ON CHOWLA'S CONJECTURE FOR $\mathbf{Q}(\sqrt{4u^2+1})$

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MARK WATKINS

Abstract. Chowla conjectured that the instances of $\mathbf{Q}(\sqrt{4u^2+1})$ of class number 1 (with $4u^2+1$ squarefree and $u>0$) are $u \in \{1,2,3,5,7,13\}$, and this was shown by Biró in 2003 by generalizing a method of Beck involving Dedekind ζ-functions of ideal classes. We give a different proof of Chowla's conjecture – our recent improved lower bound for $L_\chi(1)$ immediately reduces the problem to a finite computation, and herein we present the details of such.

The main technique is well-known, viz. eliminating a large "intermediate-sized" range via a battery of auxiliary Dirichlet L-functions that have low-height zeros significantly smaller than the average, and then handling the range of small discriminants by a computational sieve. For the intermediate range we work in some generality, and in particular remove the condition of $D \ge 4\pi^2\exp(10^6)$ that appeared in our previous work.

We also consider another family $\mathbf{Q}(\sqrt{25u^2+14u+2})$ with small fundamental unit, again proving the list $u \in \{-5,-1,0,1,3\}$ with class number 1 (and $25u^2+14u+2$ squarefree) is complete. In fact, we prove the completeness of the lists for class numbers up through 5 for these families.

As another example of our methods, we additionally show there are exactly 22 fundamental discriminants $D>0$ with fundamental unit $(A+B\sqrt{D})/2$ that have $B \le D^{1/4}$ and class number 1 (the largest is 1253).

1. History and introduction

Recently we showed [32] an improved (effective) lower bound for $L_\chi(1)$ based upon relating such an L-value to the precision to which we could compute various third central derivatives of specific elliptic curves of rank 5.

1.1. More precisely, writing $\Lambda_E(s)$ for the completed L-function of E, in [32] we showed the following theorem,[1] where $\mathbf{Q}(\sqrt{\Delta})$ is a quadratic field with Δ a fundamental discriminant with $D=|\Delta|$, and χ is the associated quadratic character.

Theorem 1.1.1. *Suppose that $|\Lambda_E'''(1)| \le 10^{-1025}\cdot(2\pi/\sqrt{N_E})$ for the six elliptic curves for $\mathcal{E}_2^{\pm}$ in Table 1 of* [33]. *Then for $D \ge 4\pi^2\exp(10^6)$ we have*

$$\sqrt{D}L_\chi(1) \ge \min\bigl(10^{1000}\log D, (\log D)^3/10^{13}\bigr)\cdot\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

Prior bounds had the constant $1/7000$ in place of 10^{1000}, and for real quadratic fields did not beat the trivial bound that comes from the size of the regulator.

In [33] we then computed said third central derivatives to the desired precision.

1.2. One reason why this new lower bound of $L_\chi(1)$ is of note (even though it only gains a constant over previous work) is that it allows effective resolution of various class number problems for real quadratic fields.

Herein we carry out the explicit details to solve the specific case of Chowla's conjecture, which states the cases of class number 1 for $\mathbf{Q}(\sqrt{4u^2+1})$ with $4u^2+1$ squarefree and $u>0$ are given by $u \in \{1,2,3,5,7,13\}$. This was solved by Biró [3] using alternative means in 2003, and is the most venerable in the Richaud-Degert

[1]We also showed a similar theorem for D that were coprime to the conductors of various rank 5 elliptic curves; here the conductors were smaller, and the calculations could be extended to 10000 digits. The resulting constant multiplying $(\log D)$ was thus 10^{10000} while that for $(\log D)^3$ was $1/10^8$ in place of $1/10^{13}$. However, we will not use this result here.

genre of such problems (see below for more history). We also show an analogous class number result regarding the fields $\mathbf{Q}(\sqrt{25u^2+14u+2})$, which were considered already by Euler as an example where the associated continued fraction has short period (thus the field has small regulator).

1.2.1. Along the way, as we need to make similar computations in any case, we also remove the condition $D \geq 4\pi^2\exp(10^6)$ in the above Theorem by employing a battery of auxiliary L-functions in a manner similar to our work [30] on class numbers up through 100 for imaginary quadratic fields. In particular, we show the following "general" result, which allows us to replace the $D \geq 4\pi^2\exp(10^6)$ condition in the above quoted Theorem.

Proposition 3.3.1. *We have* $\sqrt{D}L_\chi(1) \geq 100\log D$ *when* $10^3 \leq \log D \leq 10^8$.

1.2.2. We then shift to the specific cases highlighted above; we could be more general, but choose these as exemplary cases with a small fundamental unit.

Lemmata 4.2.2 & 4.2.3. *Suppose we are in one of the following cases, with D fundamental and* $10^{28} \leq D \leq \exp(10^3)$*:*

(1) $D = 4u^2+1$ *with* $u > 0$,
(2) $D = u^2+4$ *with u odd and* $u > 0$,
(3) $D = 4(u^2+1)$ *with u odd and* $u > 0$,
(4) $D = 25u^2+14u+2$ *with u odd,*
(5) $D = 4(25u^2+14u+2)$ *with u even.*

Then $h_K \geq 6$ *for* $K = \mathbf{Q}(\sqrt{D})$.

Here the range $100 \leq \log D \leq 1000$ is handled by the real Dirichlet character of conductor 12461947, and the lower range by that of conductor 17923.

1.2.3. We then complete the classifications of cases with $h_K \leq 5$ in these families by a routine computational sieve up to 10^{28} (needing to consider only about 10^{14} values of u). Explicitly, we have the following result.

Theorem 1.2.4. *Suppose that D is a fundamental discriminant and is in one of the above five families. Then the cases with* $h_K \leq 5$ *are as in Tables 1 and 2.*

case	$h_K = 1$	$h_K = 2$	$h_K = 3$
(1)	$1,2,3,5,7,13$	$4,11,17,23,29$	$8,27,37,47$
(2)	$1,3,5,7,13,17$	$9,19,23,25,31,41,43,53$	$15,27,35,37,47,67,73,97$
(3)	1	$3,5,11,19$	
(4/5)	$-5,-1,0,1,3$	$-3,-2,4,15$	$-21,-9,19$

TABLE 1. Values of u that give $h_K \in \{1,2,3\}$ for the families

case	$h_K = 4$	$h_K = 5$
(1)	$6,15,25,31,43,49,53,61,71$	$10,33,55,73,103$
(2)	$21,49,55,59,71,77,79,$ $83,101,107,113,127,157$	$33,57,85,103,115,137,167,193$
(3)	$9,13,17,23,25,31,37$	
(4/5)	$-33,-17,-13,-8,-6,2,6,10,39$	$-41,9,11$

TABLE 2. Values of u that give $h_K \in \{4,5\}$ for the families

The bound of $h_K \leq 5$ is chosen to go further than any precedent result, but not be too large so as to lead to unwieldy complications.

1.3. Let us recall some history about Chowla's conjecture for the instances of class number 1 for real quadratic fields of the form $\mathbf{Q}(\sqrt{4u^2+1})$ with $D = 4u^2+1$ squarefree.[2] Both this and the companion problem of Yokoi's conjecture for $\mathbf{Q}(\sqrt{u^2+4})$ were solved in 2003 by Biró [3] by a method involving ideal class ζ-functions.

Of course, the "point" with these conjectures is that we have an explicit fundamental unit of small height, and thus (by Dirichlet's class number formula) class number 1 is equivalent to a small value of $L_\chi(1)$, namely of size $(\log D)/\sqrt{D}$. Our new lower bound on $L_\chi(1)$ allows us to solve these and other such problems in an effective manner. Herein we are concerned with the computations of making a couple of cases completely explicit.

1.3.1. The literature often puts the Chowla and Yokoi conjectures in the context of a somewhat larger class of real quadratic fields, namely those of "Richaud-Degert" type,[3] which are essentially given by $\mathbf{Q}(\sqrt{(au)^2+ka})$ where $k \in \{\pm1,\pm2,\pm4\}$, subject to various conditions to avoid trivialities and ensure the discriminant is positive. The particular cases where $|ka| \in \{1,4\}$ are the "narrow" cases, where the fundamental unit is respectively $au+\sqrt{D}$ or this divided by 2, and its norm has the opposite sign as k. More generally, the fundamental unit is $((2au^2+k)+2u\sqrt{D})/|k|$ and has norm 1 (see Degert's Satz 1 in [13], as quoted in Lemma 2.3 of [21]).

Many results for Richaud-Degert fields are phrased in terms of necessary and sufficient conditions for small class number, for instance being related to prime-producing polynomials or to special values of partial Dedekind ζ-functions (which are at the heart of Beck's method as employed by Biró). However, there are some unconditional results: for instance Biró's student Lapkova developed his method in her thesis [21], and together they explicitly solved [20, 5] the class number 1 problem for Richaud-Degert fields with $k = 4$. Another notable result is that of Byeon and Lee [7], who show that when $u^2 + 1$ squarefree with $u > 0$ odd, then $\mathbf{Q}(\sqrt{u^2+1})$ has class number 2 exactly for $u \in \{3, 5, 11, 19\}$. This uses a generalization of Biró's method, which was then further expounded by Biró and Granville [4].

1.3.2. Our method is more general in that it can (at least in theory) show a class number 1 result for any real quadratic field whose fundamental unit is of polynomial size up to large degree (roughly 10^{500}) in D. For instance, Euler [14, Ex. 1] already considered (as an illumination of an instance with short continued fraction period) examples where $D = (3u+1)^2 + (4u+1)^2 = 25u^2 + 14u + 2 = (5u+7/5)^2 + 1/25$, which has fundamental unit $(25u + 7) + 5\sqrt{D}$ when u is odd and D is squarefree. Similarly, at least for discriminants where primes $p|D$ do not intervene heavily (for instance, prime D), when the fundamental unit is of size $\sqrt{D}$ our method can (in theory) show a class number h result for h up to roughly 10^{1000} again.

As an example of a more general result, again chosen so as to be exemplary and not too computationally arduous, we will show the following Theorem.

[2]If we omit this squarefree condition, then we are interested in solutions to $4u^2 - dv^2 = -1$, and thus have a Pellian family for each d. For instance, with $d = 13$ we can take $(u, v) = (9, 5)$. Indeed, the family of Euler that we consider has $D = ((25u + 7)^2 + 1)/5^2$ (or 4 times this), and is thus the 5-imprimitive version of the $u^2 + 1$ family.

[3]The term comes from Hasse (1965), denoting a combination of Richaud's work (1866) with that of Degert (1958), though as Lemmermeyer [22] notes, the history contains many more names (including Euler of course) that could be appended.

Theorem 1.3.3. *Suppose that $D > 0$ is fundamental with $(A+B\sqrt{D})/2$ its fundamental unit, with $B \le D^{1/4}$ and $h_K = 1$. Then D is in the set*

$$\{5, 8, 12, 13, 17, 21, 24, 29, 37, 53, 77, 93, 101,$$
$$173, 197, 293, 413, 437, 677, 773, 1133, 1253\}.$$

1.4. Let us take the opportunity to expand on comments we made in the Remark to [31, §2.5.2] regarding class number 1 computations for imaginary quadratic fields.

Therein we cited Bundschuh and Hock [6] for an explicit version of Baker's method that showed an imaginary quadratic field does not have class number 1 for $D \ge \exp(160000)$, still necessitating a calculation from Stark's method to finish the problem. However, we can follow the history further forward to the 1987 paper of Cherubini and Wallisser [8], who note that: Bundschuh and Hock [6] used linear forms in three logarithms with integral coefficients; Stark's reduction to two logarithms with algebraic coefficients was made explicit by Fel'dman and Chudakov [15] (with an upper bound of 10^{40});[4] and they themselves note that later developments (namely Schneider's solution to Hilbert's seventh problem, and its quantitative version by Mignotte and Waldschmidt [24]) indicate that two logarithms in integral coefficients suffice, whereupon the remaining range $D \le 10^{34}$ is easily handled by continued fractions.

1.5. We briefly outline the contents herein. In §2 we review the relevant theory of quadratic fields and binary quadratic forms, with significant overlap to [32, §3] (here we use a slightly different Mellin transform in our weighting of an error term). In §3 we then use a battery of 60 auxiliary Dirichlet L-functions with zeros of low height to handle the range $10^3 \le \log D \le 10^8$, doing this in some generality. In §4 we specialize to our families of interest, and handle $10^{28} \le D \le \exp(1000)$, and in §5 we describe the sieving procedures for $D \le 10^{28}$ in our families. In §6 we then prove Theorem 1.3.3 via similar techniques.

1.5.1. *Notation.* We will continue our usage of "∂" with integrals instead of "d". We let $K = \mathbf{Q}(\sqrt{\Delta})$ be a quadratic field of discriminant Δ with $D = |\Delta| > 4$, and χ its associated character. We write $\zeta_K(s)/\zeta(2s) = \sum_n R_K^\star(n)/n^s$ for the primitivized Dedekind ζ-function, and then split $R_K^\star(n) = R_K^{\star s}(n) + \tilde{R}_K^{\star s}(n)$ at $\sqrt{D}/4$ (see §2.1.5).

We write ψ for a real primitive auxiliary Dirichlet character of conductor k, and $L_\psi^K(s) = L_\psi(s)L_{\psi\chi}(s)$. The approximant $E_\psi^{\mathbf{P}}(s)$ (for a given set of primes $\mathbf{P}$) is defined in §2.4, while $\zeta_u(s) = \zeta(s)P_u(s)$ with $P_u(s) = \prod_{p|u}(1 - 1/p^s)$. We also have the Mellin transform $I(x)$ as $\int x^{-s}\frac{\Gamma(s)}{s-1/2}\frac{\partial s}{2\pi i}$.

