A Database of Elliptic Curves—First Report

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1 Introduction

In the late 1980s, Brumer and McGuinness [2] undertook the construction of a database of elliptic curves whose absolute discriminant $|\Delta|$ was both prime and satisfied $|\Delta| \leq 10^8$. While the restriction to primality was nice for many reasons, there are still many curves of interest lacking this property. As ten years have passed since the original experiment, we decided to undertake an extension of it, simultaneously extending the range for the type of curves they considered, and also including curves with composite discriminant. Our database can be crudely described as being the curves with $|\Delta| \leq 10^{12}$ which either have conductor smaller than 10^8 or have prime conductor less than 10^{10}—but there are a few caveats concerning issues like quadratic twists and isogenous curves. For each curve in our database, we have undertaken to compute various invariants (as did Brumer and McGuinness), such as the Birch–Swinnerton-Dyer L-ratio, generators, and the modular degree. We did not compute the latter two of these for every curve. The database currently contains about 44 million curves; the end goal is find as many curves with conductor less than 10^8 as possible, and we comment below on this direction of growth of the database. Of these 44 million curves, we have started a first stage of processing (computation of analytic rank data), with point searching to be carried out in a later second stage of computation.

Our general frame of mind is that computation of many of the invariants is rather trivial, for instance, the discriminant, conductor, and even the isogeny structure. We do not even save these data, expecting them to be recomputable quite easily in real time. For instance, for each isogeny class, we store only one representative (the one of minimal Faltings height), as we view the construction of isogenous curves as a "fast" process. It is only information like analytic ranks, modular degrees (both of which use computation of the Frobenius traces l_p), and coordinates of generators that we save; saving the l_p themselves would take too much storage space. It might be seen that our database could be used a "seed" for other more specialised databases, as we can quickly calculate the less time-consuming information and append it to the saved data.

2 Generating the Curves.

While Brumer and McGuinness fixed the a_1, a_2, a_3 invariants of the elliptic curve (12 total possibilities) and then searched for a_4 and a_6 which made $|\Delta|$ small, we instead decided to break the c_4 and c_6 invariants into congruence classes, and then find small solutions to $c_4^3 - c_6^2 = 1728\Delta$. We write $c_4^\star$ for the least nonnegative residue of c_4 modulo 576, and $c_6^\star$ for the least nonnegative residue of c_6 modulo 1728. The work of Connell [3] gives necessary and sufficient conditions on c_4 and c_6 for an elliptic curve with such invariants to exist. We first need that $c_6 \equiv 3 \pmod 4$ (when it follows that c_4 is odd) or $2^4 \mid c_4$ and $c_6 \equiv 0, 8 \pmod{32}$, and secondly we require a local condition at the prime 3, namely that $c_6 \not\equiv \pm 9 \pmod{27}$. Using this information and the fact that $1728 \mid \left(c_4^3 - c_6^2\right)$, this leads to 288 possible $(c_4^\star, c_6^\star)$ pairs.

For each fixed such $(c_4^\star, c_6^\star)$ pair, we can simply loop over c_4 and c_6, finding the curves with $|\Delta| \le 10^{12}$. Of course, it is only under the ABC-conjecture that we would have an upper bound on c_4 to ensure that we would have found all such curves, and even then the bound would be too large. Our method was to take $c_4 \le 1.44 \cdot 10^{12}$ in this first step; in any case, curves with larger c_4 are most likely found more easily using the method of Elkies [5].

2.1 Minimal Twists

In the sequel, we shall write E_d for the quadratic twist of E by d. For each (c_4, c_6) pair (again with $c_4 \le 1.44 \cdot 10^{12}$) which satisfies the $|\Delta| \le 10^{12}$ condition, we then determine whether this curve is minimal—not only in the traditional sense for its minimal discriminant, but also whether it is has the minimal discriminant in its family of quadratic twists. For $p \ge 5$, this is rather easy to determine; unless $p^6 \mid \Delta$ and $p \mid c_4$, the curve is minimal for quadratic twists (the only difference between this and the standard notion of minimality is that the exponent here is 6 instead of 12). If both the above conditions hold, then we throw the curve out, as $E_{\tilde{p}}$, where $\tilde{p} = \left(\frac{-1}{p}\right) p$, is a curve with lesser discriminant (which will be found by our search procedure). Given that the curve is minimal at a prime divisor $p \ge 5$ of Δ, the local conductor at p is p^2 if $p \mid c_4$ and p^1 otherwise.

