Note on the Chowla conjecture

Supposedly S. Chowla conjectured that $L(1/2,\chi)\neq 0$ for real Dirichlet characters χ. A function field analogue of this could be that the zeta-function of a hyperelliptic curve over $\mathbf{F}_q$ does not have $\pm\sqrt{q}$ as an eigenvalue. If a curve has $\pm\sqrt{q}$ as an eigenvalue, we say it has the Chowla property, or is a Chowla curve.

For genus 2 curves over $\mathbf{F}_p$, the question becomes whether there is a curve with $(p+1)$ points over $\mathbf{F}_p$ and (p^2-4p+1) points over $\mathbf{F}_{p^2}$. Clearly $y^2=x^5-x$ over $\mathbf{F}_5$ has 6 points (counting the one at infinity) over both these fields; however, this curve has extra automorphisms (besides just the hyperelliptic one), and so might be considered suspect for a putative number field analogue.

The thesis of Cardona enables gives an explicit method to list all genus 2 curves with various automorphism group G. The moduli space of all genus 2 curves has dimension 3, those with $G=C_2\times C_2$ form a surface of dimension 2, while $G=D_8,D_{12}$ are lines on this surface, and contain special points corresponding to double covers of D_{12} or S_4. Finally, there is an isolated point with $G=D_{10}$ that is given by $y^2=x^5-1$ and its twists. This classification is valid as stated for $p\geq 7$.

In the most exotic cases where $|G|=48$, for primes p where -2 is not square it is possible to write down a twist of

$$y^2=(x^2-2)(x^2-4x+2)(x^2-2x+2)$$

with the Chowla property. When p is 5 mod 8 this can just be taken as $y^2=x^5-x$, but when p is 7 mod 8, we need to twist the curve[1] by $\begin{pmatrix}1 & -2\\ 1 & -1\end{pmatrix}$ and this involves finding a point on the conic $-2X^2-Y^2=1$.

Similarly, in the case where $|G|=24$, for primes where -3 is not square we twist

$$y^2=(x^2-2x-2)(x^2+4x+1)(2x^2+2x-1)$$

by $\frac{\sqrt{-3}}{3}\begin{pmatrix}-1 & -2\\ 2 & 1\end{pmatrix}$ and get a curve with Chowla property.

The first curve with the Chowla property and $G=D_{12}$ occurs with $p=13$ and is given by

$$y^2=(x^2+2)(x^2+5)(x^2+10x+7).$$

The first curve with the Chowla property and $G=D_8$ occurs with $p=19$ and is given by

$$y^2=(x^2+1)(x^2+13x+7)(x^2+9x+11).$$

The first curve with the Chowla property and $G=C_2\times C_2$ occurs with $p=29$ and is given by

$$y^2=(x^2+2)(x^2+20x+10)(x^2+27x+16).$$

For the 13 primes from 101 to 163, we find 149 Chowla curves with $G=C_2\times C_2$, 36 with $G=D_8$ and 52 with $G=D_{12}$.

The first curve with generic automorphism group and the Chowla property occurs with $p=61$ and is given by

$$y^2=(x^2+2)(x^2+6x+19)(x^2+7x+47)$$

and its quadratic twist (by a nonsquare).

Since the zeta function of a genus 2 curve is determined by the number of points over the first two fields, the Weil bounds imply that there are approximately $(8\sqrt{p})(8p)\approx 64p^{3/2}$ possible zeta-functions. Since there are about $2p^3$ possible curves, we should expect there to be many Chowla curves asymptotically.

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[1] See Chapter X of Serre's Local Fields; we essentially use the Poincaré series trick with Hilbert's Theorem 90. We want to find M with $M^\sigma M^{-1}=V$ for a matrix V of order k, and this is done by taking a random matrix A in $\mathbf{F}_{q^k}$ and summing $M=\sum_i V^iA^{\sigma^{-i}}$ over the powers of V where σ is the Galois action on $\mathbf{F}_{q^k}/\mathbf{F}_q$; there is such a A such that M is invertible, and transforming the curve by M gives the twist.