Finally, we write $\mathcal{F}(u,z) = \sum_j 2^j\binom{z}{j}\binom{u}{j}$ as appears with Lemma 2.1.7.

[4]Our citation of Gelfond's 1952 book [18] (which indeed gives the details with Theorem IX on page 38), could be enlarged to also include his 1939 paper [16] and his 1948 paper with Linnik [17]. It seems that this [17] is the "1949" referrent of Stark in [28], as replicated by Baker in [1, p. 681]. It also seems to be cited in [18, p. 10] in reference to the case of three logarithms as having been "shown by Linnik", though there is no non-joint work of Linnik listed in the Literature, and the citation in question inverts the order of authors. Indeed, the second paragraph of [17] notes that the results relevant to the effectivity problem for imaginary quadratic fields are due to Linnik (with those on the approximation of algebraic numbers being due to Gelfond). The student of history may note that in 1948 there were thus two (quite) distinct methods to solve the class number one problem ineffectively (zeros of L-functions, and linear forms in logarithms), and [17] indeed broaches the idea that there may be a single or "universal" (единую) nature of the problems therein. (I thank T. Dokchitser for his help in translating the Russian text).

2. Background with quadratic fields and binary quadratic forms

We show some assorted results in the theory of quadratic forms. Some of this appears *mutatis mutandis* in §3 of [32], with the main difference being that here we have a different Mellin transform in the weighting of an error term.

We let $K = \mathbf{Q}(\sqrt{\Delta})$ be a quadratic field of fundamental discriminant Δ, with χ its quadratic character. We write $D = |\Delta|$ for the absolute value of the discriminant. So as to avoid units in the imaginary quadratic case, we assume that $D > 4$.

The character χ is odd when $\chi(-1) = -1$ and even when $\chi(-1) = +1$, the former corresponding to the imaginary quadratic case and the latter the real quadratic.

2.1. The primitivized Dedekind ζ-function for K is defined as

$$\frac{\zeta_K(s)}{\zeta(2s)} = \frac{\zeta(s)L_\chi(s)}{\zeta(2s)} = \prod_p \frac{1+1/p^s}{1-\chi(p)/p^s} = \sum_{n=1}^{\infty} \frac{R_K^\star(n)}{n^s},$$

where in the imaginary quadratic case $R_K^\star(n)$ counts half the number of primitive (that is, coprime) representations of n by reduced binary quadratic forms of discriminant $-D$. An analogous accounting for $R_K^\star(n)$ holds true in the real quadratic case when one also requires the representations to be primary (see [12, §6 (12)]).

2.1.1. We write (a, b, c) for an integral binary quadratic form $ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = \Delta$, and the operation of composition on the set of such forms yields a group. The class group identifies forms under G-equivalence where G is $\mathbf{SL}_2(\mathbf{Z})$ for K imaginary and $\mathbf{GL}_2(\mathbf{Z})$ for K real, and this class group is indeed isomorphic to the class group of K (cf. Cox [11, p. 128ff]). We write h_K for its size.

2.1.2. In the imaginary quadratic case the class number formula of Dirichlet states that $L_\chi(1) = \pi h_K/\sqrt{D}$ (recall we are assuming $D > 4$ so only ± 1 are units).

In the real quadratic case the fundamental unit $\epsilon_0 = (u + v\sqrt{D})/2$ corresponds to a minimal solution of $u^2 - Dv^2 = \pm 4$, with $u, v > 0$. This fundamental unit thus has norm ± 1, and (since $v \geq 1$) we have the lower bound $\epsilon_0 \geq (\sqrt{D-4} + \sqrt{D})/2$. Here the class number formula states $L_\chi(1) = 2(h_K \log \epsilon_0)/\sqrt{D}$.

2.1.3. Two forms are said to be in the same genus if they are locally equivalent at all primes (including ∞). The number of genera is 2^t where in the imaginary quadratic case t is one less than the number of prime divisors of D, and in the real quadratic case it is again one less, except when the fundamental unit has norm $+1$ when it is two less.[5] The number of genera divides the class number, and indeed the genus class group is coarser than the class group.

2.1.4. A positive definite binary quadratic form is reduced when $-a < b \leq a < c$ or $0 \leq b \leq a = c$, and in the indefinite case when $0 < \sqrt{D} - b < 2|a| < \sqrt{D} + b$. In the former case each equivalence class of forms has exactly one reduced form, and a is its minimum. In the latter case there can be more one than one reduced form in an equivalence class, and we choose (up to sign) a canonical representative in each class by minimizing $|a|$ (and then $|b|$ if necessary), referring to this $|a|$ as the "minimum" of the class. We then write sums over (a, b, c), or (a) as a shorthand, as being over such canonical reduced forms, one per equivalence class.

[5] It is also this case of norm $+1$ when equivalence under $\mathbf{SL}_2(\mathbf{Z})$ differs from $\mathbf{GL}_2(\mathbf{Z})$.

2.1.5. We split $R_K^\star(n) = R_K^{\star\mathrm{s}}(n) + \tilde{R}_K^{\star\mathrm{s}}(n)$, the former supported on $n \le \sqrt{D}/4$ and the latter on $n \ge \sqrt{D}/4$. (This splitting is somewhat arbitrary, though it coincides with Goldfeld's Lemma 4).

We also introduce $R_K^{\star\mathrm{m}}(n)$, which is the number of times that n appears in in the multiset of minima of canonical reduced forms. For its complement we again write $\tilde{R}_K^{\star\mathrm{m}}(n) = R_K^\star(n) - R_K^{\star\mathrm{m}}(n)$. We can note that $\tilde{R}_K^{\star\mathrm{s}}(n)$ is 0 for $n \le \sqrt{D}/4$ and thus (trivially) bounded above by $\tilde{R}_K^{\star\mathrm{m}}(n)$ for such n, while for $n \ge \sqrt{D}/4$ it is equal to (and thus bounded by) $R_K^\star(n) = \tilde{R}_K^{\star\mathrm{m}}(n) + R_K^{\star\mathrm{m}}(n)$.

2.1.6. We give a counting result that slightly enhances Lemma 5.1.1 of [32].

Lemma 2.1.7. *Suppose there are z primes p with $\chi(p) = +1$ and $p \le X$. Then*

$$\frac{1}{4\log 2}\sqrt{D}L_\chi(1) \ge \sum_{n \le \sqrt{D}/4} R_K^\star(n) \ge \sum_{j=0}^{z} 2^j \binom{z}{j}\binom{u}{j} \quad \textit{where} \quad u = \left\lfloor \frac{\log(\sqrt{D}/4)}{\log X} \right\rfloor.$$

Proof. Denoting primes p with $\chi(p) = +1$ by p_i, the product $\prod_i p_i^{e_i}$ is $\le \sqrt{D}/4$ for every nonnegative integral vector $\vec{e}$ that satisfies $\sum_i e_i \le u$, and by the fundamental theorem of arithmetic each such vector gives rise to a unique $n = \prod_i p_i^{e_i}$ with $R_K^\star(n) = 2^{\omega(n)}$ where $\omega(n)$ is the number of distinct prime divisors of n.

We then account as follows. Given a vector $\vec{e}$, let j be the number of coordinates that are nonzero. We then want to distribute v amongst these nonzero coordinates, where $v = \sum e_i - j \le u$ (having already distributed j to ensure these coordinates are indeed nonzero). The number of ways of doing this (across all v) is the multi-choose coefficient $\left(\!\binom{j+1}{u-j}\!\right)$, namely we have $u - j$ to distribute, and can put each one into either any one of the j coordinates, or simply not distribute it at all. This multi-choose coefficient is equal to $\binom{u}{u-j} = \binom{u}{j}$, and by summing over all j (there being $\binom{z}{j}$ ways of choosing the j nonzero coordinates) we then get

$$\sum_{n \le \sqrt{D}/4} R_K^\star(n) \ge \sum_{j=0}^{z} 2^j \binom{z}{j}\binom{u}{j}$$

and the left side is $\le \sqrt{D}L_\chi(1)/(4\log 2)$ by Goldfeld [19, Lemma 4]. □

Corollary 2.1.8. *The number of split primes up to $\sqrt{D}/4$ is $\le \sqrt{D}L_\chi(1)/\log 256$.*

Proof. We apply the Lemma with $X = \sqrt{D}/4$ so $u = 1$, and isolate $j = 1$. □

If we write $\mathcal{F}(u,z) = \sum_j 2^j \binom{z}{j}\binom{u}{j}$ we symmetrically have $\mathcal{F}(u,z) = \mathcal{F}(z,u)$, and indeed these are the Delannoy numbers [2] (or tribonacci triangle, with $\mathcal{F}(0,z) = 1$ and $\mathcal{F}(u,z) = F(u-1,z) + \mathcal{F}(u,z-1) + \mathcal{F}(u-1,z-1)$ for $u,z \ge 1$).

2.2. We next recall Goldfeld's decomposition [19, Theorem 4] of $\zeta_K(s)$ for real quadratic K, namely that

$$\zeta(s)L_\chi(s) = \frac{\Gamma(s)}{\Gamma(s/2)^2}\sum_{(a,b,c)}\sum_{n=1}^{M}\left[\int_{H_{n-1}}^{H_n}\left(\frac{2\zeta(2s)}{(\alpha^\star)^s} + \frac{\pi}{\sqrt{D}}\frac{s}{s-1}\Big(\frac{D}{\alpha^\star}\Big)^{1-s}\right)\frac{\partial\varphi}{\varphi}\right] + Z_{\mathrm{r}}(s)$$

where for $\sigma > 1/2$ we have

$$|Z_{\mathrm{r}}(s)| \le \left|\frac{\Gamma(s)}{\Gamma(s/2)^2}\right| \cdot \frac{4|s|}{\sigma - 1/2}\sum_{(a,b,c)}\sum_{n=1}^{M}\left[\int_{H_{n-1}}^{H_n}\left(\frac{1}{2}\Big(\frac{\alpha^\star}{D}\Big)^\sigma + \frac{(\alpha^\star)^{\sigma-1}}{D^{\sigma-1/2}}\right)\frac{\partial\varphi}{\varphi}\right].$$

The sum over (a,b,c) is over reduced (inequivalent) canonical forms,[6] while M is the least common multiple of 2 and the length k of the primitive period of the continued fraction for $\omega = (-b+\sqrt{D})/2a = [0, \overline{b_1, b_2, \ldots, b_k}]$. Taking $A_v/B_v = [0, b_1, \ldots b_v]$ to be the vth continued fraction convergent, the limits of integration H_n are

$$H_n = \frac{|B_n\omega - A_n|}{|B_n\bar{\omega} - A_n|} \cdot \frac{1}{2}\left[\frac{\sqrt{D}}{|a|}B_n^2 + \sqrt{DB_n^4/a^2 - 4}\right].$$

The variable $\alpha^\star$ depends on φ, and indeed is

$$\alpha^\star = (\varphi + 1/\varphi) \cdot |a| \cdot |B_n\omega - A_n| \cdot |B_n\bar{\omega} - A_n|.$$

We also have $\alpha^\star \leq 5\sqrt{D}$ (see (14) and (20) of [19]).

2.2.1. In fact, we can remove the terms $\zeta(2s)\sum_{(a)} 1/a^s$ from $\zeta_K(s)$, exactly eliminating the $2\zeta(2s)/(\alpha^\star)^s$ in the integral. Indeed, with [19, (11,12)] we see the terms with $n = 0$ are accounted in the proof of [19, Theorem 4] as corresponding therein.

We write $\zeta_K(s)/\zeta(2s) - \sum_{(a)} 1/a^s = \sum_n \tilde{R}_K^{\star\mathrm{m}}(n)/n^s$ so that

$$\zeta(2s)\sum_{n=1}^{\infty}\frac{\tilde{R}_K^{\star\mathrm{m}}(n)}{n^s} = \frac{\Gamma(s)}{\Gamma(s/2)^2}\frac{\pi}{\sqrt{D}}\frac{s}{s-1}\sum_{(a,b,c)}\sum_{n=1}^{M}\left[\int_{H_{n-1}}^{H_n}\left(\frac{D}{\alpha^\star}\right)^{1-s}\frac{\partial\varphi}{\varphi}\right] + Z_{\mathrm{r}}(s). \quad (1)$$

2.2.2. In the imaginary quadratic case we could perhaps work more simply by counting lattice points in ellipses, but can also note Goldfeld's Theorem 3 (following Iseki), and again subtracting off the minima of reduced forms we get

$$\zeta(s)L_\chi(s) - \sum_{(a,b,c)}\frac{\zeta(2s)}{a^s} = \frac{\pi}{\sqrt{D}}\frac{s}{s-1}\sum_{(a,b,c)}\left(\frac{D}{4a}\right)^{1-s} + Z_{\mathrm{i}}(s)$$

where for $\sigma > 1/2$ we have

$$|Z_{\mathrm{i}}(s)| \leq \frac{|s|}{\sigma - 1/2}\sum_{(a,b,c)}\left(1 + \frac{\sqrt{D}}{a}\right)\left(\frac{D}{4a}\right)^{-\sigma}.$$

This error term is given in the last display in Goldfeld's proof of Theorem 3, which he then simplifies via $a \leq \sqrt{D/3}$ when stating the Theorem (in fact, he has a typo and says $a \leq \sqrt{D}/3$).