The case with $p = 3$ is a bit harder. By Connell's conditions, we see that if $3 \mid c_6$ and $3^9 \mid \left(c_4^3 - c_6^2\right)$ but 3^5 does not exactly divide c_6, then E_{-3} is a curve with invariants $(c_4/9, -c_6/27)$ which has the discriminant reduced by 3^6. This is the only prohibition against the curve being the minimal twist at 3. If $3 \mid\mid c_4$, the curve has good reduction (at 3), while if c_4 is not divisible by 3, the curve has either good or multiplicative reduction. In both cases, the local conductor can be computed readily, it being 3^0 for good reduction and 3^1 for multiplicative. To compute the conductor in the remaining cases of additive reduction (where we know that $3^2 \mid c_4$ and $3^3 \mid c_6$), let $\tilde{c}_4$ be the the least nonnegative residue of $(c_4/9)$ modulo 3, and $\tilde{c}_6$ be the the least nonnegative residue of $(c_6/27)$ modulo 9. Table 1 then gives us the exponent of the local conductor. Here $e = 5$ if $3^4 \mid c_4$ and $e = 4$ if $3^3 \mid\mid c_4$ (note that we must have $3^5 \mid\mid c_6$ in this case for the curve to be twist-minimal, and that the table assumes that the curve is twist-minimal).

Table 1. Local Conductors at 3

$\tilde{c}_4 \backslash \tilde{c}_6$	0	1	2	3	4	5	6	7	8
0	e	3	3	5	2	2	5	3	3
1	2	3	4	3	4	4	3	4	3
2	2	3	2	3	3	3	3	2	3

For $p = 2$, the minimality test and conductor computation is much more complicated. We include the prime at infinity (twisting by -1) in the test for $p = 2$. By Connell's conditions, if $2^6 \mid c_4$ and $2^8 \mid c_6$, we see that E_2 is a curve with invariants $(c_4/4, c_6/8)$, and has a lesser discriminant. Also if $2^6 \mid c_4$ and $2^6 \mid\mid c_6$, then one of the twists $E_{\pm 2}$ (the sign depending on whether $c_6/8$ is 8 mod 32) has lesser discriminant. And finally if we have $2^4 \mid\mid c_4$ and $2^6 \mid\mid c_6$ and $2^{18} \mid \left(c_4^3 - c_6^2\right)$, then one of $E_{\pm 1}$ (depending on whether $c_6/64$ is 3 mod 4) is nonminimal (in the standard sense) at 2, and hence can be ignored. If none of these events happens, then the curve is twist-minimal at $p = 2$ and the infinite prime. We next describe how to compute the local conductor at $p = 2$ in terms of congruence conditions. If c_4 is odd, then the local conductor is 2^0 or 2^1, depending on whether 2 divides Δ. In the case where $2^4 | c_4$, when c_6 is 8 mod 32 there is good reduction at 2, and again the local conductor is 2^0. So we are left to consider the cases of additive reduction where $2^4 \mid c_4$ and $2^5 \mid c_6$. Let $\tilde{c}_4$ be the the least nonnegative residue of $(c_4/16)$ modulo 8, and $\tilde{c}_6$ be the the least nonnegative residue of $(c_6/32)$ modulo 8. Table 2 then gives the exponent of the local conductor at 2. In this, the dashed entries simply do not occur. For the entries marked by e, let $\tilde{c}_4$ be the the least nonnegative residue of $(c_4/16)$ modulo 16, and $\tilde{c}_6$ be the the least nonnegative residue of $(c_6/32)$ modulo 16. We then use the further Table 3. All the conductor computations are exercises with Tate's algorithm [12]; again the claims on the conductor need only be valid upon assuming that the curve is twist-minimal.

