Log symplectic pairs and mixed Hodge structures

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Lehigh University

(Slides available on my website, sites.google.com/view/anh318/research)
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Kulikov–Persson–Pinkham:

$\pi : S \to \Delta$ a semistable degeneration whose smooth fibers are K3, and all of the components of $S_0 = \pi^{-1}(0)$ are Kähler, and so that $K_{S'} = 0$.

Then $S_0$ of one of the following three types:

- (Type I) smooth K3 surface.
- (Type II) a chain of surfaces meeting in smooth elliptic curves (e.g., degenerate quartic to a union of cubic and a plane, resolve).
- (Type III) a union of rational surfaces whose dual intersection complex is a triangulation of the 2-sphere (e.g., degenerate a quartic to a tetrahedron of planes, resolve).
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Let $X$ be a component of a the central fiber, $S_0$ of a semistable degeneration of K3 surfaces, let $Y$ be its intersection with the singular locus of $S_0$. Such a pair is log Calabi–Yau, which, for us, means that $Y$ is snc and anticanonical.
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- **Rational surface with anticanonical cycle** (Gross, Hacking, Keel): Blow up of toric surface pair $(X_\Delta, D_\Delta)$ in a collection of (smooth) points in $D_\Delta$. 
Summary

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Generalization to higher dimensions

Higer dimensions: K3 ⇝ hyperkähler

Degenerations of hyperkähler manifolds: there is a similar trichotomy on the level of mixed Hodge structures.

Let $V \to \Delta$ be a semistable degeneration of hyperkähler manifolds. Limit mixed Hodge structure on $H^2(V_\infty; \mathbb{Q})$ takes the following forms:

- **Type I**: Pure Hodge structure of weight 2, $h_{2,0} = 1$

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Study the geometry of the components of the central fibers of degenerations of hyperkähler manifolds.
Goals

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This (potentially) could be used to address the problem of construction of hyperkähler manifolds. If we can construct degenerate hyperkähler manifolds, we may smooth them (Hanke).

This is also interesting in its own right. This leads to “logarithmic” versions of holomorphic symplectic manifolds which appear frequently in representation theory (cluster varieties, character varieties etc.)
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Study their cohomology rings.

Mixed analogues of structural results on the cohomology of hyperkähler varieties (Verbitsky). New proofs of results of Soldatenkov, sheds light on Nagai’s conjecture.
A pair consisting of a smooth variety $X$ of dimension $2d$ and a snc divisor $Y$ is called log symplectic if there is some

$$\sigma \in H^0(X; \Omega^2_X(\log Y))$$

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Examples

- If $X$ is a surface, then the pair $(X, Y)$ is log symplectic if and only if $Y$ is anticanonical and simple normal crossings.
- If $Y = \emptyset$, then $X$ is just called holomorphic symplectic. Examples include $S[n]$ and $\text{Kum}_n(A)$ for $A$ an abelian surface, $S$ a K3 surface.
- (Ran) Resolution of Hilbert schemes of points on a surface with a smooth anticanonical divisor.
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A good degeneration is a semistable degeneration \( \mathcal{V} \to \Delta \) so that there is an element

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Let $X$ be an irreducible component of the central fiber of a good degeneration, and let $Y$ be the intersection of $X$ with the singular locus of $V_0$. Then $(X, Y)$ is a log symplectic pair.
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Remark

Not very many examples of good degenerations are known beyond dimension 2; Nagai has constructed some in dimension 4.
The Deligne decomposition

There is a functorial decomposition of any mixed Hodge structure \((V, F^\bullet, W_\bullet)\), called the Deligne decomposition, which breaks up \(V \otimes \mathbb{C}\) into pieces \(I^{p,q}\).

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We say that a log symplectic form \(\sigma\) has pure weight \(w\) if the corresponding element of \(H^2(X \setminus Y; \mathbb{C})\) is contained in \(I^{2w}\).

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Let \(\pi: V \to \Delta\) be a good degeneration of hyperkähler manifolds. Then if \(X\) is an irreducible component of \(V_0\), and \(D\) is the intersection of \(X\) with the singular locus of \(V_0\), then \((X, Y)\) admits a log symplectic form of pure weight \(w\).

