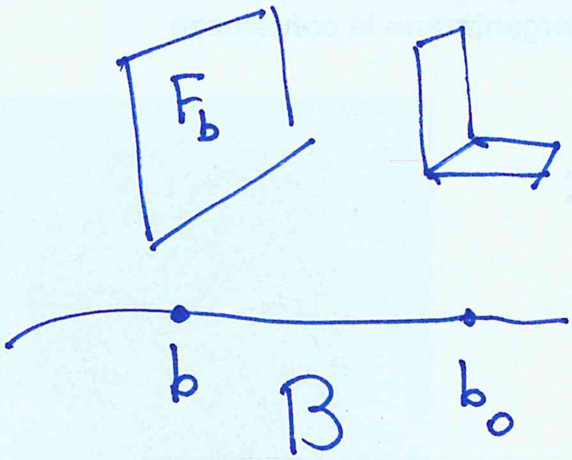
Fujita's decomposition and Massey products for fibered varieties

$f : X \to B$ fibration

- $\dim B = 1$
- $\dim X = n$
- $\dim F \geq 2$
- F_b s.s. normal crossing singularities and reduced



$p_g = h^0(F, \omega_F)$, $q(F) = h^0(F, \Omega^1_F)$

$$\omega_{X/B} = \mathcal{O}_X(K_X - f^* K_B)$$

$$\boxed{f_* \omega_{X/B}}$$

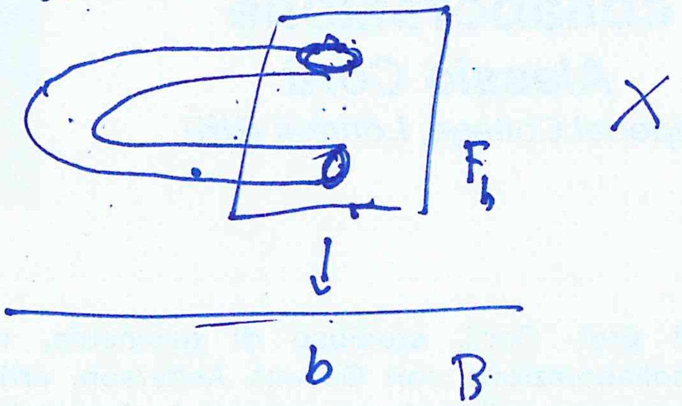
$$f_*\omega_{X/B,b} = H^0(\underline{F_b, \omega_{F_b}})$$

$b \in B$

$$\bigcup_{b\in B} H^0(F_b, \omega_{F_b})$$

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr, "f"'] & \boxed{\mathbb{P}(f_*\omega_{X/B})} \arrow[d] \\ & B \end{tikzcd}$$

$\underline{1^{st}}$ $\qquad f_*\omega_{X/B} = \mathcal{O}_B^{\oplus h} \oplus \mathcal{E} \qquad H^1(B, \mathcal{E}) = 0$



$$f_* \omega_{X/B} = \mathcal{U} \oplus \mathcal{A} \qquad r_k = p_g(F)$$

$\mathcal{A}$ is ample $\mathcal{U}$ is $\boxed{\text{unitary flat}}$

1) Unitary flat vector bundles.

2) Local system

3) Monodromy representation of $\rho: \pi_1(B) \to V$

i) $(\mathcal{U}, \nabla)$ $\nabla \circ \nabla = 0 \longleftrightarrow$ Unitary flat

ii) $\mathbb{V}$ a local system is a sheaf of vector spaces locally constant $s_\alpha = \rho_{\alpha\beta}\, s_\beta$ Gl(r) ~~$H^1(U, \mathcal{O}^*)$~~

iii) $\rho: \pi_1(B) \to V$ homomorph.

$\boxed{\mathcal{U} \text{ is unitary flat with finite monodromy}}$

$\Updownarrow$

$\mathcal{U}$ is semiample (i.e. there $\exists$ a multiple of the Tautological sheaf $\mathbb{P}(\mathcal{U})$ which is b.p.f.

Suppose you know that $\exists$ finite Galois cover

$$\begin{tikzcd} B' \arrow[r, "\pi"] & \frac{B'}{G} = B \\ X' \arrow[u] \arrow[r] & X \arrow[u] \end{tikzcd}$$

$$\underline{\pi^* \mathcal{U} = B' \times \mathbb{P}^{r-1}} \qquad r := r_K(\mathcal{U}) < p_g(F)$$

$\Rightarrow$ semiampleness?

