Tropical F-polynomials and Cluster Algebras

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The Quiver Mutation

A quiver Q without loops and 2-cycles gives rise to a cluster algebra (without coefficients). This is *not* the general setup, but is enough for the purpose of this talk.

We attach to each vertex u of Q a variable $x_u \in k(x_1, \cdots, x_n)$. We can *mutate* at each vertex u of Q, then Q together with the variable x_u transforms into a new quiver Q' and a new (cluster) variable x'_u.

1. For each pair of arrows $v \to u \to w$, introduce a new arrow $v \to w$;
2. Reverse the direction all arrows incident to u;
3. Remove oriented 2-cycles one by one until there is none left.

$$x'_u = \Big(\prod_{v \to u} x_v + \prod_{u \to w} x_w \Big) / x_u.$$

Cluster: $(x_1, \ldots, x_u, \ldots, x_n) \to (x_1, \ldots, x'_u, \ldots, x_n)$.

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The Cluster Algebras and Varieties

Definition 1

The *cluster algebra* associated to Q is the subalgebra in $k(x_1, \cdots, x_n)$ generated by all cluster variables. The *upper cluster algebra* of Q consists of all elements in $k(x_1, \cdots, x_n)$ which is universally Laurent (Laurent in any cluster).

There is also a mutation rule for the y-seeds, which describes how to glue torus in the cluster $\mathcal{X}$-variety.

In general, a matrix B with skew-symmetrizable principal part gives rise to a pair of cluster varieties $(\mathcal{A}, \mathcal{X})$, and their *Langlands dual* $(\mathcal{A}^\vee, \mathcal{X}^\vee)$.

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Fock-Goncharov Duality

Fock-Goncharov duality conjecture says that the tropical points $\mathcal{X}^\vee(\mathbb{Z}^t)$ of $\mathcal{X}^\vee$ parametrize a basis of ring of regular functions $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$, and we can interchange the roles of $\mathcal{A}$ and $\mathcal{X}$:

$$I_{\mathcal{A}} : \mathcal{A}(\mathbb{Z}^t) \hookrightarrow \mathcal{O}(\mathcal{X}^\vee) \quad \text{and} \quad I_{\mathcal{X}^\vee} : \mathcal{X}^\vee(\mathbb{Z}^t) \hookrightarrow \mathcal{O}(\mathcal{A}).$$

The duality conjecture fails in general, but can hold with some mild assumption, or if we replace it with a formal version ([GHKK]).

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Additive Categorifications

Additive Categorification: There are cluster characters CC sending a class of objects (in some additive category) onto cluster variables, satisfying $CC(M \oplus N) = CC(M)CC(N)$.

1. Quivers with Potentials [Derksen-Weyman-Zelevinsky],
2. (Generalized) Cluster Category [BMRRT, Keller-Fu, Amiot].

A *potential* $\mathcal{P}$ on a quiver Q is a linear combination of oriented cycles of Q. The Jacobian ideal $\partial\mathcal{P}$ is the two-sided (closed) ideal in $\widehat{kQ}$ generated by all "noncommutative partial derivatives" $\partial_a\mathcal{P}$. The *Jacobian algebra* $J(Q,\mathcal{P})$ is the quotient algebra $\widehat{kQ}/\partial\mathcal{P}$.

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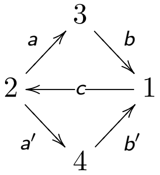
Example

Consider the quiver



with potential $\mathcal{P} = cba - cb'a'$.

Then the Jacobian ideal is generated by

$$\partial_a \mathcal{P} = cb, \qquad \partial_{a'} \mathcal{P} = cb',$$
$$\partial_b \mathcal{P} = ac, \qquad \partial_{b'} \mathcal{P} = a'c,$$
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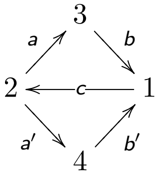
Consider the quiver 

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The Category of Presentations

For two projective representations P_- and P_+ (of some finite-dim algebra $A = kQ/I$), we call a morphism $d : P_- \to P_+$ a *presentation*.
The (additive) category of all presentations is *Krull-Schmidt*. The corresponding homotopy category $K(\text{proj-}A)$ is triangulated.

The presentation space of weight δ is the space

$$\text{PHom}(\delta) := \text{Hom}\left(P([-\delta]_+), P([\delta]_+)\right),$$

where we denote $[\delta]_+ := \max(\delta, 0)$ and $P(\beta) := \bigoplus_{i \in Q_0} \beta(i) P_i$.

We define $\text{E}(d_1, d_2) := \text{Hom}_{K(\text{proj-}A)}(d_1, d_2[1])$. We also denote the kernel and cokernel of

$$\text{Hom}(P_+, M) \to \text{Hom}(P_-, M)$$

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Fomin-Zelevinsky: Any cluster variable can be written as

$$\mathbf{x}^{\mathrm{g}} F(\mathbf{y}),$$

where $\mathbf{x}^{\mathrm{g}} = x_1^{\mathrm{g}(1)} \cdots x_n^{\mathrm{g}(n)}$ and $\mathbf{y}$ is a monomial in $\mathbf{x}$.

