Residual categories of Grassmannians

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based on joint work with Alexander Kuznetsov

Exceptional collections

X – smooth projective variety over $\mathbb{C}$

$D^b(X)$ – bounded derived category of coherent sheaves on X

1. An object E of $D^b(X)$ is called **exceptional** iff

$$\mathrm{Hom}(E,E) = \mathbb{C}id_E \quad \text{and} \quad \mathrm{Ext}^i(E,E) = 0 \quad \forall i \neq 0.$$

2. A sequence of exceptional objects $E_1, \ldots, E_n$ is called an **exceptional collection** iff for $i > j$

$$\mathrm{Ext}^k(E_i, E_j) = 0 \quad \forall k.$$

3. An exceptional collection $E_1, \ldots, E_n$ is said to be **full** iff it generates $D^b(X)$ in some sense. In this case we write

$$D^b(X) = \langle E_1, \ldots, E_n \rangle.$$

More precisely, the smallest full triangulated subcategory containing all $E_1, \ldots, E_n$ should be equivalent to $D^b(X)$.

Fullness is a very important, but somewhat technical aspect of this story and we'll mostly ignore it today.

Examples of exceptional collections

1. Projective spaces $\mathbb{P}^n$ (Beilinson, $\approx$ 1978)

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$$

2. Grassmannians $G(k, n)$ and quadrics Q^n (Kapranov, $\approx$ 1983)

   For $G(2, 4)$, which is both a Grassmannian and a quadric, Kapranov's collection becomes

$$D^b(G(2,4)) = \langle \mathcal{O}, \mathcal{U}^*, S^2\mathcal{U}^*, \mathcal{O}(1), \mathcal{U}^*(1), \mathcal{O}(2) \rangle$$

3. More examples later!

Remark. In these examples checking the exceptionality of the collection can be done relatively easily. For $\mathbb{P}^n$ this is just the standard computation of cohomology of line bundles on $\mathbb{P}^n$. For $G(k, n)$ one can apply Borel-Weil-Bott theorem. As is usual in this business, the difficult part is to prove fullness!

Simple consequences of having a FEC

Assume that $D^b(X)$ has a full exceptional collection

$$D^b(X) = \langle E_1, \dots, E_n \rangle.$$

Then we have:

1. The Hodge numbers $h^{p,q}(X) = 0$ for $p \neq q$.
2. $K_0(X)$ is a free abelian group of rank n and classes $[E_1], \dots, [E_n]$ form a basis.
3. The number of exceptional objects in any full exceptional collection in $D^b(X)$ is the same and is equal to

$$n = \operatorname{rk} K_0(X) = \dim_{\mathbb{C}} H^*(X, \mathbb{C}).$$

Lefschetz exceptional collections

This is a special type of exceptional collections introduced by Alexander Kuznetsov (around 2006) in his work on homological projective duality.

Let X be a smooth projective variety endowed with an (ample) line bundle $\mathcal{O}(1)$.

- A **Lefschetz collection** with respect to $\mathcal{O}(1)$ is an exceptional collection, which has a block structure

$$\underbrace{E_1, E_2, \ldots, E_{\sigma_0}}; \underbrace{E_1(1), E_2(1), \ldots, E_{\sigma_1}(1)}; \ldots; \underbrace{E_1(m), E_2(m), \ldots, E_{\sigma_m}(m)}$$

  where $\sigma = (\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m \geq 0)$ is a non-increasing sequence of non-negative integers called the **support partition** of the collection.

- If $\sigma_0 = \sigma_1 = \cdots = \sigma_m$, then the corresponding Lefschetz collection is called **rectangular**.

Examples of Lefschetz collections

1. Beilinson's collection

$$D^b(\mathbb{P}^n) = \langle \mathcal{O};\ \mathcal{O}(1);\ \dots;\ \mathcal{O}(n)\rangle$$

   is Lefschetz with the starting block $(\mathcal{O})$ and support partition $1, \dots, 1$.

2. Kapranov's collection

$$D^b(\mathsf{G}(2,4)) = \langle \mathcal{O}, \mathcal{U}^*, S^2\mathcal{U}^*;\ \mathcal{O}(1), \mathcal{U}^*(1);\ \mathcal{O}(2)\rangle$$

   is Lefschetz with the starting block $(\mathcal{O}, \mathcal{U}^*, S^2\mathcal{U}^*)$ and support partition $3, 2, 1$.

