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K-moduli stacks and K-moduli spaces of Fano varieties can be singular

Andrea Petracci (Freie Universität Berlin)

joint work with
Anne-Sophie Kaloghiros (Brunel University London)

Plan of the talk

- Fano varieties and their moduli/deformations
- Good moduli spaces and K-moduli
- An example where K-moduli have several branches
- An example where K-moduli is a fat point

$/\mathbb{C}$

Fano: normal projective variety X with $-K_X$ $\mathbb{Q}$-Cartier and ample

For $n \in \mathbb{Z}_{\geq 1}$ and $V \in \mathbb{Q}_{>0}$,
$\boxed{\mathcal{M}^{\mathrm{Fano}}_{n,V}}$ = stack of Fano n-folds with anticanonical volume V

$\mathcal{M}^{\mathrm{Fano}}_{n,V}$: $(\mathrm{Schemes})^{\mathrm{op}} \longrightarrow (\mathrm{Groupoids})$

$\forall T$ scheme

$$\mathcal{M}^{\mathrm{Fano}}_{n,V}(T) := \left\{ \begin{array}{c} \mathcal{X} \to T \text{ flat, proper, of finite presentation s.t.} \\ \bullet \text{ fibres are Fano } n\text{-folds with volume } V \\ \bullet \boxed{\text{Kollár condition / } \mathbb{Q}\text{-Gorenstein (qG) families}} \end{array} \right\}$$

↳ automatically satisfied if the fibres of $\mathcal{X} \to T$ are Gorenstein

X Fano, $\dim X = n$, $(-K_X)^n = V \quad \rightsquigarrow \quad [X] \in \mathcal{M}^{\mathsf{Fano}}_{n,V}(\mathbb{C})$.

The local structure of $\mathcal{M}^{\mathsf{Fano}}_{n,V}$ at the point $[X]$ is controlled by the action of $\mathrm{Aut}(X)$ on the functor of infinitesimal deformations of X: $\mathrm{Def}_X \colon (\text{Fat points})^{\mathrm{op}} = (\text{Local finite } \mathbb{C}\text{-algebras}) \longrightarrow (\text{Sets})$

$$\mathrm{Def}_X(T) := \left\| \left\{ \begin{array}{c} \mathcal{X} \to T \text{ flat, proper, of finite type s.t.} \\ \bullet \text{ the closed fibre is } X \\ \bullet \text{ Kollár condition / } \mathbb{Q}\text{-Gorenstein (qG) families} \end{array} \right\} \right\|$$

restriction of $\mathcal{M}^{\mathrm{Fano}}_{n,V}$

set of isom. class

$\mathbb{T}^1_X = \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ is the tangent space of Def_X $=$ $\mathrm{Def}_X(\mathbb{C}[t]/(t^2))$
$\mathbb{T}^2_X = \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X)$ is an obstruction space of Def_X

deformations of X are unobstructed

- X smooth Fano $\Rightarrow \mathrm{Def}_X$ smooth
  Proof: $n = \dim X$. $T_X = \Omega_X^{n-1} \otimes \omega_X^\vee$. $\mathbb{T}^2_X = H^2(T_X) = 0$ by Kodaira–Nakano vanishing. □

  Kodaira–Nakano

  $\mathrm{Ext}^2(\Omega_X, \mathcal{O}_X) = \mathrm{Ext}^2(\mathcal{O}_X, T_X)$

- $\dim X = 2$, X Fano with cyclic quotient singularities $\Rightarrow \mathrm{Def}_X$ smooth [Odaka–Spotti–Sun, Akhtar–Coates–Corti–Heuberger–Kasprzyk–Oneto–P.–Prince–Tveiten]

- $\dim X = 3$, X Fano with terminal singularities $\Rightarrow \mathrm{Def}_X$ smooth [Namikawa, Sano]

  $d = 8$ $\quad dP_8 \hookrightarrow \mathbb{P}^8$ $\quad \dfrac{\mathbb{C}[t]}{(t^2)}$

But:

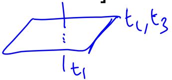
$dP_8 = \mathbb{F}_1$

anticanonically

- $\exists$ obstructed Fano 3-folds with Gorenstein canonical singularities, e.g. the projective cone over $dP_d \hookrightarrow \mathbb{P}^d$ for $d \in \{8, 7, 6\}$ [Altmann]

  del Pezzo surface of degree d

$d = 6$: $dP_6 \hookrightarrow \mathbb{P}^6$ $\quad \dfrac{\mathbb{C}[[t_1, t_2, t_3]]}{(t_1t_2, t_1t_3)}$



Artin

$\mathcal{M}$ algebraic stack of finite type over $\mathbb{C}$.

