A SPECTRAL PROOF OF CLASS NUMBER ONE

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Abstract. We continue our previous work on the subject of re-proving the Heegner-Baker-Stark theorem, giving another effective resolution of this conjecture of Gauss, namely there are exactly 9 imaginary quadratic fields $\mathbf{Q}(\sqrt{-q})$ with class number 1 (specifically the list is $q \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$). We again follow an ansatz of Goldfeld, considering a modular form L-function of analytic rank 2, and reducing the situation to showing sufficient cancellation of the Dirichlet series coefficients when restricted to the principal form.

Previously we then chose a specific (rank 2) elliptic curve with complex multiplication by $\mathbf{Q}(\sqrt{-1})$ and deduced the desired cancellation from an equidistribution result of Hooley's for roots of a (quadratic) polynomial to varying moduli. Herein we instead use spectral techniques (filling in details of work of Templier and Tsimerman) to complete the proof, relying on the Duke-Iwaniec bound for Fourier coefficients of half-integral weight Maass forms. Unlike the previous method we do not require complex multiplication, and any analytic rank 2 modular form whose level has an even number of prime factors suffices.

1. Introduction

Let $\mathbf{Q}(\sqrt{-q})$ be an imaginary quadratic field, with $-q$ a fundamental discriminant. The *class number* (commonly denoted by h) is the number of (inequivalent) reduced binary quadratic forms $ax^2 + bxy + cy^2$ with discriminant $b^2 - 4ac = -q$. This was studied by Gauss [20, §303], who noted h tended to grow with q, and indeed was of size proportional to $\sqrt{q}$ on average. However, while he was able to compute that $h = 1$ for $q \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$, he was unable to show this list was complete.

1.1. Various proofs of this conjecture have been given, the first one being that of Heegner [28] using modular functions (*à la* Weber), which was only widely accepted following work of Stark [58], who himself gave a proof by a related method [57, 60]. Other proofs in this genre include [56, 17, 34, 10, 4, 50], with the latter papers demonstrating an observation of Serre, that an imaginary quadratic field with class number one generically gives rise to an integral point on various modular curves.

A different proof was given contemporaneously to Stark's work by Baker, using transcendental methods with linear forms in logarithms to yield an effective resolution [3, 59]. A third class of proof came some years later, when Gross and Zagier [26] completed an idea of Goldfeld [22] by proving that a specific modular elliptic curve has analytic rank 3, which indeed shows that the class number diverges effectively.

1.1.1. Recently, in [64] we noted that one can use Goldfeld's idea with a degree 2 L-function of analytic rank 2 to solve the class number one problem, provided that sufficient cancellation can be shown with the Dirichlet series coefficients when restricted to integers represented by the principal form. We then exploited various arithmetic properties of a specific elliptic curve with complex multiplication by $\mathbf{Q}(\sqrt{-1})$, completing the proof via a theorem of Hooley [29] concerning equidistribution of congruential roots of a polynomial to varying moduli.

Templier does not explicitly state it, but he already shows class number 1 follows from analytic rank 2. See (1.6) of Theorem 2 of [61], wherein the left side is thus 0, while the right side is proportional to $L'_\chi(1) \sim \pi^2/6$ as $D \to \infty$. This [61] uses the δ-symbol method, while [62, (1.9)] simplifies via spectral theory. Note also that (1.9) is (1.10) in the preprint version, and α and β need a $L_\chi(1)$-factor.

As noted therein, belatedly we realized that spectral techniques had been brought to bear on the problem of Hecke eigenvalues over quadratic sequences [6, 61, 62],

Key words and phrases. Spectral theory, class number one, Maass forms, analytic rank 2.

and this paper fleshes out the details, ultimately relying on Duke's bound [18] (following Iwaniec [30]) for Fourier coefficients of half-integral weight Maass forms.

We are well aware that most of the concept of our previous work [64] was to *reduce* the amount of background needed (particularly to avoid the Gross-Zagier theorem), whereas now we are invoking spectral theory. Moreover, most of our argument therein simply follows the template of Templier and Tsimerman [62, §4]. However, we still think our method has some value, and we indicate how one might apply similar ideas to get a partial result in the situation of one class per genus.

In order not to be repetitive, we refer to reader to [64, §1] for a fuller historical context, and comments about implicit constants, etc.

1.2. **Notation.** Our most novel piece of notation is using ∂ when integrating – I suspect at one point the letter "d" was clashing, and upon switching, I've never found a reason to go back. We also denote L-functions with subscripts, for instance $L_F(s)$ for the L-function of a modular form F. More substantially, we use a different scaling in the completed L-function, namely for (say) a primitive even Dirichlet character ψ of conductor k we have $\Lambda_\psi(s) = L_\psi(s)\Gamma(s/2)(\sqrt{k/\pi})^{s-1/2}$, whereas the usual ansatz would have s for the final exponent. In general we scale by the center of the critical strip, so for a weight 2 modular form we will have $\Lambda_F(s) = L_F(s)\Gamma(s)(\sqrt{N_F}/2\pi)^{s-1}$.

We write $-q$ for the fundamental discriminant of (assumed) class number 1, with χ its quadratic character and K the corresponding imaginary quadratic field. The generic Dirichlet character will be labelled ψ, and we use ψ_s to denote the Kronecker character for s a fundamental discriminant (also allowing $s = 1$).

The number of prime factors of n counted with multiplicity is $\Omega(n)$, and the number of (positive) divisors of n is $\tau_2(n)$, while $\dot{n}$ is the largest square divisor of n.

We write $\mathfrak{h}$ for the Lobachevsky upper half-plane with the Poincaré-Beltrami metric, and $\int_{(2)}$ for (e.g.) a line integral up the 2-line, and retain the typical $s = \sigma + it$ decomposition of a complex variable where applicable.

1.3. **Acknowledgments.** . The author would like to thank A. R. Booker, D. Goldfeld, K. Khuri-Makdisi, P. Sarnak, and M. P. Young for comments regarding this work, and the anonymous referee for suggestions about exposition.

2. Brief review of binary quadratic forms and modular forms

2.1. The necessary elements from the theory of binary quadratic forms already appear in the work of Gauss [20], though in a rather different language.

We let $K = \mathbf{Q}(\sqrt{-q})$ where $-q$ is a fundamental discriminant with $q > 8$. We write χ for the quadratic character of K, and define $R_\chi(n)$ as half the number of representations of n by reduced binary quadratic forms of discriminant $-q$. This $R_\chi(n)$ is multiplicative, indeed forming the Dedekind ζ-function for K as

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{R_\chi(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \prod_p \sum_{v=0}^{\infty} \frac{1}{p^{vs}} \sum_{e=0}^{v} \chi(p^e),$$

so that for primes p:

- when $\chi(p) = -1$ we have $R_\chi(p^v) = 1$ for v even and $R_\chi(p^v) = 0$ for v odd;
- when $\chi(p) = 0$ we have $R_\chi(p^v) = 1$ for all v;
- when $\chi(p) = +1$ we have $R_\chi(p^v) = v + 1 = \tau_2(p^v)$ for all v.

When the class number is 1 the theory of genera [20, §257] implies q is prime (as $q > 8$). This assumption also implies $\chi(p) = -1$ for primes $p < q/4$, as there is only the principal form and everything it represents up to $q/4$ is square.

We also define a splitting $R_\chi(n) = R_\chi^\square(n) + \tilde{R}_\chi(n)$ where $R_\chi^\square(n)$ is 1 if n is square and 0 otherwise, corresponding to half the number of representations of n by $x^2 + xy + \frac{q+1}{4}y^2$ with $y = 0$.

2.2. Let F be a weight 2 Hecke-invariant newform for $\Gamma_0(N)$ with trivial character (one could be more general, but we take this case for simplicity). By Hecke's theory of modular form L-functions [27] we have

$$L_F(s) = \sum_{l=1}^{\infty} \frac{c(l)}{l^s} = \prod_p \Big(1 - \frac{\alpha_p}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_p}{p^s}\Big)^{-1}$$

where for $p \nmid N$ we have $\alpha_p \beta_p = p$, and from Deligne [14] we have $|c(p)| \le 2\sqrt{p}$ and $\beta_p = \bar{\alpha}_p$. When $p|N$ we have $\beta_p = 0$ and $\alpha_p \in \{-1, 0, +1\}$.

The above Euler product for $L_F(s)$ converges for $\sigma > 3/2$, while the completed L-function $\Lambda_F(s) = (\sqrt{N}/2\pi)^{s-1}\Gamma(s)L_F(s)$ has an entire continutation that satisfies $\Lambda_F(s) = \epsilon(F)\Lambda_F(2-s)$, where the root number $\epsilon(F) \in \{-1, +1\}$ can be computed explicitly (e.g. from Atkin-Lehner involutions as in [2]).

Assuming that $\gcd(N, q) = 1$ (as we always shall), the quadratic twist $F\chi$ of F by χ has level Nq^2 and root number $\epsilon(F\chi) = \chi(-N)\epsilon(F)$ (see [65, Satz 2], which also appears as [33, Proposition 14.20]; both the level and the root number can also presumably be obtained from some parts of [2], such as Lemma 30 and Theorem 6). We thus have $\Lambda_{F\chi}(s) = \epsilon(F\chi)\Lambda_{F\chi}(2-s)$ with $\Lambda_{F\chi}(s) = (\sqrt{Nq^2}/2\pi)^{s-1}\Gamma(s)L_{F\chi}(s)$ and $\epsilon(F\chi) = \chi(-N)\epsilon(F)$.

2.2.1. The symmetric-square L-function $L_{S^2F}(s)$ has an Euler product (for $\sigma > 2$)

$$L_{S^2F}(s) = \prod_p \left(1 - \frac{\tilde{\alpha}_p^2}{p^s}\right)^{-1} \left(1 - \frac{\tilde{\alpha}_p\tilde{\beta}_p}{p^s}\right)^{-1} \left(1 - \frac{\tilde{\beta}_p^2}{p^s}\right)^{-1}$$

where $\alpha_p = \tilde{\alpha}_p$ and $\beta_p = \tilde{\beta}_p$ unless $p^2|N$. The analytic properties of this (including an entire continuation, and functional equation relating s to $3-s$) were shown by Shimura [54], while the computation of bad Euler factors can be discerned from later works [21, 12, 63]. We also note that $L_{S^2F}(2) \neq 0$, as this is at the edge of the critical strip (see also [24] for a useful lower bound).

3. Goldfeld's argument

We let F be a weight 2 Hecke-invariant newform of level $\Gamma_0(N)$ with trivial character, whose L-function has analytic rank 2, so that in particular the root number is $\epsilon(F) = +1$. In order for root numbers to vary beneficially in a family of quadratic twists, we will require that N have an even number of prime factors counted with multiplicity, so that $N = 389$ (or 433) will not suffice.

Via a calculation with modular symbols (see [42] or [13, §2.8]) one can verify that the modular form associated to the elliptic curve 446d has the desired properties. (It is a significant point of our work in general that such a calculation is substantially simpler than the Gross-Zagier theorem.)

We now proceed to give an elaboration of Goldfeld's method, with a different formulation compared to our previous work [64, §6].

3.1. We assume that $K = \mathbf{Q}(\sqrt{-q})$ has class number 1 where $-q$ is a fundamental discriminant with $q > 4N$. By the theory of genera of Gauss [20, §257] we know that $2^{\omega(q)-1}$ divides the class number, and thus the class number 1 assumption implies q is a prime power, indeed a prime for $q > 8$, and in particular $\gcd(q, N) = 1$. We let χ be the quadratic character corresponding to K, so that $\chi(p) = -1$ for all primes $p \le q/4$, and thus $\chi(N) = (-1)^{\Omega(N)}$.

Writing $\Lambda_F(s) = L_F(s)\Gamma(s)(\sqrt{N}/2\pi)^{s-1}$ and similarly for the completed twisted L-function $\Lambda_{F\chi}(s)$, the central vanishing of $L_F(s)$ then implies by Cauchy's integral theorem that

$$0 = \left(\int_{(2)} - \int_{(0)}\right)\Lambda_F(s)\Lambda_{F\chi}(s)\frac{\partial s/2\pi i}{(s-1)^2}.$$

The twist of F by χ has level q^2N and a computation shows that $F\chi$ has odd parity as its root number is $\chi(-N) = -\chi(N) = -(-1)^{\Omega(N)} = -1$, so the integral on the 0-line is the additive inverse of that on the 2-line by the functional equation(s) for $\Lambda_F(s)\Lambda_{F\chi}(s)$ that relates $s \leftrightarrow 2-s$. Thus we have

$$0 = \int_{(2)} \Lambda_F(s)\Lambda_{F\chi}(s)\frac{\partial s/2\pi i}{(s-1)^2} = \int_{(2)} L_F(s)L_{F\chi}(s)\Gamma(s)^2\left(\frac{Nq}{4\pi^2}\right)^{s-1}\frac{\partial s/2\pi i}{(s-1)^2}.$$

3.2. We next show $L_F(s)L_{F\chi}(s)$ is equal to $B_F(s)L_\chi(2s-1)\sum_n c(n)R_\chi(n)/n^s$, where $B_F(s)$ is a correction for bad primes $p|N$ and $R_\chi(n)$ is half the number of representations of n by reduced binary quadratic forms of discriminant $-q$. In the case of class number 1, the only reduced form is the principal form $x^2 + xy + \frac{q+1}{4}y^2$.

