

JACOBI SUMS AND GRÖSSENCHARACTERS

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ABSTRACT. In 1952, Weil published a paper describing how to interpret Jacobi sums in terms of Hecke Grössencharacters of cyclotomic fields. We describe an explicit version of this, with reference to our previous work concerning algorithmic implementation of Grössencharacters. We correct various errors involving root numbers in the latter, and also indicate how Jacobi sum methods can be used to understand tame primes of hypergeometric motives.

1. INTRODUCTION

Starting in the 1940s, André Weil carried out a programme of study concerning numbers of solutions of equations in finite fields [16]. As a part of this, in 1952 he wrote a brief paper [17] showing that Jacobi sums were canonically related to Hecke Grössencharacters. Armed with our previous work [15] on the latter, we describe how to carry out this correspondence explicitly. In particular, we find that there are many practical advantages in dealing with Grössencharacters, as they can be computed much more readily.

We have chosen to phrase our work in the more modern language of Anderson [1] and Schappacher [13], the former of whom describes how to construct a motive attached to a Jacobi sum, thus providing an arithmetic-geometric interpretation of them as alluded to in Weil's introduction [17]. In §2 we recall both Weil's and Anderson's setup, and then after giving some basic examples, in §4 describe how to compute the Jacobi sums in two different ways, the first using complex Gauss sums, and the second p -adic Gauss sums and the Gross-Koblitz formula. We then indicate (§5) how Kummer twists (as dubbed by D. P. Roberts) can easily be added to Jacobi sum picture, and describe (§6) how to distinguish the associated Grössencharacter to a Kummer-twisted Jacobi sum.

We can then use the Grössencharacter to read off various information about the Jacobi sum motive, such as its conductor and local root numbers. Unfortunately, our previous work [15] had a number of inaccuracies regarding root numbers, and we take the opportunity (§6.1) to correct them here, following work of Rohrlich [12].

We then describe some formal operations on Jacobi sum motives in §7, which are one vestige of various identities for Gauss and Jacobi sums (such as the Gross-Koblitz formula again). Finally in §8 we show how Jacobi sums can be used to understand tame prime behavior of hypergeometric motives, which was actually the original motivation for this work.

1.1. Magma implementation. The methods we describe here have been implemented in the Magma computer algebra system [2] maintained by the University of Sydney. The Hypergeometric Motives chapter of the Magma Handbook [3] provides more information about usage.

1.2. Acknowledgments. The author wishes to thank D. P. Roberts for frequent help over the timeline of this article (including an initial Magma implementation of some Jacobi sum computations), and both him and F. Rodriguez Villegas for the context of the section regarding hypergeometric motives. T. Dokchitser also provided useful comments regarding some L -function questions. The referee helped clarify a discrepancy between Weil's definition of Jacobi sums and ours.

2. BACKGROUND

2.1. Weil's setup. Given an integer $m > 1$, Weil [17] first considers power sum characters. For prime ideals \mathfrak{p} coprime to m in $\mathbf{Q}(\zeta_m)$ and x with $\mathfrak{p} \nmid x$, we define $\chi_{\mathfrak{p}}(x)$ as the root of unity in $\mathbf{Q}(\zeta_m)$ that satisfies

$$\chi_{\mathfrak{p}}(x) \equiv x^{(q-1)/m} \pmod{\mathfrak{p}},$$

where q is the norm of \mathfrak{p} . This $\chi_{\mathfrak{p}}$ is a multiplicative character, and we extend the definition so that $\chi_{\mathfrak{p}}(x) = 0$ for $\mathfrak{p}|x$.

Weil takes a vector \vec{a} of r integers (a_j) modulo m , with at least one of the a_j nonzero, and for a prime ideal \mathfrak{p} coprime to m in $\mathbf{Q}(\zeta_m)$ defines the Jacobi sum¹

$$J_{\vec{a}}(\mathfrak{p}) = (-1)^{r+1} \sum_{\substack{x_1, \dots, x_r \pmod{\mathfrak{p}} \\ \sum_i x_i \equiv -1 \pmod{\mathfrak{p}}}} \prod_j \chi_{\mathfrak{p}}(x_k)^{a_j}.$$

After using multiplicativity to extend the definition of $J_{\vec{a}}$ to all ideals coprime to m , Weil proves this a Hecke Grössencharacter on $\mathbf{Q}(\zeta_m)$, with m^2 a defining ideal for it (though not necessarily minimal).

¹There are various normalisations of Jacobi/Gauss sums extant in the literature.