Table 2. Local Conductors at 2

$\tilde{c}_4 \backslash \tilde{c}_6$	0	1	2	3	4	5	6	7
1,5	6	4	e	3	6	4	e	3
2,6	8	3	6	4	7	3	6	4
3,7	5	2	7	4	5	2	7	4
4	6	2	-	4	3	2	-	4
0	6	2	-	4	2	2	-	4

A curve which has minimal discriminant at $p = 2$ will be of minimal conductor at $p = 2$ unless $2^4 \mid\mid N$ or $2^6 \mid\mid N$; we can throw out the curve in the first case, since E_{-1} will be found in the search process (and it has lesser conductor). But in the latter case, we cannot immediately discard the curve, as E_2 will have

Table 3. More of the Same

$\tilde{c}_4 \backslash \tilde{c}_6$	2	6	10	14
1	4	5	5	3
5	3	2	4	4
9	5	3	4	5
13	4	4	3	2

conductor smaller by a factor of 2, but the discriminant is larger by a factor of 64 (this behavior follows from the assumption that E has a twist-minimal discriminant and $2^6 \mid\mid N$). So only if $|\Delta| \leq 10^{12}/64$ do we discard the curve; in the alternative case we replace the curve by E_2, so that we have the twist of minimal conductor. Finally, if we have $2^5 \mid\mid N$ (possibly after the above twisting by 2), or $2^7 \mid N$, we make the arbitrary decision to discard the curve if $c_6 < 0$, as we will also find E_{-1} in the search, which will have the same conductor and discriminant. This positivity condition on c_6 will be part of our definition of minimal twist.

Using the above method, we can rid ourselves of all curves which are not minimal twists, and simultaneously compute the conductor. If $N > 10^{10}$, we simply ignore the curve; if $N > 10^8$ (and $N \leq 10^{10}$), we check whether N is a strong pseudoprime for 2, 13, 23, and 1662803, this being sufficient to prove primality [6]. At this point, we have a list of curves which meet our size conditions on the discriminant, and which have the minimal conductor in a family of quadratic twists (and minimal discriminant at primes other than $p = 2$).

2.2 Isogenous Curves

The next step will be to get rid of isogenous curves. The process of finding all curves isogenous to a given one is described in [4]. This is a fairly fast process, as most curves will have no nontrivial isogenies. Amongst the isogenous curves, we then take the curve of largest fundamental volume, that is, minimal Faltings height (which is unique by [11]), as our representative. Note that this curve might not have the minimal discriminant in the isogeny class. Our final set of curves is then: the set of elliptic curves E such that E has minimal height in its isogeny class, and has some isogenous curve F (possibly the same as E) for which we have $c_4 \leq 1.44 \cdot 10^{12}$ and either $N \leq 10^{10}$ with $|\Delta|$ prime, or $N \leq 10^8$ with $|\Delta| \leq 10^{12}$ for either the curve F or F_2.