3.2.1. The pth Euler factor $V_p(s)$ of the L-product $L_F(s)L_{F\chi}(s)$ can be three different options, as given in Table 1. The validity of the third line relies on $\gcd(q, N) = 1$.

condition	$V_p(s)^{-1}$
$\chi(p) = -1$	$(1-\alpha_p^2/p^{2s})(1-\beta_p^2/p^{2s})$
$\chi(p) = +1$	$(1-\alpha_p/p^s)^2(1-\beta_p/p^s)^2$
$p = q$	$(1-\alpha_p/p^s)(1-\beta_p/p^s)$

TABLE 1. Tabulation of Euler factors

3.2.2. When $\chi(p) = 0$ we have that $R_\chi(p^v)$ is 1 for all v, while the Euler factor of $L_\chi(2s-1)$ is trivial, so the Euler factor of $L_\chi(2s-1)\sum_n c(n)R_\chi(n)/n^s$ is

$$\sum_{v=0}^{\infty}\frac{c(p^v)}{p^{vs}} = \sum_{v=0}^{\infty}\sum_{e=0}^{v}\frac{\alpha_p^e\beta_p^{v-e}}{p^{vs}} = \frac{1}{(1-\alpha_p/p^s)(1-\beta_p/p^s)} = V_p(s).$$

Here the $B_F(s)$ contribution is trivial. When $\chi(p) = -1$ we have that $R_\chi(p^v)$ is 1 for v even and else 0, and so the Euler factor from $\sum_n c(n)R_\chi(n)/n^s$ is given by

$$\sum_{v=0}^{\infty}\frac{c(p^{2v})}{p^{2vs}} = \sum_{v=0}^{\infty}\sum_{e=0}^{2v}\frac{\alpha_p^e\beta_p^{2v-e}}{p^{2vs}} = \left(1+\frac{\alpha_p\beta_p}{p^{2s}}\right)\sum_{v=0}^{\infty}\sum_{e=0}^{v}\frac{\alpha_p^{2e}\beta_p^{2v-2e}}{p^{2vs}} = \left(1+\frac{\alpha_p\beta_p}{p^{2s}}\right)V_p(s).$$

The factor from $L_\chi(2s-1)$ is $(1+p/p^{2s})^{-1}$ which cancels $(1+\alpha_p\beta_p/p^{2s})$ for $p \nmid N$.

3.2.3. When $\chi(p) = +1$ the computation is slightly more involved. First we note that $V_p(s)$ here is

$$\Big(1-\frac{\alpha_p}{p^s}\Big)^{-2}\Big(1-\frac{\beta_p}{p^s}\Big)^{-2} = \Big(1+\frac{2\alpha_p}{p^s}+\frac{3\alpha_p^2}{p^{2s}}+\cdots\Big)\Big(1+\frac{2\beta_p}{p^s}+\frac{3\beta_p^2}{p^{2s}}+\cdots\Big)$$
$$= \sum_{v=0}^{\infty}\frac{1}{p^{vs}}\sum_{e=0}^{v}\alpha_p^e\beta_p^{v-e}(e+1)(v-e+1).$$

This can be seen to be equal to

$$\Big(1+\frac{\alpha_p\beta_p}{p^{2s}}+\frac{\alpha_p^2\beta_p^2}{p^{4s}}+\cdots\Big)\Big(1+\frac{2(\alpha_p+\beta_p)}{p^s}+\frac{3(\alpha_p^2+\alpha_p\beta_p+\beta_p^2)}{p^{2s}}+\cdots\Big)$$
$$= \Big(1-\frac{\alpha_p\beta_p}{p^{2s}}\Big)^{-1}\sum_{v=0}^{\infty}\frac{c(p^v)(v+1)}{p^{vs}} = \Big(1-\frac{\alpha_p\beta_p}{p^{2s}}\Big)^{-1}\sum_{v=0}^{\infty}\frac{c(p^v)R_\chi(p^v)}{p^{vs}},$$

which equals the Euler factor for $L_\chi(2s-1)\sum_n c(n)R_\chi(n)/n^s$, at least when $p\nmid N$. (In fact, due to the class number 1 assumption we know that $\chi(p) = -1$ for all $p|N$.)

3.2.4. From the above, we conclude that

$$L_F(s)L_{F\chi}(s) = \prod_{p|N}\frac{1-\chi(p)p/p^{2s}}{1-\chi(p)\alpha_p\beta_p/p^{2s}}\cdot L_\chi(2s-1)\sum_{n=1}^{\infty}\frac{c(n)R_\chi(n)}{n^s},$$

where we can remove the bad-prime denominator since $\alpha_p\beta_p = 0$ when $p|N$, and as above, note that $\chi(p) = -1$ for $p|N$, so that $B_F(s) = \prod_{p|N}(1+p/p^{2s})$.

3.3. From the above replacement we have

$$0 = \int_{(2)}\Big(B_F(s)\cdot L_\chi(2s-1)\cdot\sum_{n=1}^{\infty}\frac{c(n)R_\chi(n)}{n^s}\Big)\Gamma(s)^2\Big(\frac{Nq}{4\pi^2}\Big)^{s-1}\frac{\partial s/2\pi i}{(s-1)^2}.$$

We next wish[1] to make an approximation of $L_\chi(2s-1)$ by $\zeta(4s-2)/\zeta(2s-1)$.

3.3.1. To estimate the error involved, we use the Mellin transform of $\Gamma(s)^2/(s-1)^2$, writing the above as

$$0 = \sum_{m|N}\frac{\mu(m)^2 m}{Nq/4\pi^2}\sum_{l=1}^{\infty}l\chi(l)\sum_{n=1}^{\infty}c(n)R_\chi(n)W\Big(\frac{l^2m^2n}{Nq/4\pi^2}\Big)$$

where $W(z) = \int_{(2)} z^{-s}\,\Gamma(s)^2\frac{\partial s/2\pi i}{(s-1)^2}$ has $W(z)\sim\sqrt{\pi}\frac{e^{-2\sqrt{z}}}{z^{5/4}} \ll e^{-\sqrt{z}}$ as $z\to\infty$,

either by the general theory of Meijer G-functions (see [8], or [40, §5.7, Theorem 5]), or by noting $W(z) = 2K_0(2\sqrt{z})/z$ in terms of K-Bessel functions.

We then replace $\chi(l)$ by the completely multiplicative Liouville function $\lambda(l)$, with these being equal for $l\le q/4$ by the class number 1 assumption. Undoing the Mellin transformation then gives (upon removing a factor of $Nq/4\pi^2$) that

$$\left|\int_{(2)}B_F(s)\frac{\zeta(4s-2)}{\zeta(2s-1)}\sum_{n=1}^{\infty}\frac{c(n)R_\chi(n)}{n^s}\Gamma(s)^2\Big(\frac{Nq}{4\pi^2}\Big)^s\frac{\partial s/2\pi i}{(s-1)^2}\right| \ll e^{-\sqrt{q/N}}. \qquad (1)$$

[1]One can instead move the contour to the left already with $L_\chi(2s-1)$, using the smallness of $L_\chi(1)$ in bounding the ensuing residues. Also, Gross and Zagier [26, §IV] use Rankin's method to show an analogous functional equation for $B_F(s)L_\chi(2s-1)D_{\mathcal{F}}(s)$ where the Dirichlet series $D_{\mathcal{F}}(s)$ is $c(n)/n^s$ summed over n coming from the nonzero representations of a binary quadratic form $\mathcal{F}$.

3.4. We then split off the n-representations from $x^2+xy+\frac{q+1}{4}y^2$ with $y=0$ via

$$\int_{(2)}\sum_{n=1}^{\infty}\biggl(\frac{c(n)R_\chi^{\square}(n)}{n^s}+\frac{c(n)\tilde R_\chi(n)}{n^s}\biggr)\cdot\frac{\zeta(4s-2)}{\zeta(2s-1)}B_F(s)\Gamma(s)^2\Bigl(\frac{Nq}{4\pi^2}\Bigr)^s\frac{\partial s/2\pi i}{(s-1)^2},$$

where $R_\chi^{\square}(n)$ is 1 if n is square and 0 else.

Calling the integrals from this $R_\chi(n)$-splitting by $\mathcal{I}_1^F(q)$ and $\mathcal{I}_2^F(q)$, we will show the following proposition in this section.

Proposition 3.5. *Suppose F is a weight 2 Hecke-invariant newform for $\Gamma_0(N)$ and let*

$$\mathcal{I}_1^F(q)=\int_{(2)}\sum_{m=1}^{\infty}\frac{c(m^2)}{m^{2s}}\cdot\frac{\zeta(4s-2)}{\zeta(2s-1)}B_F(s)\Gamma(s)^2\Bigl(\frac{Nq}{4\pi^2}\Bigr)^s\frac{\partial s/2\pi i}{(s-1)^2}.$$

Then

$$\bigl|\mathcal{I}_1^F(q)\bigr|\gg_F q.$$

The remainder of the paper will then be dedicated to an estimation of the error term, for which we shall show the following.

Theorem 3.6. *Suppose F is a weight 2 Hecke-invariant newform for $\Gamma_0(N)$ and let*

$$\mathcal{I}_2^F(q)=\int_{(2)}\sum_{n=1}^{\infty}\frac{c(n)\tilde R_\chi(n)}{n^s}\cdot\frac{\zeta(4s-2)}{\zeta(2s-1)}B_F(s)\Gamma(s)^2\Bigl(\frac{Nq}{4\pi^2}\Bigr)^s\frac{\partial s/2\pi i}{(s-1)^2}$$

where $\tilde R_\chi(n)$ is the number of representations of n as $x^2+xy+\frac{q+1}{4}y^2$ with $y\ge 1$. Then

$$\bigl|\mathcal{I}_2^F(q)\bigr|\ll_F q^{1-1/29}.$$

Combining these two results with the above bound from (1) then gives the asserted effective upper bound on fundamental discriminants with class number 1. Note that (1) is contingent on the conditions that $L_F(s)$ have (at least) a double zero at $s=1$, and that $\epsilon(F)\epsilon(F\chi)=-1$. Also, various re-writings of $R_\chi(n)$ to involve only the principal form are dependent on our class number 1 assumption.

3.7. We proceed to estimate $\mathcal{I}_1^F(q)$, which is given by

$$\mathcal{I}_1^F(q)=\int_{(2)}\sum_{m=1}^{\infty}\frac{c(m^2)}{m^{2s}}\frac{\zeta(4s-2)}{\zeta(2s-1)}B_F(s)\Gamma(s)^2\Bigl(\frac{Nq}{4\pi^2}\Bigr)^s\frac{\partial s/2\pi i}{(s-1)^2}.$$

Similar to the computation in §3.2.2 we have

$$\begin{aligned}\sum_{v=0}^{\infty}\frac{c(p^{2v})}{p^{2vs}}=\sum_{v=0}^{\infty}\sum_{e=0}^{2v}\frac{\alpha_p^e\beta_p^{2v-e}}{p^{2vs}}&=\frac{1+\alpha_p\beta_p/p^{2s}}{(1-\alpha_p^2/p^{2s})(1-\beta_p^2/p^{2s})}\\&=\frac{1-\alpha_p^2\beta_p^2/p^{4s}}{(1-\alpha_p^2/p^{2s})(1-\alpha_p\beta_p/p^{2s})(1-\beta_p^2/p^{2s})}\end{aligned}$$

where the denominator recalls (§2.2.1) the symmetric-square L-function of F.

From this we get

$$\sum_{m=1}^{\infty}\frac{c(m^2)}{m^{2s}}=\frac{L_{S^2F}(2s)}{\zeta(4s-2)}\cdot\tilde B_F(s)$$

where the correction factor for bad primes is

$$\tilde{B}_F(s) = \prod_{p|N} \frac{1-\alpha_p^2\beta_p^2/p^{4s}}{1-p^2/p^{4s}} \frac{(1-\tilde{\alpha}_p^2/p^{2s})(1-\tilde{\alpha}_p\tilde{\beta}_p/p^{2s})(1-\tilde{\beta}_p^2/p^{2s})}{(1-\alpha_p^2/p^{2s})(1-\alpha_p\beta_p/p^{2s})(1-\beta_p^2/p^{2s})}.$$

3.8. We thus have

$$\mathcal{I}_1^F(q) = \int_{(2)} \frac{L_{S^2F}(2s)}{\zeta(2s-1)} B_F(s)\tilde{B}_F(s)\Gamma(s)^2\Big(\frac{Nq}{4\pi^2}\Big)^s \frac{\partial s/2\pi i}{(s-1)^2}.$$

Here we move the contour to the left.

We do so at a sufficiently high height $H = (\log q)^2$ to ensure that the tails are negligible due to the vertical decay of $\Gamma(s)$. The new path of integration is over the line segment $\sigma = 1 - 1/98 \log\log q$ up to height H. The residue from $s = 1$ is

$$2\frac{Nq}{4\pi^2} L_{S^2F}(2)B_F(2)\tilde{B}_F(2) \underset{F}{\gg} q,$$

where we used that $L_{S^2F}(2) \neq 0$, as this is the edge of the critical strip.

3.9. We are left to handle the shifted line integral

$$\int_{(\sigma_0)}^{[H]} \frac{L_{S^2F}(2s)}{\zeta(2s-1)} B_F(s)\tilde{B}_F(s)\Gamma(s)^2\Big(\frac{Nq}{4\pi^2}\Big)^s \frac{\partial s/2\pi i}{(s-1)^2}$$

where $\sigma_0 = 1 - 1/98 \log\log q$ and $H = (\log q)^2$.

