Log symplectic pairs and mixed Hodge structures

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(Slides available on my website, `sites.google.com/view/anh318/research`)

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(Type III) a union of rational surfaces whose dual intersection complex is a triangulation of the 2-sphere (ex. degenerate a quartic to a tetrahedron of planes, resolve).

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- Rational surface with anticanonical cycle (Gross, Hacking, Keel): Blow up of toric surface pair (X_Δ, D_Δ) in a collection of (smooth) points in D_Δ.

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- In types II, III, the dual intersection complex of the central fiber is of dimension $\dim V_t/2$ or $\dim V_t$ respectively.

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Mixed analogues of structural results on the cohomology of hyperkähler varieties (Verbitsky). New proofs of results of Soldatenkov, sheds light on Nagai's conjecture.

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- If $Y = \emptyset$, then X is just called holomorphic symplectic.

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- If X is a surface, then the pair (X, Y) is log symplectic if and only if Y is anticanonical and simple normal crossings.
- If $Y = \emptyset$, then X is just called holomorphic symplectic. Examples include $S^{[n]}$ and $\mathrm{Kum}^n(A)$ for A an abelian surface, S a K3 surface.

Log symplectic pairs

Definition

A pair consisting of a smooth variety X of dimension $2d$ and a snc divisor Y is called log symplectic if there is some

$$\sigma \in \mathrm{H}^0(X; \Omega^2_X(\log Y))$$

so that $\sigma^d \in \Omega^{2d}_X(\log Y) = K_X(Y)$ is nonvanishing. The dimension of a holomorphic symplectic variety is always even.

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- (Ran) Resolution of Hilbert schemes of points on a surface with a smooth anticanonial divisor.

Good degenerations

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A good degeneration is a semistable degeneration $\mathcal{V} \to \Delta$ so that there is an element

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Remark

Not very many examples of good degenerations are known beyond dimension 2; Nagai has constructed some in dimension 4.

Pure weight

The Deligne decomposition

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Let $\pi : \mathcal{V} \to \Delta$ be a good degeneration of hyperkähler manifolds. Then if X is an irreducible component of V_0, and D is the intersection of X with the singular locus of V_0, then (X, Y) admits a log symplectic form of pure weight w.

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Remark

There's a correspondence between the type of degeneration and w;

Type I $\implies$ $w = 0$, Type II $\implies$ $w = 1$, Type III $\implies$ $w = 2$.

Consequences

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Remark

There are many log symplectic pairs which are <u>not</u> of pure weight. Let S_1 is a K3 surface and (S_2, E) is a pair consisting of a smooth rational surface S_2 and E is a smooth anticanonical elliptic curve. Then $(S_1 \times S_2, S_1 \times E)$ is log symplectic with no symplectic form of pure weight.

Examples

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- Let $X_E = \mathrm{Bl}_V \mathbb{P}^4$ and let Y_E be the union of the proper transform of $\mathrm{Sec}(E)$ and the exceptional divisor. Then (X_E, Y_E) is a log symplectic pair of pure weight 1

New examples from old

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If α the adjacency matrix of an acyclic quiver, and Σ is the standard simplex, this produces the corresponding acyclic cluster variety.

Blowing up the Feigin–Odeskii example

Description of symplectic leaves (Pym)

There are two components of Y_E; a resolution of of $\mathrm{Sec}(E)$ and the exceptional divisor of the blow up of $\mathbb{P}^4$ in $\mathrm{Sym}^2(E)$.

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Blowing up the leaves

We can now choose an arbitrary number of distinct leaves in each component. Blowing up repeatedly produces an infinite number of topologically distinct log symplectic pairs of pure weight 1.

Classification?

This brings up the following question

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It seems overly optimistic to think that the situation is as simple as the 2-dimensional case; there's likely subtle phenomena occurring in codimension greater than 2.

Moreover, it seems that the normal crossings condition is too strong for any real applications, but it is used because it's easier to compute with mixed Hodge structures when the boundary is normal crossings.

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Proposition (H.) (Symmetry)

If (X, Y) is a log symplectic pair with symplectic form σ, cup product with σ induces isomorphisms.

$$\sigma^{d-p} : \mathrm{Gr}_F^p \mathrm{H}^{p+q}(X \setminus Y) \longrightarrow \mathrm{Gr}_F^{2d-p} \mathrm{H}^{2d-p+q}(X \setminus Y), \qquad \forall p, q.$$

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Theorem (H.) (Simplicity)

If (X, Y) is a log symplectic pair of pure weight 2, then $\mathrm{H}^i(X \setminus Y; \mathbb{Q})$ has Hodge–Tate mixed Hodge structure.

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Theorem (Soldatenkov)

Let $\pi : \mathcal{V} \to \Delta$ be a good degeneration of Type III. Then the limit mixed Hodge structure on $\mathrm{H}^i(\mathcal{V}_\infty; \mathbb{Q})$ is Hodge–Tate for all i.

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Corollary (Vanishing)

Let (X, Y) be a log symplectic pair of pure weight 2 so that $2d = \dim X$. Then $\mathrm{H}^i(X \setminus Y) = 0$ if $i > 2d$.

These results are largely formal, and they can be extended the the cohomology rings of limit mixed Hodge structures of good degenerations.

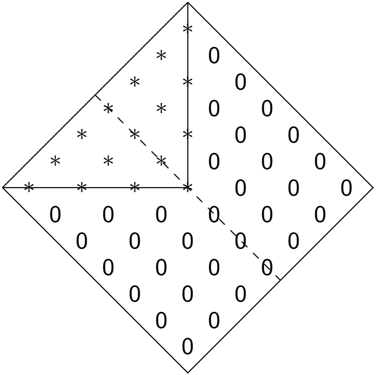
Theorem (Soldatenkov)

Let $\pi : \mathcal{V} \to \Delta$ be a good degeneration of Type III. Then the limit mixed Hodge structure on $\mathrm{H}^i(\mathcal{V}_\infty; \mathbb{Q})$ is Hodge–Tate for all i.

Remark

All of these results have analogues for pure weight 1 which are a bit more difficult to state.

Hodge diamonds



Hodge diamonds