Take a $P \in \mathbb{P}^{r-1}$ and a linear form L

such that $\cancel{L \in}$ $\boxed{(L=0) \cap GP = \emptyset}$

$$T = \prod_{g \in G} g^* L \qquad G\text{-invariant}$$

$$\deg T = \# G = q$$

$\hat{T}[P] \neq 0$ so I've found a

multiple of the $\mathcal{O}_{\mathbb{P}(\mathcal{U}')}(1)$ which is

generated

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→ U is not always semiample

Study conditions such that

→ $\boxed{U \text{ is semiample}}$ ←

Problem: study U and semiampleness conditions

The Local System $\mathbb{D}^{n-1}$

Joint Luca Rizzi

$$0 \to f^* \omega_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0$$

semi-stability $\Rightarrow$ $\Omega^1_{X/B}$ torsion free

Λ^{n-1} $$\boxed{0 \to f^* \omega_B \otimes \Omega^{n-2}_{X/B} \to \Omega^{n-1}_X \to \Omega^{n-1}_{X/B} \to 0}$$

Λ^n $$0 \to f^* \omega_B \otimes \Omega^{n-1}_{X/B} \to \Omega^n_X \to \Omega^n_{X|Z} \to 0$$

$Z :=$ scheme of critical pts.

$$\boxed{\mathbb{D}^{n-1} := \mathrm{CoKer}\left(\omega_B \otimes f_* \Omega^{n-2}_{X/B} \to f_* \Omega^{n-1}_{X,d}\right)}$$

$\mathbb{D}^{n-1}$:= sheaf of forms on the fibers liftable to closed holomorphic forms on the "Global object" X

= closed relative differential $(n-1)$-forms

$$\boxed{\mathbb{D}^{n-1} \hookrightarrow \Omega^{n-1}_{X/B, d_{X/B}}}$$

$$j : B^\circ \hookrightarrow \underline{B}$$

B° are the regular pts of values of $f : X \to B$

$$j^* \mathbb{D}^{n-1} = \text{\underline{largest} \underline{local} \underline{system}}$$

contained in $f_* \Omega^{n-1}_{X/B} \Big|_{B^\circ}$

<u>You show</u> $\underline{j_* j^* \mathbb{D}^{n-1} = \mathbb{D}^{n-1}}$

$\underline{\mathbb{D}^{n-1}}$ is the largest <u>local</u> system <u>inside</u>

$$\underline{f_* \Omega^{n-1}_{X/B}}$$

$b \in B \quad \underline{\mathbb{D}^{n-1}_b} < H^0(F_b, \omega_{F_b})$

$$f_* \Omega^{n-1}_{X/B} \hookrightarrow f_* \omega_{X/B}$$

<u>Th</u> [R, Z] $\boxed{\mathcal{U} = \mathbb{D}^{n-1} \otimes \mathcal{O}_B}$

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$$\boxed{f_* \omega_{X/B} = \mathcal{U} \oplus \mathcal{A}}$$

$n = 2$ <u>surface case</u>

Important splitting

$$\begin{tikzcd}
0 \arrow[r] & f^*\omega_B \arrow[r] \arrow[d] & \Omega^1_X \arrow[r] \arrow[d, "d"] & \Omega^1_{X/B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^*\omega_B \otimes \Omega^1_{X/B} \arrow[r] & \Omega^2_X \arrow[r] & \Omega^2_{X/B} \arrow[r] & 0
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, hook] & \Omega^1_{X,d} \arrow[r] \arrow[d, hook] & \Omega^1_{X/B,d_{X/B}} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & f^*\omega_B \arrow[r] \arrow[d] & \Omega^1_X \arrow[r] \arrow[d] & \Omega^1_{X/B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^*\omega_B \otimes \Omega^1_{X/B} \arrow[r] & \Omega^2_X \arrow[r] & \Omega^2_{X/B} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\Rightarrow 0 \arrow[r] & \omega_B \arrow[r] & \boxed{f_*\Omega^1_{X,d}} \arrow[r] & \mathbb{D} \arrow[r] \arrow[d] \arrow[l, bend right] & 0 \\
 & & & f_*\Omega^1_{X/B,d_{X/B}} \arrow[d] & \\
 & & & R^1f_*\mathbb{C} &
\end{tikzcd}$$

$\mathbb{D}$ a local system

$\mathbb{D}$ splits the sequence

$\underline{n=2}$ Sara Torelli PhD-Thesis

[P, T.] 2020

The splitting theorem

$$\boxed{\underline{\mathcal{U} \otimes K(b)} < \underline{H^0(F, \omega_F)}} \qquad g^1_2 \quad g^1_3$$

$\dim F = 1$

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Massey Product

$$\Lambda^n f_* \Omega^1_X \longrightarrow f_* \Lambda^n \Omega^1_X = \boxed{f_* w_X}$$

$$\Rightarrow (\Lambda^n f_* \Omega^1_X) \otimes T_B \longrightarrow f_* w_{X/B}$$

$$\Lambda^{n} \mathbb{D} \hookrightarrow \Lambda^n f_* \Omega^1_{X,d} \quad [\text{splitting}]$$

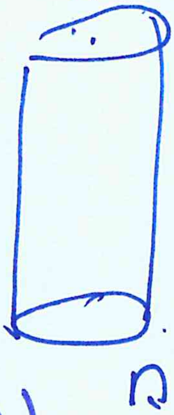
$$\Rightarrow \boxed{\Lambda^n \mathbb{D} \otimes T_B \longrightarrow f_* w_{X/B}}$$