Although the class number 1 hypothesis implies (by the Deuring-Heilbronn phenomenon) a significant zero-free region for $\zeta(s)$, we only need to recall the standard results already due to de La Valleé Poussin. Via the standard zero-free region [38] and estimates for ζ we find that $|1/\zeta(2s-1)| \ll \log\log q$ on $\sigma_0 = 1-1/98 \log\log q$ up to height $H = (\log q)^2$. By standard convexity arguments for the symmetric-square L-function we have (with $t_\star = |t| + 5$)

$$|L_{S^2F}(2s)| \underset{F}{\ll} (\log t_\star)^3 t_\star^{1-\sigma_0} \ll (\log\log q)^3,$$

with a similar factor of $(\log\log q)^2$ from $1/(s-1)^2$.

The correction $B_F(s)\tilde{B}_F(s)$ is bounded in terms of F, while $|q^s| \ll q^{1-1/98 \log\log q}$ on the new line of integration, so we find that the line integral is bounded as

$$\underset{F}{\ll} q^{1-1/98 \log\log q}(\log\log q)^6 \underset{F}{\ll} q^{1-1/99 \log\log q},$$

which is dominated by the residue of size $\gg_F q$.

From this and the previous estimates, we conclude Proposition 3.5.

Remark. Presumably, a similar argument (derived in terms of an approximate functional equation, via replacing $1/(s-1)^2$ by $1/(s-z)$ in the integrand) yields a type of "Deuring decomposition" (compare [16, Satz 1]), namely that in a suitable region around $z = 1$ we have

$$\Lambda_F(z)\Lambda_{F\chi}(z) = T_F(z) + \epsilon(F)\epsilon(F\chi)T_F(2-z) + U_F(z)$$

where

Perhaps it is more transparent to have $L_\chi(2z)/\zeta(4z-2)$ instead of $1/\zeta(2z-1)$ here.

$$T_F(z) = \Big(\frac{Nq}{4\pi^2}\Big)^{z-1}\Gamma(z)^2 \frac{L_{S^2F}(2z)}{\zeta(2z-1)} B_F(z)\tilde{B}_F(z).$$

The crude bound for $U_F(z)$ is $\ll h\sum_a 1/a$ (with some z-dependence) where the sum is over minima of reduced forms; in the case of class number 1, when there is only the principal form, we find that $U_F(z) \ll 1/q^{1/29}$ from the spectral analysis.

However, I don't know any relevant utility of such a decomposition other than at the central point (i.e., exploiting arithmetic information concerning vanishing, in our case that the first derivative of the Λ-product is zero).

4. Reduction to a spectral estimation

We are left to estimate the error term

$$\mathcal{I}_2^F(q) = \int_{(2)} \sum_{n=1}^{\infty} \frac{c(n)\tilde{R}_\chi(n)}{n^s} \cdot \frac{\zeta(4s-2)}{\zeta(2s-1)} B_F(s) \cdot \Gamma(s)^2 \Big(\frac{Nq}{4\pi^2}\Big)^s \frac{\partial s/2\pi i}{(s-1)^2}$$

where $\tilde{R}_\chi(n)$ is the number of representations of n as $x^2 + xy + \frac{q+1}{4}y^2$ with $y \geq 1$. Via the Mellin transform of $\Gamma(s)^2/(s-1)^2$, this is

$$\mathcal{I}_2^F(q) = \sum_{\kappa=1}^{\infty} \xi(\kappa^2) \sum_{l=1}^{\infty} c(l)\tilde{R}_\chi(l) W\Big(\frac{4\pi^2\kappa^2 l}{Nq}\Big)$$

where $c(l)$ is the lth Dirichlet series coefficient of $L_F(s)$ and $W(z) = 2K_0(2\sqrt{z})/z$ has rapid decay $\ll e^{-\sqrt{z}}$ as $z \to \infty$, while $|\xi(\kappa^2)| \leq \kappa$ since

$$\sum_{m=1}^{\infty} \frac{\xi(\kappa^2)}{\kappa^{2s}} = \frac{\zeta(4s-2)}{\zeta(2s-1)} B_F(s) = \prod_{p \nmid N} \Big(1 + \frac{p}{p^{2s}}\Big)^{-1},$$

We wish to show that $\mathcal{I}_2^F(q)$ is negligible compared to q as $q \to \infty$, as by the computation of §3.4 this will contradict the assumption of class number 1.

4.1. We consider $l = x^2 + xy + \frac{q+1}{4}y^2$ with $y \geq 1$. By rearrangement this implies that $4l = (2x+y)^2 + qy^2$, and so by the accounting of $\tilde{R}_\chi(l)$ the above inner sum over l is

$$E_\kappa = \sum_{\substack{y=1 \\ 4l=(2x+y)^2+qy^2}}^{\infty} \sum_{x=-\infty}^{\infty} c(l)\, W\Big(\frac{4\pi^2\kappa^2 l}{Nq}\Big).$$

4.1.1. We then split up y according to parity, putting $\tilde{y} = y/2$ in the even case, and abbreviate $\tilde{W}_\kappa(l) = W(4\pi\kappa^2 l/Nq)$ to get that

$$\begin{aligned} E_\kappa &= \sum_{\substack{y=1 \\ y\,\mathrm{odd}}}^{\infty} \sum_{x=-\infty}^{\infty} c\Big(\frac{(2x+y)^2+qy^2}{4}\Big)\tilde{W}_\kappa(l) + \sum_{\tilde{y}=1}^{\infty} \sum_{x=-\infty}^{\infty} c\Big(\frac{(2x+2\tilde{y})^2+4q\tilde{y}^2}{4}\Big)\tilde{W}_\kappa(l) \\ &= \sum_{\substack{y=1 \\ y\,\mathrm{odd}}}^{\infty} U_{\mathrm{odd}}^{\kappa,4}(qy^2) + \sum_{\tilde{y}=1}^{\infty} U^{\kappa,1}(q\tilde{y}^2) \;\text{ with }\; U_{[\mathrm{odd}]}^{\kappa,u}(h) = \sum_{\substack{n=-\infty \\ [n\,\mathrm{odd}]}}^{\infty} c\Big(\frac{n^2+h}{u}\Big)\tilde{W}_\kappa\Big(\frac{n^2+h}{u}\Big). \end{aligned}$$

4.2. It is this latter type of sum[2] that we shall estimate by spectral methods, though in practice we shall work via the associated Dirichlet series. While we are somewhat more general below, we could restrict ourselves to bounding U when either $u = 1$ and the sum is over all squares or $u = 4$ and the sum is over the odd squares, as these are the only cases that we require.

[2] Comparatively, we can note the GL_1-analogue, with $L_\psi(s)L_{\chi\psi}(s) = B_\psi(s)\sum_l \psi(l)R_\chi(l)/l^s$ for an auxiliary Dirichlet character ψ, does not have suitable cancellation on each y-slice upon expanding $\tilde{R}_\chi(l)$, as the period sums of ψ evaluated on a quadratic polynomial can be nonzero.

Simply by the Mellin transform for $\tilde{W}_\kappa$ we have

$$U^{\kappa,u}_{[\mathrm{odd}]}(h) = \int_{(2)} \Gamma(s)^2 \Big(\frac{Nq}{4\pi^2\kappa^2}\Big)^s \Bigg[\sum_{\substack{n=-\infty \\ [n\,\mathrm{odd}]}}^{\infty} c\Big(\frac{n^2+h}{u}\Big)\Big(\frac{u}{n^2+h}\Big)^s\Bigg] \frac{\partial s/2\pi i}{(s-1)^2},$$

and bounds from the spectral methods (Theorem 5.1) imply that for any $\delta > 3/14$ either bracketed Dirichlet series is bounded on any line $\sigma > 1$ (with $\dot{h}$ the largest square divisor of h) as $\ll_{\delta,F,u,\sigma} h^{3/4-\sigma+\delta}\dot{h}^{1/4} \cdot t_\star^{(\sigma-2/2)+15/4}/|\Gamma(s)|$, where $t_\star = |t| + 5$.

Although the vertical growth of $1/|\Gamma(s)|$ is undoubtedly inoptimal, it suffices here because of the $\Gamma(s)^2$ in the Mellin transform. By integrating on the 2-line (which suffices to control the tails of the κ- and y-sums, though these also could be removed by W-decay) we get

$$\big|U^{\kappa,u}_{[\mathrm{odd}]}(h)\big| \ll_{\delta,F.u} (q/\kappa^2)^2 \cdot \dot{h}^{1/4}h^{3/4-2+\delta} \ll_{\delta,F,u} \frac{q^2\dot{h}^{1/4}}{\kappa^4 h^{5/4-\delta}},$$

and so

$$|\mathcal{I}^F_2(q)| \le \sum_{\kappa=1}^{\infty} |\xi(\kappa^2)| \cdot |E_\kappa| \ll_F \sum_{\kappa=1}^{\infty} \kappa \sum_{y=1}^{\infty} \frac{q^2\sqrt{y}}{\kappa^4(qy^2)^{30/29}} \ll_F q^{1-1/29}.$$

This bound is sufficient to effectively resolve the class number 1 problem (that is, to reduce it a finite computation), and so we turn to showing Theorem 5.1.

5. Cancellation via spectral methods

Let F be a modular cuspform $F(z) = \sum_{m>0} c(m)e^{2\pi i m z}$ of integral weight $k \ge 1$ for $\Gamma_0(N)$ with real[3] character ψ. For simplicity we only consider (integral) $h > 0$, and wish to estimate the Dirichlet series (for a given integral $u \ge 1$)

$$D^{u,h}_{[\mathrm{odd}]}(s) = \sum_{\substack{n=-\infty \\ [n\,\mathrm{odd}]}}^{\infty} c\Big(\frac{n^2+h}{u}\Big)\Big(\frac{u}{n^2+h}\Big)^s,$$

where $c(x) = 0$ when x is non-integral. We wish to bound this on the line $\sigma = k/2+1$ (say) and aim to improve on the "trivial" bound $\ll_{F,u} h^{(k-1)/2}\sqrt{h}/h^\sigma = h^{k/2-\sigma}$ roughly from $\approx \sqrt{h}$ contributors with $n^2+h \approx h$ and Deligne's bound [14] (we ignore epsilonics from $\tau_2(n^2+h)$ here). Recall $\dot{h}$ is the largest square divisor of h.

Noting $D^{u,h}_{[\mathrm{odd}]}(s)$ converges absolutely for $\sigma > k/2$, we shall show the following.

Theorem 5.1. *With the above definitions and hypotheses, for $\sigma > k/2$ we have*

$$\big|D^{u,h}_{[\mathrm{odd}]}(s)\big| \ll_{\delta,F,u,\sigma} \frac{h^{k/2}}{h^\sigma} \cdot h^{\delta-1/4}\,\dot{h}^{1/4} \cdot \frac{t_\star^{(\sigma-k/2)+15/4}}{|\Gamma(s)|}$$

for any $\delta > 3/14$. (One could improve the exponent on $\dot{h}$, but we do not bother).

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[3] Perhaps more is known, but Duke's original result (see §5.4.1 below) on Fourier coefficients of Maass forms is only stated for real characters (phrased as the discriminant of the Maass form). Baruch and Mao [5] appear to allow any compatible character (cf. [5, end of §1.2.4]).

5.2. We largely follow Templier and Tsimerman [62, §4] in our proof, which itself is inspired by Sarnak [47] (and indeed Selberg [52] to some degree).

We optically get the same result as they implicitly conclude in [62, §4.8] regarding the vertically-growing $1/|\Gamma(s)|$ on the right side of the bound for $D^{u,h}_{[\mathrm{odd}]}(s)$; however (see §5.5.2 below), the appearance of this in their work is an unnecessary artifact of a sloppy simplification, while in our case it comes from bounding an inner product crudely (not attempting to get decay in the eigenvalue parameter as they achieve).

5.2.1. For a given weight $\kappa \in \frac{1}{2}\mathbf{Z}$, congruence subgroup $\mathbf{\Gamma}$, and compatible Dirichlet character β, upon notating $e(v) = e^{2\pi i v}$ we define the mth Poincaré series (see [44], used in this context in [52, §3]) for $m > 0$ and convergent in $\mathrm{Re}(s) > 1$ as

$$P_m(z, s, \mathbf{\Gamma}, \beta, \kappa) = \sum_{\gamma \in \mathbf{\Gamma}_\infty \backslash \mathbf{\Gamma}} \bar{\eta}^\beta_\kappa(\gamma)\, e(m\gamma z) J_\gamma(z)^{-\kappa}\, \mathrm{Im}(\gamma z)^s,$$

where with $(\cdot|\cdot)$ the applicable extended Jacobi symbol (see [53]) we have

$$\eta^\beta_\kappa(\gamma) = (c|d)^{2\kappa} \epsilon_d^{-2\kappa} \beta(d) \quad \text{and} \quad J_\gamma(z) = \frac{cz + d}{|cz + d|}.$$

Here ϵ_d is i for $d \equiv 3\,(4)$ and 1 otherwise. For odd integral κ the level is divisible by 4, and $\epsilon_d^{-2\kappa}$ gives a factor of $(-1|d) = \psi_{-4}(d)$ separate from the $\beta(d)$. With this in mind, we require the compatibility condition that the character β be even.