2.3. For $x > 0$ we define the positive function

$$I(x) = \int_{(2)} x^{-s}\frac{\Gamma(s)}{s-1/2}\frac{\partial s}{2\pi i} = \frac{1}{\sqrt{x}}\int_x^\infty e^{-u}\frac{\partial u}{\sqrt{u}}.$$

We also recall the splitting $R_K^\star(n) = R_K^{\star\mathrm{s}}(n) + \tilde{R}_K^{\star\mathrm{s}}(n)$ at $\sqrt{D}/4$.

Lemma 2.3.1. *For $X > 0$ we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{R}_K^{\star\mathrm{s}}(n)I(m^2n/X) \leq (2 + 3.817 + 0.983)\cdot XL_\chi(1) = 6.8\cdot XL_\chi(1).$$

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[6] Although I'm not sure he ever states it explicitly, Goldfeld appears to be using equivalence under $\mathbf{GL}_2(\mathbf{Z})$; for instance on page 633 he describes the correspondence to ideal classes.

Proof. Rather than work with $\tilde{R}_K^{\star\mathrm{s}}(n)$ it is easier with our above decompositions to consider $\zeta(2s)\sum_n \tilde{R}_K^{\star\mathrm{m}}(n)/n^s = \zeta_K(s) - \sum_{(a)}\zeta(2s)/|a|^s$.

By integrating on the 1-line we have $|I(x)| \le 0.469/x$, so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{R}_K^{\star\mathrm{s}}(n)I(m^2n/X) \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{R}_K^{\star\mathrm{m}}(n)I(m^2n/X) + \sum_{m=1}^{\infty}\sum_{\substack{(a,b,c)\\ |a|\ge\sqrt{D}/4}} I(|a|m^2/X)$$

$$\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{R}_K^{\star\mathrm{m}}(n)I(m^2n/X) + 0.469\zeta(2)h_K\frac{4X}{\sqrt{D}},$$

and by Dirichlet's class number formula $h_K/\sqrt{D} \le L_\chi(1)/\pi$, so the second term here gives the 0.983 contribution in the statement of the Lemma.

We have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{R}_K^{\star\mathrm{m}}(n)I(m^2n/X) = \int_{(2)} X^s\left[\zeta_K(s) - \sum_{(a,b,c)}\frac{\zeta(2s)}{|a|^s}\right]\frac{\Gamma(s)}{s-1/2}\frac{\partial s}{2\pi i}.$$

2.3.2. In the real quadratic case, we replace the bracketed $\zeta(2s)\sum_n \tilde{R}_K^{\star\mathrm{m}}(n)/n^s$ by (1), and have a main term from this decomposition of

$$\frac{\pi}{\sqrt{D}}\int_{(2)} X^s \sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\Big(\frac{D}{\alpha^\star}\Big)^{1-s}\frac{\Gamma(s)}{\Gamma(s/2)^2}\frac{s}{s-1}\frac{\Gamma(s)}{s-1/2}\frac{\partial\varphi}{\varphi}\frac{\partial s}{2\pi i}.$$

We use the duplication formula $\Gamma(s)/\Gamma(s/2) = 2^{s-1}\Gamma(s/2+1/2)/\sqrt{\pi}$ and divide out a factor of $(s/2+1/2)$ from $\Gamma(s/2+1/2) = \Gamma(s/2+3/2)/(s/2+1/2)$ to get

$$\frac{1/4}{\sqrt{D}}\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\int_{(2)}(4X)^s\Big(\frac{D}{\alpha^\star}\Big)^{1-s}\frac{\Gamma(s/2+1/2)}{s-1/2}\frac{\Gamma(s/2+3/2)}{s/2+1/2}\frac{s}{s-1}\frac{\partial s}{2\pi i}\frac{\partial\varphi}{\varphi}.$$

We then expand both Γ-functions to get the main term is

$$\frac{1/4}{\sqrt{D}}\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\frac{D}{\alpha^\star}\times$$

$$\times\int_{(2)}\int_0^\infty u_1^{1/2}\int_0^\infty u_2^{3/2}\Big(4X\sqrt{u_1u_2}\frac{\alpha^\star}{D}\Big)^s\frac{\partial u_1}{e^{u_1}u_1}\frac{\partial u_2}{e^{u_2}u_2}\frac{s/(s/2+1/2)}{(s-1)(s-1/2)}\frac{\partial s}{2\pi i}\frac{\partial\varphi}{\varphi}.$$

We then note that

$$0 \le \int_{(2)} u^s\frac{s/(s/2+1/2)}{(s-1)(s-1/2)}\frac{\partial s}{2\pi i} \le 2u.$$

for all $u > 0$. Indeed, when $u \le 1$ the integral is 0 by moving the contour to the right, while when $u \ge 1$ it is $2u - 4\sqrt{u}/3 - 2/3u$ by moving to the left. Thus this main term is bounded as

$$\le \frac{1/4}{\sqrt{D}}\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\frac{\partial\varphi}{\varphi}\cdot\int_0^\infty u_1^{1/2}\int_0^\infty u_2^{3/2}\big(8X\sqrt{u_1u_2}\big)\frac{\partial u_1}{e^{u_1}u_1}\frac{\partial u_2}{e^{u_2}u_2},$$

and the u-integrals give $\Gamma(1)\Gamma(2) = 1$, while

$$\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\frac{\partial\varphi}{\varphi} = \sum_{(a,b,c)}(2\log\epsilon_0) = 2h_K\log\epsilon_0 = \sqrt{D}L_\chi(1)$$

by Goldfeld's integration formula (middle of page 641) for $\partial\varphi/\varphi$ and Dirichlet's class number formula. This gives the first term (with "2") in the Lemma.

For the secondary term with $Z_{\mathrm{r}}(s)$ we move the integral to $\sigma = 1$ for a bound

$$\leq X\int_{(1)}\left|\frac{\Gamma(s)}{\Gamma(s/2)^2}\frac{\Gamma(s)}{s-1/2}\right|\frac{4|s|}{\sigma-1/2}\frac{|\partial s|}{2\pi}\cdot\sum_{(a,b,c)}\sum_{n=1}^{M}\left[\int_{H_{n-1}}^{H_n}\left(\frac{1}{2}\frac{\alpha^\star}{D}+\frac{1}{\sqrt{D}}\right)\frac{\partial\varphi}{\varphi}\right],$$

and using $\alpha^\star \leq 5\sqrt{D}$ this is

$$\leq 1.802X\cdot(1+\sqrt{5}/2)L_\chi(1) \leq 3.817XL_\chi(1).$$

2.3.3. In the imaginary quadratic case the main term is

$$\frac{\pi}{\sqrt{D}}\int_{(2)}X^s\sum_{(a,b,c)}\left(\frac{D}{4a}\right)^{1-s}\frac{s}{s-1}\frac{\Gamma(s)}{s-1/2}\frac{\partial s}{2\pi i}.$$

We replace $s\Gamma(s) = \Gamma(s+1)$ and expand this to get

$$\frac{\pi}{\sqrt{D}}\sum_{(a,b,c)}\frac{D}{4a}\int_{(2)}\int_0^\infty u\left(Xu\frac{4a}{D}\right)^s\frac{\partial u}{e^u u}\frac{\partial s/2\pi i}{(s-1)(s-1/2)}.$$

We switch the order of integration and note

$$\int_{(2)}\frac{y^s\,\partial s/2\pi i}{(s-1)(s-1/2)} = \begin{cases}2y-2\sqrt{y} & \text{for } y\geq 1,\\ 0 & \text{for } y\leq 1,\end{cases}$$

so that this integral is nonnegative and $\leq 2y$. The main term is thus bounded as

$$\leq \frac{2\pi}{\sqrt{D}}\sum_{(a,b,c)}\int_0^\infty u(Xu)\frac{\partial u}{e^u u} = \frac{2\pi h_K}{\sqrt{D}}X = 2L_\chi(1)X.$$

For the secondary term we move the contour to $\sigma = 1$ and get

$$\leq X\int_{(1)}\frac{|s|}{\sigma-1/2}\left|\frac{\Gamma(s)}{s-1/2}\right|\frac{|\partial s|}{2\pi}\cdot\sum_{(a,b,c)}\left(1+\frac{\sqrt{D}}{a}\right)\left(\frac{4a}{D}\right) \leq 1.154X\cdot\frac{h_K}{\sqrt{D}}\left(\frac{4}{\sqrt{3}}+4\right),$$

and as $1.154\cdot 6.31/\pi \leq 2.32 < 3.817$ this fits into the second term in the Lemma. □

2.4. Let ψ be a real primitive Dirichlet character of odd conductor k, and write $[\psi\chi]$ for the primitive character inducing $\psi\chi$. By comparing Euler products we have

$$\begin{aligned}\frac{L_\psi^K(s)}{\zeta(2s)} &= \prod_{p\nmid k}\frac{1+\psi(p)/p^s}{1-(\psi\chi)(p)/p^s}\prod_{p\mid k}\frac{1-1/p^{2s}}{1-[\psi\chi](p)/p^s}\\ &= \prod_{p\nmid k}\frac{1+\psi(p)/p^s}{1-(\psi\chi)(p)/p^s}\prod_{p\mid g}\left(1+\frac{[\psi\chi](p)}{p^s}\right)\prod_{p\mid(k/g)}\left(1-\frac{1}{p^{2s}}\right) = G(s)P_{k/g}(2s)\end{aligned}$$

where $g = \gcd(k,D)$, while $P_u(s) = \prod_{p\mid u}(1-1/p^s)$ and

$$G(s) = \prod_p G_p(s) = \prod_{p\nmid k}\frac{1+\psi(p)/p^s}{1-(\psi\chi)(p)/p^s}\prod_{p\mid g}\left(1+\frac{[\psi\chi](p)}{p^s}\right) = \sum_{n=1}^\infty\frac{S(n)}{n^s},$$

where we note that $|S(n)| \leq R_K^\star(n)$.

Given a set $\mathbf{P}$ of primes, we define $\hat{\mathbf{P}}$ to be the set containing all $n \le \sqrt{D}/4$ with no prime factor in $\mathbf{P}$ (including $n = 1$), and $E_\psi^{\mathbf{P}}(s)$ and $r_\psi^{\mathbf{P}}(m)$ by

$$E_\psi^{\mathbf{P}}(s) = \prod_{p\in\mathbf{P}} G_p(s) \sum_{n\in\hat{\mathbf{P}}} \frac{S(n)}{n^s}$$

and

$$\frac{L_\psi^K(s)}{\zeta(2s)} = \left(E_\psi^{\mathbf{P}}(s) + \sum_{m=1}^{\infty} \frac{r_\psi^{\mathbf{P}}(m)}{m^s}\right) \prod_{p|(k/g)} \left(1 - \frac{1}{p^{2s}}\right).$$

2.4.1. The crucial point here (for any choice of $\mathbf{P}$) is that $r_\psi^{\mathbf{P}}(m) = 0$ for $m \le \sqrt{D}/4$ and $|r_\psi^{\mathbf{P}}(m)| \le R_K^\star(m)$ in general, so that $|r_\psi^{\mathbf{P}}(m)| \le \tilde{R}_K^{\star s}(m)$. Indeed, we have

$$\sum_{m=1}^{\infty} \frac{r_\psi^{\mathbf{P}}(m)}{m^s} = \prod_{p\in\mathbf{P}} G_p(s) \cdot \left[\prod_{p\notin\mathbf{P}} G_p(s) - \sum_{n\in\hat{\mathbf{P}}} \frac{S(n)}{n^s}\right],$$

where in the bracketed term the sum over n is the series truncation to $n \le \sqrt{D}/4$ of the preceding Euler product, and upon calling this bracketed term $\sum_l b(l)/l^s$, we thus have $b(l) = 0$ for $l \le \sqrt{D}/4$ while $|b(l)| \le |S(l)| \le R_K^\star(l)$ in general, whereupon multiplication by the product over $p \in \mathbf{P}$ implies the desired $|r_\psi^{\mathbf{P}}(m)| \le R_K^\star(m)$.

3. Using auxiliary moduli to bound $L_\chi(1)$ for mid-sized discriminants

We first show a lower bound on $L_\chi(1)$ for D with $10^3 \le \log D \le 10^8$, namely that $L_\chi(1) \ge 100 \log D$ in this range. As noted in the Introduction, this is largely a matter of generating enough Dirichlet L-functions that possess an abnormally small lowest height zero. The theory is rather analogous to §5 of [30]. (One could also phrase things in terms of a Deuring decomposition as with [32] but we choose simply to work at a known zero of $L_\psi(s)$).

The the upper bound on D here is a type of "crossover" (if one ignores $p|D$) with the $(\log D)^3/10^{13}$ in Theorem 1.1.1 (though with 14, from before it was improved to 13). The lower bound on D is simply a point that is convenient numerically.

3.1. We copy over Lemma 7 from [30], and its attendant upper bound.

Lemma 3.1.1. *Suppose $\xi \ge 0$ and $x > 0$. Then*

$$\left|\int_{(2)} x^{-s}\Gamma(s) \frac{s - 1/2}{(s - 1/2)^2 + \xi^2} \frac{\partial s}{2\pi i}\right| = \left|\int_x^\infty e^{-u} \sqrt{\frac{u}{x}} \cos\left(\xi \log \frac{u}{x}\right) \partial u\right|$$
$$\le \int_x^\infty e^{-u} \sqrt{\frac{u}{x}}\, \partial u = \int_{(2)} x^{-s} \frac{\Gamma(s)}{s - 1/2} \frac{\partial s}{2\pi i} = I(x).$$

As noted in the proof in [30], there is a minor issue with convergence but otherwise this follows by unravelling the Γ-function and swapping the integration order.

3.2. We let ψ be an auxiliary primitive real[7] Dirichlet character of odd conductor k with $\psi(-1) = -1$, and put $g = \gcd(k, D)$. We write $1/2 + i\xi_0$ for a low-height zero of $L_\psi(s)$ on the half-line.