2.1.1. *Reformulation in terms of Gauss sums.* Weil also notes that one can rewrite the above Jacobi sum in terms of Gauss sums. For a prime p with $\gcd(p, m) = 1$, we take a nontrivial additive character ψ on \mathbf{F}_p , and for prime ideals \mathfrak{p} above p in $\mathbf{Q}(\zeta_m)$ we let q be the norm of \mathfrak{p} and define the a th Gauss sum (with $\mathbf{F}_{\mathfrak{p}} = \mathbf{F}_q$) as²

$$G_{a/m}^\psi(\mathfrak{p}) = - \sum_{x \in \mathbf{F}_{\mathfrak{p}}^\times} \chi_{\mathfrak{p}}(x)^a \psi\left(\mathrm{Tr}_{\mathbf{F}_{\mathfrak{p}}}^{\mathbf{F}_p} x\right).$$

As with [17, (7)] (see [16, p. 501ff]), we have that

$$J_{\vec{a}}(\mathfrak{p}) = (-1)^{r+1} q^{-1} \cdot \prod_{i=1}^r G_{a_i/m}^\psi(\mathfrak{p}) \cdot G_{-s/m}^\psi(\mathfrak{p}),$$

where s is the sum of the a_i . This alternative definition of the Jacobi sum as a product of Gauss sums is independent of the choice of ψ .

We can also (here and below) consider prime ideals \mathfrak{q} in subfields of $\mathbf{Q}(\zeta_m)$: namely when the $J(\mathfrak{p})$ are equal for all primes \mathfrak{p} in $\mathbf{Q}(\zeta_m)$ above \mathfrak{q} , then we take this common value as $J(\mathfrak{q})$ also.

2.2. **Anderson’s formalism.** We can reformulate³ Weil’s setup by replacing the vector \vec{a} by a parallel notion ([1, §2.2]). Let θ be an integral formal linear combination of elements of \mathbf{Q}/\mathbf{Z} whose projected sum to \mathbf{Q}/\mathbf{Z} is zero. Writing $\theta = \sum_j n_j \langle y_j \rangle$ with $n_j \in \mathbf{Z}$ and $y_j \in \mathbf{Q}/\mathbf{Z}$, we then let m be the least common multiple of the denominators of the y_j .

Our Jacobi sum evaluation is then the product of Gauss sums

$$J_\theta(\mathfrak{p}) = \prod_j G_{y_j}^\psi(\mathfrak{p})^{n_j},$$

and this can be shown to be independent of the choice of additive character ψ due to our requirement on θ that $\sum_j n_j y_j = 0$ in \mathbf{Q}/\mathbf{Z} . Moreover, we shall take this to be the “correct” extension of J_θ to the case where θ is empty, as we then get $J_\theta(\mathfrak{p})$ is identically 1. Similarly, for $y_j = 0$ the Gauss sum is 1, which can be ignored for computational purposes. Indeed, it is often better to simply omit such y_j in θ , as they do not contribute to the weight.

2.2.1. *Comparison to Weil’s setup.* We can make explicit the correspondence between Weil’s \vec{a} and the θ (of Anderson) that we use. Namely $\theta = \langle -s/m \rangle + \sum_i \langle a_i/m \rangle$ has $J_{\vec{a}}(\mathfrak{p}) = (-1)^{r+1} q^{-1} J_\theta(\mathfrak{p})$.

²Our definition differs from Weil’s by a minus sign.

³It may seem that we “extend” Weil’s ansatz by additionally allowing negative coefficients in θ , but in fact these can already be handled by Jacobi sum identities.

2.3. Jacobi motives. Anderson's formalism allows one to construct a motive $\mathbf{J}(\theta)$ corresponding to θ (see [1] and [13, §1]). We do not describe this here, but note some consequences.

2.3.1. Field of definition. We introduce the scaling $u \circ \theta$ of θ by u , which is defined by $u \circ \theta = \sum_j n_j \langle uy_j \rangle$. The natural field of definition K_θ is the subfield of $\mathbf{Q}(\zeta_m)$ that corresponds by class field theory to quotienting out $(\mathbf{Z}/m\mathbf{Z})^*$ by elements which leave θ fixed when scaling by them. The field K_θ is totally real when scaling by -1 fixes θ , and otherwise it is a CM field.

For example, with $\theta = \langle 1/13 \rangle + \langle 3/13 \rangle + \langle 9/13 \rangle$ there is a self-scaling by 3, and the motive is defined over the quartic subfield of $\mathbf{Q}(\zeta_{13})$.

2.3.2. Weight and Hodge structure. The weight w of the motive is given by $\sum_j n_j$ where the sum is over j with $y_j \neq 0$, and the local Hodge weight⁴ associated to a coprime residue class $u \bmod m$ is given by $\sum_j n_j \{uy_j\}$, where $\{\cdot\}$ is the fractional part. In the totally real case (when scaling by -1 fixes θ) the individual weights at each u must all be the same while the action of complex conjugation on $H^{p,p}$ is $(-1)^p$ at each infinite place. In the alternate case of complex multiplication, the Hodge structure is obtained by pairing u and $-u$, with $(p, p, +)$ and $(p, p, -)$ occurring equally when applicable.

For instance, for $\theta = 5\langle 1/5 \rangle$ the local weights for $u = (1, 2, 3, 4)$ are simply $(1, 2, 3, 4)$, so assuming that the embeddings are taken in the order $(\zeta_5, \bar{\zeta}_5, \zeta_5^2, \bar{\zeta}_5^2)$ the ∞ -type will be $(1, 4, 2, 3)$. The weight is 5, while the Hodge structure has $h^{1,4} = h^{4,1} = h^{2,3} = h^{3,2} = 1$ so that the effective weight is 3, being the largest difference between weights at conjugate embeddings.