Recall that for d negative we define $(c|d) = (c|-d)$ for $c > 0$ and $(c|d) = -(c|-d)$ for $c < 0$. In particular, $(-1|d) = -(-1|-d)$ for $d < 0$. It is an exercise of care (noting the argument of $cz+d$ when $c < 0$ and κ is half-integral) to check $\bar{\eta}^\beta_\kappa(\gamma) J_\gamma(z)^{-\kappa}$ is stable under $(c, d) \to (-c, -d)$ as desired (cf. [37, IV §I] or [53, (1.10), §2]).

There are various normalizations of automorphy factors seen in the literature,[4] and we take $J_\gamma(z)$ to always be unitary. We also note $\mathrm{Im}(\gamma z) = \mathrm{Im}(z)/|cz + d|^2$.

5.2.2. We recall our notation ψ_v for the Kronecker character of a fundamental discriminant v, with this character trivial when $v = 1$. For a coordinate of the upper half-plane $\mathfrak{h}$ we write $z = x + iy$, where we expect this should not clash with the x and y used as variables of the principal form.

Writing $\theta(z) = \sum_n e(n^2 z)$ we define $\theta_o(z) = \theta(z) - \theta(4z)$ of level 16, and abbreviate $\theta_\bullet$ to indicate either of θ or θ_o in the sequel. We then consider the automorphy scaling $\tilde{\theta}_\bullet(z) = y^{1/4}\theta_\bullet(z)$ and have that $\tilde{\theta}_\bullet(\gamma z) = \eta^{\psi_1}_{1/2}(\gamma) J_\gamma(z)^{1/2} \tilde{\theta}_\bullet(z)$ for $\gamma \in \Gamma_0(16)$. Similarly, for $f(z) = y^{k/2}F(z)$ we have $f(\gamma z) = \psi(\gamma) J_\gamma(z)^k f(z)$ for $\gamma \in \Gamma_0(N)$, and put $f_u(z) = f(uz)$ with an analogous rule for $\gamma \in \Gamma_0(uN)$.

Writing $\tilde{k} = k - 1/2$, $\tilde{\psi} = \psi\psi^k_{-4}$ and $\mathbf{\Gamma} = \Gamma_0(16uN)$, with $P^h_{\tilde{s}}(z) = P_h(z, \tilde{s}, \mathbf{\Gamma}, \tilde{\psi}, \tilde{k})$ we take the inner product (here h indicates $h$armonic, not the class number)

$$I^h_{\tilde{s}} = \langle P^h_{\tilde{s}}, f_u \bar{\tilde{\theta}}_\bullet \rangle = \int_{\mathbf{\Gamma} \backslash \mathfrak{h}} P^h_{\tilde{s}}(z) \bar{f}_u(z) \tilde{\theta}_\bullet(z)\, \partial z.$$

Expanding the Poincaré series $P^h_{\tilde{s}}(z)$ and using the above-noted automorphy relations $\bar{f}_u(\gamma z) = \bar{\psi}(\gamma) \bar{J}_\gamma(z)^k \bar{f}_u(z)$ and $\tilde{\theta}_\bullet(\gamma z) = \eta^{\psi_1}_{1/2}(\gamma) J_\gamma(z)^{1/2}\, \tilde{\theta}_\bullet(z)$, for $\gamma \in \mathbf{\Gamma}$ we can calculate

$$\bar{\eta}^{\tilde{\psi}}_{\tilde{k}}(\gamma)\psi(\gamma)\eta^{\psi_1}_{1/2}(\gamma)^{-1} = \bar{\psi}(d)\psi^k_{-4}(d)(c|d)^{2\tilde{k}}\epsilon_d^{2\tilde{k}} \cdot \psi(d) \cdot (c|d)\epsilon_d = \psi^k_{-4}(d)\epsilon_d^{2k} = 1,$$

[4] One can also vary the Poincaré series itself, e.g., Sarnak [49, (A6)] chooses to employ the kernel $\mathrm{Im}(\gamma z)^s e(-m\mathrm{Re}(\gamma z))$, which is not in L^2, but is integrable against rapidly decaying functions.

and so we achieve an unfolding relation

$$I_{\tilde s}^h = \int_{\mathbf{\Gamma}\backslash\mathfrak{h}} \sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} \bar\eta_{\tilde k}^{\tilde\psi}(\gamma)e(h\gamma z)J_\gamma(z)^{-\tilde k}\operatorname{Im}(\gamma z)^{\tilde s}\times$$
$$\times\,\psi(\gamma)\bar J_\gamma(z)^{-k}\bar f_u(\gamma z)\times\eta_{1/2}^{\psi_1}(\gamma)^{-1}J_\gamma(z)^{-1/2}\tilde\theta_\bullet(\gamma z)\,\partial z$$
$$= \int_{\mathbf{\Gamma}\backslash\mathfrak{h}} \sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} e(h\gamma z)\operatorname{Im}(\gamma z)^{\tilde s}\bar f_u(\gamma z)\tilde\theta_\bullet(\gamma z)\,\partial z = \int_{\mathbf{\Gamma}_\infty\backslash\mathfrak{h}} e(hz)\operatorname{Im}(z)^{\tilde s}\bar f_u(z)\tilde\theta_\bullet(z)\,\partial z.$$

Transforming to (x,y)-coordinates, we thus get (compare [47, (2.14)] or [62, (4.3)])

$$I_{\tilde s}^h = \int_0^\infty y^{\tilde s}e^{-2\pi hy}\int_0^1 e(hx)\cdot(uy)^{k/2}\sum_{l=1}^\infty \bar c(l)\,\overline{e(ulz)}\cdot y^{1/4}\sum_{\substack{n=-\infty\\ [n\text{ odd}]}}^\infty e(n^2z)\,\frac{\partial x\partial y}{y^2}$$

One really doesn't need to use θ_{o} for our application; we only apply the result therein when h is odd and $u=4$, in which case $h-ul+n^2=0$ already implies n is odd .

$$= u^{k/2}\sum_{l=1}^\infty \bar c(l)\sum_{\substack{n=-\infty\\ [n\text{ odd}]}}^\infty \int_0^\infty y^{k/2-3/4}y^{\tilde s}e^{-2\pi(h+ul+n^2)y}\int_0^1 e\big((h-ul+n^2)x\big)\,\partial x\,\frac{\partial y}{y}$$
$$= u^{k/2}\sum_{\substack{n=-\infty\\ [n\text{ odd}]}}^\infty \bar c\Big(\frac{n^2+h}{u}\Big)\int_0^\infty y^{\tilde s+k/2-3/4}e^{-4\pi(n^2+h)y}\,\frac{\partial y}{y}$$
$$= \frac{\Gamma(\tilde s+k/2-3/4)}{(4\pi)^{\tilde s+k/2-3/4}}\sum_{\substack{n=-\infty\\ [n\text{ odd}]}}^\infty \bar c\Big(\frac{n^2+h}{u}\Big)\frac{u^{k/2}}{(n^2+h)^{\tilde s+k/2-3/4}}, \tag{2}$$

This last series converges for $\mathrm{Re}(\tilde s)>3/4$ by Deligne's bound, and thus provides an analytic continuation of $I_{\tilde s}^h$, even though $P_{\tilde s}^h(z)$ only converges for $\mathrm{Re}(\tilde s)>1$. We shall write $s=\tilde s+k/2-3/4$ as a shorthand in the sequel, so that (for $\sigma>k/2$)

$$I_{\tilde s}^h = \big\langle P_{\tilde s}^h, f_u\bar{\tilde\theta}_\bullet\big\rangle = \frac{\Gamma(s)}{(4\pi)^s}\sum_{\substack{n=-\infty\\ [n\text{ odd}]}}^\infty \bar c\Big(\frac{n^2+h}{u}\Big)\frac{u^{k/2}}{(n^2+h)^s}.$$

5.3. We now follow [62, §4.3-6], first recalling the underlying spectral theory for $L^2(\mathbf{\Gamma}\backslash\mathfrak{h})$, see [46]. We consider the Laplacian of weight κ given by

$$\Delta_\kappa = y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big) - i\kappa y\frac{\partial}{\partial x},$$

and the Maass forms ϕ of weight κ and character β on $\mathbf{\Gamma}$ are eigenfunctions ϕ of this Laplacian that satisfy the transformation law $\phi(\gamma z)=\eta_\kappa^\beta(\gamma)J_\gamma(z)^\kappa\phi(z)$. There are discrete and continuous spectra for this Laplacian.

5.3.1. The continuous spectrum is given by Eisenstein series $E_w^{\mathfrak{a}}$ for $\mathrm{Re}(w)=1/2$, indexed by singular (sometimes called essential, or open) cusp representatives $\mathfrak{a}$ of $\mathbf{\Gamma}$ for the multiplier system given by κ and β, namely in our case $\eta_\kappa^\beta(\gamma)=1$ for γ in the stabilizer $\mathbf{\Gamma}_{\mathfrak{a}}$. These correspond to the "$m=0$" case of Poincaré series, defined (originally for $\mathrm{Re}(w)>1$) for the cusp at ∞ by

$$E_w^\infty(z) = \sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} \bar\eta_\kappa^\beta(\gamma)J_\gamma(z)^{-\kappa}\operatorname{Im}(\gamma z)^w \quad\text{where}\quad J_\gamma(z)=\frac{cz+d}{|cz+d|},$$

and for a general singular cusp $\mathfrak{a}$ via a scaling matrix $\sigma_{\mathfrak{a}}$ (see [32, (2.1)]) as

$$E_w^{\mathfrak{a}}(z) = \sum_{\gamma \in \boldsymbol{\Gamma}_{\mathfrak{a}} \backslash \boldsymbol{\Gamma}} \bar{\eta}_\kappa^\beta(\gamma) J_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-\kappa} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^w.$$

(The notation is standard – it is hoped that $\sigma_{\mathfrak{a}}$ will not be confused with $\sigma = \operatorname{Re}(s)$).

Selberg showed that each $E_w^{\mathfrak{a}}(z)$ has a meromorphic continuation to $\operatorname{Re}(w) \ge 1/2$, with finitely many poles, all contained in the real interval $(1/2, 1]$.

These Eisenstein series do not have bounded L^2-norm on $\operatorname{Re}(w) = 1/2$, since in the Fourier expansion about $\mathfrak{a}$ there is a term y^w in the constant coefficient.

5.3.2. For the discrete spectrum, the Laplacian has eigenvalues λ_j and an orthonormal basis of nonzero eigenfunctions ϕ_j of finite L^2-norm with $\Delta_\kappa \phi_j + \lambda_j \phi_j = 0$. The eigenvalues (possibly with multiplicity) are typically written as $\lambda_j = 1/4 + r_j^2$, with ir_j either real or imaginary. By symmetry, we can take either $ir_j \ge 0$ or $r_j \ge 0$, and our j-ordering will have λ_j non-decreasing.

The minimal possible eigenvalue in weight κ is $(\kappa/2)(1 - \kappa/2)$, from automorphic re-scalings of holomorphic forms of weight κ (similarly with anti-holomorphic forms when κ is negative). In this case we have $ir_j = 1/2$ in weight 0 and $ir_j = 1/4$ in weight 1/2, while for $\kappa \ge 1$ we have $ir_j = (\kappa - 1)/2$. There are also possible eigenvalues $(l/2)(1 - l/2)$ for $l \ge 0$ congruent to κ modulo 2, corresponding to starting with an automorphic re-scaling of a weight l holomorphic form (including constants when $l = 0$) and iteratively applying the $(l + 2m)$th Maass raising operator $(iy\partial_x + iy\partial_y + (l + 2m)/2)$ for m with $l \le l + 2m < \kappa$ (similarly with the lowering operator for $\kappa < 0$). Here the eigenfunctions will be cuspidal when the original holomorphic form is, though note when $\kappa \ge 1$ a noncuspidal holomorphic form will not be in L^2 in the first place.

The residual spectrum comes from residues of Eisenstein series. It is claimed in [18, End of Section 2] that this spectrum is empty for positive weights congruent to 3/2 mod 2, though this is given as a consequence of the folklore fact [47, p. 304] that the (Fourier coefficients of) Eisenstein series can be written in terms of Dirichlet L-functions – to the best of my knowledge this is a minor lacuna in the literature.[5] In the integral weight case it was only recently codified by Young [66, §4ff] in desired generality[6] while in half-integral weight it is discussed by Shimura [54] at one cusp, with the technique then termed "well known" by Goldfeld and Hoffstein [23], who give explicit expressions in terms of Dirichlet L-functions for all three cusps of $\Gamma_0(4)$.

In any case, the residual eigenvalues satisfy $\lambda_j \ge 0$ and indeed $\lambda_j \ge 3/16$ in our case of κ half-integral. The corresponding eigenfunctions here are not cuspidal.

There are also other possible "exceptional"[7] eigenvalues $\lambda_j < 1/4$ with r_j imaginary, which satisfy $\lambda_j \ge 3/16$ by Selberg's bound.[8] Under our classification system,

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[5] For the purposes of the non-existence of the residual spectrum in weights 3/2 mod 2, it suffices to write merely the zeroth Fourier coefficient in terms of Dirichlet L-functions. However, even this seems missing from the literature (compare the end of [31, §13.7], where $\Gamma_0(4)$ is discussed). Shimura's statement [55, (6.2)] is suggestive, but I was unable to determine if an unwinding of notation does indeed imply the desired conclusion (I think it again only does one cusp).

[6] With enough effort, one might be able to retrieve this result from Miyake [43, Theorem 7.2.9].

[7] Some authors use this term to include any $\lambda_j < 1/4$, or perhaps any such positive λ_j, without due respect to our previously defined classes of holomorphic re-scalings and Eisenstein residues.

