\vec{v}A$ and $\vec{v}B$).

3.4.1. *A minor generalisation.* This method could be generalised to handle the case of sets A, B such that BA^{-1} as a set covers G, and so even in a case where there are no subgroups of useful size, one can still choose A and B of size about $\sqrt{\#G \log \#G}$ if desired. We can note that the expected time to find V vectors under an automorphism group G is thus roughly proportional to $V \log V/\sqrt{\#G}$.

3.4.2. *Computational data.* We chose a subgroup A of size 820 in G and so $\#B = 504$, with $-1 \in A$ reducing computations by a factor of 2. For each vector $\vec{v} \in L$ that we find, the computation of the set $\vec{v}B$ will take about $504 \cdot 80^2 \sim 3.22 \cdot 10^6$ multiply-and-add operations. We can stop computing $\vec{v}B$ immediately when we run across a saved $\vec{w}A$ vector, and this saves a factor of about two on average when an orbit is already known.

When we find a new orbit we compute $\vec{v}A$, which again requires around 3 million multiply-and-add operations. We save each vector in $\vec{v}A$ as a 64-bit hash – in the worst case we could erroneously regard two distinct orbits as equivalent (in which case we should just find this orbit later), but this hash will never incorrectly claim that a previously seen orbit is new.

Even with this hashing, we still need a storage space of $18265080 \cdot 410 \cdot 8$ bytes, or about 64 gigabytes. We chose A of the given size to push the memory limits as much as we could, so that the time to compute the $\vec{v}B$ would be as small as possible. It turns out that we can process nearly

200 vectors per cpu-second, and so distinguishing the orbits of 305 million vectors takes about 3 cpu-weeks.

We can check our proof in less time than it took in the first place, as we only need run through 18.6 million vectors rather than 305 million. We provide code[4] that can check that our list does indeed provide 7541401190400 distinct vectors of norm 10, but this still requires around 3 cpu-days (we ran it on 10 cpus in 7 hours) and 64GB of memory.

4. Inner product distributions

We are able to analyse the inner product distribution of the minimal vectors by weighting with respect to Gegenbauer polynomials (see [39], or [4, §4-5]). Given an extremal 80-dimensional lattice, for any fixed $\vec{w}$ with norm 8 and any $d = 1, 2, 3$ (and also $d = 5$, though it gives no new information in dimension 80), we have that

$$\sum_{\|\vec{v}\|=8} G_{2d}\left(\frac{\vec{v}\cdot\vec{w}}{8}\right) = 0$$

where the G_{2d} are related to the Gegenbauer polynomials. This is a special case of the more general fact that for any fixed nonzero $\vec{w}$ and positive integer d, the sum

$$\sum_{\vec{v}\neq 0} G_{2d}\left(\frac{\vec{v}\cdot\vec{w}}{\sqrt{\|\vec{v}\|\,\|\vec{w}\|}}\right) q^{\vec{v}\cdot\vec{v}/2}$$

is a modular form, and extremality forces some of the coefficients to vanish.[5]

Explicitly, in the case of dimension 80 we have $\frac{1}{(1-2xt+t^2)^{39}} = \sum_k G_k(x)t^k$, so that

$$G_2(x) = 760x^2 - 19,\ \ G_4(x) = 117040x^4 - 15960x^2 + 190,$$
$$\text{and } G_6(x) = 8614144x^6 - 2691920x^4 + 175560x^2 - 1330.$$

Furthermore, the signs of the $\vec{v}\cdot\vec{w}$ are equi-distributed, and except for the cases when $\vec{v} = \pm\vec{w}$, we have $|\vec{v}\cdot\vec{w}| \leq 4$. We have 5 unknowns, namely the number b_i of vectors $\vec{v}$ with $\vec{v}\cdot\vec{w} = i$ for $i = 0\ldots 4$, and 4 linear equations, given by the three above for $d = 1, 2, 3$ plus the accounting

$$b_0 + 2(b_1 + b_2 + b_3 + b_4) = 1250172000 - 2,$$

where this comes from noting that an extremal lattice in dimension 80 has 1250172000 vectors of norm 8. We solve these and get

$$b_0 = 2(35y + 275885775),\ \ b_1 = 301716800 - 56y,$$
$$b_2 = 28y + 45799776,\ \ b_3 = 1683648 - 8y,\ \ b_4 = y,$$

[4]This is available from `http://magma.maths.usyd.edu.au/~watkins/sl241dim80.tar.bz2`

[5]There is a slight reworking of this for extremal lattices in dimensions $24k$ and $24k+16$, where in the first case we get vanishing for $d = 1, 2, 3, 4, 5, 7$, and in the latter case only for $d = 1, 3$.

for some integer y with $0 \le y \le 210456$. We do not know if the parameter y can be related to a type of "Nachbareffekt" as in [4, §5, Example 1]. Unlike for the case of dimension 32, with extremal lattices of dimension 80 the Siegel modular form given by

$$\Theta_2(L) = \sum_{\vec{v}\in L}\sum_{\vec{w}\in L} q_{1,1}^{\vec{v}\cdot\vec{v}/2}\cdot\ q_{1,2}^{\vec{v}\cdot\vec{w}}\cdot q_{2,2}^{\vec{w}\cdot\vec{w}/2}$$

is not uniquely determined (see [30]), with the indeterminate factor being a multiple of χ_{10}^4.

Below we shall use the computation of the inner product distributions as one of the methods to show that our lattice L is not isometric to any of the previously known extremal lattices in dimension 80. None of the material in this section is strictly necessary for that, but we provide it for context.

5. Computational results

5.1. Minimal vectors. The vectors of norm 8 in the lattice L split as follows under the known automorphism group $G = \mathbf{SL}_2(\mathbf{F}_{41}) \otimes \tilde{S}_3$:

- 2788 orbits with trivial stabiliser,
- 464 orbits with stabiliser of size 2,
- 8 orbits with stabiliser of size 3,
- 14 orbits with stabiliser of size 6.

As there are 3274 orbits and 1250172000/2 minimal vectors up to sign, we need to compute about 2 trillion inner products to find the complete distribution. Each inner product can be computed in 80 multiply-and-adds upon switching the minimal vectors to a basis (over $\mathbf{R}$) in which the Gram matrix is diagonal. Our code ran in about 4 cpu-days.

Using the notation of the previous section, for each vector $\vec{v}$ we write y for the number of vectors that have inner product 4 with it. This value is preserved by automorphisms, and so is constant for all vectors in the same orbit. We find that the smallest y-value is 8092 (obtained for 2 free orbits), while the largest is 9220 (obtained for 4 orbits, all of stabiliser of size 6). The average is slightly above 8574. Each y-value appears in our data an even number of times; this is to be expected due to the 2-extension of G that is noted in [25, Lemma 4.3(i)]. Via a slight modification of the methods given in Section 5.3 below, we are able to determine the complete automorphism group $G^+ \cong \big(\mathbf{SL}_2(\mathbf{F}_{41}) \circ \tilde{S}_3\big).2$ with $[G^+ : G] = 2$. The 80-dimensional representation of G^+ corresponding to L is absolutely irreducible. We can also note that the matrix group $G^+ \subset \mathbf{SL}_{80}(\mathbf{Z})$ is uniform, that is, it fixes a unique symmetric form (up to scalars), unlike G for which the space of symmetric fixed forms has dimension 2 (see [25, §5]).