3. For $\mathsf{G}(2,4)$ one can make the starting block smaller by taking $(\mathcal{O}, \mathcal{U}^*)$ with the support partition $2, 2, 1, 1$

$$D^b(\mathsf{G}(2,4)) = \langle \mathcal{O}, \mathcal{U}^*;\ \mathcal{O}(1), \mathcal{U}^*(1);\ \mathcal{O}(2);\ \mathcal{O}(3)\rangle$$

   Lefschetz collections with the smallest possible starting block are called **minimal**.

Lefschetz exceptional collections on G/P

G is a simple simply connected algebraic group
$P \subset G$ is a maximal parabolic subgroup

Many people have worked on this topic. Here is a surely incomplete list: Beilinson, Faenzi, Fonarev, Guseva, Kapranov, Kuznetsov, Manivel, Novikov, Polishchuk, Samokhin ...

Yet a complete answer for arbitrary G/P is still lacking. The most general result is the construction by Kuznetsov and Polishchuk of a candidate for a full exceptional collection on G/P in the classical types $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$. Fullness of these collections is only known in a few special cases.

In this talk we are interested in (minimal) Lefschetz collections and even less is known in this case. Essentially until recently the only known series of examples were $\mathrm{G}(k, n)$, $\mathrm{IG}(2, 2n)$ and $\mathrm{OG}(2, 2n+1)$ due to Fonarev and Kuznetsov.

Residual category of a Lefschetz collection

Let X and $\mathcal{O}(1)$ be as before, and consider a Lefschetz exceptional collection

$$E_1, E_2, \dots, E_{\sigma_0};\ E_1(1), E_2(1), \dots, E_{\sigma_1}(1);\ \dots;\ E_1(m), E_2(m), \dots, E_{\sigma_m}(m)$$

We can take its rectangular part

$$E_1, E_2, \dots, E_{\sigma_m}; \dots; E_1(m), E_2(m), \dots, E_{\sigma_m}(m).$$

and define the **residual category** of this Lefschetz collection to be the subcategory of $D^b(X)$ left orthogonal to the rectangular part:

$$\mathcal{R} = \left\langle E_1, E_2, \dots, E_{\sigma_m}; \dots; E_1(m), E_2(m), \dots, E_{\sigma_m}(m) \right\rangle^{\perp}.$$

Thus, we have a semiorthogonal decomposition

$$D^b(X) = \left\langle \mathcal{R}\ ;\ E_1, E_2, \dots, E_{\sigma_m}; \dots; E_1(m), E_2(m), \dots, E_{\sigma_m}(m) \right\rangle.$$

The residual category is zero if and only if $(E_\bullet, \sigma)$ is full and rectangular.

Residual category for G(2, 4)

Consdier the minimal Lefschetz collection on G(2, 4)

$$D^b(\mathrm{G}(2,4)) = \langle \mathcal{O}, \mathcal{U}^*; \mathcal{O}(1), \mathcal{U}^*(1); \mathcal{O}(2); \mathcal{O}(3) \rangle.$$

Objects not belonging to the rectangular part are highlighted in red. Projecting them into the residual category $\mathcal{R}$ we obtain the exceptional collection

$$D^b(\mathrm{G}(2,4)) = \langle A, B; \mathcal{O}; \mathcal{O}(1); \mathcal{O}(2); \mathcal{O}(3) \rangle \quad \text{and} \quad \mathcal{R} = \langle A, B \rangle.$$

General feature: Projecting the objects not belonging to the rectangular part into $\mathcal{R}$ gives rise to an exceptional collection in $\mathcal{R}$. Technical name for this is *mutation of exceptional collections*.

Interesting phenomenon for G(2, 4): Since A, B form an exceptional pair, we necessarily have $\mathrm{Ext}^\bullet(B, A) = 0$. **Surprisingly** we also have

$$\mathrm{Ext}^\bullet(A, B) = 0.$$

Thus, A and B are completely orthogonal!

Residual category for $G(k, n)$

Minimal Lefschetz collections for $G(k, n)$ have been studied by Anton Fonarev ($\approx$ 2011) generalising earlier results for $G(2, n)$ by Alexander Kuznetsov ($\approx$ 2005).