[8] In half-integral weight, via the Shimura lift we have $\lambda_j \ge 15/64 = (1/4) - (1/8)^2$ for exceptional eigenvalues, as noted by Goldfeld and Sarnak [47, p. 304]. Additionally, the bound of $\lambda_j \ge 1/4-(7/64)^2$ (compared to $3/16 = 1/4-(1/4)^2$) is now available by other means [36], which

the corresponding eigenfunctions for "exceptional" eigenvalues are necessarily cuspidal [46, Satz 11.3], as are those for eigenvalues $\lambda_j \geq 1/4$.

One can note that the Maass lowering/raising operators preserve eigenvalues (and have only the (anti-)/holomorphic rescalings in their kernels), which implies that the spectra for weights that are congruent modulo 2 are essentially the same.

It was shown by Selberg that the discrete eigenvalues satisfy a Weyl law as to the linear growth of the number of λ_j up to height T as $T \to \infty$ (and thus the number of r_j grows quadratically); however, we only need an upper bound, which follows somewhat more easily from a Bessel's inequality argument as given by Iwaniec (see [32, Proposition 7.2ff]).

5.3.3. The desired spectral consequence from the above is (see [46, Lemma 5.2]) that we can write any $A \in L^2(\mathbf{\Gamma}\backslash\mathfrak{h})$ as (with the j-sum converging in the L^2-sense)

$$A(z) = \sum_{j=1}^{\infty} \langle A, \phi_j\rangle \phi_j(z) + \sum_{\mathfrak{a}} \int_{(\frac{1}{2})} \langle A, E_w^{\mathfrak{a}}\rangle E_w^{\mathfrak{a}}(z)\, \frac{\partial w}{4\pi i}$$

so that

$$\langle P_{\tilde{s}}^h, f_u\bar{\tilde{\theta}}_\bullet\rangle = \sum_{j=1}^{\infty} \langle P_{\tilde{s}}^h, \phi_j\rangle\langle \phi_j, f_u\bar{\tilde{\theta}}_\bullet\rangle + \sum_{\mathfrak{a}} \int_{(\frac{1}{2})} \langle P_{\tilde{s}}^h, E_w^{\mathfrak{a}}\rangle\langle E_w^{\mathfrak{a}}, f_u\bar{\tilde{\theta}}_\bullet\rangle\, \frac{\partial w}{4\pi i}. \qquad (3)$$

5.4. The Fourier expansions of the orthonormal basis ϕ_j are given by [46, (2.13ff)] [9] (here ε_n is the sign of n)

$$\phi_j(z) = \Bigl(\rho_j^+(0)y^{1/2+ir_j} + \rho_j^-(0)y^{1/2-ir_j}\Bigr) + \sum_{n\neq 0} \rho_j(n)\mathbf{W}_{\kappa\varepsilon_n/2, ir_j}(4\pi|n|y)e(nx),$$

where the Whittaker function $\mathbf{W}_{p,\mu}(y)$ is (up to scaling) the solution of exponential decay to $W''(y) + \bigl(-1/4 + p/y + (1/4+\mu^2)/y^2\bigr)W(y) = 0$. When ir_j is imaginary we must have $\rho_j^{\pm}(0) = 0$ for ϕ_j to be in L^2, so that such eigenfunctions are cuspidal.[10] On the other hand, for $\lambda_j < 1/4$ we can have $\rho_j^-(0) \neq 0$ and still remain in L^2.

The most important feature of the Whittaker function is the (weighted) Mellin transform [25, (7.621-11)], valid for $\mathrm{Re}(s) > -1/2 + |\mathrm{Re}(\mu)|$ as

$$\int_0^{\infty} y^s e^{-y/2}\mathbf{W}_{p,\mu}(y)\frac{\partial y}{y} = \frac{\Gamma(1/2+s+\mu)\Gamma(1/2+s-\mu)}{\Gamma(1+s-p)}. \qquad (4)$$

For our specific case of $p = \kappa/2$ and $\mu = ir_j$, the possible eigenvalue $ir_j = (\kappa-1)/2$ in weight $\kappa \geq 1$ (see Paragraph 2 of §5.3.2) then has $\mu = p - 1/2$. We then can note that $\Gamma(1/2+s-\mu) = \Gamma(1+s-p)$ so the right side of (4) is $\Gamma(1/2+s+\mu)$. Indeed, the Whittaker function in this case is just $y^{\kappa/2}e^{-y/2}$. Moreover, a similar simplification occurs when $\mu = p - 1/2 - l$ for some integral $l > 0$. Also, in weights 0 and 1/2 with μ respectively 1/2 and 1/4 such cancellation occurs to leave $\Gamma(1/2+s-\mu)$.

The Fourier expansions (about ∞) of the Eisenstein series are denoted by

$$E_w^{\mathfrak{a}}(z) = \Bigl(\delta_\infty y^w + \tilde{\alpha}_w^{\mathfrak{a}}(0)y^{1-w}\Bigr) + \sum_{n\neq 0} \alpha_w^{\mathfrak{a}}(n)\mathbf{W}_{\kappa\varepsilon_n/2, w-1/2}(4\pi|n|y)e(nx) \qquad (5)$$

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then gives $\lambda_j \geq 1/4 - (7/128)^2$ in half-integral weight. In odd integral weight we have $\lambda_j \geq 1/4$, via the weight 1 lower bound of 1/4 in conjunction with the Maass operators.

[9] There is a tradition in weight 0 is to use K-Bessel functions instead of Whittaker functions, which multiplies $\rho(n)$ by $2|n|^{1/2}$ for nonzero n, even though the notation $\rho(n)$ is still often used.

[10] In fact, for $ir_j = 0$ (which is $\lambda_j = 1/4$) one needs to replace one of the $y^{1/2+ir_j}$ by $y^{1/2}\log y$.

where δ_∞ is 1 when the cusp $\mathfrak{a}$ is equivalent to ∞ and 0 otherwise. We do not need it, but the $\alpha_w^{\mathfrak{a}}$ can be given in terms of Γ-functions and singular series [18, (2.8)].

5.4.1. From Duke [18], for half-integral weights $\tilde{k}$ and squarefree n we have

$$|\rho_j(n)| \ll_{k,\delta} |\lambda_j|^{5/4-\tilde{k}\varepsilon_n/4} e^{\pi|r_j|/2}|n|^{-1/2+\delta}$$

for $\delta > 1/4-1/28 = 3/14$. This is improved[11] by Bao and Maruch [5, Theorem 1.5] who allow any $\delta > 1/4-(1-2\cdot 7/64)/16 = 1/4-25/512$, but we shall not use this.

This follows (via the usage of Proskurin's version of Kuznetsov's trace formula) from an analogous bound of Iwaniec [30, Theorem 1] for coefficients of half-integral weight holomorphic cuspforms. Note that this is independent of the level. For non-squarefree n, the analogous result is (with $\dot{n}$ the largest square divisor of n)

$$|\rho_j(n)| \ll_{k,\delta} |\lambda_j|^{5/4-\tilde{k}\varepsilon_n/4} e^{\pi|r_j|/2}|n|^{-1/2+\delta}\dot{n}^{1/4-\delta}\tau_2(\dot{n}). \tag{6}$$

We need a similar bound for the Eisenstein coefficients, and Duke's Theorem 5 already contains this case[12] due to his definition (end of Section 2) of a "spectral" Maass form to include Eisenstein coefficients (in (5.1) he majorizes both the discrete and continuous spectra). In particular, for $\mathrm{Re}(w) = 1/2$ and any $\delta > 3/14$ we have

$$|\alpha_w^{\mathfrak{a}}(n)| \ll_{k,\delta} |w^2|^{5/4-\tilde{k}\varepsilon_n/4} e^{\pi|w|/2}|n|^{-1/2+\delta}\dot{n}^{1/4-\delta}\tau_2(\dot{n}). \tag{7}$$

Remark. An alternative phrasing of the n-behavior for these Duke bounds is to write $|n| = lm^2$ with l squarefree, with the n-part of the each bound then being given as $\ll_\delta l^{-1/2+\delta}(m^2)^{-1/4}\tau_2(m^2)$. Also, unless l is significantly smaller than m, we can elide the $\tau_2(m^2)$ by increasing δ by an arbitrarily small amount.

Furthermore, by using a Hecke action one can soften the squarefree requirement, as the effect of square divisors can be related to coefficients of the Shimura-lift Maass form, which are then bounded in terms of an exponent toward the Ramanujan-Petersson conjecture. This is standard in the holomorphic case [48, §4 Notes], and has been used for Maass forms[13] in (e.g.) [1, Theorem 8.1, (8.8ff)] and [39, §10].

5.5. We proceed to calculate the discrete spectrum contribution to $\langle P_{\tilde{s}}^h, f\bar{\tilde{\theta}}\rangle$ in (3). From $\bar{\phi}_j(\gamma z) = \bar{\eta}_{\tilde{k}}^{\psi}(\gamma)\bar{J}_\gamma(z)^{\tilde{k}}\bar{\phi}_j(z)$ via unfolding $P_{\tilde{s}}^h$ for $\tilde{\sigma} > 1$ we get (see [62, §4.4])

$$\begin{aligned}\langle P_{\tilde{s}}^h, \phi_j\rangle &= \int_{\boldsymbol{\Gamma}\backslash\mathfrak{h}} P_{\tilde{s}}^h(z)\bar{\phi}_j(z)\,\partial z = \int_{\boldsymbol{\Gamma}_\infty\backslash\mathfrak{h}} e(hz)\,\mathrm{Im}(z)^{\tilde{s}}\bar{\phi}_j(z)\,\partial z\\ &= \sum_{l=-\infty}^{\infty} \bar{\rho}_j(l)\int_0^\infty y^{\tilde{s}}e^{-2\pi hy}\,\bar{\mathbf{W}}_{\tilde{k}\varepsilon_l/2, ir_j}(4\pi|l|y)\int_0^1 e(hx)e(-lx)\,\frac{\partial x\partial y}{y^2}\end{aligned}$$

[11]Some interpretative care is needed, as [5] prefaces §1.2.5 by allowing any level, though previously in §1.1.2 they restricted to $4M$ with M odd. The result is also stated only for weight $\pm 1/2$, though one can use Maass raising/lowering operators to obtain a result for all half-integral weights. Related results appear in Blomer and Harcos [7, Corollary 2] and Bykovskii [9].

[12]Baruch and Mao [5, Theorem 1.5] don't mention this case, and indeed restrict to cuspforms.

[13]Khuri-Makdisi [35] gives a general version of the Shimura lift for (Hilbert-)Maass forms.

and by orthogonality only the $h = l$ term contributes, giving

$$\begin{aligned}\big\langle P^h_{\tilde s}, \phi_j\big\rangle &= \bar\rho_j(h)\int_0^\infty y^{\tilde s-1}e^{-2\pi hy}\,\mathbf{W}_{\tilde k/2,\overline{ir_j}}(4\pi hy)\,\frac{\partial y}{y}\\ &= \bar\rho_j(h)(4\pi h)^{1-\tilde s}\,\frac{\Gamma(\tilde s-1/2+ir_j)\Gamma(\tilde s-1/2-ir_j)}{\Gamma(\tilde s-\tilde k/2)}\end{aligned} \tag{8}$$

where we used $h > 0$ and (4), and exploited that $\{ir_j, -ir_j\}$ is closed under complex conjugation (independent of whether these are real/imaginary).[14] Moreover, in half-integral weight (8) holds by analytic continuation for $\tilde\sigma > 3/4$, upon noting that $\lambda_j \geq 3/16$ implies $ir_j \leq 1/4$ with the exceptional/residual spectra, while eigenvalues $\lambda_j < 3/16$ from Maass raisings of holomorphic forms are associated to ir_j whose poles from $\Gamma(\tilde s - 1/2 - ir_j)$ are cancelled by $\Gamma(\tilde s - \tilde k/2)$ in the denominator.[15]

5.5.1. Next we turn to the "triple-product" calculation [62, §4.5], where we operate in the crudest method possible, simply bounding

$$\Big|\big\langle \phi_j, f_u\bar{\tilde\theta}_\bullet\big\rangle\Big| \leq \|\phi_j\|\cdot\|f_u\bar{\tilde\theta}_\bullet\| = \|f_u\bar{\tilde\theta}_\bullet\| \underset{F,u}{\ll} 1.$$

This has no decay in the eigenvalue parameter, but will suffice below.[16]

5.5.2. However, we still take the opportunity here to discuss various difficulties that I had with my understanding of [62, §4.5].

Their main result (4.7) therein is that for the θ-series they obtain the inner product bound (with the desired exponential decay in the eigenvalue) of

$$\Big|\big\langle \phi_j, f\bar{\tilde\theta}\big\rangle\Big| \underset{F}{\ll} |r_j|^k e^{-\pi|r_j|/2},$$

via a residue computation with an Eisenstein series.

However, there are various infelicities compounding [62, §4.5], and sundry surrounding claims. The first line on page 706 is already misleading to me (though not wrong), as it seems that only $\mathrm{Re}(s) > 1+\delta$ is required rather than $\mathrm{Re}(s) > K/2$ (this appears to come from shifting s the wrong way). Also, the last line of (4.6) fails to shift the Γ-factors $s \to s + K/2 - 1$.

