Version of 21 Dec 2019.

A NEW EFFECTIVE LOWER BOUND FOR $L_\chi(1)$

MARK WATKINS

Abstract. Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with $-D$ a fundamental discriminant and $D \geq 4\pi^2 \exp(10^6)$. The best known lower bound on the class number h_K is due to Oesterlé: following work of Goldfeld in the 70s and Gross and Zagier in the 80s, he showed that for D prime (in particular) we have $h_K \geq (\log D)/55$. We improve this to

$$\pi h_K \geq \min\bigl(10^{10000} \log D, (\log D)^3/10^8\bigr)$$

for prime D. For general D there is a product over $p|D$ of $(1 - 2\sqrt{p}/(p+1))$, but also the constant is smaller, namely $1/7000$ instead of $1/55$ for Oesterlé, and 10^{1000} instead of 10^{10000} for us.

We also show an analogous result for real quadratic fields; this provides an alternative method of resolution to the conjectures of Chowla and Yokoi on the cases of class number 1 for $\mathbf{Q}(\sqrt{4u^2+1})$ and $\mathbf{Q}(\sqrt{u^2+4})$, which were solved by Biró in 2003 by other means.

Our method starts from a given elliptic curve E of rank 5 whose twist by $-D$ has odd parity. The analytic rank of $L_E(s)$ is at least 3 by Kolyvagin's theorem, and by computation we can show that this L-function has exactly two other zeros close to the central point; thus by symmetry these are either on the central line or on the real axis. On the other hand, we derive a "Deuring decomposition" for the completed product L-function $\Lambda_E^K(s)$, showing it is locally approximated around $s = 1$ in suitable co-ordinates as $\sim c\xi(\sin\xi - \xi)$, which in particular has its noncentral zeros *off* the axes.

The stated constant follows by high-precision computation of $L_E'''(1)$ to 10025 digits (for 6 different curves), taking about a month and a half. However, for the more general version this task is quite nontrivial due to the conductor size necessitated by the rank and parity conditions, and took 3 core-months to reach 1025 digits. We give the details of such calculations in a companion paper, and in another companion paper show Chowla's conjecture explicitly (incidentally removing the restriction $D \geq 4\pi^2 \exp(10^6)$ that appears here).

1. History and introduction

Let $K = \mathbf{Q}(\sqrt{\Delta})$ be an imaginary quadratic field, with Δ a negative fundamental discriminant, and $D = |\Delta|$. The *class number* (commonly denoted by h_K) is the number of (inequivalent) primitive reduced binary quadratic forms $aX^2+bXY+cY^2$ with discriminant $b^2-4ac = -D$. Indeed, the primitive binary quadratic forms of a given discriminant form a group under composition as shown by Gauss [30, §234ff], and under equivalence by $\mathbf{SL}_2(\mathbf{Z})$ the reduced forms are canonical representatives of the classes.

1.1. Gauss catalogued a list of class numbers for various small D, and suggested that the class number should tend to infinity with D. In other words, any list of imaginary quadratic fields with a given class number should be finite.

Gauss also noted [30, §302ff] that the class number for orders of negative discriminants (including nonfundamental ones) was on average[1] $(2\pi/7\zeta(3)) \cdot \sqrt{D}$.

Jacobi [46] was the first to publicize a conjectural version of Dirichlet's class number formula (though it was possibly known to Gauss) that linked the class

[1] This is the correct constant for modern conventions (Lagrange/Eisenstein) on discriminants. See Lipschitz [53, (24)] or the end of Mertens [56, §4]. This paper of Mertens does not seem to mention the (obscure?) precursor 1865 work of Lipschitz (though it does improve the error term).

number to the value of the associated Dirichlet L-function at 1, phrased as

$$h_K = -\frac{w_K}{2D}\sum_{m=1}^{D} m\chi(m)$$

where χ is the quadratic Dirichlet character associated to K and w_K is the number of units in K. This was then shown a few years later by Dirichlet [22] and is more commonly written in the form $L_\chi(1) = 2\pi h_K/w_K\sqrt{D}$.

The link of the class number to L-functions was then strengthened by Gronwall [35] who in 1913 related a type of zero-free real interval for $L_\chi(s)$ near $s = 1$ to the largeness of h_K, and this was subsequently improved by Hecke (as published by Landau [49]) to state that a zero-free interval of the form $1 - c/\log D \le \sigma \le 1$ implies $h_K \gg \sqrt{D}/\log D$ (whereas Gronwall had an extra $(\log\log D)^{3/8}$ in the denominator). The papers [49, 50] of Landau showed[2] in a statistical sense that small class numbers were rare: specifically, if $-D_n$ is the putative nth fundamental discriminant with $h(-D_n) \ll \sqrt{D_n}/\log D_n$ then $D_{n+1} \gg_C D_n^C$ for any C. Landau exploited an idea he attributes to Remak of using an auxiliary imaginary quadratic field (often called an auxiliary modulus) thereby showing (under suitable size constraints) that at least one of the two has class number near the expected size.

It was shown by Littlewood [54, §9] in 1928 that under the Generalized Riemann Hypothesis for real χ we have (where $\gamma \approx 0.577$ is Euler's constant)

$$\frac{1}{2}\cdot\frac{\pi^2}{6}\frac{e^{-\gamma}}{\log\log D} \lesssim L_\chi(1) \lesssim 2\cdot e^{\gamma}\log\log D \quad \text{as } D \to \infty,$$

and these bounds are best possible [54, §10] except for the factors of 2. In particular, the class number is roughly of size $\sqrt{D}$, varying at most by a factor of $(\log\log D)$.

1.1.1. The next wave of results took matters in a different direction, the first being Deuring's 1933 paper [20] that noted that if there are infinitely many imaginary quadratic fields with class number 1, then the Riemann Hypothesis is true. He did this by deriving an approximation for the Epstein ζ-function of the principal form, which in this case is equal to the Dedekind ζ-function of the field. We view this as the initial example of a "Deuring decomposition" and will have much to say about such below, but roughly it says that completed Dedekind ζ-function satisfies

$$\zeta_K(s)\Gamma(s)\Big(\frac{\sqrt{D}}{2\pi}\Big)^s = T(s) + T(1-s) + U(s)$$

where $T(s) = \zeta(2s)\Gamma(s)(\sqrt{D}/2\pi)^s$ and $U(s)$ is an error term that is exponentially small (in terms of $\sqrt{D}$) in this case of the principal form. One then sees that $T(s)$ dominates $T(1-s)$ and $U(s)$ when $\sigma > 1/2$ (and $|s-1| \gg 1/D^{1/4}$), at least up to some height depending on D, and this implies that the region has no nonexceptional zeros of $\zeta_K(s)$, so in particular of $\zeta(s)$ itself. The assumption that there are infinitely many D with class number 1 then yields the Riemann Hypothesis.

This was soon followed by Heilbronn [40] who gave an analogue in the conductor aspect and was able to conclude that the class number diverged as $D \to \infty$. However, this result was *ineffective* in that (from the method) one could not say at what rate the divergence occurred. With Linfoot [41] he showed that there are at most 10 imaginary quadratic fields with class number 1 (the known nine

[2]The literature often cites one of these papers of Landau for the general quadratic case, but the phrasing is only for the imaginary quadratic case. The same is true with [51].

being $D \in \{3,4,7,8,11,19,43,67,163\}$), but again they could not determine an upper bound on the size of the discriminant of a putative 10th such field.

Meanwhile, Deuring [21] expanded his zero-distribution analysis for the $h = 1$ case, showing the nonexceptional zeros of $\zeta_K(s)$ up to height roughly $\sqrt{D}$ lie on the half-line, and moreover enjoy a regular spacing, corresponding (with small error) to solutions of $T(s)+T(1-s) = 0$, the behaviour therein dominated by $\sqrt{D}^{it} + \sqrt{D}^{-it}$. This used his decomposition in conjunction with Rouché's theorem.

Soon after, Landau [51] revisited his exploitation of Remak's idea of an auxiliary modulus to show that $h_K \gg_\epsilon D^{1/8-\epsilon}$ for all $\epsilon > 0$, while Siegel [71] gave a proof that showed $h_K \gg_\epsilon D^{1/2-\epsilon}$ and moreover the general result $L_\chi(1) \gg_\epsilon 1/D^\epsilon$ applicable also to real quadratic fields. These results were again ineffective.

1.1.2. The class number 1 problem was solved by Heegner [39] in 1952, though over a decade intervened before his proof was widely accepted, in part due to his idiosyncratic style, and in part due to its seeming reliance on a part of Weber's *Lehrbuch der Algebra* (1908) that was known to be in error.[3]

In 1966 Baker [3] used his newly developed methods of linear forms in logarithms to show an effective bound on the 10th putative field of class number 1, while soon after Stark [72] used modular functions (though with a more analytic flavor, involving Kronecker's limit formula, than Heegner had followed) to show the same, and moreover resolved the "gap" in Heegner's work [73, 74].[4]

These methods are strongly dependent on the fact that there is only the principal form when the class number is 1. Baker [4] and Stark [77] each gave proofs that used transcendence theory to solve the class number 2 problem, but there is scant hope of going further by such methods.

1.2. A slightly different approach to class number problems was described in 1976 by Goldfeld [33].[5] He adapted Landau's usage of the auxiliary modulus to the case of degree 2 L-functions, and showed that a (modular) elliptic curve with sufficiently high analytic rank (the order of central vanishing of its L-function) would imply that the class number of imaginary quadratic fields diverges effectively – in other words, for any fixed class number h, one could (provably) make lists in the manner of Gauss of all the imaginary quadratic fields with that class number.

From Goldfeld's work, given an elliptic curve $E/\mathbf{Q}$ of analytic rank r (or more), meeting technical conditions so that the $-D$th quadratic twist of E has odd parity, one gets roughly that $h_K \gg_r (\log D)^{r-2}$. Unfortunately, showing that a given elliptic curve had analytic rank more than 2 was not possible at the time.

Contrary to the previous methods, there were no special properties of the principal form used. Moreover, there was a reasonable expectation of finding an elliptic curve of analytic rank 3 or more via the Birch and Swinnerton-Dyer conjecture [7].

[3]The subject of modular functions *à la* Weber seems to have dropped almost totally out of fashion over the preceding decades. We can do no better than to quote Birch [6]: "... it is hardly an exaggeration to say that for half a century most mathematicians hardly knew that the theory of modular functions had ever existed ... Unhappily, in 1952 there was noone left who was sufficiently expert in Weber's *Algebra* to appreciate Heegner's achievement."

[4]He also noted [76] that (retrospectively) Baker's method was somewhat overkill, as via a Deuring decomposition for a twisted Dedekind ζ-function (which he had recently shown [75]) one only needs linear forms in *two* logarithms, whence a prior result of Gelfond [32] already suffices.

[5]A contemporaneous paper of Friedlander [28] also considers the effect of a central zero (from parity effects of Armitage L-functions) on class number problems.

Goldfeld also handled the real quadratic case, and his Theorem 5 states that

$$L_\chi(1) \underset{\tilde r}{\gg} \frac{1}{\sqrt{D}} \frac{(\log D)^{\tilde r-3}}{(\log\log D)^{\tilde r+4}} \prod_{\substack{\chi(p)\neq -1\\ p<(\log D)^{8\tilde r}}} \Bigl(1+\frac{1}{\sqrt{p}}\Bigr)^{-4}$$

where $\tilde r$ is the analytic rank (or a lower bound therein) of the degree 4 L-function formed by the "Landau product" of $L_E(s)$ and its twist $L_{E\chi}(s)$.

One can then note that there are few small split primes and use genus theory to bound the number of $p|D$, thusly arriving at the statement of his Theorem 1, that

$$L_\chi(1) \underset{\tilde r}{\gg} \frac{(\log D)^{\tilde r-3}}{\sqrt{D}} \exp\bigl(-21\sqrt{\tilde r \log\log D}\bigr).$$

1.2.1. The existence of a modular elliptic curve (with suitable technical properties) of analytic rank 3 was shown in 1986 by Gross and Zagier [36], using the theory of Heegner points. Although their work has been generalized in various directions, this is still the highest known analytic rank.

The suitable technical conditions include results about root numbers that imply we can then take $\tilde r = 4$. Oesterlé [58] used the rank 3 elliptic curve $[0,0,1,-7,6]$ of conductor 5077 and got the resulting lower bound of $h_K \geq (\log D)/55$ for prime D.[6] This is quite small compared to the GRH expectation of $h_K \gg \sqrt{D}/\log\log D$, but still effectively resolves the class number problem for any fixed h (indeed, much of Oesterlé's motivation was the $h = 3$ case). However, the problem remains quite difficult computationally, and it is only for $h \leq 100$ that the "expected" lists (easily derivable by computing to $2100(h \log 13h)^2$ as per Tatuzawa [78]) have been proven complete [82]. (This [82] is from 15 years ago, and likely $h \leq 1000$ is doable today).

Note that $\tilde r = 4$ does not give a useful lower bound on $L_\chi(1)$ in the real quadratic case, as $L_\chi(1) \gtrsim (\log D)/\sqrt{D}$ already follows from Dirichlet's class number formula and the fact that the fundamental unit is $\gtrsim \sqrt{D}$ (hence the regulator is $\gtrsim \log\sqrt{D}$).

1.3. We improve the constant in Oesterlé's version of Goldfeld's result. To ensure the root number condition (namely $E\chi$ must have odd parity) we use a fixed selection of three elliptic curves $E/\mathbf{Q}$ of rank 5, for each of which we know the analytic rank is at least 3 by Kolyvagin's work (in conjunction with the theorem of Gross and Zagier). Furthermore, for each E we can show by computational means that there are exactly two additional zeros of $L_E(s)$ with $|s-1| \leq 10^{-510}$ (say) and by symmetry these zeros $1 \pm i\kappa$ must be either on the central line or the real axis. To show this bound on zeros of $L_E(s)$ near $s = 1$ we will compute its third derivative to high precision and use Rouché's theorem.

For notational reasons we write $L_f(s)$ instead of $L_E(s)$, with f the weight 2 modular newform associated to E.

We then use Lavrik's general method [52] for approximate functional equations to derive a "Deuring decomposition" for $\Lambda_E^K(s) = \Lambda_f^K(s)$ that accentuates the behavior around $s = 1$. The fact that the product L-function has analytic rank at least 4 allows us to show (assuming $\gcd(N_f, D) = 1$ for notational simplicity)

$$\tilde\Lambda_f^K(s) = L_f^K(s)\Gamma(s)^2\Bigl(\frac{N_f D}{4\pi^2}\Bigr)^{s-1} = T_f(s) + T_f(2-s) - T_f''(1)(s-1)^2 + O_f(\tilde h_K|s-1|^4)$$

[6] For composite D there is also a mild product over $p|D$, but more annoying is that when $5077|D$ the constant $1/55$ is replaced by $1/7000$. See §10 below for more on this.

where $\tilde{h}_K = h_K \sum_a 1/a$ includes a reciprocal sum over minima and

$$T_f(s) = \frac{L_{S^2 f}(2s)}{\zeta(2s-1)}\Gamma(s)^2\Big(\frac{N_f D}{4\pi^2}\Big)^{s-1} E_f(s),$$

with N_f the conductor, while $L_{S^2 f}(s)$ is the symmetric-square L-function of f, and E_f is an adjustment for bad primes of f and small noninert primes of K.

In particular, we have an approximation of $\tilde{\Lambda}_f^K(s)$ by $c\xi(\sin\xi - \xi)$ where $i\xi$ is essentially $(s-1)(\log D)$. The noncentral zeros of this approximation are off the axes, and by Rouché's theorem this can be transferred to $\tilde{\Lambda}_f^K(s)$, contradicting our previous observation concerning the known zeros $1 \pm i\kappa$ of $L_f(s)$. A comparison then gives us that

$$\tilde{h}_K|\kappa|^4 \gg_f E_f(1)\cdot \min\Big((\log D)|\kappa|^2, (\log D)^3|\kappa|^4\Big),$$

from which we obtain a lower bound for h_K that is quadratic in $1/|\kappa|$.

1.3.1. The same methods also work for real quadratic fields, with a different selection of three curves of rank 5 to start. Here Goldfeld already exhibited some counting arguments that relate small values of $L_\chi(1)$ to the splitting of small primes in K, though we still need some effort to adapt our results to this setting.

Our ultimate result provides another way to effectively resolve the conjectures of Chowla and Yokoi. These respectively give complete lists of squarefree $4m^2+1$ or m^2+4 such that $\mathbf{Q}(\sqrt{4m^2+1})$ or $\mathbf{Q}(\sqrt{m^2+4})$ has class number 1, and were shown by other means by Biró [8] in 2003.

1.4. Let us catalogue what we actually show.

1.4.1. Firstly we have bounds for $L_\chi(1)$ under coprimality restrictions on D (which occur because of the difficulty in controlling root numbers when $\gcd(D, N_f)\neq 1$).

Theorem 10.2.1. *Suppose $K = \mathbf{Q}(\sqrt{-D})$ is an imaginary quadratic field with $-D$ a fundamental discriminant and $D \geq 4\pi^2\exp(10^6)$, while*

$$\gcd(D, 19047851\cdot 64921931\cdot 67445803) = 1.$$

Then the class number h_K is bounded below as

$$\pi h_K = \sqrt{D}L_\chi(1) \geq \min\big(10^{10000}\log D, (\log D)^3/10^8\big)\cdot \prod_{p|D}\Big(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Big).$$

Theorem 10.2.2. *Suppose $K = \mathbf{Q}(\sqrt{D})$ is a real quadratic field with D a fundamental discriminant and $D \geq 4\pi^2\exp(10^6)$, while*

$$\gcd(D, 3089\cdot 6599\cdot 647\cdot 86131\cdot 409\cdot 146099) = 1.$$

Then we have

$$\sqrt{D}L_\chi(1) \geq \min\big(10^{10000}\log D, (\log D)^3/10^8\big)\cdot \prod_{p|D}\Big(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Big).$$

These are proven by employing a battery of three elliptic curves of rank 5 (for each case) and computing $L_f'''(1)$ to 10025 digits – of course we find it is 0 to this precision. As the conductors are $\leq 10^8$ this takes approximately a week or two per curve. We give the details of the computation in a companion paper [87].

1.4.2. We then have a version unrestricted by the gcd-condition.

Theorem 10.4.1. *For $D \geq 4\pi^2 \exp(10^6)$ we have*

$$\sqrt{D}L_\chi(1) \geq \min\bigl(10^{1000}\log D, (\log D)^3/10^{13}\bigr) \cdot \prod_{p|D}\Bigl(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

We again use three rank 5 elliptic curves each for the real and imaginary cases. However, here the curves must be of a specific type to ensure $f\chi$ has odd parity, and the conductors are much larger (nearly 10^{16}). Thus the $L_f'''(1)$-computation consumes more time, with merely 1025 digits taking a couple weeks for each curve.

1.4.3. The constants on the $(\log D)$ terms in the above Theorems can of course be increased by further computation. More delicate are the constants with the $(\log D)^3$, which depend on a number of factors.

These constants come about from a comparison of a main term in the Deuring decomposition to the error term therein. The latter contains a factor of $\sqrt{N_f}$ from the extra length of the approximate functional equation, and any substantial improvement on this would be quite notable. Meanwhile, for the main term, there is a factor of $L_{S^2f}(2)$ that occurs naturally.

The error term also contains factors from what could be termed as coming from lattice point counting in ellipses in the imaginary quadratic case. However, it turns out to be convenient to handle both this and the real quadratic case simultaneously via L-function techniques as first considered by Goldfeld, and it seems the real quadratic case (even with its greater complications) does not lose a great deal here.

Then there are factors that are somewhat dependent on the specific elliptic curves used, though not initially to the extent of the $\sqrt{N_f}$ factor. One example is the effect of ramified primes, and another is the size of $\sum_l |b_l|/l^3$ where b_l are the coefficients of the symmetric-square (see §4.1.6). In particular, for small l the trivial bound $|b_l| \leq l\tau_2(l)$ can often be improved. A similar observation holds for a contribution from the possibility of small split primes in $E_f(s)$: rather than the bounds $c_f(2) \geq -2\sqrt{2}$ and $c_f(3) \geq -2\sqrt{3}$, one already has a good improvement from using $c_f(2) \geq -2$ and $c_f(3) \geq -3$, and I do not discount that one could get an overall improvement by choosing curves (even with larger conductor by a decent factor) for which the initial $c_f(p)$ are not so small.

It might be that one could reduce 10^8 and 10^{13} by 1 each in the exponents, but I doubt any greater gains could be made without a notable theoretical improvement.

1.4.4. The restriction of $D \geq 4\pi^2 \exp(10^6)$ is convenient to our proof but somewhat annoying in the final result. We can remove it by a process similar to our computational work for class numbers up through 100 in [82] (which can be suitably adapted to the real quadratic case too). Therein we used a battery of auxiliary (primitive odd) Dirichlet L-functions of moduli k_i each of whose lowest zero has height significantly smaller than the typical $2\pi/\log k_i$. When such a modulus is coprime to D, we get periodic ranges in $\log D$ for which $L_\chi(1)$ is not too small. By using many such L-functions the "miss ranges" of said moduli do not overlap for $\log D \leq 4+10^6$, while the coprimality condition can be handled by genus theory (e.g., if 50 small primes divide a large D, then the class number is at least $2^{50} \geq 10^{15}$).

Indeed, in [82, §3] the range $2^{162} \leq D \leq \exp(298368000)$ was fairly mechanical, and it was only with $2^{52} \leq D \leq 2^{162}$ that things became difficult.

The details will be given in our companion paper [88].

1.4.5. For the real quadratic case, the bound in Theorem 10.4.1 fails to beat the trivial bound (from the regulator) essentially when the sum over $p|D$ of $\sum 2/\sqrt{p}$ exceeds $1000 \log 10$. This is a finite proportion of D, and indeed likely contains a proportional amount of small regulators (small-(u,v) solutions to $u^2 - Dv^2 = \pm 4$).

1.5. It does not seem that our method gives an analogous result starting from elliptic curves of rank 4 (where we would know $\tilde{r} \geq 3$). The problem is that the local approximation is now of the form $c\xi(\cos\xi - 1)$; this has double zeros at twice the expected zero-spacing of $\tilde{\Lambda}_f^K(s)$, but they do indeed occur on the real line.

We can also note that the device we employ in §10.3 to ensure $E\chi$ has odd parity might not work so well with higher ranks. Indeed, for the above Theorem 10.4.1 we need to exhibit an elliptic curve of rank 5 in various quadratic twist families; a heuristic of Granville [89, §11] suspects 5 is generically the maximal rank in a quadratic twist family, and [59, §8.3] concurs. On the other hand, more recently the twist of 11a by -203145767 was found by Daniels and Goodwillie [17, Table 4] to have rank 6, and similarly they noted the twist of 550k by 4817182 has rank 6.

1.5.1. Finally, we can note that the present work has almost no effect on the computational problem of proving class number lists for imaginary quadratic fields are complete (as in [82]), as therein the upper bound on the size of D (coming from Goldfeld's work and its completion by Gross and Zagier) is not the bottleneck.

1.6. Though it is not immediately relevant to the current work (and largely we mention it to expand on the literature that we later became aware of), recently in [84, 85] we noted that it is possible to use Goldfeld's idea with an L-function (of degree 2) with analytic rank merely 2 to re-solve the class number 1 problem, provided that sufficient cancellation can be shown with the Dirichlet series coefficients when restricted to integers represented by the principal form.

In particular, for small fixed y one wishes to show that

$$\sum_{x \sim \sqrt{D}y} c_f(x^2 + Dy^2) = o(Dy^2)$$

where c_f denotes the L-series coefficients of the weight 2 modular newform f.

1.6.1. In our first proof [84], we exploited various arithmetic properties of a specific elliptic curve with complex multiplication by $\mathbf{Q}(\sqrt{-1})$ to rewrite $c_f(x^2 + Dy^2)$ in terms of solutions to $a^2 + b^2 = x^2 + Dy^2$, completing the proof via a theorem of Hooley [43] concerning the equi-distribution of congruential roots of a polynomial to varying moduli.

I was unaware of the work of Friedlander and Iwaniec [29] on small solutions of indefinite ternary quadratic forms (such as $a^2 + b^2 - x^2 = Dy^2$ for small fixed y), which could likely (with sufficient effort) be turned into a similar proof, though using (ultimately) the Iwaniec bound [44] rather than Hooley's result. In particular, the change of variables they use following (2.2) is similar to that in [84]. This [29] depends on their joint work [24] with Duke on Weyl sums for quadratic roots.

While in [84] we only saved a small power of $(\log D)$ (which suffices for the class number 1 problem), a rendition involving such Weyl sums would likely instead be able to save a small power of D.