Note that scaling θ changes the local weights by permuting them, which as in [15] corresponds to choosing embeddings, and so keeps the same motive over \mathbf{Q} .

3. SMALL EXAMPLES

There are various special cases of Jacobi sum motives, particularly when small numbers occur, and we recall a few of these. The case of θ empty gives the unital motive (over \mathbf{Q}) corresponding to the Riemann ζ -function. The first interesting case is $2\langle 1/2 \rangle$, when the weight is 2, the Hodge type is $(1, 1, -)$, the field of definition is the totally real subfield of $\mathbf{Q}(\zeta_4)$ (namely \mathbf{Q}), and computation gives us that we have

⁴The u th residue class corresponds to sending $\zeta_m \rightarrow \zeta_m^u$, and in a computer implementation of Grössencharacters one must specify the corresponding weight for each embedding when evaluating as complex numbers.

a Tate twist of the Kronecker character χ_{-4} . Similarly $\langle 1/3 \rangle + \langle 2/3 \rangle$ gives a Tate twist of χ_{-3} , and $\langle 1/4 \rangle + \langle 3/4 \rangle$ gives a Tate twist of χ_{-8} , while $\langle 1/5 \rangle + \langle 4/5 \rangle$ gives a Tate twist of the nontrivial Hecke character of modulus $\mathfrak{p}_5 \infty_1 \infty_2$ for $\mathbf{Q}(\sqrt{5})$.

Schappacher notes that $2\langle 2/3 \rangle - \langle 1/3 \rangle$ corresponds to a canonical Grössencharacter of modulus \mathfrak{p}_3^2 over $\mathbf{Q}(\zeta_3)$ which in turn also corresponds to an elliptic curve (isogeny class) of conductor 27, and similarly $\langle 1/2 \rangle + \langle 3/4 \rangle - \langle 1/4 \rangle$ for the Grössencharacter of modulus \mathfrak{p}_2^3 over $\mathbf{Q}(\zeta_4)$, or an elliptic curve of conductor 32. The final example of this nature is $\langle 5/8 \rangle + \langle 7/8 \rangle - \langle 1/2 \rangle$ with stability upon scaling by 3, and the field of definition is thus $\mathbf{Q}(\sqrt{-2})$, with the Grössencharacter of modulus \mathfrak{p}_2^5 , and the corresponding elliptic curve of conductor 256.

4. COMPUTING EULER FACTORS AT GOOD PRIMES

In this section we describe a couple of methods to compute $J_\theta(\mathfrak{p})$ reasonably efficiently for a given \mathfrak{p} . However, the method is still slow compared to Grössencharacter evaluation, which can be done in polynomial time. While in a later section we will indicate how to identify \mathbf{J}_θ as a Grössencharacter (which then allows faster methods to be used), we will still need to use the methods given here as a “bootstrap” at sufficiently many primes of small norm.

We first give a complex method for computing good Euler factors, and then also a p -adic method (communicated to us by D. P. Roberts). There are advantages and disadvantages for each.

4.1. Complex method. This method simply computes the Gauss sums in §2.1.1 over \mathbf{C} . There is still a choice of how to identify the ζ_m from the power residue symbols into \mathbf{C} , though this just involves fixing the embeddings. This gives a way of mapping a (good) prime ideal to a complex number, and thus a pre- L -function⁵ for θ exhibited from GL_1/K_θ , explicitly given as

$$L^*(\mathbf{J}_\theta, s) = \prod_{\mathfrak{p} \nmid m} (1 - J_\theta(\mathfrak{p})/q^s)^{-1},$$

where the prime ideals \mathfrak{p} are from $K_\theta \subseteq \mathbf{Q}(\zeta_m)$. This method essentially takes time proportional to q , namely the length of the Gauss sums.⁶

⁵We don't know the Euler factors at bad primes $\mathfrak{p}|m$, and they can be nontrivial; below we note that $\theta = \langle 1/5 \rangle + \langle 4/5 \rangle - \langle 2/5 \rangle - \langle 3/5 \rangle$ corresponds to the Dedekind ζ -function of $\mathbf{Q}(\sqrt{5})$, for which the Euler factor at \mathfrak{p}_5 is nontrivial.

⁶When $q = p$ is prime, one might hope to use functional equation methods to reduce the time to \sqrt{p} , but this is not an important issue for us.

4.2. p -adic method. ⁷ We can alternatively determine the Euler factor for primes $p \nmid m$ as follows.⁸ Determine the smallest positive f with $m \mid (p^f - 1)$, and consider the splitting by scaled orbit sets as

$$\left\{ \{ap^e \circ \theta\}_{e=0}^{f-1} : 1 \leq a \leq m-1, \gcd(a, m) = 1 \right\}.$$

For each orbit, sum a representative as $\sum_e ap^e \circ \theta = \sum_e \sum_j n_j \langle ap^e y_j \rangle$ and compute

$$R(a) = (-p)^v \prod_{e=0}^{f-1} \prod_j \Gamma_p(\{ap^e y_j\})^{n_j} \quad \text{where} \quad v = \sum_{e=0}^{f-1} \sum_j n_j \{ap^e y_j\},$$

with each Γ_p -evaluation being invertible in \mathbf{Z}_p . The $R(a)$ are roots of the Euler factor at p , which is given by the product $\prod_a (1 - R(a)T^z)^{-1}$, where z is the degree of K_θ divided by the number of a -orbits. As usual, one computes $R(a)$ by p -adic methods to sufficient precision to be able to recognise the coefficients of the Euler factors as integers in a suitable Weil interval (depending on the weight), after scaling by the known valuation. Also, in the case where $z = \deg(K_\theta)$ so there is just one orbit, we find that $R(a)$ is a scaled product of trivial Gauss sums.