Due to the lack of time we do not reproduce their construction here. In the case of $G(2, 4)$ it gives the collection considered on the previous slide.

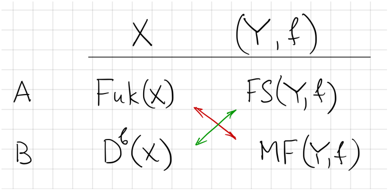
Conjecture (Kuznetsov – S., 2018)**.** The residual category of Fonarev's minimal Lefschetz collection on $G(k, n)$ is generated by a completely orthogonal exceptional collection.

Theorem (Kuznetsov – S., 2018)**.** The above conjecture is true if k is a prime number.

This behaviour can be motivated/explained via quantum cohomology and mirror symmetry!

Motivation from Homological Mirror Symmetry I

Let X be a Fano variety and (Y, f) its LG model. Then we have the following conjectural equivalences of triangulated categories



Let us also for simplicity assume that $Pic\, X = \mathbb{Z}$ and all the critical points of f are isolated. Then we have the following:

- The Fukaya–Seidel category $FS(Y, f)$ has a full exceptional collection, whose objects are given by Lefschetz thimbles associated with the critical points of f.
- Under the green equivalence of categories it gives a full exceptional collection in $D^b(X)$.

Motivation from Homological Mirror Symmetry II

Intuition:

- Thimbles corresponding to the critical points of f with non-zero critical values correspond to the rectangular part of a Lefschetz collection in $D^b(X)$.
- Thimbles corresponding to the critical points in $f^{-1}(0)$ and the subcategory generated by them correspond to the residual category of the Lefschetz collection in $D^b(X)$.

Examples:

1. If there are no critical points in $f^{-1}(0)$, then we expect $D^b(X)$ to have a full rectangular Lefschetz collection. Its residual category vanishes. This happens for $\mathbb{P}^n$, for example.
2. If $f^{-1}(0)$ has only non-degenerate critical points, then the corresponding thimbles (one for each critical point) do not intersect and, therefore, are completely orthogonal as objects of $FS(Y, f)$. So we expect $D^b(X)$ to have a Lefschetz collection, whose residual category is generated by a completely orthogonal exceptional collection. This happens for $\mathrm{G}(k, n)$, for example.

Motivation from Homological Mirror Symmetry III

3. If $f^{-1}(0)$ has several isolated critical points (possibly degenerate), then the thimbles corresponding to distinct critical points do not intersect (as above). However, now we have several thimbles attached to each critical point, and the subcategory that they generate is the Fukaya–Seidel category of the respective singularity.

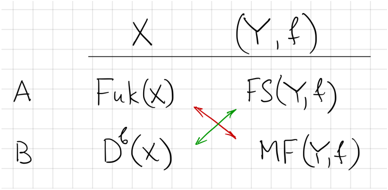
   Hence, we expect $D^b(X)$ to have a Lefschetz collection, whose residual category has a completely orthogonal decomposition into several components, each of which is equivalent to the Fukaya–Seidel category of the corresponding singularity.

   If $f^{-1}(0)$ has a unique critical point and this critical point is of ADE type, then the above discussion suggests

$$\mathcal{R} \simeq D^b(Q),$$

   where Q is the corresponding ADE quiver and $D^b(Q)$ its bounded derived category of representations (by a theorem of Seidel).

Relation to quantum cohomology



- Taking Hochschild cohomology of $Fuk(X)$ you get the **small quantum cohomology** $\mathrm{QH}(X)$.
- Using the red equivalence of categories you get

$$\mathrm{QH}(X) = HH^*(Fuk(X)) = HH^*(MF(Y,f)) = Jac(Y,f),$$

under which f in $Jac(Y,f)$ corresponds to $-K_X$ in $\mathrm{QH}(X)$.

- By looking at the finite scheme $\mathrm{Spec}(\mathrm{QH}(X))$ we can read-off the structure of the critical points in $f^{-1}(0)$.

Residual category for $\mathrm{IG}(2, 2n)$

The simplest example of X for which $\mathrm{QH}(X)$ has an interesting singularity is the symplectic isotropic Grassmannians $\mathrm{IG}(2, 2n)$.