1.6.2. Our second proof [85] then brought spectral techniques similar to those of Templier and Tsimerman [80] into the picture, considering the problem of Hecke eigenvalues over quadratic sequences, and ultimately relying on Duke's bound [23] (following Iwaniec [44]) for Fourier coefficients of Maass forms of half-integral weight to show the desired cancellation.

I should emphasize that Templier in [79] did "everything but" point out that class number 1 follows from analytic rank 2. See (1.6) of Theorem 2 of [79], wherein the left side is the average of $L'(f\otimes\theta,1)$ over class group characters θ and is thus 0 under our assumptions, while the right side is proportional to $L'_\chi(1)\sim\pi^2/6$ as $D\to\infty$.[7]

In retrospect, it seems I was a bit cavalier in [85, §5.8], when proposing that one could show cancellation in $\sum_x c_f(x^2+Dy^2)$ by: bounding a Poincaré series via its norm, writing said norm in terms of a sum of Kloosterman sums by Proskurin's version of the Kuznetsov trace formula, and then utilizing progress toward the Linnik-Selberg conjecture to obtain an adequate result. In particular, given the size of the argument(s) of the Kloosterman sum, one needs to beat the power 1/4 in the mn-dependence in a result such as that of Sarnak and Tsimerman [67]. They do this in weight zero (and it has been generalized to nontrivial level), but in half-integral weight this was only recently considered by Dunn [25], who adapts work of Ahlgren and Andersen for the purpose. Though his result is for the η-multiplier, the same strategy should readily apply to the Θ-multiplier case.

In [25] Dunn only beats the 1/4 barrier for the opposite sign case (Theorem 1.1). The difficulty in the same-sign case is in the holomorphic contribution (see (7.2) and (10.2)), where one wants to use the Iwaniec bound for Fourier coefficients of half-integral weight modular forms in order to beat the 1/4 exponent, but there are difficulties involved with uniformity in the weight. This was subsequently overcome in his later work [1] with Ahlgren. (I thank Dunn for explaining his results herein to me, and Andersen for his help too).

One can also follow Blomer's initial work [9] concerning sums of Hecke eigenvalues over quadratic sequences more closely and get to a similar juncture: in the middle of (3.5) he writes the relevant expression as

$$\sum_h\sum_c \frac{K(4m-h^2,-\Delta,c)}{c}\cdot\frac{g(h;c/4)}{\sqrt{c}}e(-2sh/c),$$

and again one needs to be able to beat the 1/4 exponent in the mn-dependence to be able to profitably apply sums of Kloosterman sums (with partial summation).

I would thus say that there are now as many as six ways to show adequate cancellation in $\sum_x c_f(x^2+Dy^2)$ and thereby obtain a resolution of the class number 1 problem via a suitable L-function of analytic rank 2. Three of these are extant in the literature: Templier's proof [79] using the δ-symbol method; our proof [84] controlling solutions to $a^2+b^2-x^2=Dy^2$ by using Hooley's root equi-distribution; and the spectral methods of Templier and Tsimerman [80] as simplified in [85].

The other three methods are: analysis of solutions to $a^2+b^2-x^2=Dy^2$ by Weyl sums for quadratic roots from [29] and [24]; spectral methods involving the Poincaré series norm ([85, §5.8]) in combination with the work of Dunn and Ahlgren [25, 1] to show sufficient cancellation in sums of Kloosterman sums; and Blomer's transformation (via Poisson summation) of sums of Kloosterman sums over a quadratic

[7] This [79] uses the δ-symbol method, while [80, (1.9)] simplifies via spectral theory. Note also that (1.9) is (1.10) in the preprint version, and α and β need a $L_\chi(1)$-factor.

sequence into sums of Kloosterman sums with Θ-multiplier [9], again relying on the work of Dunn and Ahlgren to bound the latter.

Of the six, all of them except [84] depend on Iwaniec's seminal bound [44] for Fourier coefficients of modular forms of half-integral weight.

1.6.3. The comments on root numbers in [85, §5.9.1] admit some more discussion. Firstly, at least in weight 2 with the goal of an even parity curve all of whose twists by negative fundamental discriminants have odd parity, one can take an elliptic curve of conductor p^2 with $p \equiv 7\,(12)$ and Kodaira symbol either II or IV at p. Examples already exist in the list [26] given by Edixhoven, de Groot, and Top, namely with $p \in \{43, 307, 739, 1999, 2251, 3331, 4423, \ldots\}$. One still has to achieve the rank 2 condition, but this can be done simply by taking one such twist in the family, for instance, the twist by -4 of 1849b. More directly, one can take the rank 2 curve $[1, -11, -492, 4302]$ of conductor 7867^2.

On the other hand, it seems that the second part of [85, Footnote 24] concerning non-selfdual L-functions might be somewhat wishful thinking, in part due to the fact that I misparsed [45, §23.5], failing to note they *assume* that "$P(1, f)$ is also real" on page 537 (I had thought they were asserting this followed from ε being real, particularly as the intervening clause about $\mathrm{Sym}^2 f$ having real coefficients *does* in fact follow from this). Without this condition on $P(1, f)$, one would require a replacement for (23.40) along the lines that the product of the root numbers is not close to $-e^{i \arg S}(-1)^g$ (which is $-(-1)^g$ in the selfdual case) and in general one would need to figure out a way to show the argument of S does not interact with the root number product in this way. For the case of class number 1 this follows readily since there are no small noninert primes, but in general I don't think (for instance) one so easily gets a bound of roughly $\gg (\log D)$ from a nonselfdual L-function with a central zero of order 2 (or more); see also their comments at the start of §23.7.

1.7. The Deuring decomposition around $s = 1$ given in §6 grew out of currently unpublished work [86] on the Deuring-Heilbronn phenomenon (or Deuring's zero-spacing phenomenon) that was done in 2016 or earlier. The idea of using it to obtain the current result(s) was conceived at Bill Duke's 60th birthday conference at ETH Zürich in June 2019. I thank H. Iwaniec for his interest in this topic.

1.8. **Notation.** We use the "∂"-symbol rather than "d" for measures in integrals. We write L-functions with subscripts and consider the objects involved always to be primitive.

We let $K = \mathbf{Q}(\sqrt{\Delta})$ be our quadratic field of interest, with Δ a fundamental discriminant and $D = |\Delta|$. Its associated character is χ, whose L-function is $L_\chi(s)$.

We will be particularly interested in L-functions of weight 2 newforms f of level N_f, writing $L_f(s)$. The completed version is $\tilde{\Lambda}_f(s) = \Gamma(s) L_f(s) (\sqrt{N_f}/2\pi)^{s-1}$ where we have chosen to scale the conductor factor to be 1 at the central point $s = 1$. The product L-function $L_f(s) L_{f\chi}(s)$ will be notated as an "induced" L-function as $L_f^K(s)$, and similarly with $\tilde{\Lambda}_f^K(s)$. The sign of the functional equation for $\tilde{\Lambda}_f^K(s)$ is ϵ_f^K, and the symmetric-square L-function of f is denoted by $L_{S^2 f}(s)$.

The Deuring approximant $T_f(z)$ is $\Gamma(z)^2 (MD)^{z-1} E_f(z) L_{S^2 f}(2z)/\zeta(2z-1)$, with here $4\pi^2 M = \sqrt{N_f N_{f\chi}}$ while $E_f(z)$ is defined in §4.1.4, along with the related coefficients $r_f^K(n)$. The product $\mathcal{P}_\sigma(D)$ is defined in §5.2 and the convenient $\mathcal{R}(\chi)$ appears in §4.1.6.

The Mellin transform I_j of $\Gamma(s)^2/(s-1)^j$ is described in §3.2.1. The Dirichlet series coefficients of $\zeta_K(s)/\zeta(2s)$ are denoted by $R_K^\star(n)$, and this is split at $\sqrt{D}/4$ according to $R_K^\star = R_K^{\star\mathrm{s}} + \tilde{R}_K^{\star\mathrm{s}}$. We use the $\ll$- and O-notation, and also with the latter the Θ-notation, which represents a constant bounded by 1.

2. Deuring decompositions

We first discuss the history of Deuring decompositions for Dirichlet L-functions, and then explain how we will generalize this to modular form L-functions.

2.1. Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field and χ its character. For the case where the class number $h_K = 1$, in 1933 Deuring [20] derived a decomposition of the Dedekind ζ-function that stated

$$\Gamma(z)\Big(\frac{\sqrt{D}}{2\pi}\Big)^{z-1/2}\zeta(z)L_\chi(z) = T(z) + T(1-z) + U(z)$$

where

$$T(z) = \Gamma(z)\Big(\frac{\sqrt{D}}{2\pi}\Big)^{z-1/2}\zeta(2z)$$

and $U(z)$ is an error term. Although one can be more general, it is probably easiest to restrict to the line $z = 1/2 + iy$, where the error term can be shown to be $\ll \sqrt{y_\star}|\Gamma(1/2+iy)|/D^{1/4}$ where $y_\star = |y| + 5$. (The principal form (with its small minimum) allows different techniques in analysis, and Deuring obtains the alternative error bound $\ll |\Gamma(1/2+iy)|\exp(-\pi\sqrt{D}/y_\star)$; the above version comes more readily from Lavrik's methods [52]).

This was later generalized in various directions, notably to other class numbers by Selberg and Chowla [69] (via an Epstein ζ-function decomposition), with related results by Rankin [62] and Bateman and Grosswald [5]. An advance of Stark [75] made clear how to generalize to twists by nontrivial Dirichlet characters (he previously used characters modulo 8 and 12 in his proof [72] of class number 1); for prime moduli this had been utilized by Heilbronn and Linfoot [41].

2.1.1. Let us make a general statement concerning such decompositions. Consider a primitive real character θ of conductor k coprime to D. Then we have

$$\tilde{\Lambda}_\theta^K(z) = \Gamma(z)\Big(\frac{k\sqrt{D}}{2\pi}\Big)^{z-1/2}L_\theta(z)L_{\theta\chi}(z) = T_\theta(z) + T_\theta(1-z) + U_\theta(z) \qquad (1)$$

where

$$T_\theta(z) = \Gamma(z)\Big(\frac{k\sqrt{D}}{2\pi}\Big)^{z-1/2}\zeta(2z)E_\theta(z)$$

and $E_\theta(z)$ takes into account ramified primes for θ and small noninert primes for K. We split this up as $E_\theta = E_\theta^{\mathrm{r}}E_\theta^{\mathrm{m}}$ where in our case of θ quadratic we have the ramified contribution as

$$E_\theta^{\mathrm{r}}(z) = \prod_{p|k}\big(1 - 1/p^{2z}\big).$$

There is some flexibility in the choice of E_θ^{m}; one natural choice is to write it terms of the multiset $\mathcal{M}_D$ of minima of reduced binary quadratic forms of discriminant $-D$ as

$$E_\theta^{\mathrm{m}}(z) = \sum_{a\in\mathcal{M}_D}\frac{\theta(a)}{a^z}.$$

Another possibility is to write it in terms of an Euler product as

$$E_\theta^{\mathrm{m}}(z) = \prod_{p\le\sqrt{D/4}} \frac{1+\theta(p)/p^z}{1-\chi\theta(p)/p^z}.$$

In particular, this does not vanish for $\mathrm{Re}(z) > 0$. One can show that the Dirichlet series and Euler product are good approximations to each other for $\mathrm{Re}(z) \ge 1/2$ roughly when class number is $\ll D^{1/6}$ (it appears that Montgomery and Weinberger [57, Paragraph 2] placed this change of aspect at $D^{1/8}$ rather than $D^{1/6}$).

The bound on the error term on the central line is now

$$\bigl|U_\theta(1/2+iy)\bigr| \ll \sqrt{y_\star}\bigl|\Gamma(1/2+iy)\bigr| \cdot \frac{h_K\sqrt{k}}{D^{1/4}} \log Dy_\star.$$

2.1.2. For real quadratic fields the Γ-factor is either $\Gamma(z/2)^2$ or $\Gamma(z/2+1/2)^2$ depending on whether θ is even or odd, and the analog of the sum-version for E_θ should be a sum over coefficients up to $\sqrt{D}/4$ of $\prod_p (1+\theta(p)/p^z)/(1-\chi\theta(p)/p^z)$. The class number h_K will be replaced by $\sqrt{D}L_\chi(1)$.

For nonreal characters θ one should replace $\zeta(2z)$ by $L_{\theta^2}(2z)$ and $T_\theta(1-z)$ by $T_{\bar\theta}(1-z)$ while including a sign ϵ_θ^K. Moreover, imprimitive $\theta\chi$ can be handled by suitable modifications to the Euler factor and conductor for primes dividing $\gcd(D,k)$. One can also include class group characters if desired.

2.1.3. A principal application of such a decomposition is, following Deuring [21], to show that the product L-function has a regular distribution of zeros, corresponding to solutions of $T_\theta(z) + T_\theta(1-z) = 0$ with small error by Rouché's theorem. In particular, the nonexceptional zeros up to height roughly $\sqrt{D}/kh_K^2$ lie on the central line. However, such results seem rarely to be recorded in the literature. We refer the reader to our draft manuscript [86] from a few years ago.

We can note that the result of Conrey and Iwaniec [14], involving sufficiently many suitably subnormal gaps of L-function zeros implying $L_\chi(1) \gg 1/(\log D)^{99}$, softens the effect of E_θ by not needing to control it on a pointwise basis, but only "on average", which indeed allows them much more leeway.

2.1.4. We should make a few comments about the methodology used to prove such Deuring decompositions. Typically the error term $U_\theta(z)$ comes about from an exact formula involving K-Bessel functions (see [5, (4)]), which are then estimated in a suitable manner. However, the derivation of such an exact formula in the first place often involves (e.g.) exploiting the periodic nature of the Dirichlet series coefficients, for instance by Poisson summation (cf. [75, (18)]).

Rather than follow this path, we instead choose to use a method coming from work of Lavrik [52], using a general schema for approximate functional equations.

Our reference text is Rubinstein [66]. Writing $\tilde\Lambda_\theta^K(s)$ as in (1), the idea is to consider

$$\left(\int_{(2)} - \int_{(-1)}\right) \tilde\Lambda_\theta^K(s)\, Y(s)\, \frac{\partial s}{2\pi i}$$

with $Y(s) = 1/\alpha^s(s-z)$ where $\alpha = z/|z|$, and after substituting $s \to 1-s$ and applying the functional equation to the second integral we find

$$\frac{\tilde\Lambda_\theta^K(z)}{\alpha^z} = \int_{(2)} \tilde\Lambda_\theta^K(s)\, \frac{\alpha^{-s}}{s-z}\, \frac{\partial s}{2\pi i} - \int_{(2)} \tilde\Lambda_\theta^K(s)\, \frac{\alpha^{s-1}}{1-s-z}\, \frac{\partial s}{2\pi i},$$

and can then expand the L-series as Dirichlet series and the resulting integrals as inverse Mellin transforms. The effect of $1/\alpha^z$ is to offset the vertical decay of $\Gamma(z)$.

One then writes the L-function product in $\tilde{\Lambda}_\theta^K(z)$ as a Dirichlet series of the main term (for instance $\zeta(2z)$ in the simplest case) plus an error term, and moves the contour to the left for the main term while estimating the error term with bounds for the incomplete Γ-function. The residues from the contour movement do indeed give $T_\theta(z)+T_\theta(1-z)$, while the complementary integral is adequately bounded by controlling $E_\theta(z)$ (which is one reason its sum-version is to be preferred).

2.2. The above Lavrik/Rubinstein methodology can also be used for modular form L-functions. For simplicity we take f as a primitive modular eigenform of weight 2 and level $\Gamma_0(N_f)$ with trivial character.

We then consider the integral

$$\left(\int_{(2)}-\int_{(0)}\right)\tilde{\Lambda}_f^K(s)\,Y(s)\,\frac{\partial s}{2\pi i}$$

with $Y(s)=1/\alpha^s(s-z)$ with now $\alpha=(z/|z|)^2$, and obtain

$$\tilde{\Lambda}_f^K(z)=\Gamma(z)^2\Big(\frac{\sqrt{N_fN_{f\chi}}}{4\pi^2}\Big)^{z-1}L_f(z)L_{f\chi}(z)=T_f(z)+\epsilon_f^K T_f(2-z)+U_f(z)$$

where $\epsilon_f^K\in\{-1,+1\}$ has $\tilde{\Lambda}_f^K(z)=\epsilon_f^K\tilde{\Lambda}_f^K(2-z)$ and

$$T_f(z)=\Gamma(z)^2\Big(\frac{\sqrt{N_fN_{f\chi}}}{4\pi^2}\Big)^{z-1}\frac{L_{S^2f}(2z)}{\zeta(2z-1)}E_f(z).$$

Again $E_f(z)$ has terms involving ramified primes of f and small noninert primes of K. In this degree 2 case, the effective length of the approximate functional equation is now $Dy_\star^2$ instead of $\sqrt{D}y_\star$, and consequently the bound on the error $U_f(z)$ is significantly worse, as on the central line we only obtain

$$\big|U_f(1+iy)\big|\ll y_\star\log y_\star\cdot\big|\Gamma(1+iy)\big|^2\cdot\sqrt{N_f}\cdot\,h_K\sum_{a\in\mathcal{M}_D}\frac{1}{a}.$$

This is sufficient to reobtain Goldfeld's result: assuming that $L_f^K(s)$ has a central zero of order $\tilde{r}\geq 4$, by taking a circle of radius $1/8$ about $s=1$ and applying Cauchy's derivative theorem we get

$$0=(\tilde{\Lambda}_f^K)^{(\tilde{r}-2)}(1)=2T_f^{(\tilde{r}-2)}(1)+O\Big(\tilde{r}!8^{\tilde{r}}\sqrt{N_f}\cdot h_K\sum_a\frac{1}{a}\Big),$$

while $T_f^{(\tilde{r}-2)}(1)\approx 2L_{S^2f}(2)\cdot E_f(1)(\log D)^{\tilde{r}-3}$, showing h_K to be non-negligible.

There is some issue here with the $\zeta(2z-1)$ in the denominator of $T_f(z)$, though when the class number is small we can (for instance) use the Deuring decomposition for $\zeta_K(z)$ to show an adequate zero-free region for it.[8]

When K is real quadratic a similar result holds, with the class number being replaced by $\sqrt{D}L_\chi(1)$ while the reciprocal sum over minima should be taken over coefficients of $\zeta_K(s)/\zeta(2s)$ up to $\sqrt{D}/4$.

[8]Alternatively, one can start with an integral of the symmetrical $\tilde{\Lambda}_f^K(s)\zeta(2s-1)\zeta(3-2s)(s-1)^2$ multiplied by $Y(s)$ as above, then transform $\zeta(3-2s)=\pi^{5/2-2s}\zeta(2s-2)\Gamma(s-1)/\Gamma(3/2-s)$ by the functional equation, and expand the ζ-terms as Dirichlet series (the Γ-quotient exhibits no new difficulties in its Mellin transform). Yet this is not a lossless procedure, especially in the y-aspect.

2.3. We shall not show the details of the above two Deuring decompositions (again referring to [86] for those), but instead concentrate on our main goal of a Deuring decomposition that takes the central vanishing behavior of $\tilde\Lambda_f^K(z)$ into account.

We let $\tilde r$ be a lower bound for the order of vanishing of $L_f^K(z)$ at the central point $z = 1$. By considering

$$\left(\int_{(2)} - \int_{(0)}\right) \tilde\Lambda_f^K(s)\, Y(s)\, \frac{\partial s}{2\pi i} \tag{2}$$

with $Y(s) = 1/(s-1)^{\tilde r}(s-z)$ we obtain the following result (see Proposition 6.1.1).

Suppose that $|z-1| \le 1/99$ and $\pi h_K \le (\log D)^3/10^6$ (to control E_f). Then

$$\tilde\Lambda_f^K(z) = T_f(z) + \epsilon_f^K T_f(2-z) - S_f^{\tilde r}(z) + O\Big(\sqrt{N_f}\cdot\, h_K \sum_{a\in\mathcal{M}_D} \frac{1}{a}\cdot 2^{\tilde r}|z-1|^{\tilde r}\Big)$$

where

$$T_f(z) = \Gamma(z)^2\Big(\frac{\sqrt{N_f N_{f\chi}}}{4\pi^2}\Big)^{z-1}\frac{L_{S^2 f}(2z)}{\zeta(2z-1)}E_f(z)$$

and

$$S_f^{\tilde r}(z) = \sum_{b=0}^{\tilde r-1}\big[1+\epsilon_f^K(-1)^b\big]\cdot\frac{T_f^{(b)}(1)}{b!}(z-1)^b.$$

The effect of $S_f^{\tilde r}(z)$ is to remove the lower-order Taylor series terms in the expansion of $T_f(z) + \epsilon_f^K T_f(2-z)$ about $z = 1$.

We will simultaneously derive an analogous result for K real quadratic, and by using a decomposition of $\zeta_K(s)$ due to Goldfeld, it turns out that the constant in the error term is not too much worse than in the imaginary quadratic case.

2.4. Let us outline the contents below. In §3 we derive some basic results about the paucity of small noninert primes when $L_\chi(1)$ is small. We then in §4 recall the theory of modular form L-functions, and adapt the previous result concerning small noninert primes to this setting, showing that the complementary term to our choice of $E_f(z)$ is suitably bounded. In §5 we then give sufficient bounds on $E_f(z)$ itself, to allow us to show the main Deuring decomposition in §6.

We then turn to a different direction in §7: given a modular form L-function coming from an elliptic curve of rank 5, it has at least a triple central zero by work of Gross and Zagier combined with that of Kolyvagin, and we show that by computing its third central derivative to high precision (and indeed getting it is numerically 0, as predicted by the conjecture of Birch and Swinnerton-Dyer) it has exactly two other zeros close to the central point. We then specialize to a collection of 12 specific elliptic curves, and describe the calculations regarding $L_f'''(1)$ for each.

We wish to juxtapose this computational result with the central behavior in our Deuring decomposition, and in §8 we prepare for this by deriving more results about $E_f(z)$ and its derivatives (again using the smallness of $L_\chi(1)$), and also for the symmetric-square L-functions of our 12 curves. In §9 we then use the explicit bounds on the zeros obtained from the computations in §7 in conjunction with the Deuring decomposition from §6, and obtain an explicit lower bound on $L_\chi(1)$ in terms of the precision to which the third central derivative is known to be zero. This requires $f\chi$ to have odd parity, and in §10 we describe how to ensure that either this parity condition holds, or there are already sufficiently many small split primes, in which case the lower bound on $L_\chi(1)$ follows readily in any event.

3. Background on quadratic fields and binary quadratic forms

We show some assorted results in the theory of binary quadratic forms. Some of this is already due to Gauss [30]. Although we have made some attempt to achieve decent constants, we make no great effort to optimize them (and indeed, have aimed for simplicity following Goldfeld's presentation [33] more than anything).

We let $K = \mathbf{Q}(\sqrt{\Delta})$ be a quadratic field of fundamental discriminant Δ, with χ its quadratic character. We write $D = |\Delta|$ for the absolute value of the discriminant. So as to avoid units in the imaginary quadratic case, we assume that $D > 4$.

The character χ is odd when $\chi(-1) = -1$ and even when $\chi(-1) = +1$, the former corresponding to the imaginary quadratic case and the latter the real quadratic.

3.1. The primitivized Dedekind ζ-function for K is defined as

$$\frac{\zeta_K(s)}{\zeta(2s)} = \frac{\zeta(s)L_\chi(s)}{\zeta(2s)} = \prod_p \frac{1+1/p^s}{1-\chi(p)/p^s} = \sum_{n=1}^{\infty} \frac{R_K^\star(n)}{n^s},$$

where in the imaginary quadratic case $R_K^\star(n)$ counts half the number of primitive (that is, coprime) representations of n by reduced binary quadratic forms of discriminant $-D$. An analogous accounting for $R_K^\star(n)$ holds true in the real quadratic case when one also requires the representations to be primary (see [18, §6 (12)]).

3.1.1. We write (a, b, c) for an integral binary quadratic form $ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = \Delta$, and the operation of composition on the set of such forms yields a group. The class group identifies forms under G-equivalence where G is $\mathbf{SL}_2(\mathbf{Z})$ for K imaginary and $\mathbf{GL}_2(\mathbf{Z})$ for K real, and this class group is indeed isomorphic to the class group of K (cf. Cox [15, p. 128ff]). We write h_K for its size.

3.1.2. In the imaginary quadratic case the class number formula of Dirichlet states that $L_\chi(1) = \pi h_K/\sqrt{D}$ (recall we are assuming $D > 4$ so only ± 1 are units).