2.3 Future Extension of the Database

As stated above, we would desire to have all minimal twists which have conductor less than 10^8. Cremona's tables have 20726 minimal twists with conductor less than 10^4, and so we might guess there are about 200–250 million minimal twists with conductor less than 10^8, while we only have about 44 million currently. There are many ways of enlarging the database. A first is extending the range on c_4 by using the algorithm of [5], but this will likely add only a small amount of curves. A better way is to find families in which we expect the conductor to be substantially less than the discriminant; for instance, curves with a rational point of order 5 will have a large 5th power dividing the discriminant, which will be reduced to a first power in the conductor. It appears that this technique will add many curves to the database — our results are as yet preliminary, and will be included in a future report on the database. For instance, Cremona's curve 174A given by $[1, 0, 1, -7705, 1226492]$ is not currently in our database, but will be found quickly with parametrisations of 3-torsion. A more simple method for enlarging the database is to extend the discriminant limit to (say) 10^{13} for certain $(c_4^\star, c_6^\star)$ pairs, especially those for which we know ahead of time that we will save significant powers of 2 and 3 in the conductor compared to the discriminant. Consideration of higher powers might allow us to find curves like 11949C (which is $[0, 1, 1, -1218949649, 16380150812351]$) where the discriminant is $-3^{41}7^2 569$. However, we will certainly not find all of Cremona's curves, as some like 11770I (which is $[1, -1, 1, -2246050998, 40972734736581]$, and has discriminant $-2^{13}5^3 11^{11} 107^4$) will not be found by any of our methods, as the absolute discriminant here is more than 10^{25}. As our database is not meant to be exhaustive, this is not a huge worry; we desire to put as much into the database as possible over as large of ranges as possible, but are not overly worried about exhaustiveness, preferring to include as much useful information as we can, without considering whether our database is "complete" in some sense.

3 Data Computed for Each Curve

One object of interest for an elliptic curve is its algebraic rank. This is hard to compute; indeed, there is no known algorithm to do this, only ones which work conditionally. By the process given in [4], we can try to determine the **analytic rank** of the curve, which is the degree of vanishing of its L-series at the central point. Of course, as there is no way to determine if a computed number is exactly zero, we can only give a good guess as to the analytic rank. The conjecture of Birch and Swinnerton-Dyer asserts that the algebraic rank and the analytic rank are equal, and that the first nonzero derivative of the L-function at the central point has arithmetic significance. For each curve in the database, we computed the suspected analytic rank and first nonzero derivative for both the curve itself and some of its quadratic twists.

Each curve in our database is the curve of minimal Faltings height in its isogeny class. A conjecture of Stevens [11] asserts that this curve should be the **optimal** curve for parametrisations from $X_1(N)$, in the sense that the parametrisations to the isogenous curves factor through the parametrisation to the strong curve (the existence of a modular parametrisation from $X_1(N)$ was proved in [1] following the methods initiated by Wiles [14]). It is sometimes the case that the optimal curve for parametrisations from $X_0(N)$ differs from the curve we find; in [13], a process is given to find the $X_0(N)$-optimal curve, assuming a technical condition, namely that the Manin constant of the optimal curve is 1 (this is similar to the Stevens conjecture). As many of the Frobenius traces were already computed for the analytic rank computation, these can be re-used at this stage. In a section below, we discuss the data obtained.

In the aforementioned paper [13], a process is given to compute the modular degree of an elliptic curve, again assuming that the Manin constant is 1. Compared to the computation of the analytic rank, which requires about the first $\sqrt{N}$ of the Frobenius traces, this method requires on the order of N of these (actually $\tilde{N}$, the symmetric-square conductor; see below). Thus for $N \geq 300000$ or so, it becomes rather time-consuming to compute the modular degree. We therefore compromised, computing the modular degree only if the symmetric-square conductor of the elliptic curve was sufficiently small (if we write $N = \prod_p p^{f_p}$ as a product of local conductors, then the symmetric-square conductor is simply $\tilde{N} = \prod_p p^{\lceil f_p/2 \rceil}$, except possibly when $f_2 = 8$, when the local symmetric-square conductor at 2 might be either 2^3 or 2^4; see [13] for details). We also computed the modular degree in some other interesting cases, for instance, when the rank is large, or in the case where there are differing optimal curves, a topic which we now discuss.

4 Differing Optimal Curves

Here we discuss the question of differing optimal curves for parametrisations from $X_0(N)$ and $X_1(N)$. Note that we do not compute the actual optimal curve for the latter, relying instead on the Stevens conjecture, and compute the optimal curve for $X_0(N)$ only under the assumption that the Manin constant is 1. But the results are still interesting.