[7] It is possible to use nonreal characters, but using them seems unnecessarily complicated. (Similarly, we fix the Γ-factor by only using odd characters). It might be noted, however, that Biró's improvement over a precursor method of Beck was largely due to using nonquadratic characters.

We define $\zeta_u(s) = \zeta(s)\prod_{p|u}(1-1/p^s)$ and

$$T_\psi^{\mathbf{P}}(s) = \Gamma(s)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{s-1/2}\zeta_{k/g}(2s)E_\psi^{\mathbf{P}}(s),$$

with $E_\psi^{\mathbf{P}}(s)$ as in §2.4, where $\mathbf{P}$ is an arbitrary set of primes.

Lemma 3.2.1. *We have* $M_1+M_2+M_3=0$ *where*

$$M_1 = 2|T_\psi^{\mathbf{P}}(1/2+i\xi_0)|\sin\bigl(\arg iT_\psi^{\mathbf{P}}(1/2+i\xi_0)\bigr),$$

$$|M_2| \le 13.6L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2},$$

and

$$|M_3| \le 3.322\cdot\tilde{E}_\psi^{\mathbf{P}}(1/4)\Bigl(\frac{2\pi g}{k\sqrt{D}}\Bigr)^{1/4}\prod_{p|(k/g)}\Bigl(1+\frac{1}{\sqrt{p}}\Bigr),$$

where $\tilde{E}_\psi^{\mathbf{P}}(1/4)$ *is a bound for* $E_\psi^{\mathbf{P}}$ *on the 1/4-line.*

Proof. We consider the integral

$$\begin{aligned}0 &= \Bigl(\int_{(2)} - \int_{(-1)}\Bigr)\Gamma(s)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{s-1/2}L_\psi(s)L_{\psi\chi}(s)\frac{(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i}\\ &= 2\int_{(2)}\Gamma(s)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{s-1/2}L_\psi^K(s)\frac{(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i},\end{aligned}$$

with the first step by Cauchy's integral theorem since the integrand is entire, and the second step following by the symmetry of the functional equation for $L_\psi^K(s)$ (here we use that ψ is odd to obtain the stated Γ-factor, and by convention $L_{\psi\chi}$ is the L-function of the primitive inducing character).

We then use the notation of §2.4 to replace

$$L_\psi^K(s) = \zeta(2s)\Biggl(E_\psi^{\mathbf{P}}(s) + \sum_{n=1}^{\infty}\frac{r_\psi^{\mathbf{P}}(n)}{n^s}\Biggr)\prod_{p|(k/g)}\Bigl(1-\frac{1}{p^{2s}}\Bigr).$$

We then move the contour to the left with $E_\psi^{\mathbf{P}}(s)$ for the main term, while bounding the contribution from the sum with $r_\psi^{\mathbf{P}}(n)$ by Mellin transforms.

3.2.2. The term induced from residues at $s=1/2\pm i\xi_0$ when moving the contour to the left is

$$\begin{aligned}&2\frac{i\xi_0}{2i\xi_0}\Gamma(1/2+i\xi_0)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{i\xi_0}\zeta_{k/g}(1+2i\xi_0)E_\psi^{\mathbf{P}}(1/2+i\xi_0)\\ &\qquad + 2\frac{-i\xi_0}{-2i\xi_0}\Gamma(1/2-i\xi_0)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{-i\xi_0}\zeta_{k/g}(1-2i\xi_0)E_\psi^{\mathbf{P}}(1/2-i\xi_0),\end{aligned}$$

which is $T_\psi^{\mathbf{P}}(1/2+i\xi_0)+T_\psi^{\mathbf{P}}(1/2-i\xi_0)$, and by symmetry this is then the real part of $2T_\psi^{\mathbf{P}}(1/2+i\xi_0)$. Going further, it is thus its absolute value times the cosine of its argument. Multiplying by i changes the cosine to the sine, and so becomes

$$2|T_\psi^{\mathbf{P}}(1/2+i\xi_0)|\sin\bigl(\arg iT_\psi(1/2+i\xi_0)\bigr).$$

3.2.3. The error term coming from $r_\psi^{\mathbf{P}}(n)$ is

$$2\int_{(2)}\Gamma(s)\zeta(2s)\prod_{p|(k/g)}\Bigl(1-\frac{1}{p^{2s}}\Bigr)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{s-1/2}\sum_{n=1}^{\infty}\frac{r_\psi^{\mathbf{P}}(n)}{n^s}\frac{(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i}$$

and by expanding $\zeta(2s)$ this is

$$2\sqrt{\frac{2\pi g}{k\sqrt{D}}}\int_{(2)}\sum_{\substack{m=1\\(m,k/g)=1}}^{\infty}\frac{1}{m^{2s}}\sum_{n=1}^{\infty}\frac{r_\psi^{\mathbf{P}}(n)}{n^s}\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^s\frac{\Gamma(s)\,(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i}.$$

We then recall that $|r_\psi^{\mathbf{P}}(n)|\le\tilde{R}_K^{\star\mathrm{s}}(n)$ from §2.4.1 and write this in terms of Mellin transforms, using Lemmata 3.1.1 and 2.3.1 to get it is

$$\begin{aligned}&\le 2\sqrt{\frac{2\pi g}{k\sqrt{D}}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\tilde{R}_K^{\star\mathrm{s}}(n)|I\Bigl(nm^2\frac{2\pi g}{k\sqrt{D}}\Bigr)\\&\le 2\sqrt{\frac{2\pi g}{k\sqrt{D}}}\cdot\Bigl(6.8L_\chi(1)\frac{k\sqrt{D}}{2\pi g}\Bigr)=13.6\cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2}.\end{aligned}$$

3.2.4. Finally, the error term from the complementary integral with $E_\psi^{\mathbf{P}}(s)$ is

$$2\int_{(1/4)}\Gamma(s)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{s-1/2}E_\psi^{\mathbf{P}}(s)\zeta(2s)P_{k/g}(2s)\frac{(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i},$$

and since

$$2\int_{(1/4)}\biggl|\frac{\Gamma(s)\zeta(2s)(s-1/2)}{(s-1/2)^2+\xi_0^2}\biggr|\frac{|\partial s|}{2\pi}\le 2\int_{(1/4)}\biggl|\frac{\Gamma(s)\zeta(2s)}{(s-1/2)}\biggr|\frac{|\partial s|}{2\pi}\le 3.322,$$

the stated bound for M_3 readily follows.[8] □

We can expand $2|T_\psi^{\mathbf{P}}(1/2+i\xi_0)|\sin\bigl(\arg iT_\psi(1/2+i\xi_0)\bigr)$ in the notation of [30] as

$$\xi_3|E_\psi^{\mathbf{P}}(1/2+i\xi_0)|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathbf{P}}(1/2+i\xi_0)\bigr]$$

where $\xi_3=2|\Gamma(1/2+i\xi_0)\zeta_{k/g}(1+2i\xi_0)|$ and

$$\xi_2=\xi_0\log\frac{k/g}{2\pi}+\arg\bigl[i\Gamma(1/2+i\xi_0)\zeta_{k/g}(1+2i\xi_0)\bigr].\qquad(2)$$

In particular, since $M_1+M_2+M_3=0$ this gives

$$\begin{aligned}&\xi_3|E_\psi^{\mathbf{P}}(1/2+i\xi_0)|\cdot\Bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathbf{P}}(1/2+i\xi_0)\bigr]\Bigr|\\&\quad\le 13.6\cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2}+3.322\cdot\tilde{E}_\psi^{\mathbf{P}}(1/4)\Bigl(\frac{2\pi g}{k\sqrt{D}}\Bigr)^{1/4}\cdot\prod_{p|(k/g)}\Bigl(1+\frac{1}{\sqrt{p}}\Bigr).\qquad(3)\end{aligned}$$

When $L_\chi(1)$ is small, say $\ll(\log D)/\sqrt{D}$, the second term will dominate, with its $1/D^{1/8}$ behavior. We can improve this by moving the contour for M_3 further to the left, but that then entails switching to a sum-based form of $E_\psi^{\mathbf{P}}$, as the Euler factors from split primes have poles on the 0-line. We revisit this in §4.3.

[8] In [30, (23)] I forgot to include the square root with the fraction involving $t^2+1/16$, and so the "11" derived there essentially used $(s-1/2)^2$ as the denominator, rather than our $(s-1/2)$.

3.3. We are now set to imitate §6 of [30], particularly the end of that section from (25) onwards. Our goal is the following result.

Proposition 3.3.1. *We have* $\sqrt{D}L_\chi(1) \geq 100\log D$ *when* $10^3 \leq \log D \leq 10^8$.

3.3.2. We let E_ψ^+ be $E_\psi^{\mathbf{P}}$ with the choice (§2.4) of $\mathbf{P}$ being all primes up to $\sqrt{D}/4$. We will show a Lemma that adequately bounds E_ψ^+. For convenience, we define

$$Y_{\rm r}(n) = 1 + 1/n^{1/4},\ \ Y_{\rm s}(n) = \frac{1+1/n^{1/4}}{1-1/n^{1/4}},$$

$$V_{\rm r}(n) = 1 - 1/\sqrt{n},\ \ V_{\rm s}(n) = \frac{1-1/\sqrt{n}}{1+1/\sqrt{n}},\ \text{ and } W(n) = \frac{\log n}{\sqrt{n}-1}.$$

Lemma 3.3.3. *Suppose that* $\sqrt{D}L_\chi(1) \leq 100\log D$*, and also* $10^3 \leq \log D \leq 10^8$*. Then we have the bounds of* $|E_\psi^+(1/2+it)| \geq 5\cdot 10^{-7}$ *and* $|E_\psi^+(1/4+it)| \leq 2\cdot 10^{13}$*, while* $|\arg E_\psi^+(1/2+it)| \leq 45.373|t|$.

Proof. We first note that if there are 37 primes dividing D, then by genus theory the class number is divisible by 2^{35}, and thus is at least this large. The class number formula then implies $\sqrt{D}L_\chi(1) \geq 2^{35}/\pi > 10^{10} \geq 100\log D$ in our range of D, contradicting our assumption. Thus there are at most 36 primes dividing D.

With Lemma 2.1.7, taking $X = 356812$ we have $u = \lfloor \log(\sqrt{D}/4)/\log X\rfloor \geq 39$ for our range of $D \geq \exp(1000)$, and then with $z = 7$ we see the Lemma implies that $\sqrt{D}L_\chi(1) \geq \mathcal{F}(39,7)\log(16) \geq 1.07\cdot 10^{10} > 100\log D$. Thus there are at most 6 split primes up to 356812. Repeating the argument with $X = 10^{43}$ has $u \geq 5$, and $z = 106$ implies $\sqrt{D}L_\chi(1) \geq \mathcal{F}(5,106)\log(16) \geq 1.01\cdot 10^{10} > 100\log D$.

Thus, somewhat crudely, under our assumptions on D and $L_\chi(1)$ we have

$$|E_\psi^+(1/2+it)| \geq \prod_{p\leq 151} V_{\rm r}(p)\prod_{p\leq 13} V_{\rm s}(p)\cdot V_{\rm s}(356812)^{105}\cdot V_{\rm s}(10^{43})^{10^{10}/2\log(16)} \geq 5\cdot 10^{-7},$$

and

$$\tilde{E}_\psi^+(1/4) \leq \prod_{p\leq 151} Y_{\rm r}(p)\prod_{p\leq 13} Y_{\rm s}(p)\cdot Y_{\rm s}(356812)^{105}\cdot Y_{\rm s}(10^{43})^{10^{10}/2\log(16)} \leq 3\cdot 10^{13}.$$

Meanwhile, we have

$$\begin{aligned}|\arg E_\psi^+(1/2+it)| &\leq |t|\max_u\left|\frac{(E_\psi^+)'}{E_\psi^+}(1/2+iu)\right| \\ &\leq |t|\Big(\sum_{p\leq 151} W(p) + 2\sum_{p\leq 13} W(p) + 210\cdot W(356812) + \frac{10^{10}}{\log 16}W(10^{43})\Big) \\ &\leq |t|(25.513 + 15.357 + 4.503).\end{aligned}$$

Of course one can make various improvements, but these shall suffice for now. □

Lemma 3.3.4. *Suppose that* $\sqrt{D}L_\chi(1) \leq 100\log D$*, and also* $10^3 \leq \log D \leq 10^8$*. Let* ψ *be a real primitive odd auxiliary Dirichlet character of odd conductor* k *with* $2^{24} \leq k \leq 2^{32}$ *and* $\gcd(k,D) = 1$*, with its lowest height zero* $1/2+i\xi_0$ *satisfying* $\xi_0 \leq 0.00225$*, with* ξ_2 *defined as in* (2)*. Then* $\xi_0\log\sqrt{D}+\xi_2$ *is within 0.103 of a multiple of* π.