In the real quadratic case the fundamental unit $\epsilon_0 = (u + v\sqrt{D})/2$ corresponds to a minimal solution of $u^2 - Dv^2 = \pm 4$, with $u, v > 0$. This fundamental unit thus has norm ± 1, and (since $v \geq 1$) we have the lower bound $\epsilon_0 \geq (\sqrt{D-4} + \sqrt{D})/2$. Here the class number formula states $L_\chi(1) = 2(h_K \log \epsilon_0)/\sqrt{D}$.

3.1.3. Two forms are said to be in the same genus if they are locally equivalent at all primes (including ∞). The number of genera is 2^t where in the imaginary quadratic case t is one less than the number of prime divisors of D, while in the real quadratic case it is also one less, except when the fundamental unit has norm $+1$ when it is two less.[9] The number of genera divides the class number, and indeed the genus class group is coarser than the class group.

3.1.4. A positive definite binary quadratic form is reduced when $-a < b \leq a < c$ or $0 \leq b \leq a = c$, and in the indefinite case when $0 < \sqrt{D} - b < 2|a| < \sqrt{D} + b$. In the former case each equivalence class of forms has exactly one reduced form, and a is its minimum. In the latter case there can be more than one reduced form in an equivalence class, and we choose (up to sign) a canonical representative in each class by minimizing $|a|$ (and then $|b|$ if necessary), referring to this $|a|$ as the "minimum" of the class. We then write sums over (a, b, c), or (a) as a shorthand, as being over such canonical reduced forms, one per equivalence class.

[9] It is also this case of norm $+1$ when equivalence under $\mathbf{SL}_2(\mathbf{Z})$ differs from $\mathbf{GL}_2(\mathbf{Z})$.

3.1.5. We split $R_K^\star(n) = R_K^{\star\mathrm{s}}(n) + \tilde{R}_K^{\star\mathrm{s}}(n)$, the former supported on $n \le \sqrt{D}/4$ and the latter on $n \ge \sqrt{D}/4$. (This splitting is somewhat arbitrary, though it coincides with Goldfeld's Lemma 4).

We also introduce $R_K^{\star\mathrm{m}}(n)$, which is the number of times that n appears in the multiset of minima of canonical reduced forms. For its complement we again write $\tilde{R}_K^{\star\mathrm{m}}(n) = R_K^\star(n) - R_K^{\star m}(n)$. We can note that $\tilde{R}_K^{\star\mathrm{s}}(n)$ is 0 for $n \le \sqrt{D}/4$ and thus (trivially) bounded above by $\tilde{R}_K^{\star\mathrm{m}}(n)$ for such n, while for $n \ge \sqrt{D}/4$ it is equal to (and thus bounded by) $R_K^\star(n) = \tilde{R}_K^{\star\mathrm{m}}(n) + R_K^{\star\mathrm{m}}(n)$.

3.1.6. It is also useful to recall Goldfeld's decomposition (Theorem 4, page 636) of $\zeta_K(s)$ in the real quadratic case. We have (for $\sigma > 1/2$)

$$\zeta(s)L_\chi(s) = \frac{\Gamma(s)}{\Gamma(s/2)^2} \sum_{(a,b,c)} \sum_{n=1}^{M} \left[\int_{H_{n-1}}^{H_n} \left(\frac{2\zeta(2s)}{(\alpha^\star)^s} + \frac{\pi}{\sqrt{D}} \frac{s}{s-1} \left(\frac{D}{\alpha^\star}\right)^{1-s} \right) \frac{\partial\varphi}{\varphi} \right] + Z_{\mathrm{r}}(s)$$

where

$$|Z_{\mathrm{r}}(s)| \le \left| \frac{\Gamma(s)}{\Gamma(s/2)^2} \right| \cdot \frac{4|s|}{\sigma - 1/2} \sum_{(a,b,c)} \sum_{n=1}^{M} \left[\int_{H_{n-1}}^{H_n} \left(\frac{1}{2}\left(\frac{\alpha^\star}{D}\right)^\sigma + \frac{(\alpha^\star)^{\sigma-1}}{D^{\sigma-1/2}} \right) \frac{\partial\varphi}{\varphi} \right].$$

The sum over (a,b,c) is over reduced (inequivalent) canonical forms,[10] while M is the least common multiple of 2 and the length k of the period of the continued fraction for $\omega = (-b+\sqrt{D})/2a = [0, \overline{b_1, b_2, \ldots, b_k}]$. Taking $A_v/B_v = [0, b_1, \ldots b_v]$ to be the vth continued fraction convergent, the limits of integration H_n are

$$H_n = \frac{|B_n\omega - A_n|}{|B_n\bar{\omega} - A_n|} \cdot \frac{1}{2} \left[\frac{\sqrt{D}}{|a|} B_n^2 + \sqrt{DB_n^4/a^2 - 4} \right].$$

The variable $\alpha^\star$ depends on φ, and indeed is

$$\alpha^\star = (\varphi + 1/\varphi) \cdot |a| \cdot |B_n\omega - A_n| \cdot |B_n\bar{\omega} - A_n|.$$

We also have $\alpha^\star \le 5\sqrt{D}$ (see (14) and (20) of [33]).

Moreover, as can be seen by following Goldfeld's (11) and (12) and the splitting off the $n = 0$ terms in his proof of Theorem 4, the terms $\zeta(2s)\sum_{(a)} 1/|a|^s$ over minima of canonical reduced forms exactly corresponds to the term $2\zeta(2s)/(\alpha^\star)^s$ in the integral. Thus we have

$$\zeta(s)L_\chi(s) - \sum_{(a,b,c)} \frac{\zeta(2s)}{|a|^s} = \frac{\Gamma(s)}{\Gamma(s/2)^2} \frac{\pi}{\sqrt{D}} \frac{s}{s-1} \sum_{(a,b,c)} \sum_{n=1}^{M} \left[\int_{H_{n-1}}^{H_n} \left(\frac{D}{\alpha^\star}\right)^{1-s} \frac{\partial\varphi}{\varphi} \right] + Z_{\mathrm{r}}(s).$$

3.1.7. In the imaginary quadratic case we could perhaps work more simply by counting lattice points in ellipses, but can also note Goldfeld's Theorem 3 (following Iseki), and again subtracting off the minima of reduced forms we get

$$\zeta(s)L_\chi(s) - \sum_{(a,b,c)} \frac{\zeta(2s)}{a^s} = \frac{\pi}{\sqrt{D}} \frac{s}{s-1} \sum_{(a,b,c)} \left(\frac{D}{4a}\right)^{1-s} + Z_{\mathrm{i}}(s)$$

[10] Although I'm not sure he ever states it explicitly, Goldfeld appears to be using equivalence under $\mathbf{GL}_2(\mathbf{Z})$; for instance on page 633 he describes the correspondence to ideal classes.

where for $\sigma > 1/2$ we have

$$|Z_{\mathrm{i}}(s)| \le \frac{|s|}{\sigma - 1/2} \sum_{(a,b,c)} \Bigl(1 + \frac{\sqrt{D}}{a}\Bigr)\Bigl(\frac{D}{4a}\Bigr)^{-\sigma}.$$

This error term is given in the last display in Goldfeld's proof of Theorem 3, then simplified via $a \le \sqrt{D/3}$ when stating the Theorem (he has a typo with $a \le \sqrt{D}/3$).

3.2. We wish to derive summation bounds for $\tilde{R}_K^{\star s}(n)$, in the spirit of Goldfeld's Lemmata 5 and 6. However, it is technically more convenient to include a weighting by a Mellin transform. (Some of this is inspired by Oesterlé's presentation [58]).

3.2.1. First we derive a bound for a Mellin transform that appears below.

Lemma 3.2.2. *For $u > 0$ and $j \ge 2$ we have*

$$0 \le \int_{(2)} u^s \frac{s-1/2}{s-3/2} \frac{\partial s/2\pi i}{(s-1)^j} \le 2^j u^{3/2}.$$

Proof. For $u \le 1$ the result follows by moving the contour to the right, as the integral is zero. Otherwise, for $j \ge 1$ by moving the contour to the left we have

$$W_j(u) = \int_{(2)} \frac{u^s}{s-3/2} \frac{\partial s/2\pi i}{(s-1)^j} = 2^j u^{3/2} - 2u \sum\sum_{l+m=j-1} 2^l \frac{(\log u)^m}{m!}$$
$$= 2^j u^{3/2} - 2u \sum_{m=0}^{j-1} 2^{j-1} \frac{(\log \sqrt{u})^m}{m!} = 2^j \Bigl(u^{3/2} - u \sum_{m=0}^{j-1} \frac{(\log \sqrt{u})^m}{m!}\Bigr),$$

so is positive by comparison to the Taylor series of $e^{\log \sqrt{u}}$. Thus the desired integral

$$W_{j-1}(u) + W_j(u)/2 = \int_{(2)} u^s \frac{(s-1)+1/2}{s-3/2} \frac{\partial s/2\pi i}{(s-1)^j}$$

satisfies the bounds given in the Lemma. □

3.2.3. Next we give an upper bound on another Mellin transform. We could arrange this somewhat differently, but the given version has some simplifications in the various computations. For $u > 0$ we define the Mellin transform pair

$$B(u) = \int_{(2)} u^{-s} \frac{\Gamma(s-1/2)\Gamma(s)}{\Gamma(s/2-1/4)^2} \frac{\partial s}{2\pi i} \quad \text{so that} \quad \frac{\Gamma(s-1/2)\Gamma(s)}{\Gamma(s/2-1/4)^2} = \int_0^\infty u^s\, B(u) \frac{\partial u}{u}.$$

Lemma 3.2.4. *We have*

$$\int_0^\infty \sqrt{u} |B(u)|\, \partial u \le 0.297.$$

Proof. Using Legendre's duplication formula and then moving the contour left (thereby picking up putative poles at $s = 1/2 - k/2$ for $k \ge 0$) we find that

$$B(u) = \int_{(2)} u^{-s} \frac{\sqrt{\pi}\Gamma(2s-1)2^{2-2s}}{\Gamma(s/2-1/4)^2} \frac{\partial s}{2\pi i} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} u^{k/2-1/2} \frac{(-1)^k}{k!} \frac{2^{1+k}}{\Gamma(-k/4)^2}.$$

By asymptotics for Meijer G-functions (see Braaksma [11], or Luke [55, (5.7.13)]) we have $B(u) \sim \exp(-u/2) \cdot \sqrt{u/32\pi}$ as $u \to \infty$, and explicitly $|B(u)| \le 10^{-10}/u^2$ for $u \ge 63$, with a unique root $r \approx 0.52458491$ satisfying $B(r) = 0$. Again by Mellin inversion we know that $\int_0^\infty \sqrt{u} B(u) \partial u = \Gamma(3/2)/\Gamma(1/2)^2 \approx 0.28209$, and thus the estimate $\int_0^r \sqrt{u} B(u) \partial u \approx -0.00708$ suffices to bound the integral as stated. □

3.2.5. Finally we get to our main weighting functions. For $j \geq 2$, we define

$$I_j(w) = \int_{(2)} w^{-s} \frac{\Gamma(s)^2}{(s-1)^j} \frac{\partial s}{2\pi i} = \int_{(2)} \int_0^\infty \int_0^\infty (u_1 u_2/w)^s e^{-u_1} \frac{\partial u_1}{u_1} e^{-u_2} \frac{\partial u_2}{u_2} \frac{\partial s/2\pi i}{(s-1)^j}$$
$$= \int_0^\infty \int_0^\infty \int_{(2)} (u_1 u_2/w)^s \frac{\partial s/2\pi i}{(s-1)^j} e^{-u_1} \frac{\partial u_1}{u_1} e^{-u_2} \frac{\partial u_1}{u_2}$$

where the inner integral is nonnegative since (with $j \geq 2$ for absolute convergence)

$$\int_{(2)} y^{-s} \frac{\partial s/2\pi i}{(s-1)^j} = \begin{cases} (\log 1/y)^{j-1}/y(j-1)! \text{ for } y \leq 1, \\ 0 \text{ for } y \geq 1, \end{cases}$$

the proof following respectively by moving the contour to the left or the right. Thus the original integral $I_j(w)$ is nonnegative too.

3.3. Next we show a summation bound for $\tilde{R}_K^{\star s}(n)$ when weighted by $\sqrt{n} I_j(n/X)$.

Proposition 3.3.1. *For $X > 0$ and $j \geq 2$, when $D \geq 100$ we have*

$$\sum_{n=1}^\infty \tilde{R}_K^{\star \mathrm{s}}(n) \sqrt{n} I_j(n/X) \leq (1.257 + 2.091 + 0.319) \cdot 2^j X^{3/2} L_\chi(1).$$

Proof. Rather than work with $\tilde{R}_K^{\star \mathrm{s}}(n)$ it is easier with our above decompositions to consider $\sum_n \sqrt{n} \tilde{R}_K^{\star \mathrm{m}}(n)/n^s = \zeta_K(s-1/2)/\zeta(2s-1) - \sum_{(a)} \sqrt{|a|}/|a|^s$ (see §3.1.5).

By integrating on the 3/2-line we have $|I_j(w)| \leq 2^j/4w^{3/2}$, so that

$$\sum_{n=1}^\infty \tilde{R}_K^{\star \mathrm{s}}(n) \sqrt{n} I_j(n/X) \leq \sum_{n=1}^\infty \tilde{R}_K^{\star \mathrm{m}}(n) \sqrt{n} I_j(n/X) + \sum_{\substack{(a,b,c) \\ |a| \geq \sqrt{D}/4}} \sqrt{|a|} I_j(|a|/X)$$
$$\leq \sum_{n=1}^\infty \tilde{R}_K^{\star \mathrm{m}}(n) \sqrt{n} I_j(n/X) + h_K \frac{2^j X^{3/2}}{\sqrt{D}}.$$

and by Dirichlet's class number formula $h_K/\sqrt{D} \leq L_\chi(1)/\pi$, so the second term here gives the 0.319 contribution in the statement of the Proposition.

3.4. We then expand out $I_j(n/X)$ to get

$$\sum_{n=1}^\infty \tilde{R}_K^{\star \mathrm{m}}(n) \sqrt{n} I_j(n/X) = \int_{(2)} X^s \frac{\Gamma(s)^2}{(s-1)^j} \left[\frac{\zeta_K(s-1/2)}{\zeta(2s-1)} - \sum_{(a,b,c)} \frac{\sqrt{|a|}}{|a|^s} \right] \frac{\partial s}{2\pi i},$$

and insert our above expressions for $\zeta_K(s) - \zeta(2s) \sum_{(a)} 1/|a|^s$.

3.4.1. In the real quadratic case (§3.1.6) this gives a main term T_j^{r} of

$$\frac{\pi}{\sqrt{D}} \sum_{(a,b,c)} \sum_{n=1}^M \int_{H_{n-1}}^{H_n} \int_{(2)} \frac{X^s}{\zeta(2s-1)} \frac{s-1/2}{s-3/2} \frac{\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2} \frac{\Gamma(s)^2}{(s-1)^j} \left(\frac{D}{\alpha^\star}\right)^{3/2-s} \frac{\partial s}{2\pi i} \frac{\partial \varphi}{\varphi}.$$

We write[11] $1/\zeta(2s-1)=\sum_m m\mu(m)/m^{2s}$ and split off one $\Gamma(s)$ in the Γ-quotient, then expand out the Mellin transforms with e^{-u} and the above $B(u)$ to get

$$T_j^{\rm r}=\frac{\pi}{\sqrt{D}}\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\Bigl(\frac{D}{\alpha^\star}\Bigr)^{3/2}\int_0^\infty\int_0^\infty\sum_{m=1}^{\infty}m\mu(m)\times$$
$$\times\int_{(2)}\Bigl(\frac{Xu_1u_2\alpha^\star}{Dm^2}\Bigr)^s\frac{s-1/2}{s-3/2}\frac{\partial s/2\pi i}{(s-1)^j}B(u_1)\frac{\partial u_1}{u_1}e^{-u_2}\frac{\partial u_2}{u_2}\frac{\partial\varphi}{\varphi}.$$

The inner s-integral is bounded by Lemma 3.2.2 to get

$$|T_j^{\rm r}|\le\frac{\pi}{\sqrt{D}}\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\Bigl(\frac{D}{\alpha^\star}\Bigr)^{3/2}\sum_{m=1}^{\infty}m|\mu(m)|\times$$
$$\times\int_0^\infty\int_0^\infty 2^j\Bigl(\frac{Xu_1u_2\alpha^\star}{Dm^2}\Bigr)^{3/2}\bigl|B(u_1)\bigr|\frac{\partial u_1}{u_1}e^{-u_2}\frac{\partial u_2}{u_2}\frac{\partial\varphi}{\varphi},$$

where the m-sum is $\zeta(2)/\zeta(4)$ so that

$$|T_j^{\rm r}|\le 2^j\frac{\pi}{\sqrt{D}}\frac{\zeta(2)}{\zeta(4)}X^{3/2}\cdot\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\frac{\partial\varphi}{\varphi}\cdot\Gamma(3/2)\int_0^\infty\sqrt{u}\,|B(u)|\,\partial u.$$

The previous Lemma 3.2.4 then bounds the u-integral by 0.297, and upon writing ϵ_0 for the fundamental unit, by Goldfeld's integration formula (middle of page 641) for $\partial\varphi/\varphi$ and Dirichlet's class number formula we have

$$\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\frac{\partial\varphi}{\varphi}=\sum_{(a,b,c)}(2\log\epsilon_0)=2h_K\log\epsilon_0=\sqrt{D}L_\chi(1),$$

and inserting this into the above gives the term with 1.257 in the Proposition.

3.4.2. For the secondary term $S_j^{\rm r}$ with $Z_{\rm r}(s)$ we move the contour to $\sigma=3/2$ and bound it there, getting

$$|S_j^{\rm r}|\le\int_{(3/2)}\frac{X^\sigma}{|\zeta(2s-1)|}\left|\frac{\Gamma(s)^2\Gamma(s-1/2)}{\Gamma(s/2-1/4)^2}\right|\cdot\frac{4|s-1/2|}{\sigma-1}\frac{|\partial s|/2\pi}{|s-1|^j}\times$$
$$\times\sum_{(a,b,c)}\sum_{n=1}^{M}\left[\int_{H_{n-1}}^{H_n}\left(\frac{1}{2}\Bigl(\frac{\alpha^\star}{D}\Bigr)^{\sigma-1/2}+\frac{(\alpha^\star)^{\sigma-3/2}}{D^{\sigma-1}}\right)\frac{\partial\varphi}{\varphi}\right]$$

and from $\alpha^\star\le 5\sqrt{D}$ we see the second line is $\le\sqrt{D}L_\chi(1)\cdot(1+\sqrt{5}/2)/\sqrt{D}$. Direct computation bounds the first line as $\le 0.987\cdot 2^jX^{3/2}$, and multiplying these gives the contribution with 2.091 in the Proposition.

3.5. In the imaginary quadratic case (§3.1.7), we similarly have the main term as

$$T_j^{\rm i}=\frac{\pi}{\sqrt{D}}\int_{(2)}\frac{X^s}{\zeta(2s-1)}\frac{s-1/2}{s-3/2}\sum_{(a,b,c)}\Bigl(\frac{D}{4a}\Bigr)^{3/2-s}\frac{\Gamma(s)^2}{(s-1)^j}\frac{\partial s}{2\pi i}.$$

[11]This unrolling of $1/\zeta$ is unneeded in [88, §2.3], as the summation over squares can be accounted more directly. I don't know if there are similar simplifications available here.

We expand both Γ-factors as integrals and $1/\zeta(2s-1)$ as a sum to get

$$T_j^{\mathrm{i}} = \frac{\pi}{\sqrt{D}} \sum_{(a,b,c)} \Big(\frac{D}{4a}\Big)^{3/2} \sum_{m=1}^{\infty} m\mu(m) \times$$
$$\times \int_0^\infty \int_0^\infty \int_{(2)} \Big(\frac{Xu_1u_2}{m^2}\frac{4a}{D}\Big)^s \frac{s-1/2}{s-3/2}\frac{\Gamma(s)^2}{(s-1)^j}\frac{\partial s}{2\pi i}\, e^{-u_1}\frac{\partial u_1}{u_1} e^{-u_2}\frac{\partial u_2}{u_2}.$$

We then bound the inner s-integral by Lemma 3.2.2 to get

$$|T_j^{\mathrm{i}}| \leq \frac{\pi}{\sqrt{D}} \sum_{(a,b,c)} \Big(\frac{D}{4a}\Big)^{3/2} \sum_{m=1}^{\infty} m|\mu(m)| \int_0^\infty \int_0^\infty 2^j \Big(\frac{Xu_1u_2}{m^2}\frac{4a}{D}\Big)^{3/2} e^{-u_1}\frac{\partial u_1}{u_1} e^{-u_2}\frac{\partial u_2}{u_2}$$
$$= 2^j X^{3/2} \cdot \frac{\pi}{\sqrt{D}} h_K \frac{\zeta(2)}{\zeta(4)} \Gamma(3/2)^2 = 2^j X^{3/2} \cdot L_\chi(1) \frac{\pi^2/6}{\pi^4/90}(\pi/4).$$

The multiplier here is $15/4\pi \approx 1.19366$, which is smaller than the 1.257 from the real quadratic case.

3.5.1. For the secondary term S_j^{i} with $Z_{\mathrm{r}}(s)$ we move the contour to $\sigma = 3/2$ and bound it there, getting

$$|S_j^{\mathrm{i}}| \leq X^\sigma \int_{(3/2)} \left|\frac{\Gamma(s)^2}{\zeta(2s-1)}\right| \cdot \frac{|s-1/2|}{\sigma - 1} \frac{|\partial s|/2\pi}{|s-1|^j} \times \sum_{(a,b,c)} \Big(1 + \frac{\sqrt{D}}{a}\Big)\Big(\frac{D}{4a}\Big)^{1/2-\sigma},$$

where the integral is bounded by $0.593 \cdot 2^j$, and by $a \leq \sqrt{D/3}$ and $L_\chi(1) = \pi h_K/\sqrt{D}$ the sum is $\leq 4h_K(1+\sqrt{3}/3)/\sqrt{D} \leq 2.009 L_\chi(1)$. Since $0.593 \cdot 2.009 \leq 2.091$ the statement of the Proposition follows. □

3.6. We can also derive bounds for the summation simply of $\sqrt{n}\tilde{R}_K^{\star\mathrm{s}}(n)$, and as this will only appear in a secondary term, we use the above in conjunction with positivity, and make no attempt to reduce the constant.

Lemma 3.6.1. *For $X \geq 1$ and $D \geq 100$ we have*

$$\sum_{n \leq X} \frac{\tilde{R}_K^{\star\mathrm{s}}(n)}{n} \leq 111 L_\chi(1) \log X.$$

Proof. We can note that $I_2(u) = 2K_0(2\sqrt{u})/u$ in terms of K-Bessel functions, so that $I_2(u) \geq 0.227 \geq 1/5$ for $0 \leq u \leq 1$, and thus by the Proposition we have

$$\sum_{n \leq X} \sqrt{n}\tilde{R}_K^{\star\mathrm{s}}(n) \leq 5 \sum_{n \leq X} \sqrt{n}\tilde{R}_K^{\star\mathrm{s}}(n) I_2(n/X) \leq 5 \cdot 14.668 X^{3/2} L_\chi(1).$$

The result follows by partial summation, using that $\tilde{R}_K^{\star\mathrm{s}}(n) = 0$ for $n \leq \sqrt{D}/4$. □

4. Background on modular form L-functions

We now recall some basic information about modular form L-functions. We do this only in weight 2, though the generalization to other weights should be clear if desired. We then consider an approximation to the Landau product $L_f(s)L_{f\chi}(s)$ and show bounds on the error therein (in terms of $L_\chi(1)$).

4.1. Let f be a modular newform of weight 2 for $\Gamma_0(N_f)$ with trivial character, and put $f\chi$ for its quadratic twist by χ. Although it is not strictly necessary to do so, it will be convenient to assume that f has integral coefficients, and is thus associated to an elliptic curve isogeny class by modularity theorems.

Following Hecke [38], the L-function of f is defined by the Euler product

$$L_f(s) = \prod_p (1-\alpha_p/p^s)^{-1}(1-\beta_p/p^s)^{-1}$$

where $\alpha_p + \beta_p = c_f(p)$ is the pth coefficient of the Fourier expansion (at ∞) of f, and $\alpha_p\beta_p = p$ for all $p \nmid N_f$. We have $|\alpha_p| = |\beta_p| = \sqrt{p}$ for $p \nmid N_f$, which follows from Hasse's bound [37] in the guise of elliptic curves, and (in weight 2) is due to Eichler [27] on the modular forms side. This implies that $|c_f(p)| \le 2\sqrt{p}$, and thus $|c_f(p)| \le \lfloor 2\sqrt{p} \rfloor$ when the coefficients are integral.