5.2. Comparison to other lattices. We can make the same computation with the other known 80-dimensional extremal lattices. For the Bachoc-Nebe lattice with automorphism group $2.M_{22}.2 \otimes 2.A_7$, there are 10 orbits, and we compute the following data:

- an orbit with stabiliser of size 6 and $y = 8728$,
- an orbit with stabiliser of size 16 and $y = 9400$,
- an orbit with stabiliser of size 48 and $y = 9688$,
- an orbit with stabiliser of size 96 and $y = 8728$ (as above),
- an orbit with stabiliser of size 112 and $y = 13336$,
- an orbit with stabiliser of size 192 and $y = 14872$,
- an orbit with stabiliser of size 384 and $y = 12184$,
- an orbit with stabiliser of size 432 and $y = 8248$,
- an orbit with stabiliser of size 8064 and $y = 24088$,
- and an orbit with stabiliser of size 24192 and $y = 15256$.

As can be seen, the average of the y-value is a bit over 9247.

For the second Bachoc-Nebe lattice [3, Lemma 4.11], the known automorphism group of size $2^{12}3^4 5^2$ yields 333 orbits. The smallest y-value is 8268 (from a free orbit), and the largest is 24088 (the same as in the above data), coming from three orbits whose stabilisers are of sizes 288, 384, and 576. The second largest y-value is 17944, from an orbit with stabiliser of size 96. We find that the average y-value is a little above 8855. All of the y-values are divisible by 4.

Finally, for the lattice proven extremal in [38] with known automorphism group $\mathbf{SL}_2(\mathbf{F}_{79})$, there are 2555 minimal orbits, with a minimal y-value of 8048, a maximum of 9406, and an average of nearly 8537.

This gives one proof that the four lattices are all distinct up to isometry. The complete data for the inner products are included in the download from the address given in Footnote 4.

5.3. Maximality of automorphism groups. We can also use the above y-value distributions to show in each case that the known automorphism group is the full automorphism group. The idea is simple.[6] We assume that σ is an unknown automorphism and that we know the images of the vectors $\vec{v}_i \in S$ under σ. Then we use the fact that σ preserves inner products. We will either show that the set of images is inconsistent, or that σ fixes all the vectors in S. In the latter case, when S is so large that it generates the lattice, we conclude that σ fixes every vector, and so must be the identity.

The only difficulty is in getting a large enough set S of vectors for which we know the image. Here is a probabilistic argument on what we might expect. First we take an orbit whose y-value is unique, and choose a vector $\vec{v}$

[6] It is also well-known — see [31] for improvements that can be applied in more difficult cases.

in it. We know that σ must map this orbit to itself, and so $\vec{v}\sigma = \vec{w}$ for some $\vec{w}$ in the orbit. There is also some known g such that $\vec{v}g = \vec{w}$. Thus by considering $g\sigma^{-1}$, we can assume that $\vec{v}$ is fixed by a new automorphism.

This gives us one fixed vector $\vec{v}$. We then use the rarity of vectors with inner product 4 to break up the current orbit classes. For instance, in the case of our lattice L, we can take $y = 8048$ and expect each of the 2528 free orbits to have maybe 3 or 4 vectors whose inner product with $\vec{v}$ is 4. In particular, this should be true for orbits whose y-value is unique (including $y = 8048$). Then we iterate through each of these possible image vectors, seeing if it can preserve inner products. We have no control over inner products except the first, but each additional member of S should only give approximately a $1/4$ chance of having a matching inner product. Thus once S has more than just a few elements, there is little chance that we will accidentally get the inner products to match.

5.3.1. *Results for the four lattices.* The automorphism group for the first Bachoc-Nebe automorphism group was proven maximal in [3, Theorem 3.2].

Their second lattice has 81 free orbits under the known automorphism group, and 22 of them have unique y-values ($y = 8268, 8292, \ldots, 8852$). This is the toughest case for our procedure, as a given free orbit would typically have about 30 vectors of inner product 4 with our initial fixed vector. However, we can exploit the classes with unique y-value and non-trivial stabiliser in this case. In particular, if the stabiliser is of size about 30, there is a decent chance of obtaining a unique vector of inner product 4. For instance, given a vector $\vec{v}$ with $y = 8268$, there is a unique vector with each $y \in \{9808, 9976, 16152\}$ whose inner product with $\vec{v}$ is 4. The stabilisers here are respectively of sizes 36, 24, and 32. This then gives us 4 fixed vectors, and the process is fairly mechanical after that. For instance, the vectors with $y = 8272$ (stabiliser of size 6) which have inner product 4 with $\vec{v}$ then split, giving us 7 new fixed vectors, then $y = 8292$ gives 56 more, and so on. We conclude that the group listed by Bachoc and Nebe is indeed the full automorphism group.

The extremal lattice associated to the length 80 extended quadratic residue code has 2528 free orbits of which 51 have a unique y-value. Fixing a vector with $y = 8048$, there is a unique vector with inner product 4 in each of the $y = 8120, 8126, 8130$ classes (and indeed with some other classes). These four vectors then yield three more with $y = 8154$, and in this manner we quickly generate the whole lattice. Thus we conclude that $\mathbf{SL}_2(\mathbf{F}_{79})$ is the full automorphism group.

As noted above, for the new extremal lattice L we first need to find the 2-extension G^+. This is expedited by taking a y-value with exactly 2 orbits under G, and (as above) composing with a known automorphism to get that a specific vector $\vec{v}$ in one of them maps to a specific vector $\vec{w}$ in the

other. Then we proceed as above, enlarging the set S until we have definite images for a set S that generates L; this then gives us an automorphism of the lattice that was not previously known. Upon finding this 2-extension, we then prove it is the full automorphism group in the same manner as above. For instance, a fixed vector in the $y = 8092$ class yields a unique vector with inner product 4 in both the $y = 8098$ and $y = 8138$ classes, which then split the four such vectors with $y = 8180$, etc.

6. A different proof that L is not isometric to the known lattices

Imitating and expanding on the appendix of [38], we can give a second proof of the non-isometry of L with the previously known lattices via the Classification of Finite Simple Groups. The problem is that we need to rule out the possibility of a finite matrix group in $\mathbf{GL}_{80}(\mathbf{Z})$ that contains both a copy of G and a copy of one of the other groups.[7] Here and in the below, when we write G as a matrix group we mean the representation corresponding to the lattice L.

For the first lattice of Bachoc and Nebe they show [3, Theorem 3.2] that the associated group $2.M_{22}.2 \otimes_\alpha 2.A_7$ (where $\alpha = \sqrt{-7}$) is maximal finite, and so it suffices to note that $\#G$ does not divide the order of that group. Similarly, the appendix of [38] shows that $\mathbf{SL}_2(\mathbf{F}_{79})$ is maximal as a finite subgroup of $\mathbf{GL}_{80}(\mathbf{Z})$ except possibly for a small-index extension, and again we easily conclude that there is no common finite supergroup of G and $\mathbf{SL}_2(\mathbf{F}_{79})$ in $\mathbf{GL}_{80}(\mathbf{Z})$.