The computation time is dominated by computing Γ_p at all multiples of $1/m$, each evaluation of which takes time proportional to p and the desired precision. When q is significantly larger than p , this is thus superior than the complex method. However, with this method it is more difficult to exhibit the resulting Euler product over K_θ . By making suitable choices, it is possible to associate to each prime ideal \mathfrak{p} a p -adic number in a coherent manner, in particular so that one can take tensor products of such objects over K_θ , but there is still some choice involved with identifying such p -adic numbers to complex evaluations.

4.3. Conductors. The conductor of Jacobi sum motives is known to differing levels of explicitness in various cases [11, 5, 9], whereas Weil [17] already gives the general upper bound⁹ that the conductor divides m^2 as an ideal in K_θ .

⁷As noted above, this was first communicated to us as a Magma implementation by D. P. Roberts. See also [1, §2] and [13, §0.8.2.2] for similar considerations.

⁸This implicitly uses the Gross-Koblitz formula [8] to rewrite the above Gauss sums in terms of the p -adic Γ -function.

⁹See [11] for references to later works that improve Weil's bound, though it must be noted that (for simplicity) some authors consider a restricted class of θ .

5. KUMMER TWISTS

We can also consider (Kummer) twists of Jacobi motives. We twist by t^ρ with $t \in \mathbf{Q}^*$ and $\rho \in \mathbf{Q}/\mathbf{Z}$, where we canonicalise by taking t not to be a nontrivial power and $\rho \in [0, 1)$. The m -value must now also include denominator of ρ , and the field of definition $K_\theta^{t^\rho}$ similarly requires the u -scalings to additionally fix $\rho \in \mathbf{Q}/\mathbf{Z}$. The Hodge structure stays the same, except that the central $h^{p,p}$ signs are switched when $\rho = 1/2$ and t is negative.

For \mathfrak{p} with $v_{\mathfrak{p}}(t) = v_{\mathfrak{p}}(m) = 0$ the Kummer-twisted Jacobi sum is

$$J_\theta^{t^\rho}(\mathfrak{p}) = \chi_{\mathfrak{p}}(t^{\rho m}) \cdot J_\theta(\mathfrak{p}) = \chi_{\mathfrak{p}}(t^{\rho m}) \cdot \prod_j G_{y_j}^\psi(\mathfrak{p})^{n_j}.$$

When using the p -adic method, we multiply $R(a)$ by $\omega_p(t)$ raised to the $\rho(p^f - 1) \sum_e a p^e$ power, where here ω_p is the Teichmüller character.

The introduction of the character $\chi_{\mathfrak{p}}(t^{\rho m})$ implies that the conductor bound (as a $K_\theta^{t^\rho}$ -ideal) is now m^2 multiplied by the primes at which t has nonzero valuation.

5.1. Twists of the Fermat cubic, and congruent number curve.

Recall from above that $\theta = 2\langle 2/3 \rangle - \langle 1/3 \rangle$ corresponds to an elliptic curve of conductor 27. We can take various twists of this by choosing t and ρ appropriately. In particular, with $\rho = 1/2$ we will obtain quadratic twists (note that these do not enlarge the field K_θ), while $\rho = \pm 1/3$ will give cubic twists, with $\rho = \pm 1/6$ for sextic twists. Similarly, for $\theta = \langle 1/2 \rangle + \langle 3/4 \rangle - \langle 1/4 \rangle$ we can take quartic twists of the associated congruent number curve (of conductor 32) via $\rho = \pm 1/4$.

5.2. Various examples.

One can Kummer-twist the unital Jacobi motive $\mathbf{J}(\theta)$ for θ empty and already obtain non-trivial results. For instance, twisting by $t^{1/2}$ gives the Kronecker character for t , and $2^{1/3}$ will yield the nontrivial 2-dimensional Artin representation of $\mathbf{Q}(2^{1/3})$. Similarly, with $\theta = \langle 1/5 \rangle + \langle 4/5 \rangle$ taking the Kummer twist by $5^{1/5}$ results in a Tate twist of the irreducible 4-dimensional Artin representation for $\mathbf{Q}(5^{1/5})$, with the Grössencharacter having conductor \mathfrak{p}_5^6 in $\mathbf{Q}(\zeta_5)$. This same conductor appears with the Kummer twist by $5^{1/5}$ of $\theta = 2\langle 3/5 \rangle - \langle 1/5 \rangle$. Moreover, the four conjugate Kummer twists here all have congruent Euler factors mod 5.