For $p \nmid N_f$ we thus have $\alpha_p = \bar{\beta}_p$, and as an arbitrary convention we take α_p to be in the upper half-plane. When $p|N_f$ we take $\beta_p = 0$ and have $\alpha_p \in \{-1, 0, +1\}$.

Recalling our convention that L-function subscripts are for the primitive inducing object, the L-function $L_{f\chi}(s)$ is similarly defined as

$$L_{f\chi}(s) = \prod_p (1-\tilde{\alpha}_p/p^s)^{-1}(1-\tilde{\beta}_p/p^s)^{-1},$$

where for $p \nmid DN_f$ we have $\tilde{\alpha}_p = \chi(p)\alpha_p$ and $\tilde{\beta}_p = \chi(p)\beta_p$.

Hecke's work [38] implies $L_f(s)$ and $L_{f\chi}(s)$ satisfy functional equations with

$$\tilde{\Lambda}_f(s) = \left(\frac{\sqrt{N_f}}{2\pi}\right)^{s-1}\Gamma(s)L_f(s) \text{ with } \tilde{\Lambda}_f(s) = \epsilon_f \tilde{\Lambda}_f(2-s),$$

and

$$\tilde{\Lambda}_{f\chi}(s) = \left(\frac{\sqrt{N_{f\chi}}}{2\pi}\right)^{s-1}\Gamma(s)L_{f\chi}(s) \text{ with } \tilde{\Lambda}_{f\chi}(s) = \epsilon_{f\chi} \tilde{\Lambda}_{f\chi}(2-s),$$

where $N_{f\chi}$ is the (minimal) level of $f\chi$, and $\epsilon_f, \epsilon_{f\chi} \in \{-1, +1\}$. In §10 we shall have more to say about root number variation in quadratic twist families. By a result of Atkin and Lehner [2, §6] we have $N_f N_{f\chi} = \eta D^2$ with $1 \le \eta \le N_f^2$.

4.1.1. The (motivic) symmetric square L-function for f is defined as

$$L_{S^2 f}(s) = \prod_{p \nmid N_f} (1-\alpha_p^2/p^s)^{-1}(1-\alpha_p\beta_p/p^s)^{-1}(1-\beta_p^2/p^s)^{-1} \prod_{p|N_f} Y_p(s),$$

where $Y_p(s)$ is defined and/or computed in various works such as [31], [13], and [81]. We shall say more about $Y_p(s)$ below. By work of Shimura [70] and Gelbart and Jacquet [31] the symmetric-square L-function has an analytic continuation to an entire function, and upon writing $N_{S^2 f}$ for the symmetric-square conductor the completed L-function $\tilde{\Lambda}_{S^2 f}(s) = (N_{S^2 f}/4\pi^3)^{s/2} L_{S^2 f}(s)\Gamma(s)\Gamma(s/2)$ satisfies the functional equation $\Lambda_{S^2 f}(s) = \Lambda_{S^2 f}(3-s)$. In particular, $s = 2$ is at the edge of the critical strip and $L_{S^2 f}(2) > 0$. When f has integral coefficients (and thus is associated to an elliptic curve isogeny class) we have $N_{S^2 f} \le N_f^2$, and indeed $N_{S^2 f} \le N_g^2$ where g is the minimal quadratic twist of f. (These hold in more generality).

The (motivic) alternating square L-function for f is simply

$$L_{A^2 f}(s) = \prod_p (1-p/p^s)^{-1} = \zeta(s-1).$$

4.1.2. We will compare the Euler factors of $L_f^K(s)$ to those of $L_{S^2f}(2s)/L_{A^2f}(2s)$, and for this we find it convenient to define five types of primes for f. For odd primes p we write $p^\star = p\cdot(-1)^{(p-1)/2}$ so that $p^\star$ is thus a fundamental discriminant.

The first type of primes are good primes p, for which $p\nmid N_f$. The second type of primes are potentially good primes p, for which $p|N_f$, but p does not divide the level of f twisted by the Kronecker character $\psi_{p^\star}$. For $p=2$ we only need 2 not to divide the level of one of twists by -4, 8, or -8.

The third type is multiplicative for which $p\|N_f$, and the fourth is potentially multiplicative where p exactly divides the level of the twist of f by $\psi_{p^\star}$ (the theory of Atkin and Lehner implies that p^2 divides either the level of f or the level of $f\psi_{p^\star}$). The fifth type is "strictly" additive, for which p^2 divides both the level of f and its twist by $\psi_{p^\star}$ (we admit that this nomenclature comes from elliptic curves, but the p-divisibility conditions are valid for modular forms in general).

For good primes, the symmetric-square Euler factor is as above. For multiplicative or potentially multiplicative primes it is $Y_p(s) = (1-1/p^s)^{-1}$. For potentially good primes the symmetric-square L-factor comes from the twist of f by $\psi_{p^\star}$ as

$$Y_p(s) = (1-\hat{\alpha}_p^2/p^s)^{-1}(1-\hat{\alpha}_p\hat{\beta}_p/p^s)^{-1}(1-\hat{\beta}_p^2/p^s)^{-1}$$

where the Euler factor at p of $f\psi_{p^\star}$ is $(1-\hat{\alpha}_p/p^s)^{-1}(1-\hat{\beta}_p/p^s)^{-1}$. For strictly additive primes, we have $Y_p(s) = (1-\gamma_p/p^s)^{-1}$ where $\gamma_p \in \{-p, 0, +p\}$.

4.1.3. We define $V(s) = \prod_p V_p(s)$ to be $L_f^K(s)$ divided by $L_{S^2f}(2s)/L_{A^2f}(2s)$, and proceed to compare Euler factors as in Table 1. (For potentially good p with $p|D$ we use that $\hat{\alpha}_p = \pm\tilde{\alpha}_p$). Here the main point is that $L_f^K(s)$ and $L_{S^2f}(2s)/L_{A^2f}(2s)$ have the same Euler factors for good primes p that have $\chi(p)=-1$; in other words $V_p(s)$ is trivial for these primes. Moreover, for the other types of good primes the Dirichlet coefficients are bounded by those of $\zeta_K(s-1/2)^2/\zeta(2s-1)^2$, allowing us to bound them in terms of $L_\chi(1)$.

$\chi(p)$	type	$L_f^K(s)$ Euler factor	$V_p(s)$
-1	good	$(1-\alpha_p^2/p^{2s})^{-1}(1-\beta_p^2/p^{2s})^{-1}$	1
0	good	$(1-\alpha_p/p^s)^{-1}(1-\beta_p/p^s)^{-1}$	$(1+\alpha_p/p^s)(1+\beta_p/p^s)$
$+1$	good	$(1-\alpha_p/p^s)^{-2}(1-\beta_p/p^s)^{-2}$	$\frac{(1+\alpha_p/p^s)(1+\beta_p/p^s)}{(1-\alpha_p/p^s)(1-\beta_p/p^s)}$
-1	mult	$(1-1/p^{2s})^{-1}$	$(1-p/p^{2s})^{-1}$
0	mult	$(1-\alpha_p/p^s)^{-1}$	$(1+\alpha_p/p^s)(1-p/p^{2s})^{-1}$
$+1$	mult	$(1-\alpha_p/p^s)^{-2}$	$\frac{(1+\alpha_p/p^s)}{(1-\alpha_p/p^s)}(1-p/p^{2s})^{-1}$
±1	pot mult	1	$(1-1/p^{2s})(1-p/p^{2s})^{-1}$
0	pot mult	$(1-\tilde{\alpha}_p/p^s)^{-1}$	$(1+\tilde{\alpha}_p/p^s)(1-p/p^{2s})^{-1}$
±1	pot good	1	$(1-\tilde{\alpha}_p^2/p^{2s})(1-\tilde{\beta}_p^2/p^{2s})$
0	pot good	$(1-\tilde{\alpha}_p/p^s)^{-1}(1-\tilde{\beta}_p/p^s)^{-1}$	$(1+\tilde{\alpha}_p/p^s)(1+\tilde{\beta}_p/p^s)$
any	additive	1	$(1-\gamma_p/p^{2s})(1-p/p^{2s})^{-1}$

TABLE 1. Comparison of Euler factors

The effect from bad primes can be suitably bounded simply by their finiteness, though we shall make a minor improvement on this. In Table 2, for bad primes we list a factor that can be included in a comparison with $\zeta_K(s-1/2)^2/\zeta(2s-1)^2$, and note that the remaining factor $\tilde{V}_p(s)$ depends only on $1/p^{2s}$ (not $1/p^s$).

$\chi(p)$	type	comparable part	$\tilde{V}_p(s)$
-1	mult	1	$(1-p/p^{2s})^{-1}$
0	mult	$(1+\alpha_p/p^s)$	$(1-p/p^{2s})^{-1}$
$+1$	mult	$(1+\alpha_p/p^s)/(1-\alpha_p/p^s)$	$(1-p/p^{2s})^{-1}$
±1	pot mult	1	$(1-1/p^{2s})(1-p/p^{2s})^{-1}$
0	pot mult	$(1+\tilde{\alpha}_p/p^s)$	$(1-p/p^{2s})^{-1}$
±1	pot good	1	$(1-\tilde{\alpha}_p^2/p^{2s})(1-\tilde{\beta}_p^2/p^{2s})$
0	pot good	$(1+\tilde{\alpha}_p/p^s)(1+\tilde{\beta}_p/p^s)$	1
any	additive	1	$(1-\gamma_p/p^{2s})(1-p/p^{2s})^{-1}$

TABLE 2. Bad Euler factor parts comparable to $\bigl(\zeta(s)L_\chi(s)/\zeta(2s)\bigr)^2$

We then define $\alpha'_p=\tilde{\alpha}_p$ and $\beta'_p=\tilde{\beta}_p$ when $p|D$ and is potentially good or potentially multiplicative, and $\alpha'_p=\alpha_p$ and $\beta'_p=\beta_p$ otherwise. We thus have

$$\frac{L_f^K(s)}{L_{S^2f}(2s)/L_{A^2f}(2s)}=\prod_{p|N_f}\tilde{V}_p(s)\cdot\prod_p\frac{1+\alpha'_p/p^s}{1-\alpha'_p\chi(p)/p^s}\frac{1+\beta'_p/p^s}{1-\beta'_p\chi(p)/p^s}.$$

We write $Z_p(s)$ for this latter Euler factor, and with $\sum_n z_n/n^s=\prod_p Z_p(s)$, by a comparison to $(\zeta_K(s)/\zeta(2s))^2=\bigl(\sum_n R_K^\star(n)/n^s\bigr)^2$ we have

$$|z_n|\le\sqrt{n}\sum_{n_1n_2=n}\sum R_K^\star(n_1)R_K^\star(n_2). \tag{3}$$

4.1.4. We are now ready to define $E_f(s)$. When the class number is sufficiently small, it will not matter whether we do so by an Euler product or by a sum. As it simplifies some arguments, we opt for the former, namely

$$E_f(s)=E_f^{\mathrm{r}}(s)\cdot E_f^{\mathrm{m}}(s)=\prod_{p|N_f}\tilde{V}_p(s)\cdot\prod_{p\le\sqrt{D}/4}Z_p(s). \tag{4}$$

We then define $r_f^K(n)$ from $\sum_n r_f^K(n)/n^s=L_f^K(s)-E_f(s)L_{S^2f}(2s)/L_{A^2f}(2s)$, i.e.

$$\sum_{n=1}^\infty\frac{r_f^K(n)}{n^s}=\frac{L_{S^2f}(2s)}{L_{A^2f}(2s)}E_f^{\mathrm{r}}(s)\cdot\Biggl[\prod_p Z_p(s)\;-\prod_{p\le\sqrt{D}/4}Z_p(s)\Biggr].$$

Finally, we notate $\sum_l b_l/l^{2s}=L_{S^2f}(2s)/\zeta(2s-1)$, noting that $|b_l|\le l\tau_2(l)$.

Lemma 4.1.5. *With notation as above and* $\prod_{p|N_f}\tilde{V}_p(s)=\sum_d\tilde{v}_d/d^{2s}$ *we have*

$$|r_f^K(n)|\le\sum_{d|N_f^\infty}|\tilde{v}_d|\sum_{l^2m=n/d^2}\sum|b_l|\cdot C(m)\sqrt{m}$$

where

$$C(m)=\Biggl(2\sum_{w_1w_2=m}\sum R_K^{\star\mathrm{s}}(w_1)\tilde{R}_K^\star(w_2)+\sum_{w_1w_2=m}\sum\tilde{R}_K^\star(w_1)\tilde{R}_K^\star(w_2)\Biggr). \tag{5}$$

Proof. This is largely accounting at this point. Firstly, the $|b_l|$ term simply comes from $L_{S^2f}(2s)/L_{A^2f}(2s)$ and the d-sum from $E_f^{\mathrm{r}}(s)$. The mth coefficient of

$$\Big[\prod_p Z_p(s) - \prod_{p\le\sqrt{D}/4} Z_p(s)\Big]$$

is zero unless m has a prime divisor exceeding $\sqrt{D}/4$, and in general is bounded by $|z_m|$, which in turn is bounded as in (3). To obtain the form of $C(m)$ in the Lemma, we recall the splitting $R_K^\star = R_K^{\star\mathrm{s}} + \tilde{R}_K^{\star\mathrm{s}}$, and note that the putative terms with $R_K^{\star\mathrm{s}}(w_1)R_K^{\star\mathrm{s}}(w_2)$ can be omitted, since $m = w_1w_2$ would not have a prime divisor exceeding $\sqrt{D}/4$ in this case. This gives the statement in the Lemma. □

4.1.6. It is convenient to define

$$\mathcal{R}(\chi) = \sum_{n=1}^{\infty} \frac{R_K^{\star\mathrm{s}}(n)}{n} = \sum_{n\le\sqrt{D}/4} \frac{R_K^\star(n)}{n} \le \prod_{p\le\sqrt{D}/4} \frac{1+1/p}{1-\chi(p)/p}. \qquad (6)$$

We then define $\mathcal{U}(f) = \sum_d |\tilde{v}_d|/d^3$ and by Table 2 have

$$\mathcal{U}(f) = \sum_{d|N_f^\infty} \frac{|\tilde{v}_d|}{d^3} \le \prod_{p|N_f} \frac{1+p/p^3}{1-p/p^3} \le \frac{\zeta(2)^2}{\zeta(4)} = 5/2.$$

Similarly, we define $\mathcal{V}(f) = \sum_l |b_l|/l^3$ and since $|b_l| \le l\tau_2(l)$ we have

$$\mathcal{V}(f) = \sum_{l=1}^{\infty} \frac{|b_l|}{l^3} \le \zeta(2)^2.$$

For individual f these bounds are improvable (perhaps only slightly for $\mathcal{V}(f)$), so we will retain them in the stated error term for our Deuring decomposition below.

4.2. We now show our main bound for a weighted summation of $|r_f^K(n)|$.

Proposition 4.2.1. *For $X \ge 1$ and $j \ge 2$ and $D \ge 100$ we have*

$$\begin{aligned} J_j(X) &= \sum_{n=1}^{\infty} |r_f^K(n)| I_j(n/X) = \sum_{n=1}^{\infty} |r_f^K(n)| \int_{(2)} \Big(\frac{X}{n}\Big)^s \frac{\Gamma(s)^2}{(s-1)^j} \frac{\partial s}{2\pi i} \\ &\le 7.334 \cdot \mathcal{U}(f)\mathcal{V}(f) \cdot 2^j X^{3/2} \cdot L_\chi(1)\mathcal{R}(\chi) + 3000 \cdot 2^j X^{3/2} L_\chi(1)^2 \cdot \log(D^2X^6). \end{aligned}$$

Proof. We bound $|r_f^K(n)|$ by Lemma 4.1.5. With the first term appearing in (5) we write $n = l^2md^2 = l^2w_1w_2d^2$ as in that Lemma, and have

$$\big|J_j^{(1)}(X)\big| \le 2\sum_{d|N_f^\infty} |\tilde{v}_d| \sum_{l=1}^{\infty} |b_l| \sum_{w_1\le\sqrt{D}/4} \sqrt{w_1} R_K^{\star\mathrm{s}}(w_1) \sum_{w_2\ge\sqrt{D}/4} \sqrt{w_2}\tilde{R}_K^{\star\mathrm{s}}(w_2) I_j\Big(\frac{w_1w_2l^2d^2}{X}\Big).$$

We then apply Proposition 3.3.1 to bound the w_2-sum and get

$$\begin{aligned} \big|J_j^{(1)}(X)\big| &\le 2\sum_{d|N_f^\infty} |\tilde{v}_d| \sum_{l=1}^{\infty} |b_l| \sum_{w_1\le\sqrt{D}/4} \sqrt{w_1} R_K^{\star\mathrm{s}}(w_1) \cdot 3.667 L_\chi(1) \cdot 2^j \Big(\frac{X}{w_1l^2d^2}\Big)^{3/2} \\ &= 7.334 \cdot \mathcal{U}(f)\mathcal{V}(f) \cdot L_\chi(1)\mathcal{R}(\chi) \cdot 2^j X^{3/2}. \end{aligned}$$

4.2.2. With the secondary term from (5), we first make a crude bound for large n, noting Lemma 4.1.5 implies that

$$|r_f^K(n)| \le \sqrt{n}\sum\sum\sum\sum_{l^2w_1w_2d^2=n} \tau_2(l)\tau_2(d)R_K^\star(w_1)R_K^\star(w_2) \le \sqrt{n}\tau_8(n)$$

while $\tau_8(n) \le (8^4n/210) \le 20n$. By integration on the 3-line the $n \ge B = X^6D^2$ thus contribute an amount bounded as $\le 12X^3/\sqrt{B} = 12/D$.

For the $n \le B$ we then imitate the above, getting

$$\begin{aligned}
&\le \sum_{d|N_f^\infty} d\tau_2(d)\sum_{l=1}^\infty l\tau_2(l)\sum_{\sqrt{D}/4<w_1<B}\sqrt{w_1}\tilde{R}_K^{\star\mathrm{s}}(w_1)\sum_{w_2\ge\sqrt{D}/4}\sqrt{w_2}\tilde{R}_K^{\star\mathrm{s}}(w_2)I_j\Big(\frac{w_1w_2l^2d^2}{X}\Big)\\
&\le 3.667\cdot 2^jX^{3/2}L_\chi(1)\sum_{d|N_f^\infty}\frac{\tau_2(d)}{d^2}\sum_{l=1}^\infty\frac{\tau_2(l)}{l^2}\sum_{\sqrt{D}/4<w_1<B}\frac{\tilde{R}_K^{\star\mathrm{s}}(w_1)}{w_1}\\
&\le 3.667\zeta(2)^4\cdot 2^jX^{3/2}L_\chi(1)\cdot 111L_\chi(1)\log B,
\end{aligned}$$

where for the w_1-sum we used Lemma 3.6.1. □

5. Bounds on $E_f(z)$ when $L_\chi(1)$ is small

There are two aspects in our choice of $E_f(z)$. We want a bound like Lemma 4.1.5 on the residual $r_f^K(n)$, but we also want to have control over the size of $E_f(z)$, at least on the central line (and somewhat to the left of it).

In this section we give enough results to allow us to derive our Deuring decomposition in §6, while we postpone additional results (regarding derivatives and lower bounds for E_f and L_{S^2f}) until later. We will make the convenient assumptions that $D \ge 4\pi^2\exp(10^6)$ and $\sqrt{D}L_\chi(1) \le (\log D)^3/10^6$ to ease our task.

5.1. We first bound the number of small split primes when $L_\chi(1)$ is small. This suffices for us when z is small compared to u (indeed, we only use it for $z \le 6$ and $u \gg \sqrt{\log D}$), though results are possible when z is larger ([88, Lemma 2.1.7]).

Lemma 5.1.1. *Suppose there are z primes p with $\chi(p) = +1$ and $p \le X$. Then*

$$\frac{1}{4\log 2}\sqrt{D}L_\chi(1) \ge 2^z\binom{u}{z} \quad \text{where} \quad u = \left\lfloor\frac{\log(\sqrt{D}/4)}{\log X}\right\rfloor.$$

Proof. Denoting primes p with $\chi(p) = +1$ by p_i, the product $\prod_i p_i^{e_i}$ is $\le \sqrt{D}/4$ for every nonnegative integral vector $\vec{e}$ that satisfies $\sum_i e_i \le u$, and by the fundamental theorem of arithmetic each such vector gives rise to a unique $n = \prod_i p_i^{e_i}$ with $R_K^\star(n) = 2^{\omega(n)}$ where $\omega(n)$ is the number of distinct prime divisors of n. We thus find

$$\sum_{n\le\sqrt{D}/4} R_K^\star(n) \ge 2^z\binom{u}{z},$$

and the left side is $\le \sqrt{D}L_\chi(1)/(4\log 2)$ by Goldfeld [33, Lemma 4]. □

5.2. We then use this to bound E_f. First we consider the ramified primes of f.

Lemma 5.2.1. *For $\sigma > 1/2$ we have*

$$\left|E_f^{\mathrm{r}}(s)\right| \leq \prod_{p|N_f} \frac{1+p/p^{2\sigma}}{1-p/p^{2\sigma}}.$$

Proof. Table 2 gives the possible Euler factors $\tilde{V}_p(s)$ with $E_f^{\mathrm{r}}(s)$, and each possibility is dominated coefficient-wise by $(1+p/p^{2s})/(1-p/p^{2s})$. The result follows. □

We proceed to bound the remaining $E_f^{\mathrm{m}}(s)$ part, which we recall from §4.1.4 is

$$E_f^{\mathrm{m}}(s) = \prod_{p\leq\sqrt{D}/4} \frac{1+\alpha'_p/p^s}{1-\alpha'_p\chi(p)/p^s}\frac{1+\beta'_p/p^s}{1-\beta'_p\chi(p)/p^s},$$

where $|\alpha'_p|, |\beta'_p| \leq \sqrt{p}$. We define

$$\mathcal{P}_\sigma(D) = \prod_{p|D}\Big(1+\frac{\sqrt{p}}{p^\sigma}\Big)^2.$$

Lemma 5.2.2. *Suppose that $D \geq 4\pi^2\exp(10^6)$ and $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$. Then we have $|E_f^{\mathrm{m}}(s)| \leq 7500\mathcal{P}_\sigma(D)$ for $\sigma \geq 4/5$ and $|E_f^{\mathrm{m}}(1+it)| \leq 600\mathcal{P}_1(D)$.*

Proof. The primes that divide D contribute the $\mathcal{P}_\sigma(D)$ term.

For the split primes, we first note there are at most 2 such primes up to 10000. Indeed, in Lemma 5.1.1 with $X = 10^4$ we have $u \geq 0.054\log D$ for $D \geq 4\pi^2\exp(10^6)$, and with $z = 3$ this would give $0.37\sqrt{D}L_\chi(1) \geq 0.0002(\log D)^3$, contradicting our assumption that $\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$.

Next we write $Y = \log(\sqrt{D}/4)$ and note there are at most 5 split primes p with $p \leq X_1 = \exp(\sqrt{Y})$. Indeed with Lemma 5.1.1 this X_1 gives us $u_1 \geq \sqrt{Y}-1$, and assuming there are l split primes with $p \leq X_1$ implies

$$2^l\cdot\frac{(\sqrt{0.5\log D-\log 4}-l)^l}{l!} = 2^l\cdot\frac{(\sqrt{Y}-l)^l}{l!} \leq 0.37\sqrt{D}L_\chi(1) \leq \frac{(\log D)^3}{10^6},$$

which is a contradiction for $l = 6$ when $D \geq 800$.

These 5 putative split primes contribute to $|E_f^{\mathrm{m}}(s)|$ no more than

$$\Big(\frac{1+\sqrt{2}/2^\sigma}{1-\sqrt{2}/2^\sigma}\Big)^2\Big(\frac{1+\sqrt{3}/3^\sigma}{1-\sqrt{3}/3^\sigma}\Big)^2\Big(\frac{1+\sqrt{10^4}/(10^4)^\sigma}{1-\sqrt{10^4}/(10^4)^\sigma}\Big)^6,$$

and for $\sigma \geq 4/5$ this is ≤ 7456 while on $\sigma = 1$ it is ≤ 534.

The number of split primes with $\exp\Big(\sqrt{\log(\sqrt{D}/4)}\Big) = X_1 < p \leq \sqrt{D}/4$ is no more than $0.37\sqrt{D}L_\chi(1) \leq (\log D)^3/10^6$, which contribute to $E_f^{\mathrm{m}}(s)$ no more than

$$\exp\Big(2\cdot\frac{(\log D)^3}{10^6}\cdot 3\frac{\sqrt{X_1}}{X_1^\sigma}\Big),$$

which is bounded by $1+10^{-79}$ for $\sigma \geq 4/5$. Multiplying these gives the Lemma. □

5.3. In our derivation of the Deuring decomposition in the next section we will require a bound for L_{S^2f} that follows from convexity. We write $t_\star = |t| + 5$.