We are left to show $H = (2^5 : S_6) \otimes_\beta (\mathbf{SL}_2(\mathbf{F}_5) \times C_3).2$ (where $\beta = \sqrt{-15}$) and G have no common finite supergroup M in $\mathbf{GL}_{80}(\mathbf{Z})$. The only facts that we use about H are that $3^4 | \#H$, and that there is no chain of subgroups $H'' \triangleleft H' \triangleleft H$ with $[H : H'] \leq 2$ and $[H' : H''] \leq 4$ such that the centre of H'' contains $C_3 \times C_3$. As with G, we associate H to a specific 80-dimensional representation, determined up to conjugacy in $\mathbf{GL}_{80}(\mathbf{Z})$.[8]

As with the appendix of [38], the fact there is a prime of "large" size (here 41, with the comparison being to the degrees of the matrix groups) dividing the order of G helps ease the proof, as does the fact that $\mathbf{SL}_2(\mathbf{F}_{41})$ is itself a classical group.

6.1. Overview of argument. By Minkowski's Lemma [24], we can inject $\iota_p : \mathbf{GL}_{80}(\mathbf{Z}) \hookrightarrow \mathbf{GL}_{80}(\mathbf{F}_p)$ for odd primes p (with a variation at 2). This helps us two ways, the first being that we can take a gcd of $\#\mathbf{GL}_{80}(\mathbf{F}_p)$ over all odd p, and get a divisibility condition on the order of any finite

[7] We worked with G rather than the 2-extension G^+ due to our belated knowledge of the latter. There is little difference in the argument, as we consider normal subgroups in any event.

[8] We could instead work in $\mathbf{SL}_{80}(\mathbf{Z})$, which has various plusses and minuses for our argument.

subgroup of $\mathbf{GL}_{80}(\mathbf{Z})$, namely that the order must divide

$$2^{198}3^{58}5^{24}7^{14}11^{8}13^{6}17^{5}19^{4}23^{3}29^{2}31^{2}37^{2}41^{2}\cdot 43\cdot 47\cdot 53\cdot 59\cdot 61\cdot 67\cdot 71\cdot 73\cdot 79.$$

6.1.1. *Choosing a prime.* Secondly, we can use this injection for a specific prime and then apply Aschbacher's theorem on subgroups of finite classical groups [2]. We use $p = 101$, though other choices are feasible. We have a variety of conditions, some more for convenience than necessity.

- p is inert in $\mathbf{Q}(\sqrt{41})$ so that $\iota_p(G)$ is irreducible,
- $3\|\#\mathbf{GL}_2(\mathbf{F}_p)$ and $3^2\|\#\mathbf{GL}_2(\mathbf{F}_{p^2})$,
- $41 \nmid \#\mathbf{GL}_n(\mathbf{F}_p)$ for $n < 40$ with $n|40$.

Finally, choosing $p > 80$ is convenient so as to make the use of Minkowski's bound completely trivial in some cases.

6.1.2. *Bounding M.* Suppose $M \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$ with $\iota_p(G), \iota_p(H) \Subset M$. Here and in the following the $\Subset$ symbol will mean that the right side contains an isomorphic copy of the left side. The $\subseteq$ symbol will usually be an inclusion into a matrix group, where the natural representation is assumed for the group on the left. The only requirement typically is that the -1 elements must behave naturally; for instance in $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset M \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$ the -1 element in $\mathbf{SL}_2(\mathbf{F}_{41})$ should map to -1 in $\mathbf{GL}_{80}(\mathbf{F}_p)$.

The continual idea of the proof shall be that: either p divides M (in fact, usually to a large power) contradicting the Minkowski bound so that M cannot have a pre-image (under ι_p) in $\mathbf{GL}_{80}(\mathbf{Z})$; or M is small, indeed too small to contain $\iota_p(H)$, for which we have $3^4|\#H$.

6.2. A reformulation of Aschbacher's theorem. We now use Aschbacher's theorem [2] (see also [20]). This is typically phrased in such a way to emphasise the characterisation of maximal subgroups, while our direction is somewhat different. In particular, it can be hard for a non-expert to know exactly what "class 9" means in some cases, as it is a catch-all class to some extent, and thus all the others must be well-understood. Therefore we first reformulate the principal result of [2] in a form more suitable for our use, following the outline of [19, §3].

While the result can be phrased in terms of semilinear representations, we chose to use (general) linear representations for simplicity. This amounts to taking a normal subgroup at various junctures.

Proposition 6.1. *Suppose that $M \subseteq \mathbf{GL}_d(\mathbf{F}_{p^r})$ with the natural action. Assume that d is not a perfect power. Then one of the following is true:*

- M is reducible. This case never occurs for us. We can mention that M is contained in a maximal parabolic subgroup (class 1). We can also include writing M over a smaller field here (class 5).

- M is irreducible but imprimitive, that is, there is some $N \triangleleft M$ that is reducible with M/N permuting the constituents fixed by N (class 2). Here the index $t = [M : N]$ must divide d and M is contained in $\mathbf{GL}_{d/t}(\mathbf{F}_{p^r}) \wr S_t$. The extension G^+/G is an example of this.
- M is irreducible and primitive, but there is some noncentral $N \triangleleft M$ (we include the possibility $N = M$ here) that is irreducible but not absolutely irreducible (class 3). In this case we can consider the set C of matrices in $\mathbf{GL}_d(\mathbf{F}_{p^r})$ that commute with N, and then C can be given a field structure $\mathbf{F}_{p^{ru}}$, where the hypothesis on N implies $u > 1$. Furthermore, we can choose a basis for $\mathbf{F}_{p^r}^d$ so that N is writeable as block matrices in dimension d/u, where each block of size u gives an element of $\mathbf{F}_{p^{ru}}$. We get that $M \subseteq \mathbf{\Gamma L}_{d/u}(\mathbf{F}_{p^{ru}})$, and when $M = N$ there is no semi-linear action induced here – but in general one can come from M/N. Finally, by possibly taking an intermediate N we can ensure that u is prime.
- M is absolutely irreducible and primitive, and all noncentral normal subgroups are either reducible or absolutely irreducible, but there is some noncentral $N \triangleleft M$ with N reducible. Here we can decompose $\mathbf{F}_{p^r}^d = \oplus_j V_j$ so that N acts irreducibly on the V_j in block-diagonal form, and the action of each element of N is given by $\mathrm{diag}(A, \ldots, A)$ for some $A \in \mathbf{GL}_a(\mathbf{F}_{p^r})$ with $a|d$. As above, the set of matrices that commute with all the A so obtained can be given a field structure $\mathbf{F}$, and the assumptions on M and N imply $\mathbf{F} \cong \mathbf{F}_{p^r}$. With this case we get a tensor product decomposition and find $M \subseteq \mathbf{GL}_a(\mathbf{F}_{p^r}) \circ \mathbf{GL}_{d/a}(\mathbf{F}_{p^r})$ (class 4).
- M is primitive and all its (noncentral) normal subgroups act absolutely irreducibly. Then M modulo its centre is almost simple (class 9, including class 8).