\X/e also

$$\Lambda^{\boxed{n-1}} \mathbb{D} \longrightarrow \Lambda^{n-1} f_* \Omega^1_{X,d} \longrightarrow f_* \Omega^{n-1}_{X/B}$$

$$\boxed{\mathbb{D}^n \otimes \upsilon_B = \mathcal{U}}$$

Two Homomorphisms



$$\boxed{\Lambda^{n}\mathbb{D}\otimes T_B \longrightarrow f_*\omega_{X/B}}$$

$$\Lambda^{n-1}\mathbb{D} \longrightarrow f_*\omega_{X/B}$$

$A \hookrightarrow B$ open $\quad \langle \eta_1, \eta_2, \dots, \eta_n \rangle < \Gamma(A, \mathbb{D})$

$$\Lambda^{n-1}\langle \eta_1, \dots, \eta_n \rangle \otimes \mathcal{O}_A \longrightarrow f_*\omega_{X/B}|_A$$

$$\Lambda^{n-1}\langle \eta_1, \dots, \eta_n \rangle \otimes \mathcal{O}_A \xrightarrow{\lambda} \boxed{\mathcal{W}_n}$$

$$\omega_i = \lambda(\eta_1 \wedge \eta_2 \wedge \cdots \wedge \hat{\eta}_i \wedge \cdots \wedge \eta_{n-1} \wedge \eta_n)$$

$$\omega = \mathrm{Contr}\left(\eta_1 \wedge \cdots \wedge \eta_n \otimes \frac{\partial}{\partial t}\Big|_A\right)$$

↳ **MASSEY PRODUCT of $\eta_1, \dots, \eta_n$**

$\eta_1, \dots, \eta_n$ **Massey Trivial**

$\Updownarrow$

$\omega \in \mathcal{W}_n$.

$W \subset \Gamma(A, \mathbb{D})$ Massey trivial

$$\Rightarrow \exists\ \widetilde{W} < \Gamma(A, f_* \Omega^1_{X,d}) :$$

$$\Lambda^n \widetilde{W} \longrightarrow \Gamma(A, f_* \omega_X)$$

is <u>Zero</u>

$$W < \Gamma(A, \mathbb{D}) \qquad H = \operatorname{Ker} \rho_{\mathbb{D}} : \pi_1(B) \to U(r, \mathbb{C})$$

$$H_W = \{ g \in H \mid g^* w = id_W \}$$

$K < H_W$ <u>subgroup</u>

<u>Theorem</u> Let A be an open subset of B and $f : X \to B$ a s.s.-fibr.

If $W \subset \Gamma(A, \mathbb{D})$ is a strict Massey Trivial subspace $\Rightarrow$

$$\begin{tikzcd} X_K \arrow[r] \arrow[d, "f_K"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ B_K \arrow[r] & B_K/K = B \end{tikzcd}$$

$\exists\ h_K : X_K \to Y_K$

Y normal of general type $W := h_K^* H^0(Y, \Omega^1_Y)$

The Local system generated.

$A \subset B$ A an open set, $W \subset \Gamma(A, \mathbb{D})$

$\rho_{\mathbb{D}} : \Pi_1(B) \to U(\gamma_K(\mathbb{D}), \mathbb{C})$

$G = \mathrm{Im}\,\rho_{\mathbb{D}}$, $\widehat{W} = \sum_{g \in G} g.W$

$\mathbb{W}$:= Local system inside $\mathbb{D}$ given by $\widehat{W}$

$\mathbb{W}$ Massey Trivial generated if W is M. Trivial

Teorem $f : X \to B$ s.s.

$\mathbb{W}$ a strict $\boxed{\text{Massey trivial generated L.S.}}$

$\Rightarrow$ The associated monodromy group is finite and the fiber is isomorphic to

$$\mathbb{W} = \sum_g K_g^* H^0(Y, \Omega^1_Y)$$

where $\rho_{\mathbb{W}} : \Pi_1(B) \to \mathrm{Aut}(\widehat{W})$ $\quad H_{\mathbb{W}} = \mathrm{Ker}\,\rho_{\mathbb{W}}$

$$\begin{tikzcd}
F_0 \arrow[r, hook] \arrow[rrd, bend right, "K_g"'] & X_{\mathbb{W}} \arrow[r, "g"] \arrow[rd, bend right, "h_g"] & X_{\mathbb{W}} \arrow[d, bend left, "h"] \\
 & & Y
\end{tikzcd}$$

Corollary $\mathbb{D}$ Massey trivial generated

$\longrightarrow \Lambda^{n-1}\mathbb{D} \longrightarrow\!\!\!\!\to \mathbb{D}^{n-1}$

$\Rightarrow \mathbb{D}^{n-1}$ has finite monodromy

(i.e. $\mathcal{U}$ is semiample)

Prop 1) $X \xrightarrow{f} B$ s.s.

2) $\dim F = n-1 =$ odd number (n even number)

3) $\exists\ \sigma : F \to F \quad \sigma^2 = id_F : F/_{\langle\sigma\rangle}$ has $\boxed{p_g(F/_{\langle\sigma\rangle}) = 0}$

4) $\mathbb{D}$ is generated by anti-invariants 1-forms

Then $\mathbb{D}$ is Massey trivial generated

Proof

$n=2$ $\quad$ F hyperelliptic $\longrightarrow f_* \omega_{S/B}$ is semiample