Lemma 5.3.1. *In* $1 \le \sigma \le 2$ *we have*

$$\bigl|L_{S^2f}(s)\bigr| \le 3.3 \cdot (1 + \log N_{S^2f} t_\star^3)^3 \cdot \Bigl(\frac{N_{S^2f} t_\star^3}{8\pi^3}\Bigr)^{1-\sigma/2}.$$

Proof. Recalling that the 2-line is the edge of the critical strip for $L_{S^2f}(s)$, we consider $\sigma_0 = 2 + 1/(\log N_{S^2f} t_\star^3)$. Comparing the degree 3 Euler product to $\zeta(s-1)$ on this line gives a bound of $(1 + \log N_{S^2f} t_\star^3)^3$. By the functional equation we have

$$\frac{L_{S^2f}(3-\sigma_0-it)}{L_{S^2f}(\sigma_0+it)} = \frac{(\sqrt{N_{S^2f}}/2\pi^{3/2})^{\sigma_0+it}}{(\sqrt{N_{S^2f}}/2\pi^{3/2})^{3-\sigma_0-it}} \frac{\Gamma(\sigma_0+it)\Gamma(\sigma_0/2+it/2)}{\Gamma(3-\sigma_0-it)\Gamma\bigl((3-\sigma_0-it)/2\bigr)},$$

and by an explicit Stirling's approximation (with constant 2) we thus have

$$\bigl|L_{S^2f}(3-\sigma_0+it)\bigr| \le 2 \cdot |L_{S^2f}(\sigma_0+it)| \cdot \Bigl(\frac{N_{S^2f} t_\star^3}{8\pi^3}\Bigr)^{\sigma_0-3/2}.$$

Applying the theorem of Phragmén and Lindelöf [60] for $3-\sigma_0 \le \sigma \le \sigma_0$ gives

$$\bigl|L_{S^2f}(\sigma+it)\bigr| \le 2 \cdot (1 + \log N_{S^2f} t_\star^3)^3 \cdot \Bigl(\frac{N_{S^2f} t_\star^3}{8\pi^3}\Bigr)^{(\sigma_0-\sigma)/2}$$

and replacing σ_0 by 2 induces a factor $\le \exp(1/2)$, giving the Lemma. □

6. A Deuring decomposition around $z = 1$ for $\tilde\Lambda_f^K(z)$

In this section we obtain a decomposition of $\tilde\Lambda_f^K(z)$ suited to a neighborhood of its central point $z = 1$.

6.1. We write ϵ_f^K for the sign of the functional equation with $\tilde\Lambda_f^K(z) = \epsilon_f^K \tilde\Lambda_f^K(2-z)$, and with $MD = \sqrt{N_f N_{f\chi}}/4\pi^2$ define

$$T_f(z) = \frac{L_{S^2f}(2z)}{\zeta(2z-1)} E_f(z)\Gamma(z)^2(MD)^{z-1}$$

with $E_f(s)$ as in (4) of §4.1.4 as

$$E_f(s) = E_f^{\mathrm{r}}(s) \cdot E_f^{\mathrm{m}}(s) = \prod_{p|N_f} \tilde V_p(s) \cdot \prod_{p \le \sqrt{D}/4} \frac{1+\alpha_p'/p^s}{1-\alpha_p'\chi(p)/p^s} \frac{1+\beta_p'/p^s}{1-\beta_p'\chi(p)/p^s}.$$

As noted at the end of §4.1, we have $1 \le 4\pi^2 M \le N_f$. We also introduce

$$S_f^{\tilde r}(z) = \sum_{b=0}^{\tilde r-1} \bigl[1 + \epsilon_f^K(-1)^b\bigr] \cdot \frac{T_f^{(b)}(1)}{b!}(z-1)^b,$$

which corresponds to the lower-order Taylor series terms of $T_f(z) + \epsilon_f^K T_f(2-z)$.

Proposition 6.1.1. *Suppose that* $D \ge 4\pi^2\exp(10^6)$ *with* $\sqrt{D}L_\chi(1) \le (\log D)^3/10^6$ *and* $N_f^9 \le D$. *Write* $\tilde r$ *for a lower bound for the order of vanishing of* $L_f^K(z)$ *at* $z = 1$ *and assume* $1 \le \tilde r \le (\log D)/100$ *with* $(-1)^{\tilde r} = \epsilon_f^K$. *Then for* $|z-1| \le 1/99$ *we have*

$$\tilde\Lambda_f^K(z) = L_f^K(z)\Gamma(z)^2(MD)^{z-1} = T_f(z) + \epsilon_f^K T_f(2-z) - S_f^{\tilde r}(z) + U_f(z) \qquad (7)$$

where

$$\bigl|U_f(z)\bigr| \le 30\sqrt{M} \cdot \mathcal{U}(f)\mathcal{V}(f) \cdot \sqrt{D}L_\chi(1)\mathcal{R}(\chi) \cdot 2^{\tilde r}|z-1|^{\tilde r}.$$

Here $\mathcal{U}(f)$ and $\mathcal{V}(f)$ are defined in §4.1.6 and are each $\le \zeta(2)^2$, and $\mathcal{R}(\chi)$ is also defined in §4.1.6 (reciprocal sum over small representations).

Proof. We use $Y(s) = 1/(s-z)(s-1)^{\tilde r}$ in (2), which accentuates the activity around the central point. By Cauchy's residue theorem and substituting $s \to 2-s$ and using the functional equation in the second integral we have

$$\frac{\tilde\Lambda_f^K(z)}{(z-1)^{\tilde r}} = \left(\int_{(2)} - \int_{(0)}\right) \frac{\tilde\Lambda_f^K(s)}{(s-z)(s-1)^{\tilde r}} \frac{\partial s}{2\pi i}$$
$$= \int_{(2)} \frac{(MD)^{s-1}\,\Gamma(s)^2 L_f^K(s)}{(s-z)(s-1)^{\tilde r}} \frac{\partial s}{2\pi i} - \epsilon_f^K (-1)^{\tilde r} \int_{(2)} \frac{(MD)^{s-1}\,\Gamma(s)^2 L_f^K(s)}{(2-s-z)(s-1)^{\tilde r}} \frac{\partial s}{2\pi i},$$

and we then insert (see (4)ff)

$$L_f^K(s) = E_f(s)\frac{L_{S^2 f}(2s)}{\zeta(2s-1)} + \sum_{n=1}^{\infty} \frac{r_f^K(n)}{n^s}$$

to get

$$\frac{\tilde\Lambda_f^K(z)}{(z-1)^{\tilde r}} = \int_{(2)} \frac{L_{S^2 f}(2s)}{\zeta(2s-1)} (MD)^{s-1} \frac{E_f(s)\Gamma(s)^2}{(s-z)(s-1)^{\tilde r}} \frac{\partial s}{2\pi i} +$$
$$- \epsilon_f^K (-1)^{\tilde r} \int_{(2)} \frac{L_{S^2 f}(2s)}{\zeta(2s-1)} (MD)^{s-1} \frac{E_f(s)\Gamma(s)^2}{(2-s-z)(s-1)^{\tilde r}} \frac{\partial s}{2\pi i} +$$
$$+ \int_{(2)} \frac{(MD)^s}{MD} \sum_{n=1}^{\infty} \frac{r_f^K(n)}{n^s} \left[\frac{1}{s-z} - \epsilon_f^K \frac{(-1)^{\tilde r}}{2-s-z}\right] \frac{\Gamma(s)^2}{(s-1)^{\tilde r}} \frac{\partial s}{2\pi i}. \quad (8)$$

6.1.2. We move the first two integrals appearing in (8) to the left, after truncating the integrals at height $H_2 = (\log D)^2$ at negligible cost due to the Γ-decay. The poles at $s = z$ and $s = 2-z$ then contribute respective residues of $T_f(z)/(z-1)^{\tilde r}$ and $-(-1)^{\tilde r} T_f(2-z)/(z-1)^{\tilde r}$, and multiplying through in (8) by $(z-1)^{\tilde r}$ gives their contribution as in the statement of the Proposition. The pole at $s = 1$ for the first integral gives a residue of

$$\sum\sum_{b+c=\tilde r-1} \frac{T_f^{(b)}(1)}{b!} \frac{(-1)^c}{(1-z)^{c+1}},$$

and by multiplying through by $(z-1)^{\tilde r}$ this gives

$$-\sum_{b=0}^{\tilde r-1} \frac{T_f^{(b)}(1)}{b!}(z-1)^b.$$

A similar statement holds for the second integral, though now the alternation of sign with respect to c is omitted, giving a contribution of

$$-\epsilon_f^K(-1)^{\tilde r} \cdot \sum\sum_{b+c=\tilde r-1} \frac{T_f^{(b)}(1)}{b!} \frac{1}{(1-z)^{c+1}},$$

and multiplying through by $(z-1)^{\tilde r}$ gives

$$-\epsilon_f^K \sum\sum_{b+c=\tilde r-1} (-1)^{c+1-\tilde r} \frac{T_f^{(b)}(1)}{b!}(z-1)^b = -\epsilon_f^K \sum_{b=0}^{\tilde r-1} (-1)^b \frac{T_f^{(b)}(1)}{b!}(z-1)^b.$$

Adding these two contributions gives $-S_f^{\tilde{r}}(z)$ as in the statement of the Lemma.

6.1.3. The error contributions from the integral on the new contour will be small due to the decay of D^σ, though it is a bit tedisome in details to codify this.

This new contour consists of the vertical segment $4/5 \pm iH_1$, the vertical segments $\sigma_0 \pm iH_1$ to $\sigma_0 \pm iH_2$ where $\sigma_0 = 1 - 1/70 \log\log D$, and the associated horizontal connecting segments, where we take $H_1 = 10^{10}$.

In particular, the lower bulge ensures the pole at $s = 2 - z$ is to the right of the new contour since $|z-1| \le 1/99$. Moreover, by the computations of Platt [61] and the zero-free region of de La Vallée Poussin [48, (53)],[12] the above contour shift does not encounter any zeros of $\zeta(2z-1)$. Similarly, we have adequate control over $1/\zeta(2z-1)$ on the new contour.

The contribution from the segment on the 4/5-line is bounded as

$$\le \frac{2}{(MD)^{1/5}} \int_{-H_1}^{H_1} \left| \frac{L_{S^2f}(8/5+2it)}{\zeta(3/5+2it)} E_f(4/5+it) \frac{\Gamma(4/5+it)^2}{(0.189+it)(1/5+it)^{\tilde{r}}} \right| \frac{\partial t}{2\pi}.$$

The convexity bound for the symmetric-square (Lemma 5.3.1) and $N_{S^2f} \le N_f^2$ yield

$$\begin{aligned} \left|L_{S^2f}(8/5+2it)\right| &\le 5(N_f^2 t_\star^3)^{1/5}(1+\log N_f^2 t_\star^3)^3 \\ &\le 5N_f^{2/5} t_\star^{3/5} (\log N_f^2)^3 (\log t_\star^3)^3 \le 135 \cdot 87000 N_f^{2/5+1/20} \cdot t_\star^{3/5} (\log t_\star)^3, \end{aligned}$$

while with Lemma 5.2.1 we have

$$\left|E_f^{\mathrm{r}}(4/5+it)\right| \le \prod_{p|N_f} \frac{1+1/p^{3/5}}{1-1/p^{3/5}} \le 233 N_f^{1/20}$$

since the multiplicands are $\le p^{1/20}$ for $p \ge 49$. Our assumption that $D \ge N_f^9$ then implies that the product of these two terms is

$$\le 10^{10}\sqrt{N_f} \le 10^{10} D^{1/18}.$$

Our assumption that $\sqrt{D} L_\chi(1) \le (\log D)^3/10^6$ implies by genus theory that there are no more than $5\log\log D$ primes dividing D, and by Lemma 5.2.2 we have

$$\begin{aligned} |E_f^{\mathrm{m}}(4/5+it)| &\le 7500 \cdot \mathcal{P}_{4/5}(D) \le 7500 \cdot \exp\Big(3 \sum_{n \le 5\log\log D} \frac{1}{n^{3/10}}\Big) \\ &\le 7500 \cdot \exp\big(14(\log\log D)^{7/10}\big) \le 10^{-22800} D^{1/19}, \end{aligned}$$

using $\log D \ge 10^6$. Our assumption of $\tilde{r} \le (\log D)/100$ implies that $5^{\tilde{r}} \le D^{0.0161}$, and upon pulling this out we can directly integrate the remaining parts of the integrand and find that

$$\int_{-H_1}^{H_1} \left| \frac{\Gamma(4/5+it)^2 \cdot t_\star^{3/5} (\log t_\star)^3}{\zeta(3/5+2it)(0.149+it)} \right| \frac{\partial t}{2\pi} \le 12.$$

Since $1/18 + 1/19 + \log(5)/100 \le 1/8$ we get a total error contribution bounded by

$$\frac{2}{(D/4\pi^2)^{1/5}} \cdot 10^{-22000} D^{1/8},$$

and upon multiplying by $|z-1|^{\tilde{r}}$, this easily fits into the stated error term.

[12]We could obtain a much superior zero-free region under our assumption of small $L_\chi(1)$, but this is unnecessary so we do not include it.

Meanwhile, the the horizontal segments at height H_1 contribute

$$\le 4\int_{4/5}^{\sigma_0}\left|\frac{L_{S^2f}(2\sigma+2iH_1)}{\zeta(2\sigma-1+2iH_1)}\frac{E_f(\sigma+iH_1)(MD)^\sigma}{MD}\frac{\Gamma(\sigma+iH_1)^2}{(H_1-1)^{1+\tilde r}}\right|\frac{\partial\sigma}{2\pi},$$

while the contribution from the segments on the σ_0-line is bounded as

$$\le 4\int_{H_1}^{H_2}(MD)^{\sigma_0-1}\left|\frac{L_{S^2f}(2\sigma_0+2it)}{\zeta(2\sigma_0-1+2it)}E_f(\sigma_0+it)\frac{\Gamma(\sigma_0+it)^2}{(H_1-1)^{1+\tilde r}}\right|\frac{\partial t}{2\pi}.$$

For both of these, the Γ-factor contributes $\approx \exp(-\pi H_1)=\exp(-\pi\cdot 10^{10})$ and will thus dominate any constants. In the first integral, Lemma 5.3.1 bounds L_{S^2f} as $\le 3.3(1+\log 2N_f^2H_1^3)^3(N_f^2H_1^3)^{1-\sigma}$, while $1/\zeta$ is (crudely) bounded by 10^{10}, and upon bounding the E_f-factor as above, we note that using $N_f^9\le D$ yields the integrand is bounded by $\exp(-\pi H_1)$ times

$$\begin{aligned}3.3\cdot 10^{10}(1+\log 2N_f^2H_1^3)^3(N_f^2H_1^3)^{1-\sigma}\cdot 7500\,\exp\bigl(15(\log\log D)^{3/2-\sigma}\bigr)\cdot(MD)^{\sigma-1}\\ \le 10^{20}\cdot\exp\bigl(15(\log\log D)^{3/2-\sigma}\bigr)(D^{2/3})^{\sigma-1}(\log D)^3\le 10^{-230}\end{aligned}$$

the last step since $\sigma\le\sigma_0=1-1/70\log\log D$ and $D\ge 4\pi^2\exp(10^6)$. This suffices for the error term. A similar bound holds on the remaining vertical segment (for indeed $\sigma=\sigma_0$ dominated the previous analysis), though now a harmless extra term of $10^5\log\log D$ appears from the reciprocal ζ-factor.

6.1.4. We are left with the integrals involving $r_f^K(n)$ in (8). We make[13] the series expansion $1/(s-z)=\sum_l (z-1)^l/(s-1)^{l+1}$ in powers of $1/(s-1)$ and consider

$$\begin{aligned}G_{\tilde r}(z)&=\int_{(2)}\sum_{n=1}^\infty\frac{r_f^K(n)}{n^s}(MD)^{s-1}\left[\frac{1}{s-z}-\frac{\epsilon_f^K(-1)^{\tilde r}}{2-s-z}\right]\frac{\Gamma(s)^2}{(s-1)^{\tilde r}}\frac{\partial s}{2\pi i}\\ &=\int_{(2)}\sum_{n=1}^\infty\frac{r_f^K(n)}{n^s}\frac{(MD)^s}{MD}\left[\sum_{l=0}^\infty\frac{(z-1)^l}{(s-1)^{l+1}}+\epsilon_f^K(-1)^{\tilde r}\sum_{l=0}^\infty\frac{(1-z)^l}{(s-1)^{l+1}}\right]\frac{\Gamma(s)^2}{(s-1)^{\tilde r}}\frac{\partial s}{2\pi i}\\ &=\sum_{n=1}^\infty\frac{r_f^K(n)}{MD}\sum_{l=0}^\infty(z-1)^l\bigl[1+\epsilon_f^K(-1)^{l+\tilde r}\bigr]\int_{(2)}\left(\frac{MD}{n}\right)^s\frac{\Gamma(s)^2}{(s-1)^{l+\tilde r+1}}\frac{\partial s}{2\pi i}.\end{aligned}$$

As with §4.2 the integral here is positive, so (we do not bother to restrict to l even)

$$|G_{\tilde r}(z)|\le\frac{2}{MD}\sum_{l=0}^\infty|z-1|^l\sum_{n=1}^\infty|r_f^K(n)|\int_{(2)}\left(\frac{MD}{n}\right)^s\frac{\Gamma(s)^2}{(s-1)^{l+\tilde r+1}}\frac{\partial s}{2\pi i}.$$

We then use Proposition 4.2.1 with $X=MD$ (here $\tilde r\ge 1$ ensures "$j\ge 2$"), with the secondary term being negligible. This gives

$$\begin{aligned}|G_{\tilde r}(z)|&\le\frac{2}{MD}\sum_{l=0}^\infty|z-1|^l\cdot 7.334\cdot\mathcal{U}(f)\mathcal{V}(f)\cdot L_\chi(1)\mathcal{R}(\chi)\cdot 2^{l+\tilde r+1}(MD)^{3/2}\\ &\le 29.336\sqrt{M}\cdot\mathcal{U}(f)\mathcal{V}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\cdot 2^{\tilde r}\sum_{l=0}^\infty 2^l|z-1|^l,\end{aligned}$$

[13]It may be possible to avoid this series expansion; e.g. right before (20) in [82] we instead used that the Mellin transform of $\Gamma(s)/(s-1/2)$ exceeds that of $\Gamma(s)(s-1/2)/\bigl((s-1/2)^2+\xi_0^2\bigr)$, as the latter just has an additional cosine term that is bounded by 1.

and since $|z-1| \le 1/99$ the final l-sum converges and is $\le 99/97$. Replacing into (8) and multiplying $G_{\tilde r}(z)$ by $|z-1|^{\tilde r}$ gives the Proposition. □

7. Restricting zeros by computing $L_f'''(1)$

We will use the above Deuring decomposition specifically when $\tilde r = 4$ and we know there is at least a triple central zero of $L_f(s)$ along with exactly two additional zeros of $L_f(s)$ nearby. Such an f will be associated to an elliptic curve of rank 5.

In this section we will describe how we can adduce such facts about $L_f(s)$, via computing its third central derivative to a large precision. As a convenience, we will then specialize to a set of 12 specific elliptic curves, each of rank 5, that will suffice to show our main result. Although we could be more general both here and in §9, I really don't see much other application of the results herein except to our specific case (e.g., it seems rather pointless to repeat the same calculations for significantly larger conductors, and putative larger analytic ranks would involve a rather different schema in any event).

By the work of Wiles [90] and others we know that elliptic curves over the rationals are modular, and thus we can interchangeably speak of an elliptic curve or its associated weight 2 modular newform with integral coefficients.

7.1. Firstly, given an elliptic curve of rank 5 whose L-function has odd parity, a lower bound of 3 for the analytic rank can be obtained either: by combining the results of Gross and Zagier [36] with those of Kolyvagin [47], which together imply that if the analytic rank is 1 then the algebraic rank is also 1; or by computing $L_f'(1)$ to sufficient precision and using explicit bounds on the quantities in the Gross-Zagier formula (see Delaunay and Roblot [19, Proposition 3.1] for a special case).

7.1.1. It turns out to be easier to work with $\tilde\Lambda_f(s)$ rather than $L_f(s)$. Recall that our scaling is $\tilde\Lambda_f(s) = (\sqrt{N_f}/2\pi)^{s-1}\Gamma(s)L_f(s)$. Assuming f has odd parity and a central zero of order 3 or more, by the Taylor expansion about $s=1$ we have

$$\tilde\Lambda_f(s) = \frac{\tilde\Lambda_f'''(1)}{3!}(s-1)^3 + \frac{\tilde\Lambda_f^{(5)}(1)}{5!}(s-1)^5 + \Theta\Big(\frac{\hat\Lambda_f^{(7)}}{7!}|s-1|^7\Big)$$

for $|s-1| \le 1/10^5$ where $\hat\Lambda_f^{(7)}$ is a bound for the seventh derivative in this region and Θ is analogous to Landau's O-notation with an implicit constant of 1.

In our cases of interest, we can computationally show that $\tilde\Lambda_f'''(1)$ is zero to a (large) desired precision, while simultaneously a computation will show that the fifth central derivative is not close to zero. By choosing a circle small enough about $s=1$ so that

$$\left|\frac{\tilde\Lambda_f^{(5)}(1)}{5!}(s-1)^5\right| > \left|\frac{\tilde\Lambda_f'''(1)}{3!}(s-1)^3\right| + \left|\frac{\hat\Lambda_f^{(7)}}{7!}(s-1)^7\right|$$

on the boundary of the circle, by Rouché's theorem we can then conclude that $\tilde\Lambda_f(s)$ has exactly five zeros inside such a circle. As we know that at least three of these zeros are at the central point $s=1$, by symmetry the other two must lie on the central line or the real axis (or indeed, possibly both).

We formulate this a precise result, writing $b_5 > 0$ as a lower bound for $|\tilde\Lambda_f^{(5)}(1)/5!|$ and B_7 for an upper bound for $|\tilde\Lambda_f^{(7)}(s)/7!|$ in the disk $|s-1| \le 1/10^5$.

Lemma 7.1.2. *Suppose f has odd parity and $\tilde\Lambda_f'(1)=0$, and that $|\tilde\Lambda_f'''(1)/3!|\le B_3$ where $|3B_3/b_5|\le \min(b_5/3B_7, 1/10^{10})$. Then $\tilde\Lambda_f(s)$ has exactly five zeros in the disk $|s-1|^2\le 3B_3/b_5$.*

Proof. On the circle $|s-1|^2 = 3B_3/b_5$ we have

$$\Bigl|\frac{\hat\Lambda_f^{(7)}}{7!}(s-1)^7\Bigr| \le B_7|s-1|^7 = \frac{3B_3}{b_5}B_7|s-1|^5 \le \frac{b_5}{3}|s-1|^5$$

and

$$\left|\frac{\tilde\Lambda_f'''(1)}{3!}(s-1)^3\right| \le B_3|s-1|^3 = \frac{b_5}{3}|s-1|^5,$$

while $|\tilde\Lambda_f^{(5)}(1)(s-1)^5/5!|\ge b_5|s-1|^5$, so that the result follows by the above Taylor expansion for $\tilde\Lambda_f(s)$ in conjunction with Rouché's theorem [64]. □

We can compute $\tilde\Lambda_f^{(5)}(1)$ to (say) five digits by the method below, and thus bound it; e.g., for the rank 5 curve of conductor 19047851 it is approximately $5!\cdot 30.286$.

A suitable bound on the 7th derivative, which is for a region rather than just at one point, can easily be obtained by crude estimates (it plays little rôle anyway). Indeed in general we have $|\tilde\Lambda_f(s)|\le 5N_f^{1/4}(1+\log 5N_f)^2$ for $|s-1|\le 1/2$ by convexity bounds, and Cauchy's bound on derivatives then implies the ninth derivative is bounded in $|s-1/2|\le 1/10^5$ as $\le 9!\cdot 2.001^9\cdot 5N_f^{1/4}(1+\log 5N_f)^2$.

As an example then, for the curve of conductor 19047851 we can bound the seventh central derivative by computation as $\le 65\cdot 7!$, so that by the Taylor series expansion of the seventh derivative about $s=1$ we have $B_7\le 65+10^8/(10^5)^2\le 66$.