There appear to be three families in which the optimal curves differ by a 2-isogeny. One of these, the so-called Setzer-Neumann curves (see [10], [8, 9]), was considered by Mestre and Oesterlé in [7]. These curves are parametrised by $c_4 = P - 16$ and $c_6 = u\,(P + 8)$, with the discriminant $P = u^2 + 64$ being a prime and u being taken to be congruent to 3 mod 4 to make c_6 be congruent to 3 mod 4 (other authors have taken u to be 1 mod 4). The second family corresponds to taking $c_4 = 16P - 16$ and $c_6 = 4v\,(16P + 8)$ with here v being 3 mod 4 and $P = v^2 + 4$ being prime. Here the conductor is $4P$ and the discriminant is $16P$; the differing optimal curves property appears to be preserved upon twisting by -1, which corresponds to negating c_6 (or v). If we take $u = 0$ or $v = 0$, we get the minimal Faltings height curve $[0, 0, 0, -1, 0]$ in the isogeny class 32A, which differs from the $X_0(32)$-optimal curve $[0, 0, 0, 4, 0]$ by a 2-isogeny. Noting that P in this case is a prime power, we can further expand the families to include the isogeny classes 128B/128D which come about from taking $v = \pm 2$ in the second family, and also $u = 15$ in the first family and $v = 11$ in the second family, giving the isogeny classes 17A and 20A respectively. Note that taking $v = -1$ in the second family also gives the isogeny class 20A. Indeed the curve obtained from $v = -1$ is the minimal Faltings height curve $[0, 1, 0, -1, 0]$, while the curve obtained from $v = 11$ differs by a 3-isogeny (since 125 is a third power). Taking $v = 1$ and $v = -11$ leads to similar behavior with the isogeny class 80B.

The class 17A will reappear in our third family; here the curve obtained from taking $u = 15$ differs from the minimal Faltings height curve $[1, -1, 1, -1, 0]$ by a 2-isogeny, and the $X_0(17)$-optimal curve is $[1, -1, 1, -1, -14]$, differing from the $X_1(17)$-optimal curve by a 4-isogeny.

The third family we have found is parametrised by $c_4 = PQ + 16$ and $c_6 = (P + 8)(PQ - 8)$ of discriminant PQ with $Q = P + 16$ and P congruent to 3 mod 4, and with both $|P|$ and $|Q|$ being prime powers, at least one of them being a power of a prime which is congruent to 3 mod 4 (so that $P = 11$ or $P = -2417$ works, but $P = -641$ does not). Upon taking $P = -17$, we obtain the $X_1(17)$-optimal curve for 17A. The isogeny class 15A (where the optimal curves differ by a 4-isogeny) comes about from both $P = -25$ and $P = -1$, the latter giving the minimal Faltings height curve even though $Q = P + 16 = 15$ is not a prime power. Similar to this are some cases where P is even, namely $P = -4$ and $P = -20$, which give 24A and 40A, and the corresponding quadratic twists $P = -12$ and $P = 4$, giving 48A and 80A. Finally there is $P = -8$, which gives 64A, the quadratic twist of 32A. These are all the known examples where the optimal curves differ by a 2-isogeny (and the two examples where they differ by a 4-isogeny); the above-cited work [7] contains the only partial results toward a proof of this classification.

Ignoring the 5-isogeny example of 11A as being spurious, this leaves just the occasions of the optimal curves differing by a 3-isogeny. Here, all known examples are parametrised by

$$c_4 = (n + 3)\left(n^3 + 9n^2 + 27n + 3\right) = (n + 3)^4 - 24\,(n + 3)$$

and

$$\begin{aligned} c_6 &= -\left(n^6 + 18n^5 + 135n^4 + 504n^3 + 891n^2 + 486n - 27\right) \\ &= -\,(n + 3)^6 + 36\,(n + 3)^3 - 216 \end{aligned}$$

where the discriminant is $n\left(n^2 + 9n + 27\right)$. The n's for which the optimal curves differ are (experimentally) precisely those for which $n^2+9n+27$ is a prime power and n has no prime factors congruent to 1 mod 6; else the optimal curves are the same. We have no proof of this.