6. GRÖSSENCHARACTER RECIPROCITY

Given Weil's above bound for the conductor and the knowledge of the ∞ -type as in §2.3.2, it is a finite problem to identify a Jacobi-Kummer motive with a Hecke Grössencharacter, as the set of possibilities for the latter is limited. The process can be carried out by choosing enough¹⁰ good primes of small norm and comparing Euler factors.

Note there can be more than one Hecke Grössencharacter over $K_\theta^{t^\rho}$ that corresponds to a motive over \mathbf{Q} . For instance with θ empty and $t^\rho = 2^{1/3}$, there are two conjugate Hecke characters that can be distinguished over $K_\theta^{t^\rho}$ but not over \mathbf{Q} . In this case, we find it more useful to work with the complex method to compute the Euler factors.

In practise, it is definitely best to identify the Jacobi motive as a Grössencharacter, as the Euler factors at good primes can be computed in time polynomial in $(\log p)$, and moreover one can obtain the relevant information about local conductors at bad primes using the method of [15, §2.2 and §5.3].

6.1. Root numbers. Unfortunately, the author's paper [15] contains many infelicities in its description of Grössencharacter root numbers. For instance, the formula in [15, §3.6.1] for the local root number is garbled (at best), and the statement in [15, §5.3.1] also contains some mystification. Fortunately, a parallel analysis has been given by Rohrlich, with a correct expression for root numbers being found in [12, (2.42)]. We take the opportunity to give corresponding corrections to [15] here.

We begin by first recalling the notation of [15, §5.2]. For a given Grössencharacter \mathbf{G} , we have an ∞ -type $\mathbf{T} = (n_\sigma)_\sigma$ where σ runs over complex embeddings and n_σ are integral, and a Hecke-Dirichlet pair (ψ, χ) of characters¹¹ with the Dirichlet character inverse to the evaluation of the ∞ -type on units, so that $\chi(\epsilon)^{-1} = \mathbf{T}(\epsilon) = \prod_\sigma (\epsilon^\sigma)^{n_\sigma}$ for units ϵ . Furthermore, for elements $\alpha \in K$ of the base field, we have $\mathbf{G}(\alpha) = \mathbf{T}(\alpha) \cdot \chi(\alpha) \cdot \psi(\alpha)$, so that $\psi\chi$, or more properly $\tilde{\psi}\chi$, namely the product of χ with the Dirichlet restriction $\tilde{\psi}$ of ψ , can be said to have the same local components at finite primes as \mathbf{G} ; indeed this is how one computes the conductor of \mathbf{G} (cf. [15, §5.3]). We write the Dirichlet decomposition of the character as $\tilde{\psi}\chi = \prod_{\mathfrak{p}} (\tilde{\psi}\chi)_{\mathfrak{p}}$. However, note that

¹⁰One can give an explicit bound (if desired) via estimates from analytic number theory, but this is likely much larger than is needed in practise.

¹¹The terminology is that ψ is a Hecke character on ideals, while χ is a Dirichlet character on K -elements. Note that a Dirichlet character has a Chinese remainder decomposition into prime power moduli, while a Hecke character need not, and that a Hecke character can always restrict to a Dirichlet character, but a Dirichlet character only lifts to a Hecke character if it is trivial on the units.

with Proposition 2.1 of Rohrlich [12] (coming from equation 2.17) the adelic components are *reciprocal* to our Dirichlet components, which can cause some confusion. In particular, we have that $\mathbf{G}_{\mathfrak{p}} = (\check{\psi}\chi)_{\mathfrak{p}}^{-1}$ on $\mathbf{Z}_{\mathfrak{p}}^{\times}$, where the former is an adelic component of \mathbf{G} *à la* Rohrlich.

At a given prime ideal \mathfrak{p} we write d for the exponent of the different ideal and e for that of the conductor. Letting u be a uniformiser, from [12, (2.42)] the local root number is

$$W_{\mathfrak{p}}(\mathbf{G}) = \tilde{\mathbf{G}}_{\mathfrak{p}}(u^{d+e}) \cdot \frac{1}{\sqrt{q^e}} \sum_{a \in (\mathbf{Z}_{\mathfrak{p}}/\mathfrak{p}^e)^{\times}} (\tilde{\psi}\chi)_{\mathfrak{p}}(a) \cdot \exp\left(2\pi i \operatorname{Tr}(a/u^{d+e})\right).$$

Here the tilde in $\tilde{\mathbf{G}}_{\mathfrak{p}}$ indicates the unitary part is taken, while $\mathbf{G}_{\mathfrak{p}}(u^{d+e})$ itself is determined by pulling back u^{d+e} to a K -element α coprime to the modulus $\mathfrak{m}\Omega$, or equivalently to a principal ideal $(\alpha) = \mathfrak{p}^{d+e}\mathfrak{a}$ satisfying $\gcd(\mathfrak{p}, \mathfrak{a}) = \gcd(\mathfrak{a}, \mathfrak{m}) = 1$. More explicitly, by the adelic formulation (see [12, §2.2]) and product formula for α we have