7.2. In the sequel, we will consider the following sets of elliptic curves:

$$\mathcal{E}_1^- = \{[0,0,1,-79,342],\ [0,0,1,-169,930],\ [0,1,1,-30,390]\}$$

of respective conductors 19047851, 64921931, and 67445803;

$$\mathcal{E}_1^+ = \{[1,0,0,-22,219],\ [0,1,1,-100,110],\ [0,0,1,-139,732]\}$$

of respective conductors 20384311, 55726757, and 59754491;[14]

$$\mathcal{E}_2^- = \{(11\text{a})\otimes -25351367,\ (17\text{a})\otimes -19502039,\ (19\text{a})\otimes -16763912\}$$

where these are quadratic twists as indicated; and

$$\mathcal{E}_2^+ = \{(91\text{b})\otimes 6350941,\ (123\text{a})\otimes 5467960,\ (209\text{a})\otimes 3217789\}.$$

Each of these 12 elliptic curves (provably) has rank 5. For instance, the computer algebra system MAGMA [10] takes only a few seconds to upper bound each rank as 5 by a 2-Selmer computation, and find five independent points on each curve.

[14]Here we could use $[0,0,1,-247,1476]$ of conductor $22966597 = 47\cdot 488651$, but I preferred to select curves for which all the bad primes exceeded 100.

7.2.1. We recall that $\tilde\Lambda_f^{(l)}(1)$ can be approximated by a method detailed by Buhler, Gross, and Zagier [12, §4] (see also Cremona [16, §2.13]). For integral $l \geq 1$ we have

$$\frac{\tilde\Lambda_f^{(l)}(1)}{l!} = [1+\epsilon_f(-1)^l]\cdot\sum_{n=1}^{\infty}\frac{c_f(n)}{n}G_l\Big(\frac{2\pi n}{\sqrt{N_f}}\Big)$$

where

$$G_l(x) = \frac{1}{(l-1)!}\int_1^\infty e^{-xy}(\log y)^{l-1}\frac{\partial y}{y}$$

satisfies $G_l'(x) = -G_{l-1}(x)/x$, with $G_0 = e^{-x}$.

It is critical here that $G_l(x) \sim e^{-x}/x^l$ decays exponentially as $x \to \infty$, implying that we can calculate to d digits using approximately $(\sqrt{N_f}/2\pi)\cdot\log 10^d$ coefficients of the series, which is thus linear in d. The $c_f(p)$ can be computed in time polynomial in $\log p$ via Schoof's algorithm [68], though in practice using baby/giant steps (taking roughly $p^{1/4}$ time) is likely just as good for our range.

For $l = 1$ this G_l is a familiar exponential integral, and in general (e.g.) since G_l is holonomic its values can be computed in quasi-linear time (in the precision) via a binary splitting method given by van der Hoeven [42]. However, as noted to us by A. R. Booker, there is also the possibility to exploit the equi-spacedness of the evaluation points of G_l and use a multi-point polynomial evaluation scheme, again giving a quasi-quadratic running time overall. We analyze the situation more fully in our companion paper [87], finding that neither of these particularly displays the asymptotic quadratic behavior in our ranges of 1000 and 10000 digits. Meanwhile, the best method appears to be to compute batches of G_3-values by local power series, using the differential equation satisfied by G_3 to efficiently compute high-order derivatives (via recursion) at suitable demarcation points.[15]

7.2.2. For curves in $\mathcal{E}_1^{\pm}$ the conductors are sufficiently small that we can rather easily compute to high precision. For instance, for the curve of conductor 19047851 it takes about 20s to compute to 200 digits using the off-the-shelf Magma implementation [10].[16] Using a more rigourous implementation based on Arb (see [87] for details), we were able to compute to 10050 digits in around 4.6 days.

7.2.3. The conductors for $\mathcal{E}_2^{\pm}$ are much larger, with that for the twist of 11a by -25351367 being about $4\cdot 10^8$ times larger than 19047851. This means the length of the approximating sum is ≈ 20000 times longer, and thus reaching 10000 digits would presumably[17] take around 250 core-years. We instead chose to aim for 1000 digits, which took approximately 3 core-weeks for this curve.

7.2.4. The value of $L_{S^2f}(2)$ can be computed as in [81]. In Table 3, for each of our 12 curves we list an approximation to this value, given as a lower bound. Additionally we list an approximation for the fifth central derivative (again as a lower bound), We also list upper bounds for the expressions $\mathcal{U}(f)$ and $\mathcal{V}(f)$ that appear in §4.1.6.

[15] Note that this method is only quasi-cubic complexity even in theory, and in practice the growth exponent is more likely to be behave like $2+\rho$ where ρ is the effective exponent for multiplication (e.g., $\log(3)/\log(2) \approx 1.585$ in the range where Karatsuba multiplication applies).

[16] This does more work than necessary, including computing the first and fifth derivatives.

[17] The relative density of evaluation points also plays a rôle, and indeed one interpolation method would predict an expected time of $250/10^3$ years, but in practice it is about 1/4 of this.

curve label	$L_{S^2f}(2)$	$\tilde\Lambda_f^{(5)}(1)/5!$	$\mathcal{U}(f)$	$\mathcal{V}(f)$
$[0,0,1,-79,342]$ 19047851	8.7980	30.286	1.001	1.239
$[0,0,1,-169,930]$ 64921931	8.7354	38.850	1.001	1.242
$[0,1,1,-30,390]$ 67445803	5.7669	43.531	1.001	1.256
$[1,0,0,-22,219]$ 20384311	3.7504	41.941	1.001	1.689
$[0,1,1,-100,110]$ 55726757	4.8495	41.975	1.001	1.255
$[0,0,1,-139,732]$ 59754491	7.0086	41.009	1.001	1.247
(11a) $\otimes$ -25351367	1.0576	1525.2	1.009	1.484
(17a) $\otimes$ -19502039	0.7850	2379,6	1.005	2.048
(19a) $\otimes$ -16763912	0.9279	2918.6	1.583	2.094
(91b) $\otimes$ 6350941	1.2096	2000.4	1.028	2.053
(123a) $\otimes$ 5467960	2.7044	5620.4	1.256	1.286
(209a) $\otimes$ 3217789	1.0517	2743.0	1.012	2.271

TABLE 3. Data about our 12 rank 5 curves

Remark. We can note that $L_{S^2f}(2) \geq 0.78$ for each of the curves in $\mathcal{E}_1^\pm \cup \mathcal{E}_2^\pm$. In general, we could use an explicit version of the lower bound $L_{S^2f}(2) \gg 1/\log N_f$ of Goldfeld, Hoffstein, and Lieman [34] as detailed in [83, Lemma 3.4] (similar calculations appear in work of Rouse [65, Proposition 11]).

7.2.5. It is also useful to comment that $E_f^{\mathrm{r}}(1) \geq 1$ for each of the 12 curves.

Here we enumerate bad primes with the factors $\tilde V_p(s)$ as in Table 2. For the curves in $\mathcal{E}_1^\pm$ the Euler factors are $1/(1-p/p^{2s})$, and each is bounded below by 1 at $s=1$. For the curves in $\mathcal{E}_2^\pm$ we must be a bit more careful, as for a prime p of potentially good reduction the factor $\tilde V_p(s)$ depends on whether $p|D$ or not. There are only two cases where an individual $\tilde V_p(1)$ at a potentially good prime can be smaller than 1: for the twist of 91b by 6359041, at $p=191$ the factor is $\tilde V_p(s) = (1-194/p^{2s}+p^2/p^{4s})$ (where $194 = \tilde a_p^2 - 2\cdot 191$) which has $\tilde V_p(1) \geq 0.994$; and for the twist of 123a by 5467960, at $p=5$ the factor is $\tilde V_p(s) = (1-6/p^{2s}+p^2/p^{4s})$ which has $\tilde V_p(1) = 4/5$. However, we still find that $E_f^{\mathrm{r}}(1)$ is at least 1 in these cases, as in the former case the multiplicative primes contribute $1/(1-1/7)(1-1/13) = 91/72 \geq 1.26$, and in the latter they give $1/(1-1/3)(1-1/41) = 123/80 \geq 1.53$, easily outweighing the above factors.

7.3. Finally, let us state an explicit lemma.

Lemma 7.3.1. *Suppose that f is associated to one of the 12 curves in $\mathcal{E}_1^\pm \cup \mathcal{E}_2^\pm$ and that $|\tilde\Lambda_f'''(1)/3!| \leq \lambda \leq 1/10^9$. Then $\tilde\Lambda_f(s)$ has (at least) a triple central zero and exactly two other zeros inside $|s-1|^2 \leq \lambda/10$.*

Proof. We use Kolyvagin's result to show each curve has analytic rank at least 3, and then apply Lemma 7.1.2 with the explicitly tabulated lower bounds for the fifth central derivative of each of the 12 curves in question. □

8. MORE BOUNDS ON E_f AND L_{S^2f} AND THEIR DERIVATIVES

We now wish to give assorted technical preparations that will allow us to bound various secondary terms in our usage of the Deuring decomposition.

8.1. First we give bounds on the symmetric-square L-functions. Similar to the previous section, it seems easier to simply state computationally verifiable results for our specific 12 curves, as opposed to working in more generality.

Lemma 8.1.1. *For each of the curves in* $\mathcal{E}_1^\pm \cup \mathcal{E}_2^\pm$*, for* $|s-2| \le 2/10^5$ *we have*

$$0.999 L_{S^2 f}(2) \le \bigl|L_{S^2 f}(s)\bigr| \le 1.001 L_{S^2 f}(2),$$

and for $l \le 6$ *in* $|s-2| \le 2/10^5$ *we have*

$$|L^{(l)}_{S^2 f}(s)| \le 1.001 L_{S^2 f}(2) \cdot 100^l.$$

Remark. Note that the given bounds on $L_{S^2 f}(2)$ will *not* hold in general; e.g., when the logarithmic derivative at $s = 2$ is large the first statement is prone to fail. We really only need a bound of this sort in terms of the conductor, but I find it convenient to instead give a version with explicit constants for the given curves. Similarly one should generally expect the derivatives to grow roughly as $(\log N_{S^2 f})^l$.

Proof. Again this follows by computation. We can bound high derivatives of $L_{S^2 f}(s)$ in $|s-2| \le 2/10^5$ by convexity bounds, and approximate smaller derivatives to high precision by computation. Combining these via Taylor series, the result follows.

For instance, for 209a we can computationally estimate the zeroth through sixth derivatives at $s = 2$ as

$$\approx (1.0517,\ -0.7960,\ 2! \cdot 1.4184,\ -3! \cdot 1.7968,\ 4! \cdot 1.9123,\ -5! \cdot 1.7157,\ 6! \cdot 1.2856),$$

and the seventh derivative is $\approx -7! \cdot 0.7927$. We have the crude bound from convexity of $|L_{S^2 f}(s)| \le 5\sqrt{N_{S^2 f}}(1 + 5 \log N_{S^2 f})^3$ in $|s-2| \le 1$, so that Cauchy's bound on derivatives implies that the kth derivative is bounded in $|s-2| \le 2/10^5$ as $\le (1.001)^k k! \cdot 5\sqrt{N_{S^2 f}}(1 + 5 \log N_{S^2 f})^3$, and upon using $k = 8$ the bound stated in the Lemma readily follows. ∎

8.1.2. We also give a result for the effect of ramified primes on E_f.

Lemma 8.1.3. *For each of the curves in* $\mathcal{E}_1^\pm \cup \mathcal{E}_2^\pm$ *we have* $|E_f^{\mathrm{r}}(s)| \le 1.003 E_f^{\mathrm{r}}(1)$ *for* $|s-1| \le 1/1000$*, and also* $\bigl|\log(E_f^{\mathrm{r}}(s)/E_f^{\mathrm{r}}(1))\bigr| \le 2.714|s-1|$ *in this range.*

Proof. We can simply enumerate the prime factors of N_f in each case, and use Table 2. For the curves in $\mathcal{E}_1^\pm$ we only have multiplicative primes and

$$E_f^{\mathrm{r}}(s) = \prod_{p \mid N_f} \Bigl(1 - \frac{p}{p^{2s}}\Bigr)^{-1}.$$

We then note

$$\log E_f^{\mathrm{r}}(s) - \log E_f^{\mathrm{r}}(1) = \int_1^s \frac{(E_f^{\mathrm{r}})'}{E_f^{\mathrm{r}}}(z)\, \partial z$$

and proceed to bound the logarithmic derivative for $\sigma \ge 999/1000$ as

$$\le \sum_{p \mid N_f} \frac{p \log(p^2)/p^{2\sigma}}{1 - p/p^{2\sigma}} \le 0.031.$$

For the curves in $\mathcal{E}_2^\pm$ we have both the multiplicative primes from the initial curve and primes involved in the twisting. For instance, with (123a) $\otimes$ 5467960 we have

$$E_f^{\mathrm{r}}(s) = \prod_{p\in\{3,41\}} \Bigl(1-\frac{p}{p^{2s}}\Bigr)^{-1} \prod_{\substack{p\in\{2,5,223,613\}\\ p\nmid D}} \bigl(1-b_p/p^{2s}+p^2/p^{4s}\bigr)$$

where $(b_2, b_5, b_{223}, b_{613}) = (0, 6, -190, -550)$, for indeed we have $b_p = \tilde{a}_p^2 - 2p$ in terms of the $\tilde{a}_p$ for the minimal twist 123a. In this worst case we see the logarithmic derivative is bounded as ≤ 2.714 for $\sigma \ge 999/1000$, and the result follows. □

8.2. Next we turn to bounding the effect of noninert primes on E_f. We introduce the notation $\tilde{D} = D/4\pi^2$, noting that $\tilde{D} \le MD$.

Lemma 8.2.1. *Suppose that $D \ge 4\pi^2\exp(10^6)$ and $\sqrt{D}L_\chi(1) \le (\log D)^3/10^6$. Then for $|s-1| \le 1000/\log\tilde{D}$ we have $|E_f^{\mathrm{m}}(s)| \le 1.051 E_f^{\mathrm{m}}(1)$.*

Remark. It is this result in particular that is a bottleneck against taking D too much smaller in our main Theorems.

Proof. We will use

$$\log E_f^{\mathrm{m}}(s) - \log E_f^{\mathrm{m}}(1) = \int_1^s \frac{(E_f^{\mathrm{m}})'}{E_f^{\mathrm{m}}}(z)\,\partial z,$$

and bound the logarithmic derivative.

We recall (§4.1.4) that $E_f^{\mathrm{m}}(s)$ is given by

$$E_f^{\mathrm{m}}(s) = \prod_{p\le\sqrt{D}/4} \frac{1+\alpha_p'/p^s}{1-\alpha_p'\chi(p)/p^s}\,\frac{1+\beta_p'/p^s}{1-\beta_p'\chi(p)/p^s}.$$

Taking logarithms, we have

$$\log E_f^{\mathrm{m}}(s) = \sum_{p\le\sqrt{D}/4}\sum_{l=1}^\infty \bigl[\chi(p)^l - (-1)^l\bigr]\frac{(\alpha_p')^l+(\beta_p')^l}{lp^{ls}},$$

and taking the derivative of this appends $-l\log p$ to the right side.

As with Lemma 5.2.2, there are at most 2 split primes up to 10000, and at most 5 split primes up to $e^{\sqrt{Y}}$ where $Y = \log(\sqrt{D}/4)$. For $\sigma \ge 999/1000$ these small split primes contribute to the logarithmic derivative no more than

$$F(2)+F(3)+3F(10^4) \le 13.86, \ \text{ where } F(p) = \sum_{l=1}^\infty 4\frac{p^{l/2}\log p}{p^{0.999l}}.$$

The split primes exceeding $X_1 = e^{\sqrt{Y}}$ contribute no more than

$$\le \frac{(\log D)^3}{10^6}\sum_{l=1}^\infty 4\frac{X_1^{l/2}\log X_1}{X_1^{0.999l}} \le 10^{-135}.$$

For the primes $p|D$, we note that by genus theory and the class number formula that our assumption of $\sqrt{D}L_\chi(1) \le (\log D)^3/10^6$ implies there are no more than $A = \lfloor 5\log\log D\rfloor$ such primes. Writing B for the Ath prime, we can thus bound their contribution for $\sigma \ge 999/1000$ to the logarithmic derivative as

$$\le \sum_{p\le B}\sum_{l=1}^\infty 2\frac{p^{l/2}\log p}{p^{0.999l}} \le 4.2\cdot(5\log\log D)^{0.501},$$

where we used partial summation and the explicit bound that the nth prime exceeds $n \log n$ (cf. [63, (3.12)]).

Integrating then gives that for $|s-1| \leq 1000/\log \tilde{D}$ we have

$$\left|\frac{E_f^{\mathrm{m}}(s)}{E_f^{\mathrm{m}}(1)}\right| \leq \exp\biggl(\frac{1000}{\log \tilde{D}} \cdot \bigl[13.9 + 4.2 \cdot (5 \log\log D)^{0.501}\bigr]\biggr),$$

and our assumptions on D imply this is $\leq \exp(0.049) \leq 1.051$. □

Now we can state our desired bound on the derivatives of E_f.

Lemma 8.2.2. *Suppose $D \geq 4\pi^2 \exp(10^6)$ and $\sqrt{D} L_\chi(1) \leq (\log D)^3/10^6$ and f is associated to one of the 12 curves above. Then for $|s-1| \leq 2/\log \tilde{D}$ we have*

$$|E_f^{(l)}(s)| \leq 1.055 E_f(1) \cdot l! \Bigl(\frac{\log \tilde{D}}{998}\Bigr)^l.$$

Proof. For $|s-1| \leq 1000/\log \tilde{D}$ and $l = 0$ we can combine Lemmata 8.1.3 and 8.2.1, and then in $|s-1| \leq 2/\log \tilde{D}$ for the higher derivatives we can apply Cauchy's bound on derivatives in a circle about s of radius $998/\log \tilde{D}$. □

We also have a version that bounds $E_f(s)$ for $|s-1| \leq 1/1000$.

Lemma 8.2.3. *Suppose $D \geq 4\pi^2 \exp(10^6)$ and $\sqrt{D} L_\chi(1) \leq (\log D)^3/10^6$ and f is associated to one of the 12 curves above. Then for $|s-1| \leq 1/1000$ we have*

$$|E_f(s)| \leq E_f(1) \cdot \exp\bigl(15|s-1| \cdot (\log\log D)^{0.501}\bigr)$$

and

$$|E_f(s)| \geq E_f(1) \cdot \exp\bigl(-15|s-1| \cdot (\log\log D)^{0.501}\bigr).$$

Proof. Combining the proofs of Lemmata 8.1.3 and 8.2.1, for $\sigma \geq 999/1000$ we have

$$\left|\log\biggl(\frac{E_f^{\mathrm{r}}(s)}{E_f^{\mathrm{r}}(1)} \frac{E_f^{\mathrm{m}}(s)}{E_f^{\mathrm{m}}(1)}\biggr)\right| \leq |s-1| \cdot \bigl[2.8 + 13.9 + 4.2 \cdot (5 \log\log D)^{0.501}\bigr],$$

and the assumption that $D \geq 4\pi^2 \exp(10^6)$ then gives the result. □

Finally, we can improve Lemma 8.2.2 for $l = 0$ when considering a smaller circle.

Corollary 8.2.4. *Suppose $D \geq 4\pi^2 \exp(10^6)$ and $\sqrt{D} L_\chi(1) \leq (\log D)^3/10^6$ and f is as above. Then for $|s-1| \leq 2/\log D$ we have $|E_f(s)| \leq 1.001 E_f(1)$.*

Proof. This follows from the previous Lemma and $D \geq 4\pi^2 \exp(10^6)$. □

9. Consequences of the above Deuring decomposition

We next use the Deuring decomposition in Proposition 6.1.1 to show a lower bound for $\sqrt{D} L_\chi(1)$ under an assumption about zeros of $\tilde{\Lambda}_f^K(s)$.

9.1. Let us first record the derivatives of $T_f(z)$, which we recall itself is

$$T_f(z) = \Gamma(z)^2 \frac{L_{S^2 f}(2z)}{\zeta(2z-1)} E_f(z)(MD)^{z-1}.$$

In general, with $F(z) = \Gamma(z)^2/\zeta(2z-1)$ we have

$$T_f^{(l)}(z) = \sum\sum\sum\sum_{a+b+c+d=l} \frac{l!}{a!b!c!d!} F^{(a)}(z) L_{S^2 f}^{(b)}(2z) E_f^{(c)}(z)(\log MD)^d (MD)^{z-1}.$$

In particular, we will apply this formula to the fourth derivative ($l = 4$) at $z = 1$, and upon noting that $T_f(1) = F(1) = 0$, we wish for the $(a, b, c, d) = (1, 0, 0, 3)$ term to be dominant. This will follow from bounding the derivatives of E_f by our assumption of small $L_\chi(1)$, and by also bounding the derivatives of $L_{S^2 f}$. We do the latter computationally for our specific 12 elliptic curves, though it could be done in terms of N_f and in turn bounding this in terms of D.

We can note that $F'(1) = 2$ and $F''(1) = -16\gamma$, while

$$|F''(1)| \leq 10,\ |F'''(1)| \leq 57,\ |F^{(4)}(1)| \leq 368,\ |F^{(5)}(1)| \leq 2731,\ |F^{(6)}(1)| \leq 22510.$$

Similar bounds hold on these derivatives hold in $|z - 1| \leq 1/10^5$ upon being multiplied by 1.001, while $|F'(z)| \leq 2.001$ and $|F(z)| \leq 2.001|z - 1|$ in this range.

9.2. We next show that for sufficiently large D the expected term is dominant in the second and fourth central derivatives, while the sixth derivative is adequately bounded near the central point. We recall that M is defined by $4\pi^2 MD = \sqrt{N_f N_{f\chi}}$ with thus $1 \leq 4\pi^2 M \leq N_f$, and also our notation $\tilde{D} = D/4\pi^2$.

Lemma 9.2.1. *Suppose that $D \geq 4\pi^2 \exp(10^6)$ and $\sqrt{D} L_\chi(1) \leq (\log D)^3/10^6$, while f is associated to one of the 12 curves considered above. Then with Θ representing a factor bounded by 1, we have*

$$T_f''(1) = \frac{2!}{1!} \cdot 2L_{S^2 f}(2) E_f(1)(\log MD) \cdot \big[1 + \Theta(1/750)\big]. \qquad \textbf{(A)}$$

and

$$T_f^{(4)}(1) = \frac{4!}{3!} \cdot 2L_{S^2 f}(2) E_f(1)(\log MD)^3 \cdot \big[1 + \Theta(1/200)\big], \qquad \textbf{(B)}$$

while for $|z - 1| \leq (5/4)/\log MD$ we have

$$\big|T_f^{(6)}(z)\big| \leq 4.5 \cdot \frac{6!}{5!} \cdot 2L_{S^2 f}(2) E_f(1)(\log MD)^5. \qquad \textbf{(C)}$$

Proof. We explicitly have that (using $F'(1) = 2$ and $F''(1) = -16\gamma$)

$$T_f''(1) = \frac{2!}{1!} \cdot 2L_{S^2 f}(2) E_f(1) \Big[\log MD - 4\gamma + 2\frac{L'_{S^2 f}}{L_{S^2 f}}(2) + \frac{E'_f}{E_f}(1)\Big],$$

where we have that $|E'_f(1)/E_f(1)| \leq (\log \tilde{D})/950$ and $|L'_{S^2 f}(2)/L_{S^2 f}(2)| \leq 101$ from Lemma 8.2.2 and Lemma 8.1.1 respectively. Thus the bracketed term in $T_f''(1)$ is $\log MD + \Theta\big(204 + (\log \tilde{D})/950\big) = (\log MD) \cdot \big[1 + \Theta(1/795)\big]$.

With the fourth derivative, the main term is $B_4 = \frac{4!}{3!} \cdot 2L_{S^2 f}(2)E_f(1)(\log MD)^3$ (from $d = 3$), and the terms for $d = 2$ are bounded as

$$\le \frac{4!}{2!2!} \cdot 10 \cdot L_{S^2 f}(2) \cdot E_f(1) \cdot (\log MD)^2 + \frac{4!}{2!} \cdot 2 \cdot 201 L_{S^2 f}(2) \cdot E_f(1) \cdot (\log MD)^2 + \\ + \frac{4!}{2!} \cdot 2 \cdot L_{S^2 f}(2) \cdot E_f(1) \frac{\log \tilde{D}}{950} \cdot (\log MD)^2 \le B_4 \cdot \Bigl(\frac{15/2 + 3 \cdot 201}{10^6} + \frac{3}{950}\Bigr).$$

The terms with $d \le 1$ contribute considerably less, for instance the bound for the bound for the $(1, 0, 2, 1)$-term is $\approx 2/998$ of the bound for the $(1, 0, 1, 2)$-term. More explicitly, their contribution is

$$\le 1.06 L_{S^2 f}(2)E_f(1) \sum_{\substack{a+b+c+d=4 \\ a \ge 1,\, d \le 1}}\!\sum\sum\sum \frac{4!}{a!b!c!d!} 10^a (200)^b c! \Bigl(\frac{\log \tilde{D}}{998}\Bigr)^c (\log MD)^d$$

$$\le 1.06 L_{S^2 f}(2)E_f(1)(\log MD)^3 \sum_{\substack{a+b+c+d=4 \\ a \ge 1,\, d \le 1}}\!\sum\sum\sum \frac{4!}{a!b!c!d!} \frac{10^a (200)^b}{(10^6)^{(a-1)+b}} \frac{c!}{998^c},$$

which is $\le 0.001 L_{S^2 f}(2)E_f(1)(\log MD)^3$. This gives the indicated error in (**B**).