A minimal Lefschetz collection for $\mathrm{IG}(2, 2n)$ has been constructed by Alexander Kuznetsov ($\approx$ 2005). In the case $n = 3$ we have:

$$D^b(\mathrm{IG}(2,6)) = \langle \mathcal{O}, \mathcal{U}^*, S^2\mathcal{U}^*, \mathcal{O}(1), \mathcal{U}^*(1), S^2\mathcal{U}^*(1), \\ \mathcal{O}(2), \mathcal{U}^*(2), \mathcal{O}(3), \mathcal{U}^*(3), \mathcal{O}(4), \mathcal{U}^*(4)\rangle.$$

Mutating the red objects into the residual category we get

$$\mathcal{R} = \langle A, B\rangle \quad \text{and} \quad \mathrm{Ext}^i(A, B) = \begin{cases} \mathbb{C} & \text{for} \quad i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that we have $\mathcal{R} \simeq D^b(A_2)$.

Similarly, for $\mathrm{IG}(2, 2n)$ you get the quiver of type A_{n-1}.

This matches perfectly with the structure of $\mathrm{QH}(\mathrm{IG}(2, 2n))$!

Residual categories for coadjoint varieties I

Here is a list of coadjoint varieties and singularities appearing in their quantum cohomology (or in the central fiber of the LG model):

Dynkin type of G	Coadjoint variety	Singularity type in QH
A_n	$\mathsf{Fl}(1, n; n+1)$	A_n
B_n	Q^{2n-1}	A_1
C_n	$\mathsf{IG}(2, 2n)$	A_{n-1}
D_n	$\mathsf{OG}(2, 2n)$	D_n
E_n	E_n/P_i	E_n
F_4	F_4/P_4	A_2
G_2	G_2/P_1	A_1

This list of singularities is a part of the **joint work in progress with Nicolas Perrin**. One of the main goals of this project is to establish the generic semisimplicity of the big quantum cohomology for coadjoint varieties.

Residual categories for coadjoint varieties II

Conjecture (Kuznetsov – S., 2020)**.** Let X be a coadjoint variety. There exists a Lefschetz exceptional collection in $D^b(X)$, whose residual category is equivalent to the derived category of the Dynkin quiver corresponding to the singularity in $\mathrm{QH}(X)$ (as in the table on the previous slide).

Theorem (Kuznetsov, 2017)**.** The conjecture holds in type C_n.

Theorem (Kuznetsov – S., 2020)**.**

1. The conjecture holds in type D_n, i.e. for $\mathrm{OG}(2, 2n)$.
2. The conjecture holds in type A_n modulo some subtleties related to the fact that $\mathrm{Fl}(1, n; n+1)$ is of Picard rank 2.

Theorem (Belmans – Kuznetsov – S., 2020)**.** The conjecture holds in type F_4.

Remark. In particular, for $\mathrm{OG}(2, 2n)$ and the coadjoint variety in type F_4 we construct the first known full exceptional Lefschetz collections.

Remark. For types B_n and G_2 the conjecture is simple and known.

The case of semisimple small quantum cohomology

Conjecture (Kuznetsov – S., 2018)**.** Let X be a smooth Fano variety with $\operatorname{Pic} X = \mathbb{Z}$. If the small quantum cohomology $\mathrm{QH}(X)$ is generically semisimple, then $D^b(X)$ has a full Lefschetz exceptional collection, whose residual category is generated by a completely orthogonal collection.

Known cases:

1. $\mathrm{G}(k, n)$ — mentioned earlier in the talk (partially known).
2. Quadrics — follows from Kapranov's work.
3. $\mathrm{OG}(2, 2n + 1)$ — follows from Kuznetsov's work.
4. Some sporadic examples:
   - 4.1 $\mathrm{G}_2/\mathrm{P}_2$ by Kuznetsov
   - 4.2 $\mathrm{IG}(3, 8)$ by Guseva
   - 4.3 $\mathrm{IG}(3, 10)$ by Novikov
   - 4.4 Caley plane $\mathrm{E}_6/\mathrm{P}_1$ is a combination of Faenzi–Manivel and Belmans–Kuznetsov–S.
   - 4.5 $\mathrm{IG}(4, 8)$ and $\mathrm{IG}(5, 10)$ should follow from Polishchuk–Samokhin and Fonarev.

Thank you!