Note on the Chowla conjecture

Supposedly S. Chowla conjectured that \( L(1/2, \chi) \neq 0 \) for real Dirichlet characters \( \chi \). A function field analogue of this could be that the zeta-function of a hyperelliptic curve over \( \mathbb{F}_q \) does not have \( \pm \sqrt{q} \) as an eigenvalue. If a curve has \( \pm \sqrt{q} \) as an eigenvalue, we say it has the Chowla property, or is a Chowla curve.

For genus 2 curves over \( \mathbb{F}_p \), the question becomes whether there is a curve with \((p+1)\) points over \( \mathbb{F}_p \) and \((p^2 - 4p + 1)\) points over \( \mathbb{F}_p^2 \). Clearly \( y^2 = x^5 - x \) over \( \mathbb{F}_3 \) has 6 points (counting the one at infinity) over both these fields; however, this curve has extra automorphisms (besides just the hyperelliptic one), and so might be considered suspect for a putative number field analogue.

The thesis of Cardona enables gives an explicit method to list all genus 2 curves with various automorphism group \( G \). The moduli space of all genus 2 curves has dimension 3, those with \( G = C_2 \times C_2 \) form a surface of dimension 2, while \( G = D_6 \), \( D_{12} \) are lines on this surface, and contain special points corresponding to double covers of \( D_{12} \) or \( S_4 \). Finally, there is an isolated point with \( G = D_{10} \) that is given by \( y^2 = x^5 - 1 \) and its twists. This classification is valid as stated for \( p \geq 7 \).

In the most exotic cases where \( |G| = 48 \), for primes \( p \) where \(-2\) is not square it is possible to write down a twist of
\[
y^2 = (x^2 - 2)(x^4 - 4x^2 + 1)(x^2 - 2x + 2)
\]
with the Chowla property. When \( p \) is 5 mod 8 this can just be taken as \( y^2 = x^5 - x \), but when \( p \) is 7 mod 8, we need to twist the curve\(^1\) by \( \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \) and this involves finding a point on the conic \(-2X^2 - Y^2 = 1\).

Similarly, in the case where \( |G| = 24 \), for primes where \(-3\) is not square we twist
\[
y^2 = (x^2 - 2x - 2)(x^2 + 4x + 1)(2x^2 + 2x - 1)
\]
by \( \frac{\sqrt{3}}{3} \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix} \) and get a curve with Chowla property.

The first curve with the Chowla property and \( G = D_{12} \) occurs with \( p = 13 \) and is given by
\[
y^2 = (x^2 + 2)(x^2 + 5)(x^2 + 10x + 7).
\]
The first curve with the Chowla property and \( G = D_6 \) occurs with \( p = 19 \) and is given by
\[
y^2 = (x^2 + 1)(x^2 + 13x + 7)(x^2 + 9x + 11).
\]
The first curve with the Chowla property and \( G = C_2 \times C_2 \) occurs with \( p = 29 \) and is given by
\[
y^2 = (x^2 + 2)(x^2 + 20x + 10)(x^2 + 27x + 16).
\]
For the 13 primes from 101 to 163, we find 149 Chowla curves with \( G = C_2 \times C_2 \), 36 with \( G = D_6 \) and 52 with \( G = D_{12} \).

The first curve with generic automorphism group and the Chowla property occurs with \( p = 61 \) and is given by
\[
y^2 = (x^2 + 2)(x^2 + 6x + 19)(x^2 + 7x + 47)
\]
and its quadratic twist (by a nonsquare).

Since the zeta function of a genus 2 curve is determined by the number of points over the first two fields, the Weil bounds imply that there are approximately \((8\sqrt{p})(8p) \approx 64p^{3/2}\) possible zeta-functions. Since there are about \(2p^3\) possible curves, we should expect there to be many Chowla curves asymptotically.

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\(^1\) See Chapter X of Serre’s Local Fields; we essentially use the Poincaré series trick with Hilbert’s Theorem 90. We want to find \( M \) with \( M^* \overline{M}^{-1} = V \) for a matrix \( V \) of order \( k \), and this is done by taking a random matrix \( A \) in \( \mathbb{F}_q^k \) and summing \( M = \sum V^t A^{\sigma^{-1}} \) over the powers of \( V \) where \( \sigma \) is the Galois action on \( \mathbb{F}_{q^k}/\mathbb{F}_q \); there is such a \( A \) such that \( M \) is invertible, and transforming the curve by \( M \) gives the twist.