Finally, for the bound on the sixth derivative, the contribution from $d = 6$ in the region $|z - 1| \le (5/4)/\log MD \le 1/10^5$ is

$$\begin{aligned} \le \frac{6!}{6!} \cdot \bigl|F(z)L_{S^2 f}(2z)E_f(z)(\log MD)^6 \cdot (MD)^{z-1}\bigr| & \\ \le 2.001|z-1| \cdot 1.001 L_{S^2 f}(2) \cdot 1.001 E_f(1) \cdot \exp(5/4)(\log MD)^6 & \\ \le 8.75 \cdot L_{S^2 f}(2)E_f(1)(\log MD)^5 \le 0.73 \cdot \frac{6!}{5!} \cdot 2L_{S^2 f}(2)E_f(1)(\log MD)^5, & \end{aligned}$$

while the principal contribution with $d = 5$ is (using Corollary 8.2.4)

$$\begin{aligned} \le \frac{6!}{5!} \cdot \bigl|F'(z)L_{S^2 f}(2z)E_f(z)(\log MD)^5 \cdot (MD)^{z-1}\bigr| & \\ \le 6 \cdot 2.001 \cdot 1.001 L_{S^2 f}(2) \cdot 1.001 E_f(1) \cdot \exp(5/4)(\log MD)^5 & \\ \le 3.50 \cdot \frac{6!}{5!} \cdot 2L_{S^2 f}(2)E_f(1)(\log MD)^5 = 3.50 B_6. & \end{aligned}$$

The other $d = 5$ terms give a contribution

$$\le \frac{6!}{5!} \cdot 2.001|z-1| \cdot L_{S^2 f}(2)E_f(1)\exp(5/4)(\log MD)^5 \cdot \Bigl(201 + \frac{\log \tilde{D}}{950}\Bigr) \le B_6/180,$$

while the main $d = 4$ term from $(1, 0, 1, 4)$ is $\le 3.5 \cdot (4/945)B_6$, etc. Adding these up, we get the stated bound of the Lemma. □

9.3. Now we state and prove our main consequence of the Deuring decomposition.

Lemma 9.3.1. *Suppose that f and $f\chi$ are of odd parity and f has analytic rank at least 3, and that $L_f(s)$ has an additional pair of zeros $1 \pm i\kappa$ with κ either real or imaginary (with possibly $\kappa = 0$) and $|\kappa| \le 1/10^5$. Suppose that $D \ge 4\pi^2 \exp(10^6)$ with $D \ge N_f^9$ and $\sqrt{D}L_\chi(1) \le (\log D)^3/10^6$, and the conditions* (**A**), (**B**), *and* (**C**) *are met, along with the bounds of Lemmata 8.1.1 and 8.1.3 on the symmetric-square L-function of f (these bounds are met for the 12 curves in $\mathcal{E}_1 \cup \mathcal{E}_2$). Then* *with* $M=\sqrt{N_fN_{f\chi}}/4\pi^2$ *so that* $1\le 4\pi^2M\le N_f$ *we have*

$$116\sqrt{N_f}\cdot\mathcal{W}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\ge L_{S^2f}(2)E_f(1)\cdot\min\Big(\frac{(\log MD)^3}{(5/4)^2},\frac{\log MD}{|\kappa|^2}\Big).$$

Here $\mathcal{W}(f)=\mathcal{U}(f)\mathcal{V}(f)$ *and the latter are defined in §4.1.6 (each being* $\le\zeta(2)^2$*), and* $\mathcal{R}(\chi)$ *is also defined in §4.1.6 (as a reciprocal sum over small representations).*

Proof. We will apply the Deuring decomposition of Proposition 6.1.1 with $\tilde r=4$.

9.3.2. We first consider the case where $|\kappa|\le(5/4)/\log MD$. In this range we will essentially use that

$$\sin\xi-\xi=-\frac{\xi^3}{3!}\big[1+\Theta(\xi^2/20)\big]$$

where $\xi=\kappa\log MD$, so that the relative error term is $\Theta(5/64)$ (which however will be multiplied by $\exp(5/4)$, and other small factors). Indeed, it does not matter whether κ is on either the real or imaginary axis or not (also, $\kappa=0$ can be seen to be allowable). Upon replacing $0=\tilde\Lambda_f^K(1+i\kappa)$ by the Deuring decomposition of Proposition 6.1.1 and using the Taylor expansion of T_f about 1 we obtain

$$0=2\frac{T_f^{(4)}(1)}{4!}\kappa^4+\Theta\Big(2\cdot\frac{\hat T_f^{(6)}}{6!}|\kappa|^6\Big)+\Theta\big(16\cdot 30\sqrt{M}\cdot\mathcal{W}(f)\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\kappa^4\big)$$

where $\hat T_f^{(6)}$ is a bound for the sixth derivative of T_f in $|z-1|\le(5/4)/\log MD$. By the above bounds from (**B**) and (**C**) we find that the fourth derivative dominates between the two first terms, and thus the final term must be large, namely

$$\Big(2\cdot\frac{1-1/200}{3!}-2\cdot\frac{4.5}{5!}\cdot(5/4)^2\Big)\cdot 2L_{S^2f}(2)E_f(1)(\log MD)^3\kappa^4 \le 77\sqrt{N_f}\cdot\mathcal{W}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\kappa^4.$$

The pre-prepending term in parentheses is ≥ 0.214, so by rearrangement we achieve the first part of the minimum in the Lemma, since $77/0.428/(5/4)^2\le 116$.

9.3.3. When $|\kappa|\ge(5/4)/\log MD$ we split into cases depending on whether κ is real or not. In the former we will use $|\sin\xi|\le 4\xi/5$ for $\xi\ge 5/4$, and in the latter (essentially) that $\sinh\xi\ge 1.28\xi$ in this range.

For κ real we can take $\kappa\ge 0$ by symmetry and we have

$$|T_f(1+i\kappa)|=|\Gamma(1+i\kappa)|^2\cdot\Big|\frac{L_{S^2f}(2+2i\kappa)}{\zeta(1+2i\kappa)}\Big|\cdot|E_f(1+i\kappa)|.$$

We have $|\Gamma(1+i\kappa)|\le 1$ and $|1/\zeta(1+2i\kappa)|\le 2\kappa$, while $|L_{S^2f}(2+2i\kappa)|\le 1.001L_{S^2f}(2)$ by Lemma 8.1.1, while by Lemma 8.2.3 writing $\beta=0.501$ we have

$$\frac{|E_f(1+i\kappa)|}{E_f(1)}\le\exp\Big(\int_0^\kappa\Big|\frac{E_f'}{E_f}(1+it)\Big|\,\partial t\Big)\le\exp\big(15\kappa(\log\log D)^\beta\big).$$

When $\kappa\le 1/\sqrt{\log D}$ and $D\ge 4\pi^2\exp(10^6)$ this is bounded as

$$|E_f(1+i\kappa)|/E_f(1)\le 1+16\kappa(\log\log D)^\beta\le 1+0.00006\kappa\log D.$$

Otherwise we use $\kappa\le 1/10^5$ and have $|E_f(1+i\kappa)|/E_f(1)$ is

$$\le\exp\Big(\frac{15}{10^5}(\log\log D)^\beta\Big)\le 0.001001\sqrt{\log D}\le 0.001001\kappa\log D.$$

We have that $1 \le (4/5)\kappa \log MD$, and thus

$$\frac{|E_f(1+i\kappa)|}{E_f(1)} \le 0.800\kappa \log MD + 0.001061\kappa \log D \le 0.802\kappa \log MD.$$

Putting these together, we find that

$$\big|T_f(1+i\kappa)\big| \le 2.002\kappa L_{S^2 f}(2)E_f(1)\cdot 0.802\kappa \log MD.$$

On the other hand, by definition $S_f^4(1+i\kappa) = T_f''(1)\kappa^2$ and by (**A**) we have

$$|T_f''(1)| \ge 3.994 L_{S^2 f}(2)E_f(1)\log MD.$$

From this, since $3.994 - 2(2.002\cdot 0.802) \ge 0.782$ we get

$$|U_f(1+i\kappa)| \ge |T_f''(1)||\kappa|^2 - 2|T_f(1+i\kappa)| \ge 0.782\kappa^2 L_{S^2 f}(2)E_f(1)\log MD,$$

and since $|U_f(1+i\kappa)| \le 77\sqrt{N_f}\cdot\mathcal{W}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\kappa^4$ as above, the second part of the minimum in the Lemma statement follows as before since $77/0.782 \le 99 < 116$.

9.3.4. Finally, if the pair of zeros of $L_f(s)$ with $|\kappa| \ge (5/4)/\log MD$ is on the real axis, we can take $i\kappa \ge 0$ by symmetry. By the second part of Lemma 8.2.3 we have the lower bound (again writing $\beta = 0.501$)

$$\begin{aligned}|T_f(1+i\kappa)| &= (MD)^{|\kappa|}\cdot\left|\Gamma(1+i\kappa)^2\frac{1}{\zeta(1+2i\kappa)}L_{S^2 f}(2+2i\kappa)E_f(1+i\kappa)\right|\\ &\ge 0.999^2\cdot 1.999|\kappa|\cdot 0.999L_{S^2 f}(2)\cdot(MD)^{|\kappa|}\cdot E_f(1)\exp\bigl(-15\kappa(\log\log D)^\beta\bigr)\\ &\ge 1.993|\kappa|L_{S^2 f}(2)\cdot E_f(1)\cdot e^{5/4}\frac{|\kappa|\log MD}{5/4}\Bigl[1-\frac{19(\log\log D)^\beta}{\log\tilde{D}}\Bigr],\end{aligned}$$

the latter since $(MD)^{|\kappa|} = e^{5\lambda/4} \ge e^{5/4}\lambda$ for $\lambda \ge 1$ where $\lambda = (|\kappa|\log MD)/(5/4)$, and similarly an upper bound for $|T_f(1-i\kappa)|$ of

$$\begin{aligned}&\le 1.001^2\cdot 2.001|\kappa|\cdot 1.001L_{S^2 f}(2)\cdot(MD)^{-|\kappa|}\cdot E_f(1)\exp\bigl(15\kappa(\log\log D)^\beta\bigr)\\ &\le 2.008|\kappa|\cdot L_{S^2 f}(2)E_f(1)\cdot e^{-5/4}\frac{|\kappa|\log MD}{5/4}\Bigl[1+\frac{19(\log\log D)^\beta}{\log\tilde{D}}\Bigr].\end{aligned}$$

Subtracting the second from the first and using $D \ge 4\pi^2\exp(10^6)$ gives us

$$|T_f(1+i\kappa)| - |T_f(1-i\kappa)| \ge (5.565 - 0.461)|\kappa|^2(\log MD)\cdot L_{S^2 f}(2)E_f(1).$$

Meanwhile, from (**A**) we have

$$|S_f^4(1+i\kappa)| \le 4.006|\kappa|^2\cdot L_{S^2 f}(2)E_f(1)\log MD,$$

so that $\tilde{\Lambda}_f(1+i\kappa) = 0$ implies

$$|U_f(1+i\kappa)| \ge 1.098|\kappa|^2(\log MD)\cdot L_{S^2 f}(2)E_f(1)$$

and we conclude as before since $77/1.098 \le 71 < 116$. □

10. Root numbers of modular forms and main results

We now describe various conditions that ensure that $f\chi$ has odd parity, so that we can then apply the above Lemma 9.3.1.

10.1. Let us first recall Oesterlé's explicit result [58] for class numbers of imaginary quadratic fields, which he deduces from the rank 3 elliptic curve of conductor 5077.

There are three possibilities for $\chi(5077)$. When this is $+1$ the class number is bounded below by $(\log\sqrt{D/4})/(\log 5077) \geq (\log D)/55$. When it is -1 the twist $f\chi$ has odd parity and he obtains[18]

$$h_K \geq \frac{\log D}{55}\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

However, when $5077|D$ we have limited control over the root number, and it appears difficult to obtain a similar result. The above can be replicated for any rank 3 curve whose conductor has an odd number of prime factors (counting multiplicity), but will generically have a condition about D being coprime to the conductor.

10.1.1. One way to get a uniform result is to exploit the greater control over the root number that exists in quadratic twist families. For instance, Gross and Zagier point out that one can use the -139th quadratic twist of the elliptic curve 37b (this has rank 3).[19] As above, when $\chi(37) = +1$ we are done, and when $f\chi$ has odd parity we achieve [58, §5.1]

$$h_K \geq \frac{\log D}{7000}\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

Moreover, a calculation with root numbers [58, §4.3] shows that this condition of odd parity holds whenever $\chi(37) \neq +1$ (including when $37|D$ or $139|D$). The smallness of $1/7000$ compared to $1/55$ is principally because of the larger conductor.

10.1.2. Our lower bound will be much bigger than $(\log\sqrt{D/4})/(\log N_f)$, and so we must necessarily employ at least two such curves (rather than just one) in exploiting the possible splitting of small primes; we actually will utilize three curves (in each case) to make cleaner statements of our results. We also need our curves to have rank 5, forcing the conductors to be larger. The case of real quadratic fields flips the parity of the number of prime factors of the conductor, but is otherwise analogous.

10.2. Having done the calculations indicated in §7.2.2, we can conclude as follows.

Theorem 10.2.1. *Suppose that $-D$ is a negative fundamental discriminant (of an imaginary quadratic field) with $D \geq 4\pi^2\exp(10^6)$ and*

$$\gcd(D, 19047851\cdot 64921931\cdot 67445803) = 1.$$

Then the class number h_K of $K = \mathbf{Q}(\sqrt{-D})$ is bounded below as

$$\pi h_K = \sqrt{D}L_\chi(1) \geq \min\bigl(10^{10000}\log D, (\log D)^3/10^8\bigr)\cdot\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

[18] One can be more clever and include the explicit a_p for the rank 3 curve rather than $2\sqrt{p}$, but this has little effect on the worst case when there are $\approx (\log\log D)/(\log 2)$ primes dividing D.

[19] This curve also has some very specific additional properties which ease the proof that the Heegner point is torsion (and thus the analytic rank is 3).

Proof. Firstly, if we have that $\chi(N_f)=+1$ for all three conductors of the curves in $\mathcal{E}_1^-$, then we already have $h_K\ge(\log\sqrt{D/4})^3/(\log 67445803)^3$ and are done.

On the other hand, if one of the three given prime conductors is inert in K, then $f\chi$ has odd parity for the corresponding f in $\mathcal{E}_1^-$ (see Lemma 10.3.2 below), and we can use Lemma 9.3.1. This gives us

$$116\sqrt{N_f}\cdot\mathcal{W}(f)\cdot\sqrt{D}L_\chi(1)\mathcal{R}(\chi)\ge L_{S^2f}(2)\cdot E_f(1)\cdot\min\Bigl(\frac{(\log\tilde D)^3}{(5/4)^2},\frac{\log\tilde D}{|\kappa|^2}\Bigr).$$

By Table 3 we have $\mathcal{W}(f)\le 1.258$, and the comments in §7.2.5 imply $E_f^{\mathrm{r}}(1)\ge 1$, while $L_{S^2f}(2)\ge 5.75$ for the three curves in question.

The effect on $E_f^{\mathrm{m}}(1)$ from split primes is bounded similarly to Lemma 5.2.2, though using the explicit bounds $c_f(2)\ge -2$ and $c_f(3)\ge -3$ we see they contribute

$$\ge 0.999\cdot\frac{1-2/2+2/2^2}{1+2/2+2/2^2}\frac{1-3/3+3/3^2}{1+3/3+3/3^2}\biggl(\Bigl(\frac{1-1/\sqrt{10^4}}{1+1/\sqrt{10^4}}\Bigr)^2\biggr)^3\ge\frac{1}{39.51}.$$

This implies

$$E_f^{\mathrm{m}}(1)\ge\frac{1}{39.51}\prod_{p|D}\Bigl(1+\frac{c_f(p)}{p}+\frac{p}{p^2}\Bigr).$$

Meanwhile, as in (6) of §4.1.6 we have

$$\mathcal{R}(\chi)\le\prod_{p\le\sqrt{D}/4}\frac{1+1/p}{1-\chi(p)/p},$$

and here the split primes have an effect bounded as ≤ 6.01. We can combine the primes with $p|D$ in $\mathcal{R}(\chi)$ with the analogous bound for $E_f^{\mathrm{m}}(1)$ upon noting the identity $(1+c_f(p)/p+p/p^2)/(1+1/p)=1+c_f(p)/(p+1)$, while $|c_f(p)|\le\lfloor 2\sqrt{p}\rfloor$.

The computations outlined in §7.2.2 show that $|\tilde\Lambda_f'''(1)/3!|\le 1/10^{10025}$ and thus by Lemma 7.3.1 we have $|\kappa|\le 10^{-5010}$ and so by Lemma 9.3.1 we get

$$\sqrt{D}L_\chi(1)\ge\frac{1}{116}\frac{1/6.01}{39.51}\frac{5.75/1.258}{\sqrt{67445803}}\cdot\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr)\cdot\min\Bigl(\frac{(\log\tilde D)^3}{(5/4)^2},10^{10020}\log\tilde D\Bigr).$$

Rearrangement then gives the statement of the Theorem, since $7.74\cdot 10^7<10^8$. □

Theorem 10.2.2. *Suppose that D is a positive fundamental discriminant (for a real quadratic field) with $D\ge 4\pi^2\exp(10^6)$ and*

$$\gcd(D,3089\cdot 6599\cdot 647\cdot 86131\cdot 409\cdot 146099)=1.$$

Then we have

$$\sqrt{D}L_\chi(1)\ge\min\bigl(10^{10000}\log D,(\log D)^3/10^8\bigr)\cdot\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

Proof. We first note that when $\chi(3089)=\chi(6599)=-1$ the associated $f\chi$ has odd parity, and similarly when $\chi(647)=\chi(86131)=-1$ or $\chi(409)=\chi(146099)=-1$. In these cases, we can apply Lemma 9.3.1 to the associated curve in $\mathcal{E}_1^+$ and obtain (similar to above, with $L_{S^2f}(2)/\mathcal{W}(f)\sqrt{N_f}$ minimized for the first curve)

$$\sqrt{D}L_\chi(1)\ge\frac{1}{116}\frac{1/6.01}{39.51}\frac{3.75/1.691}{\sqrt{20384311}}\cdot\prod_{p|D}\Bigl(1-\frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr)\cdot\min\Bigl(\frac{(\log\tilde D)^3}{(5/4)^2},10^{10020}\log\tilde D\Bigr),$$

and rearrangement gives the statement of the Theorem since $8.76 \cdot 10^7 < 10^8$.

Otherwise, in each of $\{3089, 6599\}$, $\{647, 86131\}$, and $\{409, 146099\}$ there is at least one p with $\chi(p) = +1$, and Lemma 5.1.1 gives a lower bound for $\sqrt{D}L_\chi(1)$. □

As noted in the Introduction, this lower bound gives an alternative effective resolution of the conjectures of Chowla and Yokoi concerning the cases of class number 1 for $\mathbf{Q}(\sqrt{4u^2+1})$ and $\mathbf{Q}(\sqrt{u^2+4})$. In these families, we know (by genus theory) that D can be taken as prime, so the gcd condition is easily met, while the product over D is harmless.

10.3. We now turn to stating results that are free from the gcd-restriction. For f associated to curves in $\mathcal{E}_2^\pm$ we can show that $f\chi$ has odd parity under suitable conditions, so as to again place ourselves in the situation where either there are three small split primes, or $f\chi$ indeed has odd parity and we can use Lemma 9.3.1.

10.3.1. We assemble various lemmata concerning root numbers of L-functions of modular forms. Alternatively, as we only work in weight 2, this could also be done in terms of root numbers of elliptic curves. We readily admit that our presentation is somewhat specific to our case of interest where f itself has odd parity and we wish $f\chi$ to also have odd parity.

We write $\epsilon_p(f)$ for the local root number of f at p. This is trivial when p is a prime of good reduction. As we are in weight 2, we have $\epsilon_\infty(f) = -1$.

For odd primes p we define $p^\star$ as $p \cdot (-1)^{(p-1)/2}$ so that $p^\star$ is always a fundamental discriminant. In the context of an even fundamental discriminant t, we define $2^\star$ by dividing t out by $p^\star$ for all its odd prime factors p. We write ψ_u for the quadratic character corresponding to $\mathbf{Q}(\sqrt{u})$.

Lemma 10.3.2. *Let g be a primitive eigenform of weight 2 and level l, and t be a fundamental discriminant. Then we have that $\epsilon_p(g\psi_t) = \psi_{p^\star}(-1)$ for $p|t$, while $\epsilon_p(g\psi_t) = \epsilon_p(g)\psi_t(p)$ for $p|l$ with $p \nmid t$.*

Proof. This appears in Atkin and Lehner [2, Theorem 6]. □

Lemma 10.3.3. *Let g be a primitive eigenform of weight 2 and level l. Let B be a fundamental discriminant coprime to l with the same sign as D, and let $f = g\psi_B$. Then with $G = \gcd(D, l)$ and $[\psi_B\chi]$ the primitive inducer of $\psi_B\chi$, we have*

$$\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-l/G)\frac{\epsilon(g)}{\prod_{p|G}\epsilon_p(g)}.$$

Proof. When $p|B$ and $p|D$ and p is odd we have $\epsilon_p(f\chi) = \epsilon_p(g\psi_B\chi) = +1$ since $f\chi$ has good reduction at p. When B and D are both even, we use the above Lemma to get $\epsilon_2(f\chi) = \epsilon_2(g\psi_B\chi) = \psi_u(-1)$ where $u \in \{1, 8\}$ (since B and D have the same sign), so that $\epsilon_2(f\chi) = +1$ also in this case.

When $p|B$ or $p|D$ (but not both) we have $\epsilon_p(f\chi) = \epsilon_p(g\psi_B\chi) = \psi_{p^\star}(-1)$, where this is suitably interpreted for $p = 2$ as above. Otherwise when $p|l$ and $p \nmid BD$ we have $\epsilon_p(f\chi) = \epsilon_p(g)[\psi_B\chi](p)$. These again follow from the previous Lemma.

The primes with $p|BD$ thus give a factor of $\psi_{(BD)^\star}(-1) = [\psi_B\chi](-1)$, and the primes with $p|l$ and $p \nmid BD$ yield $\prod_p \epsilon_p(g) \cdot [\psi_B\chi](l/G)$ where the product is over $p|l$ with $p \nmid D$. Upon including the prime at infinity, this gives the Lemma. □

10.3.4. We now want to set up situations where we know that various twists will have odd parity. The case of imaginary quadratic K is slightly easier.

Lemma 10.3.5. *Let f be the Bth quadratic twist of a newform g of weight 2 and prime level q with $\epsilon(g) = +1$, where B is a negative fundamental discriminant with $\psi_B(-q) = -1$. Suppose $\chi(q) \neq +1$ and $\chi(-1) = -1$. Then $f\chi$ has odd parity.*

Proof. There are two cases. When $\chi(q) = -1$ then we have $G = \gcd(D, q) = 1$ and the previous Lemma gives

$$\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-q)\epsilon(g) = -\chi(-q)\epsilon(g) = \chi(q)\epsilon(g) = -\epsilon(g) = -1.$$

When $\chi(q) = 0$ we have $G = \gcd(D, q) = q$ and we get

$$\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-1)\frac{\epsilon(g)}{\epsilon_q(g)} = (-1)^2 \cdot \epsilon_\infty(g) = -1,$$

since $\epsilon_q(g) = -1$ for g to originally have even parity. □

Here $\mathcal{E}_2^-$ uses $(g, B) \in \{(11a, -25351367), (17a, -19502039), (19a, -16763912)\}$.