Proof. This essentially follows from (3) and the previous Lemma. Indeed, under our assumptions we have

$$13.6 \cdot L_\chi(1)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{1/2} \le 5.43 \cdot 2^{16}\frac{100\log D}{D^{1/4}} \le 10^{-98}$$

and

$$3.322 \cdot \tilde{E}_\psi^+(1/4)\Big(\frac{2\pi g}{k\sqrt{D}}\Big)^{1/4} \cdot \prod_{p|(k/g)}\Big(1+\frac{1}{\sqrt{p}}\Big) \le \frac{1.58\cdot 10^{14}}{D^{1/8}} \le 10^{-40}.$$

Thus we have (using the crude $\xi_3 \ge 1$)

$$\Big|\sin\big[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\big]\Big| \le \frac{10^{-39}}{\xi_3|E_\psi^+(1/2+i\xi_0)|} \le 10^{-31},$$

where

$$\xi_2 = \xi_0\log\frac{k}{2\pi} + \arg\big[i\Gamma(1/2+i\xi_0)\zeta_k(1+2i\xi_0)\big].$$

Lemma 3.3.3 also says $|\arg E_\psi^+(1/2+it)| \le (45.373)(0.00225) \le 0.103 - 10^{-30}$, which then implies that $\xi_0\log\sqrt{D}+\xi_2$ is within 0.103 of a multiple of π. □

One can thus note that each such auxiliary character misses approximately 1/15 of the D values (on a logarithmic scale), and this fraction is dependent on the height of the lowest zero multiplied by the bound on the logarithmic derivative of $E_\psi^{\mathbf{P}}$.

3.3.5. We complete the proof that $\sqrt{D}L_\chi(1) \ge 100\log D$ for $10^3 \le \log D \le 10^8$.

Proof. (Proposition 3.3.1). We list in Tables 3 and 4 a selection of 60 auxiliary real primitive characters, all of them with prime conductor (for convenience). These Tables list a "miss period" $p = 2\pi/\xi_0$ and a "shift" interval $[l,h]$ for each, with the point being that when $(\log D) \notin [bp+l, bp+h]$ for any integer b, then $\sqrt{D}L_\chi(1)$ is large by the above Lemma, assuming $\gcd(k,D)=1$ (and $10^3 \le \log D \le 10^8$).

We then routinely verify that for any D with $10^3 \le \log D \le 10^8$ there are at most 23 auxiliary characters[9] which miss D. Thus every D is hit by at least 37 of them, so either (at least) one of these 37 is coprime to D when Lemma 3.3.4 suffices, or D is divisible by 37 primes, when the result follows by genus theory. □

Remark. The main features of these moduli are that they are prime (for convenience), not too large, and have a zero of relatively small height.

With the class number 100 problem for imaginary quadratic fields, we generated the list in [30, Table 1] from difficult examples that were found with an earlier investigation (concerning real zeros of real odd[10] Dirichlet L-functions [29]). These used a "Low score" [23] given by $\sum_a \log(8\pi a/e^\gamma\sqrt{k})$ summed over minima for $\mathbf{Q}(\sqrt{-k})$, whose small values are weakly correlated to $L_\chi(1/2)$, whose smallness is in turn correlated to a low-height zero. While it perhaps might be more trenchant to compute $L_\chi(1/2)$ directly as a surrogate, for Tables 3 and 4 we generated Low scores for

[9] The worst case is for $\log D \approx 90290383$, when there are 19 that miss.

[10] Chua [10] has a version of Low's method for the even (real quadratic) case, albeit it takes time linear in k, so for our purposes approximating $L_\chi(1/2)$ in time $O(\sqrt{k}\log k)$ would be superior.

k	ξ_0	$p = 2\pi/\xi_0$	$[l, h]$	$2\pi/\xi_0 \log k$
2798913571	0.0020159837308	3116.684530266	$[-141, 64]$	143.28
275971211	0.0022262707952	2822.291574196	$[-127, 59]$	145.21
2517922283	0.0019927337950	3153.048000104	$[-142, 66]$	145.66
3631268243	0.0019573945201	3209.973892727	$[-144, 67]$	145.82
3985600643	0.0019423166624	3234.892347205	$[-145, 68]$	146.34
1020839059	0.0020654615960	3042.024755763	$[-136, 64]$	146.65
2440122943	0.0019755475206	3180.477939203	$[-143, 67]$	147.14
162173551	0.0022455507400	2798.059823538	$[-125, 60]$	148.01
428171663	0.0021335173353	2944.989104767	$[-132, 63]$	148.18
1166402099	0.0020092504342	3127.129003042	$[-139, 67]$	149.79
921190087	0.0020296897410	3095.638303892	$[-138, 66]$	149.97
1983309763	0.0019530030558	3217.191744014	$[-144, 68]$	150.28
2045178127	0.0018840103617	3335.005706363	$[-147, 72]$	155.56
3117865243	0.0018420866501	3410.906488477	$[-151, 74]$	156.03
3853296127	0.0018155165579	3460.825118719	$[-153, 75]$	156.80
3057192779	0.0018184878465	3455.170360006	$[-152, 75]$	158.20
171459523	0.0020725257522	3031.656084643	$[-133, 67]$	159.90
3334368203	0.0017792427678	3531.381675836	$[-155, 78]$	161.05
3373766047	0.0017744236857	3540.972405743	$[-155, 78]$	161.40
2297320183	0.0018049859833	3481.016121660	$[-152, 77]$	161.49
2534311019	0.0017925518774	3505.162325566	$[-153, 77]$	161.88
754366559	0.0018955224975	3314.751112449	$[-145, 74]$	162.16
3088483259	0.0017397870859	3611.467953745	$[-157, 80]$	165.28
478282543	0.0018954710810	3314.841028219	$[-144, 75]$	165.86
1176200099	0.0018134452223	3464.778108469	$[-151, 78]$	165.89
1819421063	0.0017755545685	3538.717096501	$[-154, 79]$	165.97
1796138467	0.0017747751706	3540.271134703	$[-154, 79]$	166.14
1038929107	0.0017477665191	3594.979786178	$[-155, 82]$	173.16
2231381419	0.0016649086342	3773.891959061	$[-162, 86]$	175.32
1927312571	0.0016182273277	3882.758126501	$[-165, 90]$	181.61

TABLE 3. 30 auxiliary characters with low-height zeros

all $k \le 2^{32}$ in about 2 days using 71 threads on a E5-2699.[11] We then calculated the lowest-height zero for those with prime conductor and a negative Low score,[12] and for the Table took the smallest 60 resulting zeros (relative to the expected $2\pi/\log k$, where we record the ratio therein in the fifth column of the table).

The zeros themselves were approximated using GP/PARI and Weinberger's method [34]. The Table data are listed to sufficient precision for the Proposition.

There is also some sense we should avoid characters for which the zeros are *too* close to each other. For instance, if two ξ_0 are within 0.00001 of each other, then they will track each other rather closely for $\log \sqrt{D} \le 10^4$ (say), in particular having their miss ranges likely overlapping therein. However, this is not a worry if

[11]This limit has nothing to do with 32-bit arithmetic, but rather was a convenient demarcation point time-wise. A more relevant limit for our code would be $3 \cdot 821641^2 \approx 2 \cdot 10^{12}$, where 821641 is the 65536th prime (where our factor table entries would need to increase beyond 2 bytes each).

[12]My recollection is that Low scores are expected to become generically negative as $k \to \infty$, and indeed false positives dominated, mostly finding zeros only about 20 times smaller than expected.

k	ξ_0	$p = 2\pi/\xi_0$	$[l, h]$	$2\pi/\xi_0 \log k$
1433103107	0.0016300436112	3854.611780831	$[-164, 90]$	182.83
2031817663	0.0016017153799	3922.785150249	$[-167, 92]$	183.03
306079643	0.0017229424570	3646.776061335	$[-154, 86]$	186.64
3311502587	0.0015031181550	4180.100736790	$[-176, 99]$	190.69
4180567099	0.0014694732560	4275.807866085	$[-180, 102]$	193.01
2115632171	0.0015005148074	4187.353084586	$[-175, 100]$	195.01
1295145091	0.0015275253107	4113.310112187	$[-172, 99]$	196.04
2534772307	0.0014603339974	4302.567301978	$[-180, 104]$	198.70
1816676227	0.0014239803531	4412.410110538	$[-183, 108]$	206.96
649059211	0.0014651912369	4288.303908111	$[-176, 106]$	211.34
325757807	0.0014883725980	4221.513696044	$[-173, 105]$	215.37
3846105671	0.0013115357307	4790.708449754	$[-196, 119]$	217.07
823946743	0.0013826133868	4544.426784071	$[-185, 114]$	221.36
4238508763	0.0012770498657	4920.078280441	$[-201, 123]$	221.95
1821908299	0.0013118861223	4789.428899602	$[-195, 120]$	224.61
4112591003	0.0012621636157	4978.106823052	$[-203, 125]$	224.87
1922096731	0.0013059026683	4811.373358421	$[-196, 121]$	225.08
3025845331	0.0012740560429	4931.639657700	$[-201, 124]$	225.91
2067321943	0.0012283784751	5115.023939573	$[-206, 131]$	238.47
2168243683	0.0012039049287	5219.004555611	$[-209, 134]$	242.78
3281873687	0.0011104240748	5658.365528584	$[-225, 147]$	258.24
1409931443	0.0011370089969	5526.064722502	$[-219, 145]$	262.31
648760127	0.0011433888013	5495.230756376	$[-216, 145]$	270.83
3697896911	0.0010004324567	6280.469276116	$[-245, 168]$	285.07
1386038791	0.0009748661671	6445.177316702	$[-249, 175]$	306.19
4244770963	0.0009141780988	6873.042917593	$[-265, 187]$	310.03
2473259119	0.0009111772528	6895.678406981	$[-265, 189]$	318.82
3497154523	0.0008466775057	7420.990005328	$[-282, 205]$	337.70
1526873947	0.0008089056296	7767.513387796	$[-292, 218]$	367.32
800703083	0.0007648066394	8215.390640349	$[-306, 234]$	400.73

Table 4. 30 more auxiliary characters with low-height zeros

we ensure (e.g.) that there are not 23 auxiliary characters all with such zeros close to each other.

Undoubtedly the type of result we show here can be improved in various ways, but ultimately one needs an upper bound (here $\log D \le 10^8$) to finitize the problem.

4. Handling smaller-sized discriminants in our special cases

The above Proposition 3.3.1 reduces our considerations to $\log D \le 1000$. For these smaller D, we find it convenient to use additional information available in our situations of real quadratic fields with a small fundamental unit. We could work somewhat more generally (with Mollin's Lemma below), but largely proceed to our cases of interest.

4.1. As our main application of the methods of this paper, we shall show a class number result for the family of real quadratic fields having $D = 4u^2 + 1$. This was considered by Chowla, though it is not much more difficult to consider in parallel $D = u^2 + 4$ with u odd (Yokoi's family), and also $D = 4(u^2 + 1)$ with u odd. Assumming D is a fundamental discrminant, Chowla's family has fundamental unit $2u + \sqrt{D}$, while Yokoi's has $u + \sqrt{D}$, and for $D = 4(u^2 + 1)$ it is $u + \sqrt{D}/2$ (all have norm -1).

We shall also consider a family already known to Euler, which can be defined by $D = (3u+1)^2 + (4u+1)^2 = 25u^2 + 14u + 2 = (5u + 7/5)^2 + 1/25$ when u is odd, and 4 times this (to make D fundamental) when u is even. When $25u^2 + 14u + 2$ is squarefree, in the case where u is odd this has fundamental unit $(25u + 7) + 5\sqrt{D}$, and for u even the unit is $(25u + 7) + 5\sqrt{D}/2$ (both have norm -1).

4.1.1. For the convenience of the reader, we copy here [25, Lemma 1.1], and its corollary that we generalize from [26, Lemma 1]. As noted by Mollin, this is due to Davenport, Ankeny, and Hasse, with Yokoi providing the case where t is a square.

An initial version of this was shown[13] by Chowla and Friedlander [9, §4 (C)], namely that when $\mathbf{Q}(\sqrt{4u^2+1})$ has class number 1, we have $\chi(p) = -1$ for $p < u$.

We follow Mollin in defining a solution to $x^2 - Dy^2 = \pm 4t$ (with $t > 0$) to be *trivial* when $t = m^2$ with $m \mid \gcd(x, y)$, so that solutions with $t = 1$ are trivial. Other than this, primitive representations by the principal form give rise to nontrivial solutions.

Lemma 4.1.2. [25, Lemma 1.1]. *Let $D > 0$ be fundamental, with $(A + B\sqrt{D})/2$ the fundamental unit of $\mathbf{Q}(\sqrt{D})$, so that $A^2 - DB^2 = 4\delta$ with $\delta = \pm 1$.*

If there exists a nontrivial solution to $x^2 - Dy^2 = \pm 4t$, then $t \geq (A - \delta - 1)/B^2$.

Note that Mollin's phrasing takes D to be squarefree, while ours takes it to be a fundamental discriminant, and thereby in his notation we have $\sigma = 2$ in all cases.

Proof. Let (u, v) be a nontrivial solution to $x^2 - Dy^2 = \pm 4t$ with $u \geq 0$ and $v > 0$ minimal. Multiplying by the fundamental unit we have

$$\pm 4t\delta = (u^2 - Dv^2)(A^2 - DB^2)/4 = (uA - DvB)^2/4 - D(uB - vA)^2/4,$$

where $\big((uA - DvB)/2, (uB - vA)/2\big)$ can be seen to provide a nontrivial solution to $x^2 - Dy^2 = \pm 4$, so that by minimality we have $|uB - vA| \geq 2v$.

Breaking into cases with u, we find that either $u \geq v(A + 2)/B$ whence

$$4t = u^2 - Dv^2 \geq (v^2(A+2)^2/B^2) - Dv^2 = 4v^2(\delta + A + 1)/B^2 \geq 4(\delta + A + 1)/B^2,$$

or $u \leq v(A - 2)/B$ when $-4t$ is

$$u^2 - Dv^2 \leq (v^2(A-2)^2/B^2) - Dv^2 = v^2(A^2 - 4A + 4 - DB^2)/B^2 \leq 4(1 + \delta - A)/B^2,$$

and these give the stated bound on t. □

Corollary 4.1.3. *Suppose that $\chi(p) = +1$. Then $p^{h_K} \geq (A - \delta - 1)/B^2$.*

Proof. This follows because $\chi(p) = +1$ implies there is a prime ideal $\mathfrak{p}$ of norm p, and then $\mathfrak{p}^{h_K}$ is principal, giving a nontrivial solution to $x^2 - Dy^2 = \pm 4\delta p^{h_K}$. □

One can note that this result is only really useful when B is quite small. Indeed, we have $A \approx B\sqrt{D}$, so that $A/B^2 \approx \sqrt{D}/B$, while generically B is of size $\exp(\sqrt{D})$.