$$1 = \mathbf{G}_{\mathfrak{p}}(\alpha) \cdot \prod_{\substack{q|\mathfrak{m} \\ q \neq \mathfrak{p}}} \mathbf{G}_q(\alpha) \cdot \prod_{\mathfrak{r}|\mathfrak{a}} \mathbf{G}_{\mathfrak{r}}(\alpha) \cdot \mathbf{G}_{\infty}(\alpha),$$

and then rewriting the right side gives

$$1 = \mathbf{G}_{\mathfrak{p}}(u^{d+e}) \cdot \prod_{\substack{q|\mathfrak{m}\Omega \\ q \neq \mathfrak{p}}} (\check{\psi}\chi)_q^{-1}(\alpha) \cdot \mathbf{G}(\mathfrak{a}) \cdot \mathbf{T}(\alpha)^{-1},$$

with the last three factors directly computable. This gives $\mathbf{G}_{\mathfrak{p}}(u^{d+e})$ as a complex number, and dividing by its norm gives the unitary part. Note that in the case where \mathbf{G} is unramified at \mathfrak{p} the above local root number formula still holds with the convention that the sum is 1, and since $e = 0$ we then get $W_{\mathfrak{p}}(\mathbf{G}) = \tilde{\mathbf{G}}_{\mathfrak{p}}(u^d) = \tilde{\mathbf{G}}(\alpha)/\tilde{\mathbf{G}}(\mathfrak{a}) = \tilde{\mathbf{G}}(\mathfrak{p}^d)$ in terms of the local different.¹²

Also, in §3.6.1 of [15] the root number at a ramified real place should be reciprocated to $e^{-2\pi i/4}$, as Deligne uses $\psi(x) = \exp(2\pi i x)$ while Rohrlich in his (2.56) uses the conjugate of this. In a similar manner, with [15, §5.3.1] the local root number at a complex place should be i^{p-q} .

¹²We can note that [15, §3.6.1] appends a reciprocal on the right side, but as previously indicated, we do not consider this to be a reliable source for local root numbers. Indeed, already with the $\mathbf{Q}(\sqrt[3]{175})$ example given there one can see the discrepancy with the correct formula given here.

The confusion was likely due to inept glueing of various formulæ in the literature. For instance, when reading (7.1.60) and (7.1.61a) in [14] one must realise that $W(\rho)$ relates to $\tau(\bar{\rho})$, so $W(\rho)$ corresponds to $\rho(\partial)$, not its reciprocal.

The author wishes to thank D. P. Roberts for helping him sort out the various discrepancies in the root number formulæ. Also, note that the Magma code internally uses χ^{-1} in place of χ , so one must be careful if using more functionality than is immediately exposed.

6.1.1. *Examples.* Consider the Größencharacter \mathbf{G} of the elliptic curve isogeny class of conductor 27, which has modulus \mathfrak{p}_3^2 in $\mathbf{Q}(\sqrt{-3})$ and ∞ -type $(1, 0)$. Then

$$W_{\mathfrak{p}_3}(\mathbf{G}^o) = \begin{cases} +i & \text{when } o \bmod 12 \in \{1, 9, 11\}, \\ -i & \text{when } o \bmod 12 \in \{3, 5, 7\}, \end{cases}$$

while $W_\infty(\mathbf{G}^o) = (-i)^o$. As with [7] we find that

$$W(\mathbf{G}^o) = \begin{cases} +1 & \text{when } o \bmod 12 \in \{1, 3, 7\}, \\ -1 & \text{when } o \bmod 12 \in \{5, 9, 11\}. \end{cases}$$

The conductor is trivial when $6|o$, is \mathfrak{p}_3 when otherwise $3|o$, and else \mathfrak{p}_3^2 .

6.1.2. *Another example.* In Exercise 5.5 of [12], one starts with the Größencharacter \mathbf{G} corresponding to the elliptic curve 49a, namely with modulus \mathfrak{p}_7 in $\mathbf{Q}(\sqrt{-7})$ and ∞ -type $(1, 0)$, and twists it by the quadratic character ψ corresponding to extending $\mathbf{Q}(\sqrt{-7})$ by the square root of $-118 - 18\sqrt{-7}$. For \mathbf{G} we have the root numbers $W_{\mathfrak{p}_7}(\mathbf{G}) = +i$ and $W_\infty(\mathbf{G}) = -i$, while the conductor of ψ is $\mathfrak{p}_{11}\mathfrak{p}_{23}$, with $W_{\mathfrak{p}_7}(\psi) = 1$, $W_{\mathfrak{p}_{11}}(\psi) = +i$, and $W_{\mathfrak{p}_{23}}(\psi) = -i$. Thus both ψ and \mathbf{G} have global root number $+1$ and are self-dual.

However, the product $\mathbf{G}\psi$ is not self-dual, and while the root numbers at \mathfrak{p}_7 and ∞ are the same as with \mathbf{G} , at \mathfrak{p}_{11} and \mathfrak{p}_{23} they become roots of $11T^4 - 6T^2 + 11$ and $23T^4 + 18T^2 + 23$ respectively.¹³ But at the same time, the central L -value still vanishes,¹⁴ as the Mordell-Weil rank increases upon making this quadratic extension of $\mathbf{Q}(\sqrt{-7})$.