Lemma 10.3.6. *Let f be the Bth quadratic twist of a newform g of weight 2 and level m with $m = p_1p_2$ a product of 2 distinct odd primes, with $\epsilon_{p_1}(g) = \epsilon_{p_2}(g) = -1$, and B a positive fundamental discriminant with $\psi_B(p_1) = \psi_B(p_2) = +1$. Suppose that $\chi(p_1) \neq +1$ and $\chi(p_2) \neq +1$ and $\chi(-1) = +1$. Then $\epsilon(f\chi) = -1$.*

Proof. There are four possibilities for $G = \gcd(D, p_1p_2)$. When this is trivial, we then have $\chi(p_1) = \chi(p_2) = -1$ so that $\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-p_1p_2)\epsilon(g) = -1$, since $\epsilon(g) = -\epsilon_{p_1}(g)\epsilon_{p_2}(g) = -1$.

When $G = p_1$ we have

$$\begin{aligned}\epsilon(f\chi) = \epsilon(g\psi_B\chi) &= [\psi_B\chi](-p_2) \cdot \epsilon(g)/\epsilon_{p_1}(g)\\ &= \psi_B(p_2)\chi(p_2) \cdot (-1)/(-1) = \chi(p_2) = -1,\end{aligned}$$

and by symmetry the same calculation suffices when $G = p_2$.

Finally, when $G = p_1p_2$ we have $\epsilon(f\chi) = \epsilon(g\psi_B\chi) = [\psi_B\chi](-1)\cdot\epsilon_\infty(g) = -1$. □

Here $\mathcal{E}_2^+$ uses $(g, B) \in \{(91b, 6350941), (123a, 5467960), (209a, 3217789)\}$.

10.4. Having done the calculations of §7.2.3, we can show our ultimate result.

Theorem 10.4.1. *Suppose D is sufficiently large. Then*

$$\sqrt{D}L_\chi(1) \geq \min\bigl(10^{1000}\log D, (\log D)^3/10^{13}\bigr) \cdot \prod_{p|D}\Bigl(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr).$$

Proof. First we consider the case where K imaginary. If $f\chi$ has even parity for all of the curves in $\mathcal{E}_2^-$, by Lemma 10.3.5 we have $\chi(11) = \chi(17) = \chi(19) = +1$, and thus $h_K \geq (\log\sqrt{D/4})^3/(\log 19)^3$.

Similarly, when K is real and $f\chi$ has even parity for all three curves in $\mathcal{E}_2^+$, Lemma 10.3.6 implies $\chi(p) = +1$ for at least one p in each of the three sets $\{7, 13\}$, $\{3, 41\}$, and $\{11, 19\}$, so that $\sqrt{D}L_\chi(1)$ is large by Lemma 5.1.1.

Meanwhile, independent of whether K is real or imaginary, when $f\chi$ has odd parity we can use Lemma 9.3.1 and get

$$\sqrt{D}L_\chi(1) \geq \frac{1/238}{116}\frac{L_{S^2f}(2)}{\mathcal{W}(f)\sqrt{N_f}} \cdot \prod_{p|D}\Bigl(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\Bigr) \cdot \min\Bigl(\frac{(\log\tilde{D})^3}{(5/4)^2}, 10^{1020}\log\tilde{D}\Bigr).$$

For all the curves in $\mathcal{E}_2^\pm$ except the twist of 19a, the quotient $L_{S^2f}(2)/\mathcal{W}(f)\sqrt{N_f}$ is sufficiently large to yield the statement of the Theorem (the worse case is 17a, when the comparison is $9.1 \cdot 10^{12} < 10^{13}$). Moreover, as the twist of 19a is by the even discriminant -16763912, in this case we use $c_f(2) = 0$ and $c_f(5) = -3$ to improve the lower bound on $E_f^{\mathrm{m}}(1)$, thus obtaining the claimed constant of 10^{13}. □

References

[1] S. Ahlgren, A. Dunn, *Maass forms and the mock theta function $f(q)$*. Math. Ann. **374** (2019), 1681–1718. http://doi.org/10.1007/s00208-019-01829-0

[2] A. O. L. Atkin, J. Lehner, *Hecke operators on $\Gamma_0(m)$*. Math. Ann. **185** (1970), no. 2, 134–160. http://eudml.org/doc/161948

[3] A. Baker, *Linear forms in the logarithms of algebraic numbers*. Mathematika **13** (1966), no. 2, 204–216. http://doi.org/10.1112/S0025579300003971

[4] A. Baker, *Imaginary Quadratic Fields with Class Number 2*. Ann. Math. **94** (1971), no. 1, 139–152. http://doi.org/10.2307/1970739

[5] P. T. Bateman, E. Grosswald, *On Epstein's zeta function.* Acta Arith. **9** (1964), no. 4, 365–373. http://eudml.org/doc/207491

[6] B. Birch, *Heegner Points: The Beginnings.* In *Heegner Points and Rankin L-Series*, edited by H. Darmon and S.-W. Zhang. MSRI Publications **49** (2004), 1–10. http://library.msri.org/books/Book49/files/01birch.pdf

[7] B. J. Birch, H. P. F. Swinnerton-Dyer, *Notes on elliptic curves. II.* J. reine angew. Math. **218** (1965), 79–108. http://eudml.org/doc/150676

[8] A. Biró, *Yokoi's conjecture*. Acta Arith. **106** (2003), no. 1, 85–104.
———, *Chowla's conjecture. ibid.* **107** (2003), no. 2, 179–194.
http://doi.org/10.4064/aa106-1-6 http://doi.org/10.4064/aa107-2-5

[9] V. Blomer, *Sums of Hecke eigenvalues over values of quadratic polynomials*. Int. Math. Res. Not. **2008**, no. 16. http://doi.org/10.1093/imrn/rnn059

[10] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language.* J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. http://magma.maths.usyd.edu.au

[11] B. L. J. Braaksma, *Asymptotic expansions and analytic continuations for a class of Barnes-integrals.* Compositio Math. **15** (1962-4), 239–341. http://eudml.org/doc/88877

[12] J. P. Buhler, B. H. Gross, D. B. Zagier, *On the conjecture of Birch and Swinnerton-Dyer for an elliptic curve of rank 3.* Math. Comp. **44** (1985), 473–481. http://doi.org/10.1090/S0025-5718-1985-0777279-X

[13] J. Coates, C.-G. Schmidt, *Iwasawa theory for the symmetric square of an elliptic curve.* J. reine angew. Math. **375/376** (1987), 104–156. http://eudml.org/doc/152908

[14] B. Conrey, H. Iwaniec, *Spacing of zeros of Hecke L-functions and the class number problem.* Acta Arith. **103** (2002), no. 3, 259–312. http://eudml.org/doc/277909

[15] D. A. Cox, *Primes of the Form $x^2 + ny^2$*. John Wiley & Sons, Second Edition, 2013. http://doi.org/10.1002/9781118400722

[16] J. E. Cremona, *Algorithms for Modular Elliptic Curves.* Cambridge University Press, 1992. Second ed. 1997. http://homepages.warwick.ac.uk/~masgaj/book/fulltext/index.html

[17] H. B. Daniels, H. Goodwillie, *On the ranks of elliptic curves with isogenies.* Int. J. Number Theory **13** (2017), no. 9, 2215–2227. http://doi.org/10.1142/S1793042117501202

[18] H. Davenport, *Multiplicative Number Theory.* Springer GTM **74**. Third edition (2000), revised by H. L. Montgomery. http://doi.org/10.1007/978-1-4757-5927-3

[19] C. Delaunay, X.-F. Roblot, *Regulators of rank one quadratic twists.* J. théorie des nombres de Bordeaux **20** (2008), no. 3, 601–624. http://doi.org/10.5802/jtnb.643

[20] M. Deuring, *Imaginäre quadratische Zahlkörper mit der Klassenzahl 1.* (German) [Imaginary quadratic fields with class number 1]. Math. Z. **37** (1933), 405–415. http://eudml.org/doc/168455

[21] M. Deuring, *Zetafunktionen quadratischer Formen.* (German) [Zeta-functions of quadratic forms]. J. reine angew. Math. **172** (1935), 226–252. http://eudml.org/doc/149913

[22] G. Lejeune Dirichlet, *Recherches sur diverses applications de l'Analyse infinitésimale à la Théorie des Nombres.* (French) [Some diverse applications of the infinitesimal analysis in number theory]. J. reine angew. Math. **19** (1839), 324–369, *ibid.* **21** (1840), 1–12, 134–155.

http://doi.org/10.1515/crll.1839.19.324 http://doi.org/10.1515/crll.1840.21.1 http://doi.org/10.1515/crll.1840.21.134

[23] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms.* Invent. Math. **92** (1988), no. 1, 73–90. http://doi.org/10.1007/BF01393993

[24] W. Duke, J. B. Friedlander, H. Iwaniec, *Weyl sums for quadratic roots.* IMRN **2012**, no. 11, 2453–2549. http://doi.org/10.1093/imrn/rnr112

[25] A. Dunn, *Uniform bounds for sums of Kloosterman sums of half integral weight.* Research in Number Theory, **4** (2018), art. #45, 21pp. http://doi.org/10.1007/s40993-018-0138-6

[26] B. Edixhoven, A. de Groot, J. Top, *Elliptic curves over the rationals with bad reduction at only one prime.* Math. Comp. **54** (1990), 413–419. http://doi.org/10.1090/S0025-5718-1990-0995209-4

[27] M. Eichler, *Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion.* (German) [Quaternary quadratic forms and the Riemann hypothesis for the local ζ-function]. Arch. Math. **5** (1954), 355–366. http://doi.org/10.1007/BF01898377

[28] J. B. Friedlander, *On the class numbers of certain quadratic extensions.* Acta Arith. **XXVIII** (1976), no. 4, 391–393. http://eudml.org/doc/205399

[29] J. B. Friedlander, H. Iwaniec, *Small Representations by Indefinite Ternary Quadratic Forms.* In *Number Theory and Related Fields*, edited by J. M. Borwein, I. Shparlinski, W. Zudilin. Springer PROMS **43** (2013), 157–164. http://doi.org/10.1007/978-1-4614-6642-0_7

[30] C. F. Gauss, *Disquisitiones Arithmeticae.* (Latin) [Arithmetical investigations] (1801). From his complete works (1863): http://eudml.org/doc/202621 English translation: A. A. Clarke, *Disquisitiones Arithmeticae*, Yale Univ. Press (1965).

[31] S. Gelbart, H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$.* Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542. http://eudml.org/doc/82024

[32] А. О. Гельфонд [A. O. Gel'fond], О приближении алгебраическими числами отношения логарифмов двух алгебраических чисел. (Russian) [On the approximation of algebraic numbers by ratios of logarithms of two algebraic numbers]. Изв. АН СССР. Сер. матем. [Izv. Akad. Nauk SSSR Ser. Mat.] **5-6** (1939), 509–518. http://mi.mathnet.ru/izv3946

[33] D. M. Goldfeld, *The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), no. 4, 624–663. http://eudml.org/doc/83732

[34] D. Goldfeld, J. Hoffstein, D. Lieman, Appendix to *Coefficients of Maass forms and the Siegel zero.* Ann. of Math. (2) **140** (1994), no. 1, 177–181. http://doi.org/10.2307/2118544

[35] T. H. Gronwall, *Sur les séries de Dirichlet correspondant à des caractères complexes.* (French) [On Dirichlet series of complex characters]. Rend. Circ. Matem. Palermo **35** (1913), 145–159. http://doi.org/10.1007/BF03015596

[36] B. H. Gross, D. B. Zagier, *Heegner points and derivatives of L-series.* Invent. Math. **84** (1986), no. 2, 225–320. http://eudml.org/doc/143341

[37] H. Hasse, *Zur Theorie der abstrakten elliptischen Funktionenkörper. I-III.* (German) [On the theory of abstract elliptic function fields, I-III]. J. reine angew. Math. **175** (1936), 55–62, 69–88, 193–208. http://eudml.org/doc/149956 http://eudml.org/doc/149959 http://eudml.org/doc/149968

[38] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I-II.* (German) [On modular functions and the Dirichlet series with Euler products, I-II]. Math. Ann. **114** (1937), 1–28, 316–351. http://eudml.org/doc/159899 http://eudml.org/doc/159919

[39] K. Heegner, *Diophantische Analysis und Modulfunktionen.* (German) [Diophantine analysis and modular functions]. Math. Z. **56** (1952), 227–253. http://eudml.org/doc/169287

[40] H. Heilbronn, *On the class-number in imaginary quadratic fields.* Quart. J. Math. Oxford **5** (1934), 150–160. http://doi.org/10.1093/qmath/os-5.1.150

[41] H. Heilbronn, E. H. Linfoot, *On the imaginary quadratic corpora of class-number one.* Quart. J. Math. Oxford **5** (1934), 293–301. http://doi.org/10.1093/qmath/os-5.1.293

[42] J. van der Hoeven, *Fast evaluation of holonomic functions.* Theor. Comp. Sci. **210** (1999), no. 1, 199–215. http://doi.org/10.1016/S0304-3975(98)00102-9

[43] C. Hooley, *On the distribution of the roots of polynomial congruences.* Mathematika **11** (1964), 39–49. http://doi.org/10.1112/S0025579300003466

[44] H. Iwaniec, *Fourier coefficients of modular forms of half-integral weight.* Invent. Math. **87** (1987), 385–402. http://eudml.org/doc/143426

[45] H. Iwaniec, E. Kowalski, *Analytic Number Theory.* Colloq. Publ. AMS **53** (2004). http://doi.org/10.1090/coll/053

[46] C. G. J. Jacobi, *Observatio arithmetica de numero classium divisorum quadraticorum formae $yy + Azz$, designante A numerum primum formae $4n+3$.* (Latin) [An arithmetic observation on the class number of quadratic forms $y^2 + Az^2$ where A is a prime congruent to 3 mod 4]. J. reine angew. Math. **9** (1832), 189–192. http://doi.org/10.1515/crll.1832.9.189

[47] В. А. Колывагин [V. A. Kolyvagin], Конечность $E(\mathbf{Q})$ и Ш$(E,\mathbf{Q})$ для подкласса крибых бейля. (Russian) [Finiteness of $E(\mathbf{Q})$ and Ш$(E,\mathbf{Q})$ for a subclass of Weil curves]. Изв. Акад. Наук УССР Сер. Мат. **52** (1988), no. 3, 522–540 (Russian). English translation: Math. USSR Izv. **32** (1989), 523–541. http://mi.mathnet.ru/eng/izv1191 http://doi.org/10.1070/IM1989v032n03ABEH000779

———, *Euler systems.* In *The Grothendieck Festschrift* (Vol. II), ed. P. Cartier *et al.* Prog. in Math. **87** Birkhäuser Boston (1990), 435–483. http://doi.org/10.1007/978-0-8176-4575-5

[48] Ch.-J. de La Vallée Poussin, *Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premieres inférieurs a une limite donnée.* (French) [On the Riemann ζ-function and the number of prime numbers less than a given limit]. Mem. Courronnes et Autres Mem. Pub. Acad. Roy. Sci. Lettres Beaux-Arts Belgique **59** (1899/1900). https://archive.org/details/surlafonctionze00pousgoog

[49] E. Landau, *Über imaginär-quadratischer Zahlkörper mit gleicher Klassenzahl.* (German) [On imaginary quadratic fields with the same class number]. Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. (1918), 277–284. http://www.digizeitschriften.de/dms/resolveppn/?PID=GDZPPN002505134

[50] E. Landau, *Über die Klassenzahl imaginär-quadratischer Zahlkörper.* (German) [On the class number of imaginary quadratic fields]. Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. (1918), 285–295. http://www.digizeitschriften.de/dms/resolveppn/?PID=GDZPPN002505142

[51] E. Landau, *Bemerkungen zum Heilbronnschen Satz.* (German) [Remarks about Heilbronn's theorem]. Acta Arithmetica **1** (1935), no. 1, 1–18. http://eudml.org/doc/205049

[52] А. Ф. Лаврик [A. F. Lavrik], О функциональных уравнениях функций Дирихле. (Russian) [On functional equations of Dirichlet functions]. Изв. АН СССР. Сер. матем. [Izv. Akad. Nauk SSSR Ser. Mat.] **31** (1967), (English version: Math. USSR-Izvestiya **1**), no. 2, 431–442. http://www.mathnet.ru/eng/im2547

———, Функциональные и приближенные функциональные уравнения функций Дирихле. (Russian) [Functional and approximate functional equations for the Dirichlet function]. Матем. заметки [Mat. Zametki], **3** (1968), no. 5, 613–622. English version: Math. Notes **3** (1968), no. 5, 388–393. http://mi.mathnet.ru/eng/mz6720

———, Приближенные функциональные уравнения функций Дирихле. (Russian) [Approximate functional equations for Dirichlet functions]. Изв. АН СССР. Сер. матем. [Izv. Akad. Nauk SSSR Ser. Mat.] **32** (1968), no. 1, 134–185. English version: Math. USSR-Izv. **2** (1968), no. 1, 129–179. http://www.mathnet.ru/eng/im2696

[53] R. Lipschitz, *Über die asymptotischen Gesetze von gewissen Gattungen zahlentheoretischer Functionen.* (German) [On the asymptotics of various types of number-theoretical functions]. Sitzungsber. Preuss. Akad. Wiss. Berlin (1865), 174–185. http://babel.hathitrust.org/cgi/pt?id=ien.35556005045356;seq=188

[54] J. E. Littlewood, *On the class-number of the corpus $P(\sqrt{-k})$.* Proc. London. Math. Soc. **27** (1928), 358–372. http://doi.org/10.1112/plms/s2-27.1.358

[55] Y. L. Luke, *The special functions and their approximations.* Vol. I. Academic Press, 1969.

[56] F. Mertens, *Ueber einige asymptotische Gesetze der Zahlentheorie.* (German) [On some asymptotics in number theory]. J. reine angew. Math. **77** (1874), 289–338. http://doi.org/10.1515/crll.1874.77.289

[57] H. L. Montgomery, P. J. Weinberger, *Notes on small class numbers.* Acta Arith. **24** (1973/4), no. 5, 529–542. http://eudml.org/doc/205246

[58] J. Oesterlé, *Nombres de classes des corps quadratiques imaginaires.* (French) [Class numbers of imaginary quadratic fields]. Sém. Bourbaki **26** (1983/84), Astérisque **121–122** (1985), 309–323. http://eudml.org/doc/110032

[59] J. Park, B. Poonen, J. Voight, M. Matchett Wood, *A heuristic for boundedness of ranks of elliptic curves.* J. Eur. Math. Soc. **21** (2019), 2859–2903. http://doi.org/10.4171/JEMS/893

[60] E. Phragmén, E. Lindelöf, *Sur une extension d'un principe classique de l'analyse.* (French) [On an extension of a classical principle of analysis]. Acta. Math. **31** (1908), no. 1, 381–406. http://doi.org/10.1007/BF02415450

[61] D. J. Platt, *Isolating some non-trivial zeros of zeta.* Math. Comp. **86** (2017), 2449–2467. http://doi.org/10.1090/mcom/3198

[62] R. A. Rankin, *A minimum problem for the Epstein zeta-function.* Proc. Glasgow Math. Assoc. **1** (1953), no. 4, 149–158. http://doi.org/10.1017/S2040618500035668

[63] J. B. Rosser, L. Schoenfeld, *Approximate formulas for some functions of prime numbers.* Illinois J. Math. **6** (1962), no. 1, 64–94. http://doi.org/10.1215/ijm/1255631807

[64] E. Rouché, *Mémoire sur la série de Lagrange.* (French) [Memoir on Lagrange series]. J. de l'École Impériale polytechnique **39** (1862) [tome XXII], 193–224. http://babel.hathitrust.org/cgi/pt?id=coo.31924069342180

[65] J. Rouse, *Quadratic forms representing all odd positive integers.* Amer. J. Math. **136** (2014), no. 6, 1693–1745. http://doi.org/10.1353/ajm.2014.0041

[66] M. O. Rubinstein, *Computational methods and experiments in analytic number theory.* In *Recent perspectives in random matrix theory and number theory*, edited by F. Mezzadri and N. C. Snaith. LMS Lecture Note Series **322**, Cambridge Univ. Press (2005), 425–510. http://doi.org/10.1017/CBO9780511550492.015

[67] P. Sarnak, J. Tsimerman, *On Linnik and Selberg's Conjecture About Sums of Kloosterman Sums.* In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin,* (vol. II), edited by Yu. Tschinkel and Yu. Zarhin. Progress in Math. **270**, Birkhäuser Boston, Inc. (2011), 619–635. http://dx.doi.org/10.1007/978-0-8176-4747-6_20

[68] R. Schoof, *Elliptic curves over finite fields and the computation of square roots mod p.* Math. Comp. **44** (1985), 483–494. http://doi.org/10.2307/2007968

[69] A. Selberg, S. Chowla, *On Epstein's Zeta-function.* J. reine angew. Math. **227** (1967), 86–110. http://eudml.org/doc/150803

[70] G. Shimura, *On the holomorphy of certain Dirichlet series.* Proc. London Math. Soc. **31** (1975), no. 1, 79–98. http://doi.org/10.1112/plms/s3-31.1.79

[71] C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper.* (German) [On the class number of quadratic fields]. Acta Arith. **1** (1935), 83–86. http://eudml.org/doc/205054

[72] H. M. Stark, *A complete determination of the complex quadratic fields of class-number one.* Michigan Math. J. **14** (1967), no. 1, 1–27. http://doi.org/10.1307/mmj/1028999653

[73] H. M. Stark, *On the "gap" in a theorem of Heegner.* J. Number Theory **1** (1969), 16–27. http://doi.org/10.1016/0022-314X(69)90023-7

[74] H. M. Stark, *The role of modular functions in a class-number problem.* J. Number Theory **1** (1969), no. 2, 252–260. http://doi.org/10.1016/0022-314X(69)90044-4

[75] H. M. Stark, *L-functions and character sums for quadratic characters (II).* Acta Arith. **15** (1969), no. 3, 307–317. http://eudml.org/doc/204901

[76] H. M. Stark, *A historical note on complex quadratic fields with class-number one.* Proc. Amer. Math. Soc. **21** (1969), 254–255. http://doi.org/10.1090/S0002-9939-1969-0237461-X

[77] H. M. Stark, *A transcendence theorem for class-number problems.* Ann. Math. **94** (1971), no. 1, 153–173. http://doi.org/10.2307/1970740

[78] T. Tatuzawa, *On a Theorem of Siegel.* Jap. J. Math. **21** (1951), 163–178. http://www.jstage.jst.go.jp/article/jjm1924/21/0/21_0_163/_pdf

[79] N. Templier, *A nonsplit sum of coefficients of modular forms.* Duke. Math. J. **157** (2011), no. 1, 109–165. http://doi.org/10.1215/00127094-2011-003

[80] N. Templier, J. Tsimerman, *Non-split sums of coefficients of GL(2)-automorphic forms.* Israel J. Math. **195** (2013), no. 2, 677–723. http://doi.org/10.1007/s11856-012-0112-2

[81] M. Watkins, *Computing the modular degree of an elliptic curve.* Experiment. Math. **11** (2002), no. 4, 487–502. http://projecteuclid.org/euclid.em/1057864659

[82] M. Watkins, *Class numbers of imaginary quadratic fields.* Math. Comp. **73** (2004), no. 246, 907–938. http://doi.org/10.1090/S0025-5718-03-01517-5

[83] M. Watkins, *Explicit lower bounds on the modular degree of an elliptic curve.* Technical report (2004), Issac Newton Institute preprint NI-04026. http://magma.maths.usyd.edu.au/~watkins/papers/degphi.pdf

[84] M. Watkins, *Class number one from analytic rank two.* Mathematika **65** (2019), 333–374. http://doi.org/10.1112/S0025579318000517

[85] M. Watkins, *A spectral proof of class number one*. Math. Z. **293** (2019), no. 1, 383–406. http://doi.org/10.1007/s00209-018-2183-1
[86] M. Watkins, *Comments on Deuring's zero-spacing phenomenon*. Draft, 2015–2019. http://magma.maths.usyd.edu.au/~watkins/papers/Deuring.pdf
[87] M. Watkins, *Computing $L_E'''(1)$ to high precision*. Preprint, 2019.
[88] M. Watkins, *On Chowla's conjecture for* $\mathbf{Q}(\sqrt{4u^2+1})$. Preprint, 2019.
[89] M. Watkins, S. Donnelly, N. D. Elkies, T. Fisher, A. Granville, N. F. Rogers, *Ranks of quadratic twists of elliptic curves*. Publ. Math. Besançon, **2014/2**, 63–98. http://pmb.univ-fcomte.fr/2014/Watkins_et_all.pdf
[90] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*. Ann. of Math. (2) **141** (1995), no. 3, 443–551. http://doi.org/10.2307/2118559