Remark 6.2.1. *We comment on some differences when compared to Aschbacher's classes. The classes 6 and 7 are not possible due to our assumption that d is not a perfect power (we also avoid triality when $d = 8$).*

It is possible to permute the steps in some cases, for instance, to handle a reducible $N \triangleleft M$ (class 4) prior to enlarging the field (in class 3) [where the latter would be so as to force some other normal subgroup to have a reducible action].

We subsume the class 8 classical group inclusions into class 9. For instance, if we have that M modulo its centre is $\mathbf{PSp}_d$, then M contains $\mathbf{Sp}_d$.

Remark 6.2.2. *There are various algorithms to implement the above theorem that have been implemented in Magma by E. O'Brien, and we found these to be useful during part of this work.*

6.2.3. *Facts about G.* The natural 80-dimensional representation of G is irreducible over $\mathbf{F}_p$, and splits into two absolutely irreducible components over $\mathbf{F}_{p^2}$. There is only one proper normal subgroup $G_2 \triangleleft G$ whose index divides 80, with $G_2 \cong \mathbf{SL}_2(\mathbf{F}_{41}) \otimes C_6 \cong \mathbf{SL}_2(\mathbf{F}_{41}) \times C_3$. The 80-dimensional representation of G_2 splits into two 40-dimensional constituents over $\mathbf{F}_p$, and into four absolutely irreducible 20-dimensional components over $\mathbf{F}_{p^2}$. The centre of G_2 is cyclic of order 6.

6.3. Applying Aschbacher's theorem to our case. We next analyse the case of class 9 (including class 8) a bit further.

Lemma 6.2. *Take $p = 101$ and let $M \subseteq \mathbf{GL}_d(\mathbf{F}_{p^r})$ where $rd \in \{40, 80\}$. Suppose all the noncentral normal subgroups of M are absolutely irreducible. Assume M contains a copy of $\mathbf{SL}_2(\mathbf{F}_{41})$ (so that $20|d$). Write Y for M modulo its centre, and assume Y is almost simple. Writing $Y' \triangleleft Y$ for the associated simple group, then either $Y' \cong \mathbf{PSL}_2(\mathbf{F}_{41})$ (here we must have that $d \in \{20, 40\}$), or Y' is isomorphic to $\mathbf{C}_d(\mathbf{F}_{p^s})$ for some classical Chevalley group $\mathbf{C} \in \{\mathbf{PSL}, \mathbf{PSU}, \mathbf{PSp}, \mathbf{P\Omega}^{\pm}\}$ with $s|r$. In the first case we have $M \Subset \mathbf{Aut}\big(\mathbf{PSL}_2(\mathbf{F}_{41})\big).\mathbf{GL}_1(\mathbf{F}_{p^r})$, while in the second we have $p|\#M$.*

Proof. It is immediate that $\mathbf{PSL}_2(\mathbf{F}_{41}) \Subset Y$, and as $Y' \cap \mathbf{PSL}_2(\mathbf{F}_{41})$ must be normal in $\mathbf{PSL}_2(\mathbf{F}_{41})$, it follows that $\mathbf{PSL}_2(\mathbf{F}_{41}) \Subset Y'$ also. We let M' be a minimal cover of Y' contained in M, noting $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset M' \subseteq \mathbf{SL}_d(\mathbf{F}_{p^r})$ with $M' \triangleleft M$. Our assumption on normal subgroups implies that M' is absolutely irreducible as a matrix group in $\mathbf{GL}_d(\mathbf{F}_{p^r})$.

We have $Y' \subseteq \mathbf{PSL}_d(\mathbf{F}_{p^r})$, and by Schur's theorem [37, §2, III] we know this projective representation $\bar{\rho}$ of Y' lifts to an ordinary representation ρ of the universal cover of Y'. Since M' is contained in the universal cover, we can restrict ρ to it. By hypothesis, this representation on M' is absolutely irreducible. We now leverage the fact that the quasi-simple group M' has an absolutely irreducible d-dimensional representation. There are two cases.

Case 1. Suppose Y' is not a Chevalley group (of any flavour) in characteristic p. Here we use the list of Hiss and Malle [17], which gives the possible representation degrees for all the associated quasi-simple groups. Combined with the requirement $\#\mathbf{PSL}_2(\mathbf{F}_{41})|\#Y'$, the possibilities for Y' are:

- $\mathbf{PSL}_2(\mathbf{F}_{41})$ in dimension 20;
- $\mathbf{Alt}_{41}$, $\mathbf{PSp}_8(\mathbf{F}_3)$, and $\mathbf{PSL}_2(\mathbf{F}_{41})$ in dimension 40;
- and $\mathbf{Alt}_{81}$ in dimension 80.

The inclusion $\mathbf{PSL}_2(\mathbf{F}_{41}) \subseteq \mathbf{PSp}_8(\mathbf{F}_3)$ is not possible, for the smallest degree of a projective nontrivial representation (in characteristic not 41) of $\mathbf{PSL}_2(\mathbf{F}_{41})$ is 20 (as with ordinary representations of $\mathbf{SL}_2(\mathbf{F}_{41})$ of course).

A theorem of Galois [13] implies that the smallest degree of a permutation representation of $\mathbf{PSL}_2(\mathbf{F}_{41})$ is 42, and so we can eliminate the case

of $\mathbf{Alt}_{41}$. With $\mathbf{Alt}_{81}$ we consider possible central elements and get the inclusion chain $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset \big(\alpha \cdot \mathbf{Alt}_{81}\big) \times \mathbf{Ab} \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$. Here α is either 1 or 2 and $\mathbf{Ab}$ is an abelian group. The intersection of $\mathbf{SL}_2(\mathbf{F}_{41})$ with $\alpha \cdot \mathbf{Alt}_{81}$ must be normal in the former, and so $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset \alpha \cdot \mathbf{Alt}_{81}$. Now we can note that the smallest degree of a permutation representation of $\mathbf{SL}_2(\mathbf{F}_{41})$ is 336, so that $\alpha \neq 1$. On the other hand, for $\alpha = 2$ we find that the inclusion $\alpha \cdot \mathbf{Alt}_{81} \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$ is not possible since $2 \cdot \mathbf{Alt}_{81}$ has no faithful 80-dimensional representations.

Case 2. Next we consider the case where Y' is a (possibly twisted) Chevalley group in characteristic p. Here we use the lists of Lübeck [22], which give degrees of projective representations in defining characteristic for such simple groups. As the natural representation of M' is absolutely irreducible, the same is true for the induced projective representation of Y'.