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There’s a correspondence between the type of degeneration and \(w\):

- Type I \(\Rightarrow w = 0\),
- Type II \(\Rightarrow w = 1\),
- Type III \(\Rightarrow w = 2\).
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Remark
There are many log symplectic pairs which are not of pure weight. Let \(S_1\) is a K3 surface and \((S_2, E)\) is a pair consisting of a smooth rational surface \(S_2\) and \(E\) is a smooth anticanonical elliptic curve. Then \((S_1 \times S_2, S_1 \times E)\) is log symplectic with no symplectic form of pure weight.
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Examples

Toric varieties

• $X_{\Sigma}$ a smooth toric variety of dimension $2d$, determined by a fan $\Sigma \subseteq M \otimes \mathbb{R}$.
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Gualtieri–Pym (Feigin–Odesski)

- Let $E$ is a smooth elliptic curve, which is embedded in $\mathbb{P}^4$ and has degree 5.
- Let $\text{Sec}(E)$ be its secant variety (the closure of the union of all lines passing through pairs of points in $E$). Then $\text{Sec}(E)$ is a quintic hypersurface, which is singular along a subvariety $V$ which is biregular to $\text{Sym}^2(E)$.
- Let $X_E = \text{Bl}_V \mathbb{P}^4$ and let $Y_E$ be the union of the proper transform of $\text{Sec}(E)$ and the exceptional divisor. Then $(X_E, Y_E)$ is a log symplectic pair of pure weight 1.
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Therefore $\alpha(\rho, -) \in N: \leftrightarrow$ monomial function $f_{\alpha, \rho}$ on the big torus $\mathbb{C}^{*2d} \subseteq X_\Sigma$. 

Blowing up leaves

Choose $\Sigma, \alpha$, so that leaves intersect properly for generic fibers of $f_{\alpha, \rho}$. Blow up leaves corresponding to all $\rho$.

Each blow up gives a new "cluster chart" (Gross–Hacking–Keel); the resulting variety looks like a cluster variety.

If $\alpha$ the adjacency matrix of an acyclic quiver, and $\Sigma$ is the standard simplex, this produces the corresponding acyclic cluster variety.
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There are two components of $Y_E$; a resolution of of $\text{Sec}(E)$ and the exceptional divisor of the blow up of $\mathbb{P}^4$ in $\text{Sym}^2(E)$.
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It seems overly optimistic to think that the situation is as simple as the 2-dimensional case; there’s likely subtle phenomena occurring in codimension greater than 2.
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Moreover, it seems that the normal crossings condition is too strong for any real applications, but it is used because it’s easier to compute with mixed Hodge structures when the boundary is normal crossings.
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**Proposition (H.) (Symmetry)**

If \((X, Y)\) is a log symplectic pair with symplectic form \(\sigma\), cup product with \(\sigma\) induces isomorphisms.

\[
\sigma^{d-p} : \text{Gr}^p_F H^{p+q}(X \setminus Y) \longrightarrow \text{Gr}^{2d-p}_F H^{2d-p+q}(X \setminus Y), \quad \forall p, q.
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**Definition**

A mixed Hodge structure is *Hodge–Tate* if \(\text{Gr}^{W}_{2n+1} = 0\) for all \(n\), and if \(W\) and \(F\) are *opposed* – this means that

\[
\dim \text{Gr}^W_{2i} H^j(X \setminus Y; \mathbb{Q}) = \dim \text{Gr}^{-i}_F H^j(X \setminus Y; \mathbb{C}).
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**Theorem (H.) (Simplicity)**

If \((X, Y)\) is a log symplectic pair of pure weight 2, then \(H^i(X \setminus Y; \mathbb{Q})\) has Hodge–Tate mixed Hodge structure.
Corollary

If \((X, Y)\) is log symplectic of pure weight 2, then \(H^\ast(X \setminus Y; \mathbb{Q})\) has the curious hard Lefschetz property.
More properties

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If \((X, Y)\) is log symplectic of pure weight 2, then \(H^*(X \setminus Y; \mathbb{Q})\) has the curious hard Lefschetz property.

Corollary (Vanishing)

Let \((X, Y)\) be a log symplectic pair of pure weight 2 so that \(2d = \dim X\). Then \(H^i(X \setminus Y) = 0\) if \(i > 2d\).
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These results are largely formal, and they can be extended to the cohomology rings of limit mixed Hodge structures of good degenerations.
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Theorem (Soldatenkov)

Let \(\pi: \mathcal{V} \to \Delta\) be a good degeneration of Type III. Then the limit mixed Hodge structure on \(H^i(\mathcal{V}_\infty; \mathbb{Q})\) is Hodge–Tate for all \(i\).

Remark

All of these results have analogues for pure weight 1 which are a bit more difficult to state.
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