More critically, the unfolding argument between the first and second lines of (4.6) needs some explication here,[17] as the Eisenstein series is for $\Gamma_0(4)$ while the inner product is over a fundamental domain for $\mathbf{\Gamma}$. There is a similar problem in §3.4, where the proposed calculation[18] of $\big\langle \theta_N(z)\bar\theta_N(z), E^T_{N,0}(s,z)\big\rangle$ by unfolding seems suspect, due to the nontrivial character in their (incorrect?) definition of $E^T_{N,0}(s,z)$. Returning to bookkeeping matters, the scaling factor from the residue of the Eisenstein series to the θ-series has a discrepancy between the last display before §3.4

[14] We also did not bother to write the $l = 0$ term separately, as it does not appear in the end.

[15] In (positive) weights congruent to 3/2 modulo 2, the eigenvalues $\lambda_j = 3/16$ (which corresponds to $ir_j = 1/4$) will have their Γ-factors cancelled also.

[16] In [62, §5] they briefly indicate how to get a bound of $\ll_A 1/|r_j|^A$ for any A via applying the Laplacian A times, though I'm unclear about their proposal above (5.2) and again two lines later that (in our notation) the non-holomorphic function $f_u\bar{\tilde\theta}$ is in $S_{\tilde k}(N,\chi)$, as after (3.6) they had defined this to be holomorphic forms. A related point that misled me at one juncture is that (5.3) elides the continuous spectrum (whose consideration they omit throughout in any event).

[17] It might also be useful to use a different variable in §4.5, to distinguish from s of §4.2ff.

[18] This is not crucial to their result, as Chiera [11, Theorem 2.2ff] gives an alternative method.

and the later (3.11), while the last display on the page 698 (copying a formula from Duke) erroneously has a π^s instead of a π (perhaps the genesis of the problem).

Lastly, although it is in their next subsection, on the last line of page 706 they unnecessarily lose a Γ-factor (which will induce vertical decay in s), leading to their Remark (second paragraph of §4) concerning restriction of test functions (see also the commentary with (1.16) in their Introduction).

It can also be noted that [62] generally omits that Duke's bound (and its improvements) only holds for squarefree n in the given form, and an adjustment as in our (6) is needed in general (they briefly allude to their skirting of this technicality in the third paragraph of §1.3). There's also a couple of minor typos in §4.1, firstly that F (not f) is the newform; and also the "$2n^2+d$" appearing in (4.2) should be n^2+d, then similarly for (5.2) and (5.4) where additionally a 4π is omitted.

Note that the end result of [62, §4] is superseded by their §6 in any event, so it is perhaps not surprising that there are minor errors extant in the former.[19]

5.5.3. For $\tilde\sigma > 3/4$ we thus bound the discrete spectrum contribution to $\langle P_{\tilde s}^h, f_u\bar{\tilde\theta}_\bullet\rangle$ by using the trivial bound of §5.5.1, and Duke's bound (6) to bound $\rho_j(h)$ in (8). Writing $t_\star = |\tilde t| + 5$ and $k_\star = (5-\tilde k)/2$, and making no attempt to minimize the $\dot h$-exponent, the discrete spectrum contribution $\sum_j \langle P_{\tilde s}^h, \phi_j\rangle\langle\phi_j, f_u\bar{\tilde\theta}_\bullet\rangle$ is bounded as

$$\begin{aligned}
&\ll_{F,u,\tilde\sigma} (4\pi h)^{1-\tilde\sigma} \sum_{j=1}^\infty \left|\bar\rho_j(h)\right| \cdot \left|\frac{\Gamma(\tilde s - 1/2 + ir_j)\Gamma(\tilde s - 1/2 - ir_j)}{\Gamma(\tilde s - \tilde k/2)}\right| \\
&\ll_{\delta,F,u,\tilde\sigma} h^{1/2+\delta-\tilde\sigma}\dot h^{1/4-\delta}\tau_2(\dot h) \times \sum_{j=1}^\infty e^{\pi|r_j|/2}|r_j|^{k_\star} \cdot \left|\frac{\Gamma(\tilde s - 1/2 + ir_j)\Gamma(\tilde s - 1/2 - ir_j)}{\Gamma(\tilde s - \tilde k/2)}\right|
\end{aligned}$$

and then by using Stirling's approximation $|\Gamma(s)| \sim \sqrt{2\pi}|s|^{\sigma-1/2}e^{-\pi|t|/2}$ and Weyl's law on the quadratic growth of the number of r_j up to a given height this is

$$\begin{aligned}
&\ll_{\delta,F,u,\tilde\sigma} h^{1/2+\delta-\tilde\sigma}\dot h^{1/4}\left[\sum_{|r_j|\le t_\star} \frac{e^{\frac{\pi}{2}|r_j|}|r_j|^{k_\star}\cdot \mathbf{t}^{2\tilde\sigma-2}e^{-\pi t_\star}}{t_\star^{\tilde\sigma-1/2-\tilde k/2}e^{-\pi t_\star/2}} + \sum_{|r_j|\ge t_\star} \frac{e^{\frac{\pi}{2}|r_j|}|r_j|^{k_\star}\cdot \mathbf{t}^{2\tilde\sigma-2}e^{-\pi|r_j|}}{t_\star^{\tilde\sigma-1/2-\tilde k/2}e^{-\pi t_\star/2}}\right] \\
&\ll_{\delta,F,u,\tilde\sigma} h^{1/2+\delta-\tilde\sigma}\dot h^{1/4}\cdot t_\star^{(5-\tilde k)/2+\tilde k/2+(\tilde\sigma-3/2)+2} \ll_{\delta,F,u,\tilde\sigma} h^{1/2+\delta-\tilde\sigma}\dot h^{1/4}\cdot t_\star^{\tilde\sigma+3}. \qquad (9)
\end{aligned}$$

Here $\mathbf{t} = t_\star + |r_j|$, with the dominant term depending on whether $2\tilde\sigma - 2$ is positive. In any case, the main contribution is from when $t_\star$ and r_j are the same size.

As noted after (8), any $\lambda_j < 3/16$ will induce a cancellation of Γ-factors, so that this estimate is indeed valid for $\tilde\sigma > 3/4$.

Remark. It seems that [62, p. 706] unnecessarily lost a Γ-factor here, which in turn limited the breadth of their admissible test functions. Contrariwise, we made no attempt to save such a Γ-factor in the first place, due to our crude bound from §5.5.1 that lacked any decay in the eigenvalue parameter.

5.6. Next we handle the continuous spectrum.

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[19] An alternative method for their §4.5 could be to re-interpret $\langle\phi_j, f\bar{\tilde\theta}\rangle = \langle\phi_j\tilde\theta, f\rangle$ and then (in weight more than 2) expand f as a finite linear combination of Poincaré series – though I haven't convinced myself that this should work abstractly, let alone check any details.

We can compute $\langle P^h_{\tilde{s}}, E^{\mathfrak{a}}_w \rangle$ by unfolding the Poincaré series, and from the above Fourier expansion (5) of $E^{\mathfrak{a}}_w$, as with §5.5, upon writing $w = 1/2 + i\xi$ we find

$$\begin{aligned}\langle P^h_{\tilde{s}}, E^{\mathfrak{a}}_w \rangle &= \sum_{l=-\infty}^{\infty} \bar{\alpha}^{\mathfrak{a}}_w(l) \int_0^\infty y^{\tilde{s}} e^{-2\pi h y}\, \bar{\mathbf{W}}_{\tilde{k}\varepsilon_l/2, i\xi}(4\pi|l|y) \int_0^1 e(hx)e(-lx)\, \frac{\partial x \partial y}{y^2} \\ &= \bar{\alpha}^{\mathfrak{a}}_w(h)(4\pi h)^{1-\tilde{s}} \frac{\Gamma(\tilde{s} - 1/2 + i\xi)\,\Gamma(\tilde{s} - 1/2 - i\xi)}{\Gamma(\tilde{s} - \tilde{k}/2)}.\end{aligned}$$

Here the right side actually provides an analytic continuation to $\tilde{\sigma} > 1/2$.

5.6.1. As with §5.5.1, we could proceed to give a "trivial" estimate for $\langle E^{\mathfrak{a}}_w, f_u \bar{\tilde{\theta}}_\bullet \rangle$. One method to do this would be (compare [33, (15.15)]) to unfold the Eisenstein series for $\mathrm{Re}(w) > 1$, getting the zeroth coefficient of the Fourier expansion (at each cusp) of $f_u \bar{\tilde{\theta}}_\bullet$, which could then (presumably) be computed in terms of symmetric-square L-functions (perhaps first writing F as a linear combination of eigenforms). This gives an analytic continuation of the inner product to $\mathrm{Re}(w) = 1/2$, and should suffice to give the result. Another method would be to bound the Eisenstein series trivially for $\mathrm{Re}(w) > 1$ and then use a functional equation in conjunction with convexity to bound them on the 1/2-line. With either of these methods, the details involving various cusps of $\mathbf{\Gamma}$ require at least some care.

We instead work directly with the continuous spectrum contribution in (3) via Cauchy's inequality, noting that for each cusp $\mathfrak{a}$ we have

$$\begin{aligned}&\left| \int_{(\frac{1}{2})} \langle P^h_{\tilde{s}}, E^{\mathfrak{a}}_w \rangle \langle E^{\mathfrak{a}}_w, f_u \bar{\tilde{\theta}}_\bullet \rangle \, \frac{\partial w}{4\pi i} \right|^2 \\ &\qquad \leq \left(\int_{-\infty}^{\infty} \left| \langle P^h_{\tilde{s}}, E^{\mathfrak{a}}_{1/2+i\xi} \rangle \right|^2 (1+\xi^2)^2 \, \partial \xi \right) \left(\int_{-\infty}^{\infty} \left| \langle E^{\mathfrak{a}}_w, f_u \bar{\tilde{\theta}}_\bullet \rangle \right|^2 \frac{\partial \xi}{(1+\xi^2)^2} \right).\end{aligned}$$

The first integral $J^{\mathfrak{a}}_1(\tilde{s})$ is bounded (for $\tilde{\sigma} > 1/2$) by the above computation of $\langle P^h_{\tilde{s}}, E^{\mathfrak{a}}_w \rangle$ and Duke's bound (7) for Fourier coefficients of Eisenstein series as

$$\ll \left(h^{1/2+\delta-\tilde{\sigma}} \dot{h}^{1/4}\right)^2 \cdot \int_{-\infty}^{\infty} e^{\pi|\xi|} (1+\xi^2)^{2+5/2-\tilde{k}/2} \left| \frac{\Gamma(\tilde{s} - 1/2 + i\xi)\Gamma(\tilde{s} - 1/2 - i\xi)}{\Gamma(\tilde{s} - \tilde{k}/2)} \right|^2 \partial \xi.$$

By Stirling's approximation the integral here (dominated when $|\xi|$ is of size $t_\star$) is

$$\ll (t_\star^2)^{2+5/2-\tilde{k}/2} (t_\star^2)^{(2\tilde{\sigma}-2)-(\tilde{\sigma}-1/2-\tilde{k}/2)} \ll (t_\star^2)^{\tilde{\sigma}+3}$$

so that

$$|J^{\mathfrak{a}}_1(\tilde{s})|^{1/2} \ll_{\delta,\tilde{\sigma}} h^{1/2+\delta-\tilde{\sigma}} \dot{h}^{1/4} \cdot t_\star^{\tilde{\sigma}+3}.$$

5.6.2. For the second integral $J^{\mathfrak{a}}_2$ we apply Cauchy's inequality again to get

$$\begin{aligned}J^{\mathfrak{a}}_2 &= \int_{-\infty}^{\infty} \left| \int_{\mathbf{\Gamma}\backslash\mathfrak{h}} E^{\mathfrak{a}}_w(z) \bar{f}(uz) \tilde{\theta}_\bullet(z) \, \partial z \right|^2 \frac{\partial \xi}{(1+\xi^2)^2} \\ &\leq \int_{-\infty}^{\infty} \int_{\mathbf{\Gamma}\backslash\mathfrak{h}} \left| E^{\mathfrak{a}}_w(z) \right|^2 \frac{\partial z}{1+y_{\mathbf{\Gamma}}(z)} \frac{\partial \xi}{(1+\xi^2)^2} \cdot \int_{\mathbf{\Gamma}\backslash\mathfrak{h}} \left| \bar{f}(uz) \tilde{\theta}_\bullet(z) \right|^2 \left(1+y_{\mathbf{\Gamma}}(z)\right) \partial z,\end{aligned}$$

where we recall the invariant height (see [32, (2.42)])

$$y_{\mathbf{\Gamma}}(z) = \max_{\mathfrak{a}} \max_{\gamma \in \mathbf{\Gamma}} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z),$$

which has linear growth as z approaches a cusp.

Since f_u decays exponentially at each cusp, we find $\big|\bar{f}(uz)\tilde{\theta}_\bullet(z)\big|^2\big(1+y_{\mathbf{\Gamma}}(z)\big)$ is bounded on $\mathfrak{h}$, ergo the second z-integral above is bounded, and so we bound $J_2^{\mathfrak{a}}$ as

$$J_2^{\mathfrak{a}} \underset{F,u}{\ll} \int_{\mathbf{\Gamma}\backslash\mathfrak{h}}\int_{-\infty}^{\infty}\big|E^{\mathfrak{a}}_{1/2+i\xi}(z)\big|^2\frac{\partial\xi}{(1+\xi^2)^2}\frac{\partial z}{1+y_{\mathbf{\Gamma}}(z)}.$$

Herein the inner integral can be bounded (indeed when summed over all cusps if desired) by using the bound of [32, Proposition 7.2, (7.10)] (or more properly, its analogue in nonzero weight), namely that

$$\sum_{\mathfrak{a}}\int_{-T}^{T}\big|E^{\mathfrak{a}}_{1/2+i\xi}(z)\big|^2\,\partial\xi \ll T^2+Ty_{\mathbf{\Gamma}}(z).$$

Inserting this, the integral over the fundamental domain for $\mathbf{\Gamma}$ then converges, and we conclude that $\big|J_2^{\mathfrak{a}}\big| \underset{F,u}{\ll} 1$.