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[13] They actually thank the referee for improving their initial version of this. We might also mention (as is noted by Mollin) that in this case of the Chowla conjecture it is easy to show that u must be non-composite for $\mathbf{Q}(\sqrt{4u^2+1})$ to have class number 1. Indeed, for a prime $p \mid u$ we have $4u^2 + 1 \equiv 1 \pmod{4p}$, so that by quadratic reciprocity we have $\chi(4p) = \chi(p) = +1$, and if $p < u$ this contradicts the result of Chowla and Friedlander.

4.1.4. In all our cases we have $\delta=-1$, and Table 5 lists the relevant A/B^2. Its smallest value $\sqrt{D/10^2-1/50^2}$ is in case 4. We also have $\epsilon_0\le 10\sqrt{D}$ for our cases.

D	u	ϵ_0	A/B^2
$4u^2+1$	all	$2u+\sqrt{D}$	$4u/2^2$
u^2+4	odd	$(u+\sqrt{D})/2$	$u/1^2$
$4(u^2+1)$	odd	$u+\sqrt{D}/2$	$2u/1^2$
$25u^2+14u+2$	odd	$(25u+7)+5\sqrt{D}$	$(50u+14)/10^2$
$4(25u^2+14u+2)$	even	$(25u+7)+5\sqrt{D}/2$	$(50u+14)/5^2$

Table 5. The five cases with small regulator that we consider

4.2. We now handle the range $10^{28}\le D\le \exp(10^3)$ in our five special cases, which reduces each to a feasible sieving problem then considered in §5.

4.2.1. We split the D-range at $\exp(100)$, and for the upper range we can be somewhat lazy in bounding the various error terms. Here we use the auxiliary character of conductor 12461947.

For the lower range we make a more concerted effort with the error term, indeed using a different set $\mathbf{P}$ in our choice of $E_\psi^{\mathbf{P}}$. Here we use the character of conductor 17923, whose smallness is convenient.

We state both results together, though the proofs take somewhat different paths.

Lemma 4.2.2. *Suppose that D is fundamental with $100\le\log D\le 1000$, and we are in one of the following cases.*

(1) $D=4u^2+1$ *with* $u>0$,
(2) $D=u^2+4$ *with u odd and* $u>0$,
(3) $D=4(u^2+1)$ *with u odd and* $u>0$,
(4) $D=25u^2+14u+2$ *with u odd,*
(5) $D=4(25u^2+14u+2)$ *with u even.*

Then $h_K\ge 6$.

Lemma 4.2.3. *With D as in Lemma 4.2.2 but $10^{28}\le D\le e^{100}$ we have $h_K\ge 6$.*

Proof. (Lemma 4.2.2). We assume that $h_K\le 5$ and show a contradiction. In all five cases the fundamental unit has norm -1, and thus the theory of genera (§2.1.3) implies that D has at most 3 prime divisors (otherwise 8 divides h_K). The largest of these must be at least $D^{1/3}$.

Also, no D under consideration has a prime divisor that is 3 mod 4.

4.2.4. By using Mollin's result (Corollary 4.1.3), when $h_K\le 5$ we see that any split prime must be $\ge (D/10^2-1/50^2)^{1/10}\ge 13897$. In the notation of Lemma 2.1.8 we have

$$\mathcal{F}(u,z)=\sum_{j=0}^{z}2^j\binom{z}{j}\binom{u}{j}\le\sum_n R_K^{\star s}(n)\le\frac{\sqrt{D}L_\chi(1)}{\log(16)},$$

where $u=\lfloor\log(\sqrt{D}/4)/\log X\rfloor$ and there are z split primes up to X.

Since we have $\epsilon_0\le 10\sqrt{D}$ in our cases, Dirichlet's class number formula then implies $\sqrt{D}L_\chi(1)=2h_K\log\epsilon_0\le 10\log(10\sqrt{D})$, so that $\sqrt{D}L_\chi(1)/\log(16)\le 1812$. Since $\mathcal{F}(4,7)=2241$ there are at most 6 split primes up to $(\sqrt{D}/4)^{1/4}\ge 189743$, while $\mathcal{F}(3,11) = 2047$ implies at most 10 split primes up to $(\sqrt{D}/4)^{1/3} \geq 10^7$, and $\mathcal{F}(2,30) = 1861$ implies at most 29 split primes up to $(\sqrt{D}/4)^{1/2} \geq 10^{10}$.

In the notation of §3.3.2, this gives us that

$$|E_\psi^+(1/2+it)| \geq V_{\rm r}(2)V_{\rm r}(5)V_{\rm r}(e^{33})V_{\rm s}(13897)^6V_{\rm s}(189743)^4V_{\rm s}(10^7)^{19}V_{\rm s}(10^{10})^{877}$$

so that $|E_\psi^+(1/2+it)| \geq 0.139$, while

$$|\tilde{E}_\psi^+(1/4)| \leq Y_{\rm r}(2)Y_{\rm r}(5)Y_{\rm r}(e^{33})Y_{\rm s}(13897)^6Y_{\rm s}(189743)^4Y_{\rm s}(10^7)^{19}Y_{\rm s}(10^{10})^{877} \leq 6885$$

and $|\arg E_\psi^+(1/2+it)|$ is bounded as

$$\leq |t|\bigl(W(2)+W(5)+W(e^{33})+ \\ +12W(13897)+8W(189743)+38W(10^7)+1754W(10^{10})\bigr) \leq 4.777|t|.$$

(We could improve these slightly by noting subcases when $2 \nmid D$, but do not bother).

4.2.5. We can then apply (3) with $k = 12461947$, noting that $\gcd(k,D) = 1$ since k is 3 mod 4. This ψ has $\xi_0 \approx 0.0024972078778$, which is sufficiently small so that we never encounter a miss range for $100 \leq \log D \leq 1000$. Explicitly, with (3) we have

$$13.6 \cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi}\Bigr)^{1/2} \leq 5.43\sqrt{12461947} \cdot \frac{10\log(10\sqrt{D})}{D^{1/4}} \leq 0.00014,$$

and

$$3.322 \cdot \tilde{E}_\psi^+(1/4)\Bigl(\frac{2\pi}{k\sqrt{D}}\Bigr)^{1/4} \cdot \prod_{p|k}\Bigl(1+\frac{1}{\sqrt{p}}\Bigr) \leq \frac{610}{D^{1/8}} \leq 0.0023.$$

Thus when $h_K \leq 5$ and $100 \leq \log D \leq 1000$ we have

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr]\Bigr| \leq \frac{0.0025}{\xi_3|E_\psi^+(1/2+i\xi_0)|} \leq \frac{0.0025}{709\cdot 0.139} \leq 10^{-4},$$

where

$$\xi_2 = \xi_0\log\frac{k}{2\pi} + \arg\bigl[i\Gamma(1/2+i\xi_0)\zeta_k(1+2i\xi_0)\bigr] \approx 0.034189907.$$

4.2.6. On the other hand, we have

$$\bigl|\arg E_\psi^+(1/2+i\xi_0)\bigr| \leq 4.777\xi_0 \leq 0.012,$$

while $0.124 \leq \xi_0\log\sqrt{D} \leq 1.249$. These combine to imply

$$\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\bigr] \\ \geq \min\bigl(\sin[0.124+0.034-0.012], \sin[1.249+0.035+0.012]\bigr) \geq 0.145,$$

contradicting the above inequality, so our assumption of $h_K \leq 5$ must be wrong. □

4.3. We next provide a version[14] of Lemma 3.2.1 when taking $\mathbf{P}$ to have no split primes in our definition (§2.4) of $E_\psi^{\mathbf{P}}(s)$.

Lemma 4.3.1. *When $\mathbf{P}$ has no split primes and $\gcd(k,D) = 1$, in Lemma 3.2.1 we have*

$$|M_3| \leq 0.391\frac{\tau_2(k)}{k^{1/4}} \cdot \prod_{p\in\mathbf{P}}\bigl(1+p^{1/4}\bigr) \cdot L_\chi(1)D^{1/4}.$$

[14] We could perhaps dispense with Lemma 3.2.1 altogether, though it has some value in showing an analogy to the methodology used in [30].

Proof. Recall from §3.2.4 that M_3 is defined by the integral

$$2\int_{(1/4)}\Gamma(s)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{s-1/2}E_\psi^{\mathbf{P}}(s)\zeta(2s)P_{k/g}(2s)\frac{(s-1/2)}{(s-1/2)^2+\xi_0^2}\frac{\partial s}{2\pi i},$$

Here $g=\gcd(k,D)=1$, so $P_k(s)$ has a zero at $s=0$, to cancel the pole of $\Gamma(s)$.

We then move the contour to $\sigma=-1/4$ (this is not k-optimal) and get

$$|M_3|\le 2\Bigl(\frac{2\pi}{k\sqrt{D}}\Bigr)^{3/4}\prod_{p|k}\bigl(1+\sqrt{p}\bigr)\prod_{p\in\mathbf{P}}\bigl(1+p^{1/4}\bigr)\sum_{n\in\hat{\mathbf{P}}}R_K^\star(n)\,n^{1/4}\times$$
$$\times\int_{(-1/4)}\Bigl|\frac{\Gamma(s)\zeta(2s)(s-1/2)}{(s-1/2)^2+\xi_0^2}\Bigr|\frac{|\partial s|}{2\pi}.$$

The product over $p|k$ then is $\le\sqrt{k}\tau_2(k)$, while we see that the sum over n in $\hat{\mathbf{P}}$ is $\le(\sqrt{D}/4)^{1/4}\cdot\sqrt{D}L_\chi(1)/\log(16)$ by Goldfeld's Lemma 4 (cf. our Lemma 2.1.7).

As the integral on $\sigma=-1/4$ is ≤ 0.193, the Lemma follows. □

We proceed to show Lemma 4.2.3, which we restate for convenience. We let $\mathbf{P}$ be the set of all primes with $p|D$ and $p\le 10^4$ (there are at most two such primes when $h_K\le 5$), and with this choice write E_ψ^{m} for the notation of $E_\psi^{\mathbf{P}}$ with §2.4.

Lemma 4.2.3. *With D as in Lemma 4.2.2 but $10^{28}\le D\le e^{100}$ we have $h_K\ge 6$.*

Proof. We will use the auxiliary ψ with $k=17923$, which has $\xi_0\approx 0.0309857994985$. By Lemmata 3.2.1 and 4.3.1 we have $M_1+M_2+M_3=0$ where

$$|M_2|+|M_3|\le 13.6L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi}\Bigr)^{1/2}+0.391\frac{\tau_2(k)}{k^{1/4}}\cdot 11^2\cdot L_\chi(1)D^{1/4}\le 735\cdot L_\chi(1)D^{1/4}$$

while

$$\begin{aligned}M_1&=2|T_\psi^{\mathrm{m}}(1/2+i\xi_0)|\sin\bigl(\arg iT_\psi^{\mathrm{m}}(1/2+i\xi_0)\bigr)\\&=\xi_3|E_\psi^{\mathrm{m}}(1/2+i\xi_0)|\cdot\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathrm{m}}(1/2+i\xi_0)\bigr]\end{aligned}$$

with $\xi_3=2|\Gamma(1/2+i\xi_0)|\cdot|\zeta_k(1+2i\xi_0)|\ge 57.084$ and

$$\xi_2=\xi_0\log\frac{k}{2\pi}+\arg[i\Gamma(1/2+i\xi_0)\zeta_k(1+2i\xi_0)]\approx 0.221562909.$$

Our assumptions imply $735L_\chi(1)D^{1/4}\le 735\cdot 10(\log 10\sqrt{D})/D^{1/4}\le 0.026$, so

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathrm{m}}(1/2+i\xi_0)\bigr]\Bigr|\le\frac{0.026}{57.084}\frac{1}{|E_\psi^{\mathrm{m}}(1/2+i\xi_0)|}.\qquad(4)$$

4.3.2. We are left to bound the quantities in (4) involving $E_\psi^{\mathrm{m}}(1/2+i\xi_0)$.

From Corollary 4.1.3 every split prime satisfies $\ge(D/10^2-1/50^2)^{1/10}\ge 398$. Lemma 2.1.7 says $\sum_n R_K^{\star\mathrm{s}}(n)\le\sqrt{D}L_\chi(1)/\log(16)\le 10\log(10e^{50})/\log(16)<189$ under our assumptions $\log D\le 100$ and $h_K\le 5$. We then note that $\mathcal{F}(4,4)=321$ while $\mathcal{F}(3,5)=231$ and $\mathcal{F}(2,10)=221$, so that the fourth smallest split prime is $(\sqrt{D}/4)^{1/4}\ge 2236$, the fifth smallest is $\ge(\sqrt{D}/4)^{1/3}\ge 29240$, and the tenth smallest is $\ge(\sqrt{D}/4)^{1/2}\ge 5000000$, with there being no more than 93 total.