For ranks exceeding 11, we use [22, Table 2] to find that the only possible projective representations are from other Chevalley groups in the same degree, such as the degree 80 projective representation $\mathbf{PSU}_{80} \subset \mathbf{PSL}_{80}$. For classical groups of smaller rank, we can again use the fact that any nontrivial projective representation of $\mathbf{PSL}_2(\mathbf{F}_{41})$ in characteristic p must be of degree at least 20, and this leaves only the root systems $\{B, C, D\}_{\{10,11\}}$, $E_{\{6,7,8\}}$, F_4, and G_2. The appendices of [22] show the only feasible representations are degree 20 inclusions as above. Thus we get that Y' must be as stated in the lemma. □

6.4. A lemma of convenience. We shall find the following lemma to be useful. We prove it in a bit more generality than is necessary. The argument is also a mini-version of that for $\mathbf{GL}_{80}(\mathbf{F}_p)$.

Lemma 6.3. *Let $p = 101$ and suppose that $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset X \subseteq \mathbf{GL}_{40}(\mathbf{F}_p)$. Then either $X \Subset \mathbf{Aut}\big(\mathbf{PSL}_2(\mathbf{F}_{41})\big).\mathbf{GL}_1(\mathbf{F}_{p^2}).2$ so that $3^3 \nmid \#X$, or $p|\#X$.*

Proof. We apply the above Proposition 6.1 to $X \subseteq \mathbf{GL}_{40}(\mathbf{F}_p)$. Since p is inert in $\mathbf{Q}(\sqrt{41})$ the action of $\mathbf{SL}_2(\mathbf{F}_{41})$ is irreducible, and thus X is also irreducible, eliminating class 1. Since we need 41 to divide $\#X$ and 41 does not divide $\#\mathbf{GL}_n(\mathbf{F}_p)$ for $n \leq 20$, we can exclude classes 2 and 4.

When X and all its noncentral normal subgroups are absolutely irreducible, we use Lemma 6.2 and get: either X modulo its centre is contained in the automorphism group of $\mathbf{PSL}_2(\mathbf{F}_{41})$; or X contains some classical group of degree 40 in characteristic p. The lemma is seen to hold here.

Finally, suppose we have a class 3 reduction, which says $X \subseteq \mathbf{\Gamma L}_a(\mathbf{F}_{p^u})$ with $au = 40$. We can intersect X with $\mathbf{GL}_a(\mathbf{F}_{p^u})$, and this will have an index that divides u, so in particular no more than 40. The intersection thus must contain $\mathbf{SL}_2(\mathbf{F}_{41})$ as this group has no subgroups of index less than 42. So we have $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset \mathbf{GL}_a(\mathbf{F}_{p^u})$, and from the minimal degree of representation for $\mathbf{SL}_2(\mathbf{F}_{41})$ conclude that $a = 20$ is the only possibility.

Taking the intersection $X' = X \cap \mathbf{GL}_{20}(\mathbf{F}_{p^2})$ (of index at most 2 in X), we can apply Proposition 6.1 to $X' \subseteq \mathbf{GL}_{20}(\mathbf{F}_{p^2})$. The first four classes are excluded as above, so the only case possible is class 9. We use Lemma 6.2 and get the same dichotomy for X as previously. □

6.5. Aschbacher analysis for $M \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$. We now consider the various cases for $M \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$ in Proposition 6.1, some of which lead to a further use of this proposition. Since $\iota_p(G) \Subset M \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$ and $\iota_p(G)$ is irreducible, we cannot have a class 1 decomposition. The above Lemma 6.2 handles class 9, as it shows that M contains a characteristic p classical group, so that p divides its order, contradicting the Minkowski bound.

6.5.1. *Class 2 for $M \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$.* In this case, we have some normal subgroup $N \triangleleft M$ with N reducible. The index $[M : N]$ must divide the degree of the matrix group, and since $\iota_p(G)$ is irreducible, the reducible group $N \cap \iota_p(G)$ is a proper (normal) subgroup of $\iota_p(G)$. A brief computation shows that the only (proper) normal subgroup of G whose index divides 80 is $G_2 = \mathbf{SL}_2(\mathbf{F}_{41}) \otimes C_6 \cong \mathbf{SL}_2(\mathbf{F}_{41}) \times C_3$ (of index 2). We can note that centre of G_2 is cyclic of order 6.

Additionally in this case, there is a basis for the 80-dimensional vector space over $\mathbf{F}_p$ such that the every element of the action of M can be written as $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ or $\begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix}$ for some 40-dimensional matrices A and B. The matrices of the first form correspond to the reducible action of N. We write N_1, N_2 for the action of N on the two subsets of 40 elements, noting that $N_1, N_2 \Subset M \Subset (N_1 \times N_2).2$. The action of the subgroup G_2 on either subset of 40 elements is isomorphic to G_2, so that $G_2 \Subset N_i \subseteq \mathbf{GL}_{40}(\mathbf{F}_p)$. This allows us to use Lemma 6.3 on the N_i.

When $p|\#N_i$ for either factor, we get $p|\#M$ to contradict the Minkowski bound. We thus must have $M \Subset \big[\mathbf{Aut}\big(\mathbf{PSL}_2(\mathbf{F}_{41})\big).\mathbf{GL}_1(\mathbf{F}_{p^2}).2\big]^2.2$. We have $3^4|\#H$ implying $3^4|\#M$, and so we need $G_2^2 = (\mathbf{SL}_2(\mathbf{F}_{41}) \times C_3)^2 \Subset M$. This gives us two further facts: firstly that M and H share the common 3-Sylow subgroup C_3^4; and secondly there is some chain of normal subgroups $N' \triangleleft N \triangleleft M$ where $[M : N] = 2$ and $[N : N'] \leq 4$ with the inclusion chain $G_2^2 \Subset N' \Subset \big[\mathbf{Aut}\big(\mathbf{PSL}_2(\mathbf{F}_{41})\big).\mathbf{GL}_1(\mathbf{F}_{p^2})\big]^2$. From this we see that $C_3 \times C_3$ is contained in the centre of N'. Via intersecting this subgroup chain with H, we find a similar chain $H'' \triangleleft H' \triangleleft H$ with $[H' : H] \leq 2$ and $[H'' : H'] \leq 4$, where the centre of H'' must also contain $C_3 \times C_3$. However a brief computation shows there is no such subgroup chain for H.[9]

[9]There are normal subgroups of index 2 and 4 of H with C_3 in the centre; these correspond to the $\mathbf{SL}_2(\mathbf{F}_5) \times C_3$ part of H, but the $(2^5 : S_6)$ part does not yield any such central elements. The class 2 considerations are the only time we need to use facts about H other than its order.

6.5.2. *Class 4 for* $M \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$. This class consists of tensor product decompositions. The requirement $41|\#M$ implies the only possibility for $a|80$ is $a \in \{2, 40\}$. So we have $M \subseteq \mathbf{GL}_{40}(\mathbf{F}_p) \circ \mathbf{GL}_2(\mathbf{F}_p)$, and each of these factors is normal in the central product. In particular, the intersection of either factor with $\mathbf{SL}_2(\mathbf{F}_{41})$ must be normal too; the only nontrivial normal subgroup of $\mathbf{SL}_2(\mathbf{F}_{41})$ is the centre $\{\pm 1\}$, and so $\mathbf{SL}_2(\mathbf{F}_{41})$ must be contained in one of the factors. Since there is no 2-dimensional representation of $\mathbf{SL}_2(\mathbf{F}_{41})$ in characteristic p, we must have $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset \mathbf{GL}_{40}(\mathbf{F}_p)$ in the above factorisation.