5.6.3. Combining these and summing over singular cusps, for $\tilde{\sigma}>1/2$ the contribution from the continuous spectrum is bounded as

$$\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P^h_{\tilde{s}},E^{\mathfrak{a}}_w\rangle\langle E^{\mathfrak{a}}_w,f_u\bar{\tilde{\theta}}_\bullet\rangle\frac{\partial w}{4\pi i}\underset{F,u}{\ll}\sum_{\mathfrak{a}}\big|J_1^{\mathfrak{a}}(s)\big|^{1/2}\underset{\delta,F,u,\tilde{\sigma}}{\ll} h^{1/2+\delta-\tilde{\sigma}}\dot{h}^{1/4}t_\star^{\tilde{\sigma}+3}. \quad (10)$$

5.7. Returning to §5.2.2 and (2), by the above spectral computations from the decomposition (3), by (9) and (10) we have shown (for $\tilde{\sigma}>3/4$) that

$$\begin{aligned} I^h_{\tilde{s}}=\big\langle P^h_{\tilde{s}},f_u\bar{\tilde{\theta}}_\bullet\big\rangle &= \frac{\Gamma(s)}{(4\pi)^s}\sum_{\substack{n=-\infty\\ [n\text{ odd}]}}^{\infty}\bar{c}\Big(\frac{n^2+h}{u}\Big)\frac{u^{k/2}}{(n^2+h)^s}\\ &\underset{\delta,F,u,\sigma}{\ll} h^{1/2+\delta-\tilde{\sigma}}\dot{h}^{1/4}\cdot t_\star^{\tilde{\sigma}+3}, \end{aligned}$$

and so[20] recalling $\tilde{s}=s-k/2+3/4$ we have that

$$D^{u,h}_{[\text{odd}]}(s)=\sum_{\substack{n=-\infty\\ [n\text{ odd}]}}^{\infty}c\Big(\frac{n^2+h}{u}\Big)\Big(\frac{u}{n^2+h}\Big)^s$$

satisfies the bound

$$\big|D^{u,h}_{[\text{odd}]}(s)\big|\underset{u,k}{\ll}\frac{|I^h_{\tilde{s}}|(4\pi)^\sigma}{|\Gamma(s)|}\underset{\delta,F,u,\sigma}{\ll} h^{1/2+\delta-(\sigma-k/2+3/4)}\,\dot{h}^{1/4}\cdot\frac{t_\star^{\sigma+3-k/2+3/4}}{|\Gamma(s)|},$$

and thus for $\sigma>k/2$ we have

$$\big|D^{u,h}_{[\text{odd}]}(s)\big|\underset{\delta,F,u,\sigma}{\ll}\frac{h^{k/2}}{h^\sigma}\cdot h^{\delta-1/4}\,\dot{h}^{1/4}\cdot\frac{t_\star^{(\sigma-k/2)+15/4}}{|\Gamma(s)|}.$$

This shows Theorem 5.1.

[20] Note that we never have to worry about the nuances of the residual spectrum and/or the Ramanujan parameter θ of [62], as we don't move the contour very far to the left.

5.8. An alternative method to show the desired bound on $I^h_{\tilde{s}} = \langle P^h_{\tilde{s}}, f_u\bar{\tilde{\theta}}_\bullet\rangle$ could be simply to apply Cauchy's inequality,[21] and use the crude bound of §5.5.1 for $\|\bar{f}_u\tilde{\theta}_\bullet\|$. Then one can note that $\|P^h_{\tilde{s}}\|^2$ is computed by Proskurin [45, Lemma 1] as

$$\langle P^h_{\tilde{s}}, P^h_{\tilde{s}}\rangle = \Gamma(2\tilde{\sigma}-1)(4\pi h)^{1-2\tilde{\sigma}} - ie(-\tilde{k}/4)2^{3-2\tilde{\sigma}} \sum_{c\equiv 0\,(N)} \frac{S_c(h,h)}{c^{2\tilde{\sigma}}} J_{\tilde{s}}(c)$$

where $S_c(h,h)$ is the associated Kloosterman sum while

$$J_{\tilde{s}}(c) = \int_{-i}^{i} K_{2i\tilde{t}}\Big(\frac{4\pi hv}{c}\Big)\Big(v+\frac{1}{v}\Big)^{2\tilde{\sigma}-2} v^{\tilde{k}}\,\frac{\partial v}{v},$$

with the integration path being counter-clockwise over the semi-circle from $-i$ to i.

Assuming this integral is mild, with the K-Bessel factor effectively restricting to c of size h, the Weil bound for Kloosterman sums gives $|I^h_{\tilde{s}}| \ll h^{3/4-\tilde{\sigma}}$ which is just short of a useful result; hence, progress [45, §7] toward the Linnik-Selberg conjecture to effect cancellation amongst the $S_c(h,h)$ presumably suffices (and would replace the usage of the Iwaniec-Duke bound, though situated in the same circle of ideas).

Added in May 2019. In retrospect, it seems I was a bit cavalier here. Given the size of h as an argument of the Kloosterman sum, once seemingly needs to beat the $1/4$ in the mn-dependence in a result such as that of Sarnak and Tsimerman (2009). They do this in weight zero (and it has been generalized to nontrivial level), but in half-integral weight this was only recently considered by Dunn (*Uniform bounds for sums of Kloosterman sums of half integral weight*, 2018), who adapts work of Ahlgren and Andersen. Though his result is for the η-multiplier, the same strategy should work for the Θ-multiplier case.

Dunn only beats the $1/4$ barrier for the opposite sign case (Theorem 1.1), but my impression is that this largely due to the following. He makes a savings with (9.5) in that case which is then not recapitulated in the same-sign case, presumably because the holomorphic contribution (10.2) already has the $1/4$ exponent. However, this in turn relates back to (7.2), where (compare to (59) of Sarnak and Tsimerman) he has already lost the $(mn)^{1/4}$ in the half-integral weight case, seemingly because of the the lack of a Deligne bound. My intuition is that one should still be able to reduce the $1/4$ slightly by the Iwaniec bound. (I must admit to losing interest in completing the argument at this point, as I had hoped it would be dependent on the Selberg eigenvalue bound, rather than "again" on the Iwaniec result).

I am now told by Dunn that it is not so straightforward (as simply invoking Iwaniec) due to the arbitrarily large weights that will appear in the Kuznetsov formula, but that he and Ahlgren have fashioned a suitable version for a different application in `arxiv.org/abs/1806.01187`

One can also follow Blomer's original work more closely and get to a similar juncture: in the middle of (3.5) he writes the relevant expression as

$$\sum_h\sum_c \frac{K(4m-h^2,-\Delta,c)}{c}\cdot\frac{g(h;c/4)}{\sqrt{c}}e(-2sh/c),$$

and again one needs to be able to beat the $1/4$ exponent in the mn-dependence to be able to profitably apply sums of Kloosterman sums (with partial summation).

5.8.1. One could also try to bring the trace formula into the picture. Upon expanding $\langle\phi_j\tilde{\theta}, f\rangle$ as in Footnote 19, a sum of the form $\sum_j \bar{\rho}_j(h)\rho_j(l-n^2)W_{r_j}(\tilde{s},l,n)$ then results for some weighting function W, and by applying the "easier" direction (with the given weighting function on the Fourier coefficients, as opposed to the

[21] This should only lose a factor reflective of the effective dimension of the space of Maass forms of level $16uN$, which due to the exponential eigenvalue decay should itself be proportional to uN.

Kloosterman sums) of Kuznetsov's trace formula as generalized by Proskurin [45], one reduces the situation to an analysis of Kloosterman sums.

However, even with a further transformation to Salié sums, where extra cancellation (from congruences) when summing over y in $h = qy^2$ could well be detected, this again does not seem to ultimately offer any great advantages, even in terms of uniformity of the level (as would be desired below).

5.9. One can apply the above argumentation with $\theta(az)$ and $\theta_o(az)$, respectively related to sequences $an^2 + h$ and $a(2n+1)^2 + h$.

One possible application is the consideration of one class per genus (Euler's idoneal numbers), where every class is ambiguous (in the terminology of Gauss), and thus has a representative $(a, 0, c)$ or (a, a, c) (note that this need not be reduced). In the former case, we just let $h = cy^2$ for $y \geq 1$ and use $\theta(az)$. In the latter case, we have that $ax^2 + axy + cy^2 = \big[a(2x+y)^2 + (4c-a)y^2\big]/4$ and for odd y use $\theta_o(az)$ with $h = (4c-a)y^2$ and $u = 4$, while for even y we instead use $\theta(az)$ with $h = (4c-a)(y/2)^2$ and $u = 1$. The desired result would then detect suitable Dirichlet series cancellation when a is sufficiently small.[22]

In particular, using these ideas one should be able to show that an exceptional idoneal number has no prime factor $\gg q^{1-\alpha}$ for some[23] explicit $\alpha > 0$. However, though I think this result is new, it is a bit underwhelming as the difficulty in such class number problems is typically when the minima are near size $\sqrt{q}$, whereas here we would be restricting them all to be $\ll q^{\alpha}$.

5.9.1. For the above, one still has to ensure the root number variation works out correctly in the integral computation in §3.1. It is perhaps a superfluous exercise, but we can exhibit explicit weight 2 modular forms with analytic rank 2 for this.[24] We write F^d for the quadratic twist of F by the fundamental discriminant d. Then the following forms (labelled by associated elliptic curve isogeny classes) have analytic rank 2, and odd parity when twisted by negative fundamental discriminants $q^\star = -q$ as described:

- F_{256a}^{-35} when twisted by $q^\star \equiv 1\,(4)$;
- F_{256d}^{17} when twisted by $4\|q^\star$;

[22]A variant calculation with (2) would use the inner product $\langle P_{\bar{s}}^{ah}, f_{ua}\theta_\bullet\rangle$, still on $\Gamma_0(uaN)$.

[23]I think via a rather routine consideration of level-uniformity that one should be able to get $\alpha = \eta(1/4 - \delta)$ with $\eta = 1/2$, which could possibly be improved by the introduction of more sophisticated methods, particularly those which attempt to get cancellation over y.

In general, I expect that spectral methods will (at best) lose a factor proportional to the square root of the level, as the j-sum over the orthonormal basis will have length essentially proportional to the level aN, and detecting cancellation in such a sum seems quite difficult. (One can note that Iwaniec's argument already loses such a factor of the square root of the level at the first step, upon embedding the given modular form into an orthonormal basis.) When a is of size $q^{1/2}$, there would thus be a factor of $q^{1/4}$ lost. Comparatively, in bounding a sum like E_κ its length is effectively $q^{1/2}$ and even the most optimistic hopes (square-root cancellation) would only save $q^{1/4}$.

[24]An alternative method to handle root numbers (likely only available in weight 2) is to import an L-function of analytic rank 3 from Gross-Zagier, whereupon either parity has the main term in §3.4 bounded away from zero.

Another idea would be to use non-selfdual L-functions that centrally vanish, one example being a weight 2 level $\Gamma_1(122)$ modular form with quadratic character (there are smaller levels with nonquadratic characters, though then I expect there is additional messiness with the symmetric-square), when the root number variation becomes moot (cf. [33, §23.7]). Similarly, there is a weight 4 modular form with vanishing central L-value for $\Gamma_1(99)$ and quadratic character ψ_{33}; an example with odd squarefree level occurs with $\Gamma_1(435)$ and character ψ_{145}.

- F_{256c}^{33} when twisted by $q^\star \equiv 24\,(32)$;
- F_{32a}^{41} when twisted by $q^\star \equiv 8\,(32)$.

This follows since $\epsilon(F^d) = \epsilon(F)\psi_d(-N_F)$ when $\gcd(d, N_F) = 1$, where ϵ is the global root number and N is the level [65]. For instance for $q^\star \equiv 8\,(32)$, with $g = \gcd(q, 41)$ and $t = q^\star/8g^2 \equiv 1\,(4)$ we have

$$\epsilon\big((F_{32a}^{41})^{q^\star}\big) = \epsilon(F_{64a}^{41t}) = \epsilon(F_{64a}) \cdot \psi_{41t}(-1) = \mathrm{sgn}(41t) = -1.$$

There are similar weight 4 collections, one example being $F_{64a^3}^{-579}$, $F_{256a^3}^{401}$, $F_{256b^3}^{73}$, and $F_{32a^3}^{-395}$, with (e.g.) $64a^3$ the symmetric cube of the Grössencharacter for $64a$.

Note added later. It seems this is unnecessarily complicated. Instead, in weight 2 at least, by analyzing root number variation we just need an elliptic curve of conductor p^2 with $p \equiv 7\,(12)$ and Kodaira symbol at p of either II or IV. Examples already exist in the list given by Edixhoven, de Groot, and Top (1990), namely with $p \in \{43, 307, 739, 1999, 2251, 3331, 4423, \ldots\}$. One still has to achieve the rank 2 condition, but this is done simply by taking one such twist in the family. More directly, one can take the curve $[1, -1, 1, -492, 4302]$ of conductor 7867^2, which has rank 2, and all its twists by negative fundamental discriminants have odd parity.