[Alper] A **good moduli space** for $\mathcal{M}$ is a morphism $\phi \colon \mathcal{M} \to M$ such that

- M is an algebraic space,
- $\phi_* \colon \mathrm{QCoh}(\mathcal{M}) \to \mathrm{QCoh}(M)$ is exact,
- $\mathcal{O}_M = \phi_* \mathcal{O}_{\mathcal{M}}$.

generalisation of COARSE MODULI SPACES for Deligne-Mumford stacks.

Example

A $\mathbb{C}$-algebra of finite type with an action of a reductive group G. Then

$\underbrace{[\operatorname{Spec} A \,/\, G]}_{\text{stack theoretic quotient}} \longrightarrow \operatorname{Spec} A^G$ is a good moduli space.

This is the local structure of every good moduli space (Luna étale slice theorem [Alper–Hall-Rydh])

[Tian, Donaldson, ...] There exist notions of
K-semistable/**K-polystable**/**K-stable** klt Fano variety

Related to existence of Kähler–Einstein metrics on Fano varieties

Ksemistable open

$\mathcal{M}^{\mathrm{Kss}}_{n,V} \subseteq \mathcal{M}^{\mathrm{Fano}}_{n,V}$ substack made up of families $\mathcal{X} \to T$ where all fibres are klt and K-semistable

K-moduli stack

Theorem (Alper, Blum, Fujita, Halpern-Leistner, Li, Liu, Odaka, Spotti, Sun, Wang, Xu, Zhuang, ...)

$\mathcal{M}^{\mathrm{Kss}}_{n,V}$ is an algebraic stack of finite type over $\mathbb{C}$ and admits a good moduli space $M^{Kps}_{n,V}$, which is a separated algebraic space of finite type over $\mathbb{C}$. Moreover, $M^{Kps}_{n,V}(\mathbb{C})$ is the set of K-polystable Fano n-folds with anticanonical volume V.

K-moduli space

Question

What are the geometric properties of $\mathcal{M}^{\mathrm{Kss}}_{n,V}$ and of $\mathrm{M}^{\mathrm{Kps}}_{n,V}$?
Are these smooth?

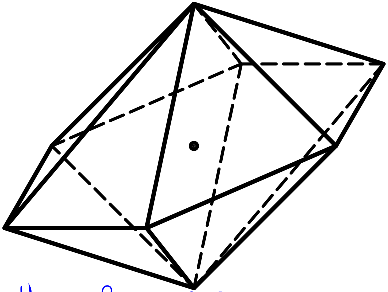
Example [Liu–Xu]

For cubic 3-folds: K-stability $=$ GIT-stability.

Goal of this talk

Via toric geometry, show examples where $\mathcal{M}^{\mathrm{Kss}}_{n,V}$ and $\mathrm{M}^{\mathrm{Kps}}_{n,V}$ are not unibranch or not reduced.

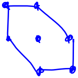
$\langle\cdot,\cdot\rangle : M \times N \to \mathbb{Z}$
$M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$

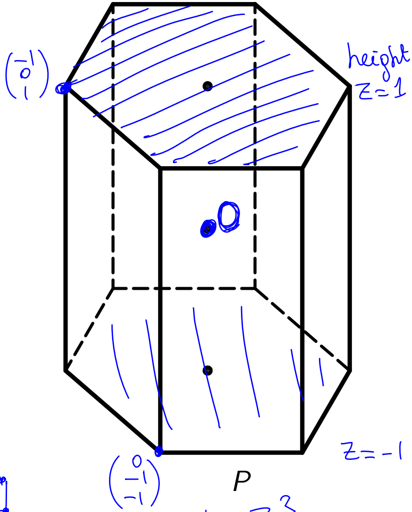


Q is the polar of P:
$Q = \{ x \in M_{\mathbb{R}} \mid \forall y \in P, \langle x, y \rangle \geq -1 \}$

Q

$M = \mathrm{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^3$





P

$\subset N = \mathbb{Z}^3$

Σ = face fan of P = normal fan of Q
X = toric variety associated to Σ
Q is the moment polytope of $(X, -K_X)$

the barycentre of Q is the origin [Berman]

Theorem [Kaloghiros–P.]

- X is a K-polystable Fano 3-fold with Gorenstein canonical singularities and degree $\underbrace{(-K_X)^3}_{\text{volume of the polytope } Q} = 12$.
- $\mathrm{Def}_X \simeq \mathrm{Spf}\, \mathbb{C}[[t_1, \dots, t_{24}]]/(t_1 t_2, t_1 t_3, t_4 t_5, t_4 t_6)$.
- X deforms to the following 3 smooth Fano 3-folds as follows.

dim 22 $\rightarrow$	On $(t_1 = t_4 = 0)$	MM_{2-6}	$\rho = 2$	$h^{1,2} = 9$
dim 21 $\rightarrow$	On $(t_1 = t_5 = t_6 = 0)$	V_{12}	$\rho = 1$	$h^{1,2} = 7$
	On $(t_2 = t_3 = t_4 = 0)$	V_{12}	$\rho = 1$	$h^{1,2} = 7$
dim 20 $\rightarrow$	On $(t_2 = t_3 = t_5 = t_6 = 0)$	MM_{3-1}	$\rho = 3$	$h^{1,2} = 8$