We now intersect M with $\mathbf{GL}_{40}(\mathbf{F}_p)$, and obtain a subgroup $S \subseteq M$ with $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset S \subseteq \mathbf{GL}_{40}(\mathbf{F}_p)$ whose index satisfies $[M : S]|\#\mathbf{GL}_2(\mathbf{F}_p)$. We can thus apply Lemma 6.3 to S, and get either $3^3 \nmid \#S$ whence $3^4 \nmid \#M$ since $3^2 \nmid \#\mathbf{GL}_2(\mathbf{F}_p)$, or $p|\#S$ so as to contradict the Minkowski bound.

6.5.3. *Class 3 for* $M \subseteq \mathbf{GL}_{80}(\mathbf{F}_p)$. This class corresponds to writing an irreducible representation of some $N \triangleleft M$ over a larger field to make it reducible. In this case we have the inclusion $M \subseteq \mathbf{\Gamma L}_a(\mathbf{F}_{p^u}) = \mathbf{GL}_a(\mathbf{F}_{p^u}).u$ where $au = 80$. The extension here is from a normaliser, being induced by the Frobenius map $x \to x^p$ (see [11] for instance). As in the proof of Lemma 6.3, the fact that $\mathbf{SL}_2(\mathbf{F}_{41})$ has a minimal representation degree of 20 implies that a is either 20 or 40. We can take u to be prime, and so need only consider $u = 2$. We take the intersection $M' = M \cap \mathbf{GL}_{40}(\mathbf{F}_{p^2})$ with $[M : M'] \leq 2$ and apply Proposition 6.1 to M'. As M' is normal in M and $\mathbf{SL}_2(\mathbf{F}_{41})$ has no normal subgroups of index 2, we get $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset M'$.

6.6. Aschbacher analysis for $M' \subseteq \mathbf{GL}_{40}(\mathbf{F}_{p^2})$**.** Again we cannot have class 1, and for class 5 we note $M' \subseteq \mathbf{GL}_{40}(\mathbf{F}_p)$ and apply Lemma 6.3. In the case of class 9, we apply Lemma 6.2 and get that either $p|\#M'$ or $M' \Subset \mathbf{Aut}\big(\mathbf{PSL}_2(\mathbf{F}_{41})\big).\mathbf{GL}_1(\mathbf{F}_{p^2})$, the latter case implying $3^3 \nmid \#M$. The analysis for the other cases will follow the same pattern as above.

6.6.1. *Class 2 for* $M' \subseteq \mathbf{GL}_{40}(\mathbf{F}_{p^2})$. As we are in class 2, there is some normal subgroup $N \triangleleft M'$ with N reducible. As above, we have $[M' : N] = 2$ and $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset N$. We write N in block form as previously, denoting the actions by N_1 and N_2, and get the inclusions $N_1, N_2 \Subset M' \Subset (N_1 \times N_2).2$ and $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset N_i \subseteq \mathbf{GL}_{20}(\mathbf{F}_{p^2}) \subseteq \mathbf{GL}_{40}(\mathbf{F}_p)$. We then conclude as with the earlier argument for class 2.

6.6.2. *Class 4 for* $M' \subseteq \mathbf{GL}_{40}(\mathbf{F}_{p^2})$. Since we have $41|\#M'$, the only possibility for $a|40$ is $a \in \{2, 20\}$, and so $M' \subseteq \mathbf{GL}_{20}(\mathbf{F}_{p^2}) \circ \mathbf{GL}_2(\mathbf{F}_{p^2})$. We can move central elements to either part of this product, and so can consider the second factor to be $\mathbf{GL}_2(\mathbf{F}_{p^2})/\mathbf{GL}_1(\mathbf{F}_{p^2})$ if we like. In particular, when we intersect M' with $\mathbf{GL}_{20}(\mathbf{F}_{p^2})$ we get S with $[M' : S]|(p^6 - p^2)$ and $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset S \subseteq \mathbf{GL}_{20}(\mathbf{F}_{p^2}) \subset \mathbf{GL}_{40}(\mathbf{F}_p)$, where the first inclusion

follows (by normality) as above. We apply Lemma 6.3 to S, and get: either $3^3 \nmid \#S$ so that $3^4 \nmid \#M$; or $p|\#S$ contradicting the Minkowski bound.

6.6.3. *Class 3 for $M' \subseteq \mathbf{GL}_{40}(\mathbf{F}_{p^2})$.* Arguing as above, the 20-dimensional representation of $\mathbf{SL}_2(\mathbf{F}_{41})$ implies that the only possibility for class 3 is for $M' \subseteq \mathbf{GL}_{20}(\mathbf{F}_{p^2}).2$. We take the intersection $M'' = M' \cap \mathbf{GL}_{20}(\mathbf{F}_{p^2})$ of index at most 2, and note $\mathbf{SL}_2(\mathbf{F}_{41}) \Subset M''$. We proceed to apply Proposition 6.1 to M''.

6.7. Aschbacher analysis for $M'' \subseteq \mathbf{GL}_{20}(\mathbf{F}_{p^4})$. As previously, neither class 1 nor class 3 is possible. Furthermore, a class 2 or class 4 splitting would yield a representation of $\mathbf{SL}_2(\mathbf{F}_{41})$ in some dimension properly dividing 20, and so these also are not possible. We handle class 9 via Lemma 6.2, getting either $p|\#M''$ or $3^3 \nmid \#M''$. When M is writeable over $\mathbf{F}_{p^2}$ (class 5), we have $M \subseteq \mathbf{GL}_{20}(\mathbf{F}_{p^2}) \subset \mathbf{GL}_{40}(\mathbf{F}_p)$ and apply Lemma 6.3.

6.8. Conclusion. The above shows that any subgroup $M \subseteq \mathbf{GL}_{80}(\mathbf{F}_{101})$ that contains a copy of $\iota_{101}(G)$ cannot also contain a copy of $\iota_{101}(H)$ unless $101|\#M$. We conclude that there is no common supergroup of G and H in $\mathbf{GL}_{80}(\mathbf{Z})$, so our lattice is distinct from the second Bachoc-Nebe lattice.

7. Extremal lattices in dimension 64

We were able to construct a new extremal lattice of dimension 64 as follows. Writing $K = \mathbf{Q}(\sqrt{-11})$ and letting $w = \frac{-1+\sqrt{-11}}{2}$, we used the unimodular matrix $M_2 = \begin{pmatrix} 2 & w \\ \bar{w} & 2 \end{pmatrix}$ and took the tensor product (over K) of it with each of Hentschel's six ϑ-lattices [16] of rank 8 over $\mathbf{Q}(\sqrt{-11})$. Upon expanding to a basis over $\mathbf{Q}$, four of these six yielded extremal lattices of dimension 32. We then took a few "random" neighbours of these 16-dimensional lattices over K, and again tensored these with M_2. One of these, namely a $(5+4w)$-neighbour of $M_2 \otimes H_3$ where H_3 is the third of Hentschel's ϑ-lattices, upon expanding the basis to $\mathbf{Q}$ yielded a 64-dimensional lattice with minimum 6. It took only about 40 cpu-minutes for an exhaustive search (after a suitable BKZ reduction) to show that the lattice had no vectors of minimum 4. The Hermitian automorphism group of the 32-dimensional K-lattice is isomorphic to the dihedral group D_6 on six symbols (which is already that for M_2).