References

[1] S. Ahlgren, N. Andersen, *Kloosterman sums and Maass cusp forms of half integral weight for the modular group.* Int. Math. Res. Not. **2018**, no. 2, 493–570. http://doi.org/10.1093/imrn/rnw234

[2] A. O. L. Atkin, J. Lehner, *Hecke operators on* $\Gamma_0(m)$. Math. Ann. **185** (1970), no. 2, 134–160. http://eudml.org/doc/161948

[3] A. Baker, *Linear forms in the logarithms of algebraic numbers.* Mathematika **13** (1966), no. 2, 204–216. http://doi.org/10.1112/S0025579300003971

[4] B. Baran, *A modular curve of level 9 and the class number one problem.* J. Number Theory **129** (2009), no. 3, 715–728. http://doi.org/10.1016/j.jnt.2008.09.013

———, *Normalizers of non-split Cartan subgroups, modular curves, and the class number one problem.* J. Number Theory **130** (2010), no. 12, 2753–2772. http://doi.org/10.1016/j.jnt.2010.06.005

[5] E. M. Baruch, Z. Mao, *A generalized Kohnen-Zagier formula for Maass forms.* J. Lond. Math. Soc. **82** (2010), no. 1, 1–16. http://doi.org/10.1112/jlms/jdq009

[6] V. Blomer, *Sums of Hecke Eigenvalues over Values of Quadratic Polynomials.* Int. Math. Res. Not. **2008**, no. 16. http://doi.org/10.1093/imrn/rnn059

[7] V. Blomer, G. Harcos, *Hybrid bounds for twisted L-functions.* J. reine angew. Math. **621** (2008), 53–79. http://doi.org/10.1515/CRELLE.2008.058

[8] B. L. J. Braaksma, *Asymptotic expansions and analytic continuations for a class of Barnes-integrals.* Compositio Math. **15** (1962-4), 239–341. http://eudml.org/doc/88877

[9] V. A. Bykovskii, *A trace formula for the scalar product of Hecke series and its applications.* J. Math. Sciences **89** (1998), no. 1, 915–932. http://doi.org/10.1007/BF02358528

[10] I. Chen, *On Siegel's modular curve of level 5 and the class number one problem.* J. Number Theory **74** (1999), no. 2, 278–297. http://doi.org/10.1006/jnth.1998.2320

[11] F. L. Chiera, *On Petersson products of not necessarily cuspidal modular forms.* J. Number Theory **122** (2007), no. 1, 13–24. http://doi.org/10.1016/j.jnt.2006.03.003

[12] J. Coates, C.-G. Schmidt, *Iwasawa theory for the symmetric square of an elliptic curve.* J. Reine Angew. Math. **375/376** (1987), 104–156. http://eudml.org/doc/152908

[13] J. E. Cremona, *Algorithms for modular elliptic curves.* Cambridge University Press, 1992. http://homepages.warwick.ac.uk/~masgaj/book/amec.html

[14] P. Deligne, *La conjecture de Weil. I.* (French) [Weil's conjecture, I]. Pub. Math. IHES **43** (1974), 273–307. http://eudml.org/doc/103930

[15] J.-M. Deshouillers, H. Iwaniec, *Kloostermann Sums and Fourier Coefficients of Cusp Forms.* Invent. Math. **70** (1982), 219–288. http://eudml.org/doc/142975

[16] M. Deuring, *Imaginäre quadratische Zahlkörper mit der Klassenzahl 1.* (German) [Imaginary quadratic fields with class number 1]. Math. Z. **37** (1933), 405–415. http://eudml.org/doc/168455

[17] M. Deuring, *Imaginäre quadratische Zahlkörper mit der Klassenzal Eins.* (German) [Imaginary quadratic fields with class number one]. Invent. Math. **5** (1968), 169–179. http://doi.org/10.1007/BF01425548

[18] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms.* Invent. Math. **92** (1988), no. 1, 73–90. http://doi.org/10.1007/BF01393993

[19] W. Duke, J. B. Friedlander, H. Iwaniec, *The subconvexity problem for Artin L-functions.* Invent. Math. **149** (2002), no. 3, 489–577. http://doi.org/10.1007/s002220200223

[20] C. F. Gauss, *Disquisitiones Arithmeticae.* (Latin) [Arithmetical investigations] (1801). From his complete works (1863): http://eudml.org/doc/202621 English translation: A. A. Clarke, *Disquisitiones Arithmeticae*, Yale Univ. Press (1965).

[21] S. Gelbart, H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$.* Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542. http://eudml.org/doc/82024

[22] D. Goldfeld, *The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), no. 4, 624–663. http://eudml.org/doc/83732

[23] D. Goldfeld, J. Hoffstein, *Eisenstein series of $\frac{1}{2}$-integral weight and the mean value of real Dirichlet L-series.* Invent. Math. **80** (1985), 185–208. http://doi.org/10.1007/BF01388603

[24] D. Goldfeld, J. Hoffstein, D. Lieman, Appendix to *Coefficients of Maass forms and the Siegel zero* by J. Hoffstein and P. Lockhart. Ann. of Math. (2) **140** (1994), no. 1, 177–181. http://www.jstor.org/stable/2118544

[25] I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, series, and products.* Eighth edition, originally translated from the Russian. Academic Press, 2015.

[26] B. H. Gross, D. B. Zagier, *Heegner points and derivatives of L-series.* Invent. Math. **84** (1986), no. 2, 225–320. http://eudml.org/doc/143341

[27] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I-II.* (German) [On modular functions and the Dirichlet series with Euler products, I-II]. Math. Ann. **114** (1937), 1–28, 316–351. http://eudml.org/doc/159899 http://eudml.org/doc/159919

[28] K. Heegner, *Diophantische Analysis und Modulfunktionen.* (German) [Diophantine analysis and modular functions]. Math. Z. **56** (1952), 227–253. http://eudml.org/doc/169287

[29] C. Hooley, *On the distribution of the roots of polynomial congruences.* Mathematika **11** (1964), 39–49. http://doi.org/10.1112/S0025579300003466

[30] H. Iwaniec, *Fourier coefficients of modular forms of half-integral weight.* Invent. Math. **87** (1987), 385–402. http://eudml.org/doc/143426

[31] H. Iwaniec, *Topics in classical automorphic forms.* Grad. Studies in Math. **17**, AMS, 1997. http://doi.org/10.1090/gsm/017

[32] H. Iwaniec, *Spectral Methods of Automorphic Forms.* Grad. Studies in Math. **53**, AMS, 2002. http://doi.org/10.1090/gsm/053

[33] H. Iwaniec, E. Kowalski, *Analytic Number Theory.* Colloq. Publ. AMS **53** (2004). http://doi.org/10.1090/coll/053

[34] M. A. Kenku, *A note on the integral points of a modular curve of level 7.* Mathematika **32** (1985), no. 1, 45–48. http://doi.org/10.1112/S0025579300010846

[35] K. Khuri-Makdisi, *On the Fourier coefficients of nonholomorphic Hilbert modular forms of half-integral weight.* Duke Math. J. **84** (1996), no, 2, 399-452. http://doi.org/10.1215/S0012-7094-96-08414-8

[36] H. H. Kim, P. Sarnak, *Refined estimates towards the Ramanujan and Selberg conjectures* (Appendix 2). J. Amer. Math. Soc. **16** (2003), 175–181. http://doi.org/10.1090/S0894-0347-02-00410-1

[37] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms.* Springer GTM **97** (1984). http://dx.doi.org/10.1007/978-1-4612-0909-6

[38] Ch.-J. de La Vallée Poussin, *Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premieres inférieurs a une limite donnée.* (French) [On the Riemann ζ-function and the number of prime numbers less than a given limit]. Mem. Courronnes et Autres Mem. Pub. Acad. Roy. Sci. Lettres Beaux-Arts Belgique **59** (1899/1900). https://archive.org/details/surlafonctionze00pousgoog

[39] S. Lester, M. Radziwiłł, *Quantum Unique Ergodicity for half-integral weight forms.* Preprint, 2016. http://www.math.mcgill.ca/radziwill/automorphic6.pdf
[40] Y. L. Luke, *The special functions and their approximations.* Vol. I. Academic Press, 1969.
[41] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen.* (German) [On a new type of nonanalytic automorphic function and the determination of Dirichlet series by functional equations]. Math. Ann. **121** (1949) 141–183. http://doi.org/10.1007/BF01329622
——, *Die Differentialgleichungen in der Theorie der elliptischen Modulfunktionen.* (German) [Differential equations in the theory of elliptic modular functions]. Math. Ann. **125** (1952), 235–263. http://doi.org/10.1007/BF01343120
[42] Ju. I. Manin, *Parabolic points and zeta-functions of modular curves.* Izv. Akad. Nauk SSSR **36** (1972), translated as Math. USSR Izvestija **6** (1972), 19–64. http://doi.org/10.1070/IM1972v006n01ABEH001867
[43] T. Miyake, *Modular Forms.* Springer-Verlag (Berlin), 2nd ed., 1989. http://doi.org/10.1007/3-540-29593-3
[44] H. Poincaré, *Fonctions modulaires et fonctions fuchsiennes.* (French) [Modular and Fuchsian functions]. Ann. Fac. Sci. Toulouse Math. **3** (1911), 125–149. http://eudml.org/doc/72826
[45] N. V. Proskurin, *On the General Kloostermann Sums.* J. Math. Sciences **129** (2005), no. 3, 3874–3889. http://doi.org/10.1007/s10958-005-0324-7
[46] W. Roelcke, *Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II.* (German) [The eigenvalue problem for automorphic forms in the hyperbolic plane, I-II]. Math. Ann. **167** (1966), 292–337, **168** (1966), 261–324. http://doi.org/10.1007/BF01364540 http://doi.org/10.1007/BF01361556
[47] P. Sarnak, *Additive number theory and Maass forms.* In *Number theory (New York, 1982)*, edited by D. V. Chudnovsky, G. V. Chudnovsky, H. Cohn, and M. B. Nathanson. Springer LNM **1052** (1984), 286–309. http://doi.org/10.1007/BFb0071548
[48] P. Sarnak, *Some applications of modular forms.* Cambridge University Press, 1990.
[49] P. Sarnak, *Estimates for Rankin-Selberg L-functions and quantum unique ergodicity.* J. Funct. Anal. **184** (2001), 419–453. http://doi.org/10.1006/jfan.2001.3783
[50] R. Schoof, N. Tzanakis, *Integral points of a modular curve of level 11.* Acta Arith. **152** (2012), no. 1, 39–49. http://doi.org/10.4064/aa152-1-4
[51] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series.* J. Indian Math. Soc. **20** (1956), 47–87.
[52] A. Selberg, *On the estimation of Fourier coefficients of modular forms.* Proc. Sympos. Pure Math. **8** (1965), 1–15. http://bookstore.ams.org/pspum-8
[53] G. Shimura, *On modular forms of half integral weight.* Ann. Math. **97** (1973), 440–481. http://doi.org/10.2307/1970831
[54] G. Shimura, *On the holomorphy of certain Dirichlet series.* Proc. London Math. Soc. **31** (1975), no. 1, 79–98. http://doi.org/10.1112/plms/s3-31.1.79
[55] G. Shimura, *On Eisenstein series of half-integral weight.* Duke Math. J. **52** (1985), no. 2, 281–314. http://dx.doi.org/10.1215/S0012-7094-85-05216-0
[56] C. L. Siegel, *Zum Beweise des Starkschen Satzes.* (German) [Some remarks on Stark's theorem]. Invent. Math. **5** (1968), no. 3, 180–191. http://eudml.org/doc/141914
[57] H. M. Stark, *A complete determination of the complex quadratic fields of class-number one.* Michigan Math. J. **14** (1967), no. 1, 1–27. http://doi.org/10.1307/mmj/1028999653
[58] H. M. Stark, *On the "gap" in a theorem of Heegner.* J. Number Theory **1** (1969), 16–27. http://doi.org/10.1016/0022-314X(69)90023-7
[59] H. M. Stark, *A historical note on complex quadratic fields with class-number one.* Proc. Amer. Math. Soc. **21** (1969), 254–255. http://doi.org/10.1090/S0002-9939-1969-0237461-X
[60] H. M. Stark, *The role of modular functions in a class-number problem.* J. Number Theory **1** (1969), no. 2, 252–260. http://doi.org/10.1016/0022-314X(69)90044-4
[61] N. Templier, *A nonsplit sum of coefficients of modular forms.* Duke. Math. J. **157** (2011), no. 1, 109–165. http://doi.org/10.1215/00127094-2011-003
[62] N. Templier, J. Tsimerman, *Non-split sums of coefficients of $GL(2)$-automorphic forms.* Israel J. Math. **195** (2013), no. 2, 677–723. http://doi.org/10.1007/s11856-012-0112-2
[63] M. Watkins, *Computing the modular degree of an elliptic curve.* Experiment. Math. **11** (2002), no. 4, 487–502. http://projecteuclid.org/euclid.em/1057864659

[64] M. Watkins, *A different proof of class number one.* Submitted, 2018.
http://magma.maths.usyd.edu.au/~watkins/papers/CLASSNO1.pdf

[65] A. Weil, *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen.* (German) [On Dirichlet series satisfying functional equations]. Math. Ann. **168** (1967), 149–156.
http://eudml.org/doc/161499

[66] M. P. Young, *Explicit calculations with Eisenstein series.* Preprint, 2017.