This will give a contribution to $E_\psi^{\mathrm{m}}(1/2+it)$ from split primes bounded by

$$\le 2\Bigl(\frac{3}{\sqrt{398}}+\frac{1}{\sqrt{2236}}+\frac{5}{\sqrt{29240}}+\frac{84}{\sqrt{5000000}}\Bigr)\le 0.477,$$

but we also need to consider contributions from products of such primes, and also from possible ramified primes exceeding 10^4. To account for the latter, we introduce the multiset $\mathcal{M} = \{1, 10^4, 10^4, 10^4, 10^8, 10^8, 10^8, 10^{12}\}$.

We let $\vec{Y} = (398, 398, 398, 2236, 29240, 29240, 29240, 29240, 29240, 5000000, \ldots)$, and for any decreasing function U (perhaps supported on $[Y_1, \sqrt{D}/4]$) we have

$$\sum_{n\in\hat{\mathbf{P}}\setminus\{1\}} R_K^{\star\mathrm{s}}(n)U(n) \le \sum_{S\neq\varnothing} 2^{c(S)} \sum_{m\in\mathcal{M}} U\Bigl(m\prod_{i\in S} Y_i\Bigr)$$

where the sum is over non-empty multisets S of integers $1 \le i \le 93$ and $c(S)$ is the number of nonzero multiplicities. Writing this in a somewhat different way, we can lump together the final 84 coordinates as one, getting a bound of

$$\sum_{n\in\hat{\mathbf{P}}\setminus\{1\}} R_K^{\star\mathrm{s}}(n)U(n) \le \sum_{\vec{e}>\vec{0}} 2^{\tilde{c}(\vec{e})} \sum_{m\in\mathcal{M}} U\Bigl(m\prod_{i=1}^{10} Y_i^{e_i}\Bigr)\cdot 2^{e_{10}}\left(\!\!\binom{84}{e_{10}}\!\!\right).$$

where $\vec{e}$ runs over nonzero vectors of length 10 with all $e_i \ge 0$, and $\tilde{c}(\vec{e})$ is the number of nonzero coordinates excluding the last. (This overcounts, as when $e_{10} \ge 2$ and a prime is multi-counted in the multi-choose, the exponent in $2^{e_{10}}$ can be reduced).

We then apply this with $U(n) = 1/\sqrt{n}$, finding that[15]

$$\sum_{n\in\hat{\mathbf{P}}\setminus\{1\}} \frac{R_K^{\star\mathrm{s}}(n)}{\sqrt{n}} \le 0.841.$$

This implies that

$$|E_\psi^{\mathrm{m}}(1/2+i\xi_0)| \ge |(1-1/2^{1/2+i\xi_0})|\cdot|(1-1/5^{1/2+i\xi_0})|\cdot(1-0.841) \ge 0.025$$

and

$$\begin{aligned}|\arg E_\psi^{\mathrm{m}}(1/2+i\xi_0)| &\le \arg(1-1/2^{1/2+i\xi_0}) + \arg(1-1/5^{1/2+i\xi_0}) + \\ &\qquad + \max_t \arg[1-0.841\cos(t)+0.841i\sin(t)] \\ &\le 0.052+0.041+1.000 = 1.093.\end{aligned}$$

From (4) the first implies $\bigl|\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathrm{m}}(1/2+i\xi_0)\bigr]\bigr| \le 0.019$, while the second combined with $0.998 \le \xi_0\log\sqrt{D} \le 1.550$ yields

$$\begin{aligned}&\sin\bigl[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^{\mathrm{m}}(1/2+i\xi_0)\bigr] \\ &\qquad\ge \min(\sin[0.998+0.221-1.093], \sin[1.550+0.222+1.093]) \ge 0.125.\end{aligned}$$

This is a contradiction, and thus $h_K \ge 6$ for D in this range. □

5. Sieving out the smallest range of discriminants

We complete our proof of Theorem 1.2.4 by handling $D \le 10^{28}$ computationally. There are roughly 10^{14} u-values to consider. For each u we will show there are sufficiently many small split primes so as to imply that the class number is large, using that the regulator $\epsilon_0 \le 10\sqrt{D}$ is small in our five families.

For the bulk of the range, the problem reduces to showing that there are (at least) four small split primes that do not exceed 421. This can then be handled by

[15]We do not win by much here (with 0.841 rather close to 1), though we could split up the D-range to obtain small gains. Note also that our accounting of $\hat{\mathbf{P}}$ throws out information about multiplicativity: for instance $-1/401^s$ and $-1/409^s$ would have a product with a positive sign.

a computational sieve, for instance by splitting the 80 smallest odd primes into 4 groups of 20, and showing for each u-value there is at least one split prime in each group. Of course this does not work quite as described, as each group fails to have a split prime for approximately $1/2^{20}$ of the u-values, but the residual set is then much smaller, and can be suitably handled.

The sieving problem for the Chowla conjecture was briefly considered by Mollin and Williams [27] in their context of proving the conjecture under GRH, handling $D \le 10^{13}$ in a few minutes with a 1986 Fortran program. They had some simplifications due to only being concerned with cases of $h_K = 1$.

We will use Lemma 2.1.7, and recall that $\mathcal{F}(2,6) = 85$ and $\mathcal{F}(4,3) = 129$.

5.1. For $D \le 10^8$ we can simply compute the class numbers directly. There are only around 10^4 relevant u-values in each case, and each computation is quite fast, with the whole range taking a few minutes.

5.1.1. For $10^8 \le D \le 10^{12}$ we again handled each u-value individually, using a lookup table of Kronecker symbol values (for small primes) to find at least 30 split primes up to $2500 \le \sqrt{D}/4$, so that Goldfeld's Lemma (Corollary 2.1.8) implies

$$2h_K \log \epsilon_0 = \sqrt{D} L_\chi(1) \ge 60 \log 16, \text{ so that } h_K \ge 60(\log 16)/2\log(10\sqrt{D}) > 5.16.$$

The computation for each u-value involves Kronecker lookups for about 60 primes on average (the worst case is $u = 400310$ for $D = 4u^2 + 1$ which takes 119 primes), and our code took less than a second in total.

5.1.2. In the range $10^{12} \le D \le 10^{17}$ we again chose to process each u-value individually. By Lemma 2.1.7 we need only find 6 split primes up to $500 \le D^{1/4}/2$, since their existence implies $h_K \ge \mathcal{F}(2,6)(\log 16)/2\log(10\sqrt{D}) \ge 5.38$. Here we have $\approx 10^9$ total u-values to process, and each on average takes around 12 Kronecker lookups (the worst case is $u = 49998347$ for $D = u^2 + 4$ which takes 57 primes), and our code took less than a minute in total.

5.2. Finally we consider the range $10^{17} \le D \le 10^{28}$. We first partition[16] the odd primes $5 \le p \le 421$ into 4 groups of size 20. If for a given u there is at least one corresponding inert prime in each of the groups, then there are at least 4 split primes $\le 421 \le (\sqrt{D}/4)^{1/3}$, and thus $h_K \ge \mathcal{F}(4,3)(\log 16)/2\log(10\sqrt{D}) > 5.17$.

Whereas in the lower ranges we processed each u individually via Kronecker lookups, here we use bitwise XOR logic (and thus cannot tell how many split primes there are within a given group, but instead only determine whether or not there are any split primes or not), with a 64-bit register thus containing data for 64 u-values, thereby allowing the $\approx 10^{14}$ u-values to be traversed reasonably quickly For each prime p, we kept a list of p 64-bit stamps, and cycled through these. The sieving of each batch of 64 u-values takes 20 XOR operations, and 20 updates of which stamp to apply next. Our code went through approximately $2.5 \cdot 10^9$ u-values per second for each group, and the entire run took a few core-days.[17]

[16] One reason not to put all the small primes in the same group is that they have slightly worse sieving ratios (for that matter, using $p = 3$ is superior to $p = 5$ for the first three families, and $p = 2$ sieves half the values for the first and fourth). Indeed, we used a "zig-zag" pattern, the 1st/8th, 9th/16th, 17th/24th primes in the first group, the 2nd/7th, 10th/15th in the next, etc.

[17] Undoubtedly the choice of "20" primes for the size of the sieving set is not an optimal balance (perhaps 10 or so is best), but our code development took various paths through parallelization schemes, etc., and as this choice was reasonably fast, I saw no reason to fiddle with it.

Each group of 20 primes misses $\approx 1/2^{20}$ of the u-values, and those that remain are handled individually. The worst examples we found were $u = 37136775445867$ for $D = 4u^2 + 1$ and $u = 48033835914287$ for $D = u^2 + 4$, both of which have the 64th prime 311 at their fourth smallest split prime. (In a previous version of the code, we found two examples with fewer than 3 split primes up through 283, the first being $u = 31205852902661$ for $D = 4(u^2 + 1)$ which indeed has but a single split prime in the range, while $u = 67555825712827$ for $D = u^2 + 4$ has only two).

5.3. By the above computations, we find the lists of cases of $h_K \le 5$ for the five families as given in Tables 1 and 2, and that there are no other cases with $D \le 10^{28}$.

By Lemma 4.2.3 we know $h_K \ge 6$ in these families when $10^{28} \le D \le \exp(100)$, and Lemma 4.2.2 implies $h_K \ge 6$ for $100 \le \log D \le 1000$. For $1000 \le \log D \le 10^8$ Proposition 3.3.1 says $\sqrt{D}L_\chi(1) \ge 100 \log D > 199 \log(10\sqrt{D})$ so that $h_K \ge 99$.

Finally, for $\log D \ge 10^8$ we use Theorem 1.1.1 to get that

$$\sqrt{D}L_\chi(1) \ge \frac{(\log D)^3}{10^{14}} \prod_{p|D}\Big(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Big) \ge (100 \log D) \prod_{p|D}\Big(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Big).$$

When $h_K \le 5$ and D is in one of our families, the theory of genera implies there are at most three primes that divide D. Moreover, no prime that is 3 mod 4 divides D. Thus (under the assumptions $h_K \le 5$ and $\log D \ge 10^8$) we find that

$$h_K = \frac{\sqrt{D}L_\chi(1)}{2\log \epsilon_0} \ge \frac{100 \log D}{2\log(10\sqrt{D})}\Big(1 - \frac{2}{3}\Big)\Big(1 - \frac{4}{6}\Big)\Big(1 - \frac{2D^{1/6}}{D^{1/3}}\Big) \ge 11.$$

This is a contradiction, and we thereby conclude as in Theorem 1.2.4.

6. A more general class number 1 result for real quadratic fields

Next we turn to our second result, namely classifying all K of class number 1 with a fundamental unit $(A + B\sqrt{D})/2$ with $B \le D^{1/4}$. We do not try too hard to aim for much generality, but instead concentrate on obtaining this specific result.

The D-exponent is chosen here to make things workable without too much difficulty. In theory one could go up to $\approx (10^{1000} - 1/2)$ from Theorem 1.1.1, though 1/2 is already a natural barrier for the use of Mollin's Lemma (Corollary 4.1.3).

6.1. We begin by giving a variant of Proposition 3.3.1. The method of proof is mostly the same, though it turns out to be more convenient to work with just one auxiliary character, and handle the gcd-condition separately.

The constant 1.51 is chosen here to be slightly larger than 3/2, as that is the natural bound coming from $B \le D^{1/4}$ when $h_K = 1$.

Lemma 6.1.1. *We have* $\sqrt{D}L_\chi(1) > 1.51 \log D$ *when* $200 \le \log D \le 1000$.

Proof. We suppose $\sqrt{D}L_\chi(1) \le 1.51 \log D$ and proceed to show a contradiction.

6.1.2. By the theory of genera, there are no more than 14 prime divisors of D, as else 2^{13} divides the class number, whence $\sqrt{D}L_\chi(1) \ge h_K/\pi \ge 2607 > 1.51 \log D$.

With $X = 1312799$ in Lemma 2.1.7 we then have $u = \lfloor \log(\sqrt{D}/4)/\log X \rfloor \ge 7$, and as $\mathcal{F}(3,7) = 575$ we find $\mathcal{F}(3,7)\log(16) > 1594 \ge 1.51 \log D \ge \sqrt{D}L_\chi(1)$; thus there are at most 2 split primes up to this X. Similarly, with $X = 10^{14}$ we have $u \ge 3$, so that there are at most 6 split primes up to 10^{14}, while there are at most 272 split primes up to $\sqrt{D}/4$.

This gives us (somewhat crudely) that

$$|E_\psi^+(1/2+it)| \geq \prod_{p\leq 43} V_{\rm r}(p)\cdot V_{\rm s}(2)V_{\rm s}(3)V_{\rm s}(1312799)^4 V_{\rm s}(10^{14})^{266} \geq 0.00018$$

and

$$\tilde E_\psi^+(1/4) \leq \prod_{p\leq 43} Y_{\rm r}(p)\cdot Y_{\rm s}(2)Y_{\rm s}(3)Y_{\rm s}(1312799)^4 Y_{\rm s}(10^{14})^{266} \leq 46858,$$

while $|\arg E_\psi^+(1/2+it)|$ is bounded as

$$\leq |t|\Bigl(\sum_{p\leq 43} W(p) + 2W(2) + 2W(3) + 8W(1312799) + 532W(10^{14})\Bigr) \leq 20.331|t|.$$

6.1.3. Next we imitate Lemma 3.3.4 and apply (3) with $k=12461947$, breaking into two cases depending on whether $k|D$ or not. (Here $\xi_0 \approx 0.0024972078778$). Independent of whether $\gcd(k,D)$ is 1 or k, when $\sqrt{D}L_\chi(1) \leq 1.51\log D$ we have

$$13.6\cdot L_\chi(1)\Bigl(\frac{k\sqrt{D}}{2\pi g}\Bigr)^{1/2} \leq 19154\frac{1.51\log D}{D^{1/4}} \leq 10^{-14}$$

and

$$3.322\cdot \tilde E_\psi^+(1/4)\Bigl(\frac{2\pi g}{k\sqrt{D}}\Bigr)^{1/4}\cdot \prod_{p|(k/g)}\Bigl(1+\frac{1}{\sqrt{p}}\Bigr) \leq \frac{5.26\cdot 46858}{D^{1/8}} \leq 10^{-5}.$$

When $\gcd(k,D)=1$, since $\xi_3 = 2|\Gamma(1/2+i\xi_0)|\cdot|\zeta_k(1+2i\xi_0)| \approx 709.76$ we have

$$\Bigl|\sin\bigl[\xi_0\log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\bigr]\Bigr| \leq \frac{1.01\cdot 10^{-5}}{709|E_\psi^+(1/2+i\xi_0)|} \leq 10^{-4}.$$

Our range of D implies that $0.249 \leq \xi_0\log\sqrt{D} \leq 1.249$, while $\xi_2 \approx 0.034189907$ and $|\arg E_\psi^+(1/2+i\xi_0)\bigr] \leq 20.331\xi_0 \leq 0.051$, so that

$$\begin{aligned}&\sin\bigl[\xi_0\log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\bigr]\\ &\qquad \geq \min\bigl(\sin[0.249+0.034-0.051], \sin[1.249+0.035+0.051]\bigr) \geq 0.229.\end{aligned}$$

This is a contradiction, so our assumption $\sqrt{D}L_\chi(1) \leq 1.51\log D$ must be incorrect.