Within these families with differing optimal curves, we also have conjectures regarding the parity of the modular degree (of the $X_0(N)$-optimal curve). In the first family, if u is 3 mod 8 then the modular degree is odd, while if u is 7 mod 8, the modular degree is even. In work joint with Matt Baker, we have been able to use the recent Refined Eisenstein Theorem of Emerton to prove this observation. In the second family, the modular degree is always odd when v is 3 mod 4 (while the quadratic twist corresponding to $-v$ will have a modular degree greater by a factor of four, and hence be even) — since the conductor here is not prime, our techniques are not applicable, and so we have no proof. In the third family, if P is 7 mod 24, then the modular degree is even, while it is odd if P is 19 mod 24 (we require $P > 0$ here); again we have no proof.

The 3-isogeny family has similar properties regarding the 3-divisibility of the modular degree. The cases where $3|n$ we shall ignore. Also, we ignore $|n| = 8$, where 3 exactly divides the modular degree. Having done this, if n is not a prime power, then 27 divides the modular degree. Else let $|n| = p^r$ and $3^k \mid\mid (p+1)$. We then have that 3^k exactly divides the modular degree, except if $k = 1$, when 3 does not divide the modular degree. We again have no proofs of these experimental data (and few examples where $r \neq 1$ or k is large).

5 Data Obtained

This may seem strange for a comprehensive database project, but we do not dwell on large-scale phemonemon; indeed, the Brumer–McGuinness work is probably already sufficient in this manner, at least for prime conductor. As noted there, telling the difference between a small power of 10^8 (or whatever the upper limit of consideration may be) and a large power of its logarithm is rather hopeless—extending their data by a factor of 5/4 on the logarithmic scale does not help matters much. The Brumer–McGuinness database had 310711 curves (five less than their stated number due to differences in their accounting), though their paper also states that they had actually found 311243 curves but threw some of them out; we have 839 curves which have prime conductor less than 10^8 which are not in their database. We have 11386955 isogeny classes of curves with prime conductor less than 10^{10} in our database (this should grow slightly when curves with $c_4 \geq 1.44 \cdot 10^{12}$ are added). Of these curves with prime conductor, of the ones we have processed, we have that 62.5% of the curves with even functional equation possess rank 0, compared to about 60% for Brumer–McGuinness. It is conjectured that asymptotically this percentage should be 100%. Similarly, 92.5% of the curves with odd functional equation have rank 1, slightly more than the previous results. The least conductor for a rank 5 curve we have found is 20384311 for $[1, 0, 0, -22, 219]$, and for rank 6 we have $[0, 0, 1, -547, -2394]$ of conductor 6756532597. These respectively fall short to the best-known (to the authors) examples of $[0, 0, 1, -79, 342]$ of conductor 19047851 and $[0, 0, 1, -7077, 235516]$ of conductor 5258110041 (the former appears in the Brumer–McGuinness database; the latter is due to Tom Womack).

Instead of concentrating on large-scale behavior, we see our database as more of a tool to be used by other mathematicians. For instance, Neil Dummigan queried us concerning examples of strong Weil curves with rank 2 and a rational point of order 5 for which the conductor is not divisible by 5, and we were able to provide him with the example $[0, 1, 1, -840, 39800]$ of conductor 13881 (and modular degree 52000), among other examples which were beyond the range of Cremona's tables (which include $[1, 1, 1, -2365, 43251]$ of conductor 5302). Though we would likely be better able to answer the question after extending our database with parametrisations from $X_0(5)$, the efficacy of our database was evinced. As another example, the second author has conjectured in [13] that 2^r divides the modular degree for any curve (where r is the rank), and perhaps higher powers of 2 should divide the modular degree when the conductor is composite, due to factorisation through Atkin–Lehner involutions. For many large-rank curves in the Brumer–McGuinness database, we verified this. With our extension to curves of composite conductor, we are able to give more evidence for this conjecture. Also, the third 2-isogeny family in the previous section was discovered after looking at our data, as was the parametrisation of the 3-isogeny family, and finally our analytic rank data concerning quadratic twists could be of use.

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