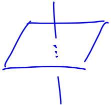
Picard rank

P hexagonal prism

Ingredients of the proof:

- the two hexagonal facets of P give two isolated singularities q_1, q_2 which are the vertex of the affine cone over the anticanonical embedding of dP_6 into $\mathbb{P}^6$

$$\mathrm{Def}_{q_i} = \mathrm{Spf}\, \frac{\mathbb{C}[[t_1,t_2,t_3]]}{(t_1t_2, t_1t_3)}$$



$\mathrm{Sing}(X) = \{q_1, q_2\} \sqcup \Gamma$

- Computation of $\mathbb{T}^1_X = H^0(\mathcal{T}^1_X)$
- $\mathrm{Def}_X \to \mathrm{Def}_{q_1} \times \mathrm{Def}_{q_2}$ is smooth of relative dimension 18

forgetful

$$\mathrm{Def}_X \simeq \mathbb{C}^{18} \times \mathrm{Spf}\, \frac{\mathbb{C}[[t_1,t_2,t_3]]}{(t_1t_2, t_1t_3)} \times \mathrm{Spf}\, \frac{\mathbb{C}[[t_4,t_5,t_6]]}{(t_4t_5, t_4t_6)}$$

- Computation with vanishing cycles to understand the topology of the 4 smoothings

Set $A = \mathbb{C}[[t_1, \ldots, t_{24}]]/(t_1 t_2, t_1 t_3, t_4 t_5, t_4 t_6)$ and
$G = \mathrm{Aut}(X) = \underbrace{(\mathbb{C}^*)^3}_{\text{big torus}} \rtimes \underbrace{(D_6 \rtimes C_2)}_{\mathrm{Aut}(P)}$.

$C_2 = \mathbb{Z}/2$ swapping top & bottom in P

The local structure of the K-moduli stack and the K-moduli space at the point $[X]$ is

$$
\begin{tikzcd}
{[\operatorname{Spec} A \ / \ G]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}^{\mathrm{Kss}}_{3,12} \\
\operatorname{Spec} A^G \arrow[r] & \mathrm{M}^{\mathrm{Kps}}_{3,12}
\end{tikzcd}
$$

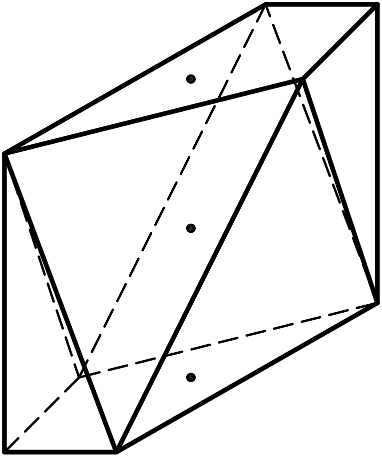
K-moduli stack

K-moduli space

where $\operatorname{Spec} A^G$ has 3 irreducible components.

Theorem [Kaloghiros–P.]

X gives a non-smooth point in $\mathcal{M}^{\mathrm{Kss}}_{3,12}$ and in $\mathrm{M}^{\mathrm{Kps}}_{3,12}$.
$\forall n \geq 4$, $X \times \mathbb{P}^{n-3}$ gives a non-smooth point in $\mathcal{M}^{\mathrm{Kss}}_{n,V}$ and in $\mathrm{M}^{\mathrm{Kps}}_{n,V}$, where $V = 2n(n-1)(n-2)^{n-2}$.



Theorem [Kaloghiros–P.]

X is a K-polystable Fano 3-fold with canonical singularities, degree $(-K_X)^3 = \frac{44}{3}$, and $\mathrm{Def}_X \simeq \mathrm{Spf}\, \mathbb{C}[[t_1, t_2]]/(t_1^2, t_2^2)$.

$A = \mathbb{C}[t_1, t_2]/(t_1^2, t_2^2)$ and
$G = \mathrm{Aut}(X) = (\mathbb{C}^*)^3 \rtimes (C_2 \times C_2)$
$A^G = \mathbb{C}[t]/(t^2)$

Local structure:

$$\begin{tikzcd}
[\mathrm{Spec}\, A \;/\; G] \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{Kss}}_{3,44/3} \arrow[d] \\
\mathrm{Spec}\, A^G \arrow[r] & \mathrm{M}^{\mathrm{Kps}}_{3,44/3}
\end{tikzcd}$$

Theorem [Kaloghiros–P.]

There exists a connected component of $\mathrm{M}^{\mathrm{Kps}}_{3,44/3}$ which is isomorphic to $\mathrm{Spec}\, \mathbb{C}[t]/(t^2)$.

Thanks for your attention!