When $g=\gcd(k,D)=k$ we can re-use the same initial bounds, while now we have $\xi_2 = \xi_0\log(1/2\pi) + \arg\bigl[i\Gamma(1/2+i\xi_0)\zeta(1+2i\xi_0)\bigr] \approx -0.0066$ so that

$$\begin{aligned}&\sin\bigl[\xi_0\log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\bigr]\\ &\qquad \geq \min\bigl(\sin[0.249-0.007-0.051], \sin[1.249-0.006+0.051]\bigr) \geq 0.189.\end{aligned}$$

This again is a contradiction, and we conclude the Lemma. □

Remark. The above bound has the convenient side feature that it overlaps intervals with our prior work [30] on imaginary quadratic fields. In particular, that result implies $h_K \geq 101$ for $D > 2383747$, and so $\sqrt{D}L_\chi(1) = \pi h_K \geq 317 \geq 1.51\log D$ for imaginary K with $2383747 < D \leq \exp(200)$.

6.2. Next we specialize to the real quadratic case, using our assumption about the size of the fundamental unit. For the larger D-range we can still use $k = 12461947$. Other than using Mollin's Lemma, the proof is the same as above.

Lemma 6.2.1. *Suppose $D > 0$ is fundamental with $h_K = 1$ and $\epsilon_0 = (A+B\sqrt{D})/2$ with $B \le D^{1/4}$. Then for $50 \le \log D \le 200$ we have $\sqrt{D}L_\chi(1) > 1.51 \log D$.*

Proof. We again suppose $\sqrt{D}L_\chi(1) \le 1.51 \log D$ and will get a contradiction. First we note that $h_K = 1$ implies D has at most 2 prime divisors by the theory of genera.

By Corollary 4.1.3 (and $B \le D^{1/4}$) the smallest split prime is

$$\ge \frac{(A-\delta-1)}{B^2} \ge \frac{(\sqrt{DB^2+4\delta}-\delta-1)}{B^2} \ge D^{1/4} - 1 \ge 268000$$

and as $\sqrt{D}L_\chi(1)/\log(16) \le 108.9$ there are at most 53 split primes up to $\sqrt{D}/4$.

Thus we find that

$$|E_\psi^+(1/2+it)| \ge V_{\mathrm{r}}(2)V_{\mathrm{r}}(\sqrt{D})V_{\mathrm{s}}(268000)^{53} \ge 0.238$$

and

$$\tilde{E}_\psi^+(1/4) \le Y_{\mathrm{r}}(2)Y_{\mathrm{r}}(\sqrt{D})Y_{\mathrm{s}}(268000)^{53} \le 196,$$

while $|\arg E_\psi^+(1/2+it)|$ is bounded as

$$|\arg E_\psi^+(1/2+it)| \le |t|\big(W(2) + W(\sqrt{D}) + 106W(268000)\big) \le 4.238|t|.$$

6.2.2. As previously, we use (3) with $k = 12461947$. When $\sqrt{D}L_\chi(1) \le 1.51 \log D$ we get (independent of $\gcd(k, D)$ for now)

$$13.6 \cdot L_\chi(1)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{1/2} \le 19154\frac{1.51 \log D}{D^{1/4}} \le 5.390$$

and

$$3.322 \cdot \tilde{E}_\psi^+(1/4)\Big(\frac{2\pi g}{k\sqrt{D}}\Big)^{1/4} \cdot \prod_{p|(k/g)}\Big(1 + \frac{1}{\sqrt{p}}\Big) \le \frac{5.26 \cdot 196}{D^{1/8}} \le 1.991.$$

For $\gcd(k, D) = 1$, as $\xi_3 = 2|\Gamma(1/2 + i\xi_0)| \cdot |\zeta_k(1 + 2i\xi_0)| \approx 709.76$ from (3) we have

$$\Big|\sin\big[\xi_0 \log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\big]\Big| \le \frac{7.381}{709|E_\psi^+(1/2+i\xi_0)|} \le 0.044.$$

Our range of D implies that $0.062 \le \xi_0 \log\sqrt{D} \le 0.250$, while $\xi_2 \approx 0.034189907$ and $|\arg E_\psi^+(1/2+i\xi_0)] \le 4.238\xi_0 \le 0.011$, so that

$$\begin{aligned}&\sin\big[\xi_0 \log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\big]\\ &\qquad \ge \min\big(\sin[0.062 + 0.034 - 0.011], \sin[0.250 + 0.035 + 0.011]\big) \ge 0.084.\end{aligned}$$

This is a contradiction, so our assumption $\sqrt{D}L_\chi(1) \le 1.51 \log D$ must be incorrect.

When $\gcd(k, D) = k$ we improve the first bound from 5.390 to 0.002, so that

$$\sin\big[\xi_0 \log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\big] \le \frac{1.993}{709|E_\psi^+(1/2+i\xi_0)|} \le 0.012,$$

while the shifted $\xi_2 \approx -0.00661$ gives us

$$\begin{aligned}&\sin\big[\xi_0 \log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\big]\\ &\qquad \ge \min\big(\sin[0.062 - 0.007 - 0.011], \sin[0.250 - 0.006 + 0.011]\big) \ge 0.043.\end{aligned}$$

Again this is a contradiction, and we conclude $\sqrt{D}L_\chi(1) \ge 1.51 \log D$. □

6.3. The remaining range $D \le \exp(50) \approx 5.18 \cdot 10^{21}$ is already quite feasible for sieving.[18] However, we can also handle the range $37 \le \log D \le 53$ via $k = 17923$, with the proof *mutatis mutandis* of the previous Lemma.[19]

Lemma 6.3.1. *Suppose $D > 0$ is fundamental with $h_K = 1$ and $\epsilon_0 = (A+B\sqrt{D})/2$ with $B \le D^{1/4}$. Then for $37 \le \log D \le 53$ we have $\sqrt{D}L_\chi(1) \ge 1.51 \log D$.*

Proof. We again suppose $\sqrt{D}L_\chi(1) \le 1.51 \log D$ and will get a contradiction. First we note that $h_K = 1$ implies D has at most 2 prime divisors by the theory of genera.

By Corollary 4.1.3 the smallest split prime is

$$\ge \frac{(A-\delta-1)}{B^2} \ge \frac{(\sqrt{DB^2+4\delta}-\delta-1)}{B^2} \ge D^{1/4}-1 \ge 10^4$$

and as $\sqrt{D}L_\chi(1)/\log(16) \le 28.9$ there are at most 13 split primes up to $\sqrt{D}/4$.

Thus we find that

$$|E_\psi^+(1/2+it)| \ge V_\mathrm{r}(2)V_\mathrm{r}(\sqrt{D})V_\mathrm{s}(10^4)^{13} \ge 0.225$$

and

$$\tilde{E}_\psi^+(1/4) \le Y_\mathrm{r}(2)Y_\mathrm{r}(\sqrt{D})Y_\mathrm{s}(10^4)^{13} \le 25.248,$$

while $|\arg E_\psi^+(1/2+it)|$ is bounded as

$$|\arg E_\psi^+(1/2+it)| \le |t|\big(W(2)+W(\sqrt{D})+26W(10^4)\big) \le 4.095|t|.$$

6.3.2. This time we use (3) with $k = 17923$ with $\xi_0 \approx 0.0309857994985$, and when $\sqrt{D}L_\chi(1) \le 1.51 \log D$ we have

$$13.6 \cdot L_\chi(1)\Big(\frac{k\sqrt{D}}{2\pi g}\Big)^{1/2} \le 727\frac{1.51\log D}{D^{1/4}} \le 3.904$$

and

$$3.322 \cdot \tilde{E}_\psi^+(1/4)\Big(\frac{2\pi g}{k\sqrt{D}}\Big)^{1/4} \cdot \prod_{p|(k/g)}\Big(1+\frac{1}{\sqrt{p}}\Big) \le \frac{5.26 \cdot 25.248}{D^{1/8}} \le 1.302.$$

For $\gcd(D,k) = 1$, as $\xi_3 = 2|\Gamma(1/2+i\xi_0)| \cdot |\zeta_k(1+2i\xi_0)| \approx 57.084$, from (3) we have

$$\Big|\sin\big[\xi_0 \log\sqrt{D} + \xi_2 + \arg E_\psi^+(1/2+i\xi_0)\big]\Big| \le \frac{5.206}{57|E_\psi^+(1/2+i\xi_0)|} \le 0.406.$$

Our range of D implies that $0.573 \le \xi_0 \log\sqrt{D} \le 0.822$, while $\xi_2 \approx 0.221562909$ and $|\arg E_\psi^+(1/2+i\xi_0)\big] \le 3.904\xi_0 \le 0.121$, so that

$$\begin{aligned}&\sin\big[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\big]\\ &\qquad \ge \min\big(\sin[0.573+0.221-0.121], \sin[0.822+0.222+0.121]\big) \ge 0.623.\end{aligned}$$

This is a contradiction, so our assumption $\sqrt{D}L_\chi(1) \le 1.51 \log D$ must be incorrect.

When $\gcd(k,D) = k$ we improve the first bound from 3.904 to 0.030, so that

$$\sin\big[\xi_0\log\sqrt{D}+\xi_2+\arg E_\psi^+(1/2+i\xi_0)\big] \le \frac{1.332}{57|E_\psi^+(1/2+i\xi_0)|} \le 0.104,$$

[18]For instance, for pseudosquares (where all the χ-values are $+1$ instead of -1), Wooding and Williams [35] reached $120120 \cdot 2^{64} \approx 2.2 \cdot 10^{24}$ (later extended by Sorenson).

[19]While we could derive a version of Lemma 4.3.1 when $k|D$ (involving accounting the residue from $s = 0$), it turns out this is not necessary.

while the shifted $\xi_2 \approx -0.0819$ gives us

$$\sin\left[\xi_0 \log \sqrt{D} + \xi_2 + \arg E_\psi^+(1/2 + i\xi_0)\right] \geq \min\bigl(\sin[0.573 - 0.082 - 0.121], \sin[0.822 - 0.081 + 0.121]\bigr) \geq 0.361.$$

Again this is a contradiction, and we conclude $\sqrt{D}L_\chi(1) \geq 1.51 \log D$. □

6.4. We thus have the range $D \leq \exp(37) \approx 1.17 \cdot 10^{16}$ left to sieve. For each D, we only need to find one split prime up to $(D^{1/4} - 1)$ to show our desired result $h_k > 1$. In particular, it is relatively easy to precondition our sieving modulo small primes, say up to 20, thus dividing into 77597520 congruence classes, of which only 114984 are viable (having no split primes and at most one ramified prime up to 20). We then used the 12 primes $23 \leq p \leq 71$ with 64-bit XOR stamps as in §5.2, and processed the unsieved D individually. This took about 50 minutes in total, with the worst example being $D = 947147572030805$, for which $p = 251$ was needed.

The range $D \leq 72^4 \leq 27 \cdot 10^6$ is still left to handle, where we simply computed the class group for $D \leq 10^4$, and again found small split primes for the remainder.

6.5. We recapitulate our proof of Theorem 1.3.3.

We have sieved the range $D \leq \exp(37)$, finding the 22 examples listed in the Theorem with $B \leq D^{1/4}$ and $h_K = 1$.

For $37 \leq \log D \leq 10^8$ we have that $2h_K \log \epsilon_0 = \sqrt{D}L_\chi(1) \geq 1.51 \log D$ by the combination of Lemmata 6.3.1, 6.2.1, and 6.1.1, and our previous Proposition 3.3.1. When $h_K = 1$ and $B \leq D^{1/4}$ we have $2h_K \log \epsilon_0 \leq 2\log(1 + D^{3/4}) \leq 1.501 \log D$, so that the above inequality implies that no such D exist in this range.

For $\log D \geq 10^8$ we use Theorem 1.1.1, noting that D has at most 2 prime factors when $h_K = 1$ by the theory of genera. This readily implies

$$2h_K \log \epsilon_0 \geq (100 \log D)\Bigl(1 - \frac{2}{3}\Bigr)\Bigl(1 - \frac{2D^{1/4}}{D^{1/2}}\Bigr) \geq 32 \log D,$$

and as above this is a contradiction when $B \leq D^{1/4}$ and $h_K = 1$.

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