Remark 7.0.1. *In some metric, this is the "best possible" case for doing such tensor products, as extremal lattices are thought to be more common in dimensions $(24k+16)$ than in dimension $24k$. The fact that the lattice has minimal automorphisms is perhaps uninteresting from the standpoint of group theory, but does show that the behaviour is "generic" in a suitable*

sense. Also, the rank 16 lattice over K we first constructed is computationally tricky to handle. For instance, it takes a couple of hours to compute the automorphism group. As already noted in [31]*, computing isometries in dimension 32 is somewhat difficult, due to the large number of extremal lattices (and a lack of easily computed invariants for isometry).*

7.1. Comparison to known extremal lattices in dimension 64. The first known extremal lattice in dimension 64 was constructed by Quebbemann [34]. As noted in [7, §8], the construction can be modified in various ways, and it not exactly clear how many non-isometric lattices can be produced. We have chosen to ignore these lattices for our discussion here.

A second extremal lattice T_{64} in dimension 64 was constructed from coding theory by Ozeki [29] (see also [15]). Finally, using an anti-identification of two maximal orders of associated quaternionic endomorphism rings, Nebe [25, Remark 5.2] constructed a (unimodular) lattice N_{64} with automorphism group containing $\big(\mathbf{SL}_2(\mathbf{F}_{17}) \circ \mathbf{SL}_2(\mathbf{F}_5)\big).2^2$, where the factors in the central product correspond respectively to quaternionic representations of degree 8 and 2. This was later proven to be extremal in [27].

In order to show the lattice constructed here differs from N_{64} and T_{64}, we can proceed by computing inner product distributions. As in Section 4, we can compute that for a given vector $\vec{v}$ of norm 6, there is some integer t with $0 \leq t \leq 17826$ such that the distribution of inner products is:

- $2(26t + 680792)$ vectors $\vec{w}$ with $\vec{v} \cdot \vec{w} = 0$,
- $-33t + 588288$ vectors $\vec{w}$ with $\vec{v} \cdot \vec{w} = 1$,
- $6t + 36519$ vectors $\vec{w}$ with $\vec{v} \cdot \vec{w} = 2$,
- t vectors $\vec{w}$ with $\vec{v} \cdot \vec{w} = 3$,
- 1 vector $\vec{w}$ with $\vec{v} \cdot \vec{w} = 6$.

7.1.1. *Nebe's lattice.* With Nebe's lattice N_{64}, there are 8 orbits of the 2611200 minimal vectors (under the automorphism group of order 1175040), divided as:

- an orbit with stabiliser of size 2 and $t = 254$,
- an orbit with stabiliser of size 2 and $t = 284$,
- an orbit with stabiliser of size 2 and $t = 318$,
- an orbit with stabiliser of size 4 and $t = 336$,
- an orbit with stabiliser of size 4 and $t = 344$,
- an orbit with stabiliser of size 12 and $t = 272$,
- an orbit with stabiliser of size 12 and $t = 452$,
- and an orbit with stabiliser of size 18 and $t = 218$.

As can be seen, the average t-value is $301\frac{7}{10}$.

7.1.2. *Ozeki's lattice.* In order to find the minimal vectors for Ozeki's lattice T_{64}, we used the method for the generic lattice given in Section 7.2

below, as it was not completely obvious whether there would be non-trivial automorphisms.[10] However, it turns out that the automorphism group for T_{64} is isomorphic to the 2-extension of $\mathbf{SL}_2(\mathbf{F}_{31})$ that is given by $2 \cdot \mathbf{Aut}\big(\mathbf{PSL}_2(\mathbf{F}_{31})\big)$; this can be computed in a few minutes with the Magma command `AutomorphismGroup` (due to W. R. Unger) once the minimal vectors are known.

There are 41 free orbits, six with a stabiliser of size 3, three with a stabiliser of size 5, four with a stabiliser of size 15, and two with a stabiliser of size 465. There are 36 distinct t-values, ranging from 158 to 308 with an average around 222.6 and a most common value of 206. See Table 1 for more complete data, which gives orbit counts weighted by automorphisms.

Table 1. t-distribution for Ozeki's lattice T_{64}

158	$\frac{1}{3}$	198	1	214	1	228	3	244	$1+\frac{1}{5}$	258	1
174	$\frac{1}{5}$	200	$2+\frac{1}{3}$	216	1	230	$\frac{1}{3}$	246	2	272	1
184	1	206	5	218	3	234	1	248	$\frac{2}{15}+\frac{2}{465}$	278	$\frac{1}{15}$
188	$\frac{1}{3}$	208	3	220	2	236	1	252	1	280	1
190	1	210	1	222	1	240	1	254	1	294	$\frac{1}{5}$
194	1	212	1	224	$2+\frac{1}{3}$	242	$\frac{1}{3}$	256	1	308	$\frac{1}{15}$

7.1.3. *The new lattice.* The new lattice H_{64} has 217600 orbits (each with trivial stabiliser), and to show its distinctness we can simply note that it has (say) a minimal vector with $t = 124$. Indeed, we found all the minimal vectors via the search strategy given in Sections 3.3 and 3.3.1, and computed the complete inner product distribution. All the t-values are even, the minimum is 124, the maximum is 304, the average is just over 214, and the most common value (8530 orbits) is 210. See Table 2 for the complete distribution.

7.2. Another extremal lattice in dimension 64. We were also able to find a generic (with only the trivial automorphisms) extremal lattice G_{64} in dimension 64 via more neighbouring. We started with the new extremal lattice of above, and then took random neighbours (over $\mathbf{Q}$). After some effort, this succeeded. One difficulty is that many of the obtained lattices had a vector of norm 4, but this was only detected near the end of the search, after taking over an hour in some instances. However, it still took less than 10 cpu-hours to find one which turned out to be extremal.

[10] For instance, the $\mathbf{F}_3$ reduction of the original $\mathbf{Z}_6$ code has C_4 as its automorphism group.