¹³Note [15, §5.3.1 (4)] again has a wrong reciprocation, as $W(\mathbf{G}\psi) = \mathbf{G}(\mathfrak{p}_{11}\mathfrak{p}_{23})$.

¹⁴Rohrlich emphasizes that the work of Coates and Wiles [4] allows one to *prove* that the central L -value vanishes in this case.

This example is slightly more complicated than those of [6], where one takes an elliptic curve and twists it by (say) a cubic Dirichlet character. For instance, with 37b and a cubic character modulo 7, the curve increases in rank over $\mathbf{Q}(\zeta_7)^+$, so there is again an expectation of central vanishing even though the motive is not self-dual and the root number is unitary. However, the cubic character is itself unitary, while the corresponding quadratic character in Rohrlich's example is not.

Another example of unforced vanishing is with (e.g.) the weight 2 newform with Dirichlet character of modulus 61 and order 6 over $\mathbf{Q}(\zeta_3)$; the root number is a root of $61T^{12} + 121T^6 + 61$ and the vanishing can be proven by modular symbols.

7. RELATIONS BETWEEN MOTIVES

We next describe how some natural operations on θ 's correspond to motivic operations. Firstly we look at the sum of elements. Assuming for simplicity that θ_1 and θ_2 have the same defining field K_θ , we then have that

$$\mathbf{J}(\theta_1 + \theta_2) = \mathbf{J}(\theta_1) \otimes_{K_\theta} \mathbf{J}(\theta_2).$$

This follows (e.g.) simply from analysis of the local degree 1 Euler factors at each prime ideal, as adding θ_1 and θ_2 multiplies the Jacobi sum evaluations (a.k.a. the local eigenvalues), which is exactly the tensor operation on the motivic side. In the general case, one must first induce the motives to a common defining field.

The negation operation on θ yields reciprocation on the eigenvalues, which can also be represented as a Tate twist of conjugation. Note that $\mathbf{J} \otimes \mathbf{J}^{-1}$ is the unital motive, as all the resulting eigenvalues are 1.

As noted previously, scaling θ retains the same motive over \mathbf{Q} , but over K_θ can result in conjugation.

7.1. An example. Consider $\theta_1 = \langle 1/5 \rangle + \langle 4/5 \rangle$ and $\theta_2 = \langle 2/5 \rangle + \langle 3/5 \rangle$. The field of definition of each of these is $\mathbf{Q}(\sqrt{5})$. We have $2 \circ \theta_1 = \theta_2$, and thus the motives are the same over \mathbf{Q} . In fact, they are the same over $\mathbf{Q}(\sqrt{5})$, as the local weights do not change under the scaling. We can then note that

$$\mathbf{J}(\theta_1 - \theta_2) = \mathbf{J}(\theta_1) \otimes_{\mathbf{Q}(\sqrt{5})} \mathbf{J}(\theta_2)^{-1} = \mathbf{J}(\theta_1) \otimes_{\mathbf{Q}(\sqrt{5})} \mathbf{J}(\theta_1)^{-1} = \mathbf{U}_{\mathbf{Q}(\sqrt{5})},$$

where the last motive is the unital motive over $\mathbf{Q}(\sqrt{5})$, whose L -function is the Dedekind ζ -function for this field.

We can also note that $\mathbf{J}(\theta_1 + \theta_2) = \mathbf{U}_{\mathbf{Q}(\sqrt{5})}^{(-2)}$, namely the Tate twist by -2 of the aforesaid unital motive. The computation here can either go through a similar formalism, or note that the Jacobi sums (at \mathfrak{p} of norm q) for $\mathbf{J}(\theta_1) = \mathbf{J}(\theta_2)$ are $\pm q$, which thus always square to q^2 when taking the tensor product.

From this starting point, we can derive more relations herein. The \mathbf{Q} -motives for $\eta_1 = 2\langle 1/5 \rangle - \langle 2/5 \rangle$ and $\eta_2 = \langle 1/5 \rangle + \langle 2/5 \rangle - \langle 3/5 \rangle$ can then be seen to be equal since $\eta_1 = 3 \circ \eta_2 + (\theta_1 - \theta_2)$. Indeed as rank 1 motives on $\mathbf{Q}(\zeta_5)$ we have

$$\mathbf{J}(\eta_1) = \mathbf{J}(3 \circ \eta_2) \otimes_{\mathbf{Q}(\zeta_5)} \mathbf{J}(\theta_1 - \theta_2),$$

and the latter motive is unital. Thus $\mathbf{J}(\eta_1) = \mathbf{J}(3 \circ \eta_2)$ as motives over $\mathbf{Q}(\zeta_5)$, and upon removing the scaling from η_2 we find that their \mathbf{Q} -realizations are the same.