TABLE 2. t-distribution for Hermitian H_{64}

124	2	162	274	192	4991	222	7691	252	1469	282	46
126	1	164	375	194	5315	224	7348	254	1276	284	30
136	3	166	461	196	5871	226	6937	256	1095	286	28
138	2	168	626	198	6460	228	6559	258	884	288	17
140	2	170	787	200	6980	230	6074	260	730	290	19
142	4	172	969	202	7452	232	5675	262	590	292	7
144	21	174	1234	204	7694	234	5125	264	505	294	12
146	13	176	1445	206	8087	236	4737	266	376	296	8
148	26	178	1760	208	8332	238	4235	268	271	298	7
150	42	180	2121	210	8530	240	3583	270	217	300	3
152	48	182	2486	212	8476	242	3214	272	165	302	1
154	85	184	2920	214	8513	244	2765	274	123		
156	110	186	3363	216	8494	246	2398	276	110		
158	171	188	3898	218	8245	248	2106	278	90		
160	210	190	4289	220	7964	250	1857	280	65		

We then turned to the listing of minimal vectors. As there are no known nontrivial automorphisms, we need to find 1305600 vectors of norm 6. The method of Section 3.3.1 above showed a few problems for this lattice. The idea is to start with a collection of norm 6 vectors, and then expand this collection via looking for pairs in it that have an inner product of size 3. From the above analysis, there is presumably around a 1/6000 chance of this happening for a random pair. At the outset, we thus need a sufficiently large "seeding" set. For instance, 1000 vectors is probably not enough, as they would only produce about $\binom{1000}{2}/6000 \approx 80$ new vectors of norm 6, and we would quickly reach a state where no new norm 6 vectors could be obtained from the current set. We used a pruning-based method as in Section 3.3 in order to start with enough vectors so as to circumvent this.

However, we can still run into problems later on. In our actual run, we hit a wall at 686824 vectors, and so returned to the device of making various perturbations of the basis, followed by reduction and pruning-based enumeration. Note that such a difficulty is much less likely to occur in a case where automorphisms are extant, as applying them to known vectors is an alternative method to generate additional vectors of norm 6.

Finally, it not altogether clear that the method of Section 3.3.1 is really faster for our 64-dimensional lattices, as we could often find more norm 6 vectors per second using the pruning method when a sufficiently sharp choice for the pruning function was used (the exact yield also depends upon the goodness of the BKZ-basis).

After obtaining all the minimal vectors, we then computed the inner product distribution, which is given in Table 3. As can be seen, the average t-value is slightly less than 212. When comparing to the previous table, recall that the numbers in Table 2 should be multiplied by 6 to account for the known automorphisms.

Table 3. t-distribution for generic G_{64}

124	1	158	757	190	28683	222	47073	254	4668	286	47
128	1	160	986	192	32565	224	44255	256	3866	288	24
130	1	162	1436	194	35707	226	40988	258	2997	290	14
132	3	164	1833	196	39516	228	37526	260	2306	292	13
134	6	166	2617	198	42830	230	33999	262	1796	294	9
136	4	168	3273	200	46093	232	30787	264	1415	296	6
138	14	170	4313	202	49257	234	27068	266	1118	298	2
140	21	172	5569	204	51197	236	24176	268	769	300	2
142	33	174	6981	206	53101	238	20569	270	594	302	2
144	46	176	8730	208	54441	240	17922	272	466	304	3
146	82	178	10804	210	55288	242	15290	274	311	306	1
148	112	180	13023	212	55679	244	13229	276	243	308	1
150	175	182	15729	214	54915	246	10955	278	174		
152	263	184	18774	216	53687	248	8858	280	121		
154	382	186	21722	218	52355	250	7340	282	89		
156	474	188	25170	220	49973	252	5825	284	61		

For each of these lattices we are able to show that the known automorphism group is complete using the method of Section 5.3.

7.3. Tensor products from quaternionic 32-dimensional lattices. The list given by Nebe [26, Theorem 18.1] contains two 32-dimensional lattices that have a quaternionic structure into which we can embed $\mathbf{Q}(\sqrt{-11})$. The first one is $L_{32} = [2_-^{1+8}.\mathbf{O}_8^-(2)]_8$, and upon tensoring with M_2, the resulting $\mathbf{Q}$-lattice splits into two copies of the $\mathbf{Q}$-expansion of L_{32}. The other compatible quaternionic lattice is $[\mathbf{SL}_2(17).2]_8$, which when tensored with M_2 gives Nebe's lattice N_{64}.

8. Sundry

8.1. No new extremal lattices of dimension 48. We were unable to find any new extremal lattices of dimension 48 via such methods. One attempt was made via $\mathfrak{p}_3$-neighbour computations starting with a rank 12 Hermitian matrix over $\mathbf{Q}(\sqrt{-2})$. It is difficult to tell how many lattices we stepped through, as we did not check isometry but at best merely counted

the number of norm 4 vectors upon tensoring with the unimodular matrix $M_2' = \begin{pmatrix} 2 & 1+\sqrt{-2} \\ 1-\sqrt{-2} & 2 \end{pmatrix}$ and expanding to a $\mathbf{Q}$-basis, though it seems that we looked at hundreds or even thousands of such examples. The "typical" resulting 48-dimensional lattice had approximately 3000 pairs of vectors of norm 4.

The only 48-dimensional extremal lattice that was found was a lattice whose Hermitian automorphism group was $\mathbf{SL}_2(\mathbf{F}_{13})$, presumably inherited from a quaternionic structure over $\mathbf{Q}(\sqrt{13})$ ramified at the two infinite places; given that M_2' has $\mathbf{GL}_2(\mathbf{F}_3)$ as its automorphism group, it seems likely that our lattice is isometric to the one already found by Nebe [25]. The arrangement is similar over $\mathbf{Q}(\sqrt{-11})$.

Additionally, in each case, any other quaternionic structure on the Leech lattice Λ_{24} which descends to the imaginary quadratic field will yield two copies of Λ_{24} upon tensoring with M_2' and expanding to a $\mathbf{Q}$-basis. It is notable that while our "random neighbouring" on Hermitian lattices would produce lattices with typically around 3000 pairs of vectors of norm 4, the quaternionic structures induced the extremes: namely, 196560 pairs of norm 4 vectors when it splits as $\Lambda_{24} \oplus \Lambda_{24}$; or zero when the lattice is extremal. The second largest number of pairs of norm 4 vectors we found was 30672 (and indeed comes from a neighbour of the quaternionic basis that induces $\Lambda_{24} \oplus \Lambda_{24}$).

8.2. Further directions. It seems possible to compute the inner product distribution of the minimal vectors for Nebe's extremal lattice [28] of dimension 72, though we have not done so. There are about 5 times as many minimal vectors as with an extremal lattice of dimension 80, but the known automorphism group has more than 10 times as many elements as G does. Via this, we could presumably show that $\big(\mathbf{PSL}_2(\mathbf{F}_7) \times \mathbf{SL}_2(\mathbf{F}_{25})\big) : 2$ is the full automorphism group.

We also have yet to consider whether extremal lattices in dimension 56 can readily be found via neighbouring. We could either start with an arbitrary even unimodular lattice in this dimension,[11] or we could take a (possibly decomposable) rank 28 ϑ-lattice over some imaginary quadratic field, and do neighbouring in this field.

Finally, there are two lattices in dimension 80 given at the end of [5] which remain candidates for extremality, namely $B_{80,1}^{(4)}$ and $B_{80,1}^{(5)}$. As the automorphism group (of either lattice) presumably only has 6560 elements, the methods used here do not seem to be readily applicable.

[11] There is no particular reason to start with an extremal lattice, though four are known in this dimension, namely $B_{56,1}^{(4)}$ in [5], T_{56} from [29], and $L_{56,2}(\mathcal{M})$ and $L_{56,2}(\tilde{\mathcal{M}})$ in [25, Table I], though I do not know if anyone has verified these are all distinct.

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