Similarly, we can note $\theta_3 = \langle 1/7 \rangle + \langle 6/7 \rangle$ and $\theta_4 = \langle 2/7 \rangle + \langle 5/7 \rangle$ has $\mathbf{J}(\theta_3 - \theta_4)$ as the unital motive over the the real cubic subfield of $\mathbf{Q}(\zeta_7)$, again since $\mathbf{J}(\theta_3) = \mathbf{J}(\theta_4)$ over this field, for the scaling does not change the local weights. From this we determine (for instance) that $\kappa_1 = 2\langle 1/7 \rangle - \langle 2/7 \rangle$ and $\kappa_3 = \langle 1/7 \rangle + \langle 3/7 \rangle - \langle 4/7 \rangle$ yield the same \mathbf{Q} -motive since $\kappa_1 - 5 \circ \kappa_3 = \theta_3 - \theta_4$, so that $\mathbf{J}(\kappa_1) = \mathbf{J}(5 \circ \kappa_3)$ over $\mathbf{Q}(\zeta_7)$. However, $\kappa_2 = \langle 1/7 \rangle + \langle 2/7 \rangle - \langle 3/7 \rangle$ must correspond to a different motive, as there is no scaling $u \pmod 7$ such that $\kappa_1 - u \circ \kappa_2$ has effective weight 0 (so in particular is not unital). One can presumably re-interpret these in terms of identities for Gauss and Jacobi sums.

8. TAME PRIMES OF HYPERGEOMETRIC MOTIVES

We conclude by describing how Jacobi sums can be used to compute Euler factors at (possibly) tame primes of hypergeometric motives. This was one of the original motivations for our project with D. P. Roberts and F. Rodriguez Villegas, though by now an alternative explanation has been given. We refer to [3] and [10] for a fuller context.

Consider a hypergeometric datum over \mathbf{Q} given by the disjoint multisets \mathcal{A} and \mathcal{B} of cyclotomic indices. We rewrite the defining quotient of cyclotomic polynomials by Möbius inversion as

$$\frac{\prod_{a \in \mathcal{A}} \Phi_a(T)}{\prod_{b \in \mathcal{B}} \Phi_b(T)} = \frac{\prod_{c \in \mathcal{C}} (T^c - 1)}{\prod_{d \in \mathcal{D}} (T^d - 1)},$$

where again \mathcal{C} and \mathcal{D} are disjoint multisets. We also recall the natural scaling parameter $M = \prod_{c \in \mathcal{C}} c^c / \prod_{d \in \mathcal{D}} d^d$.

At a rational parameter $t \neq 0, 1$, the (conjectural) hypergeometric motive has possibly tame ramification at primes p with $v_p(t) \neq 0$ which are in neither \mathcal{A} nor \mathcal{B} . In our normalization, the t of positive valuation correspond to \mathcal{C} and those of negative valuation to \mathcal{D} . Letting $v = -v_p(t) > 0$ and $t = t_0/Mp^v$, we consider the $s \in \mathcal{D}$ with $s|v$. Writing m for the multiplicity of s in $\mathcal{A} \cup \mathcal{B}$, such an s (counted only once) contributes an Euler factor of weight $w + 1 - m$, corresponding to the Kummer twisting by $t_0^{1/s}$ of the Jacobi motive given by

$$\sum_{c \in \mathcal{C}} \langle c/s \rangle - \sum_{d \in \mathcal{D}} \langle d/s \rangle,$$

where we can ignore (as noted in §2.2) the occurrences of $\langle 0 \rangle$, and must in any case take a Tate twist to get the correct (effective) weight. A similar characterisation holds true for $v_p(t) > 0$ and \mathcal{C} .

8.1. **An example.** Consider the weight 3 hypergeometric motive given by the pair $(\Phi_1^2\Phi_2^2\Phi_3^3\Phi_6, \Phi_5^3)$, so that $\mathcal{C} = \{1, 2, 3, 3, 6\}$ and $\mathcal{D} = \{5, 5, 5\}$ and $M = 2^8 3^{12} / 5^{15}$. Letting $p = 11$ and taking $t = u/Mp^5$, we find that there should be a degree 4 Euler factor of weight $3 + 1 - 3 = 1$ at p (with the local conductor being 11^{12-4}). The corresponding Jacobi-Kummer motive is the $u^{1/5}$ -Kummer twist of $2\langle 1/5 \rangle + \langle 2/5 \rangle + 2\langle 3/5 \rangle$, with the result Tate-twisted by 2 to reduce its weight to 1. There are five possibilities for the Euler factor depending on the residue class of $\pm u$ modulo 11. For instance, for $u = \pm 4$ the reciprocal Euler factor is $1 - 9T + 41T^2 - 99T^3 + 121T^4$.

To describe the situation for t with positive valuation, we take $p = 7$ (taking a prime that is 1 mod 6 simplifies somewhat). For both $s = 1, 2$ we get θ as empty, while for $s = 3$ it is $\langle 1/3 \rangle - 2\langle 2/3 \rangle$ and for $s = 6$ it is $\langle 1/6 \rangle + \langle 1/3 \rangle + 2\langle 1/2 \rangle - 3\langle 5/6 \rangle$. The weights for $s = (1, 2, 3, 6)$ are respectively $(2, 2, 1, 3)$. For instance, for $t = 5p^6/M$ we find that the reciprocal Euler factor is

$$(1 - 7T) \cdot (1 + 7T) \cdot (1 + 4T + 7T^2) \cdot (1 + 20T + 343T^2),$$

with the factors corresponding to the $s = (1, 2, 3, 6)$ contributions.

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