DWZ: We can view g-vectors as an element in $K(\text{proj-}J(Q, \mathcal{P}))$. The clusters correspond to a collection of *reachable* $\{\mathrm{g}_1, \ldots, \mathrm{g}_n\}$ such that

$$\mathrm{E}(-\mathrm{g}_i, -\mathrm{g}_j) = 0, \ \forall i, j \in [1, n].$$

Here $\mathrm{E}(-\mathrm{g}_i, -\mathrm{g}_j) = 0$ means $\mathrm{E}(d_i, d_j) = 0$ vanishes for general elements in $\mathrm{PHom}(-\mathrm{g}_i)$.

The g-vectors in Representation Theory

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The Canonical Decomposition

Definition 2 (Derksen-F)

A weight vector $\delta \in \mathbb{Z}^{Q_0}$ is called indecomposable if a general presentation in $\operatorname{PHom}(\delta)$ is indecomposable. We call $\delta = \delta_1 \oplus \delta_2 \oplus \cdots \oplus \delta_s$ the canonical decomposition of δ if a general element in $\operatorname{PHom}(\delta)$ decompose into (indecomposable) ones in each $\operatorname{PHom}(\delta_i)$.

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$\delta = \delta_1 \oplus \delta_2 \oplus \cdots \oplus \delta_s$ is the canonical decomposition of δ if and only if $\delta_1, \cdots, \delta_s$ are indecomposable, and $\mathrm{e}(\delta_i, \delta_j) = 0$ for $i \neq j$.

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The Generic Character

Definition 4 (DWZ)

For any representation M, the (dual) F-polynomial of M is

$$F_M(\mathbf{y}) = \sum_{\mathsf{e}} \chi\big(\operatorname{Gr}^{\mathsf{e}}(M)\big)\mathbf{y}^{\mathsf{e}}.$$

For each vector $\mathrm{g} \in \mathbb{Z}^{Q_0}$, there is an open subset of $\operatorname{PHom}(\mathrm{g})$ in which $\chi\big(\operatorname{Gr}^{\mathsf{e}}(\operatorname{Coker}(d))\big)$ takes constant value. The generic character CC

$$CC(\mathrm{g}) = \boldsymbol{x}^{\mathrm{g}} F_{\operatorname{Coker}(d)}(\mathbf{y})$$

maps the g-vectors to the upper cluster algebra [Dupont, Plamondon], and in many cases they form a basis [F, Qin]. For E-rigid g-vectors, the generic character gives cluster variables [CC, DWZ].

For cluster variables, $\chi\big(\operatorname{Gr}^{\mathsf{e}}(M)\big)$ is always positive [Lee-Schiffler, GHKK]. In general, $\chi\big(\operatorname{Gr}^{\mathsf{e}}(M)\big)$ can be negative.

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Let $A = kQ/I$. Let M be a finite-dimensional representation of A.

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The *tropical F-polynomial* f_M of a representation M is the function $(\mathbb{Z}^{Q_0})^* \to \mathbb{Z}_{\geq 0}$ defined by

$$\delta \mapsto \max_{L \hookrightarrow M} \delta(\underline{\dim} L).$$

If the F-polynomial F_M has non-negative coefficients, then the tropical F-polynomial f_M is the usual tropicalization of F_M.

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Representation-theoretic Interpretation of the Evaluation

We denote by $\hom(\delta, M)$ and $\mathrm{e}(\delta, M)$ the dimension of the kernel and cokernel of

$$\mathrm{Hom}(P_+, M) \to \mathrm{Hom}(P_-, M)$$

which is induced from a general presentation $P_- \to P_+$ in $\mathrm{PHom}(\delta)$.

Theorem 6 (F)

For any representation M and any $\delta \in \mathbb{Z}^{Q_0}$, there is some $n \in \mathbb{N}$ such that

$$f_M(n\delta) = \hom(n\delta, M), \qquad \check{f}_M(-n\delta) = \mathrm{e}(n\delta, M).$$

Moreover, if $\mathrm{e}(\delta, \delta) = 0$, then equalities hold for $n \in \mathbb{N}$.

Representation-theoretic Interpretation of the Evaluation

We denote by $\hom(\delta, M)$ and $\mathrm{e}(\delta, M)$ the dimension of the kernel and cokernel of

$$\operatorname{Hom}(P_+, M) \to \operatorname{Hom}(P_-, M)$$

which is induced from a general presentation $P_- \to P_+$ in $\operatorname{PHom}(\delta)$.

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A Presentation of Newton Polytopes

The tropical F-polynomial f_M is completed determined by the Newton polytope of M.

Definition 7

The *Newton polytope* $\mathrm{N}(M)$ of a representation M is the convex hull of

$$\{\underline{\dim} L \mid L \hookrightarrow M\}$$

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As an easy consequence of Theorem 6, we get a presentation of $\mathrm{N}(M)$.

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The Newton polytope $\mathrm{N}(M)$ *is defined by*

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The Case of Quiver with Potentials

Theorem 9 (F)

If M is negative reachable, then for any $\delta, \check{\delta} \in \mathbb{Z}^{Q_0}$ *we have that*

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If I_i *is negative reachable, then the dimension vector* α *of* $\mathrm{Coker}(\delta)$ *can be computed by*

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We have that $f_{\check{\delta}}(\delta) = \check{f}_{\delta}(\check{\delta})$ *for any* δ *and* $\check{\delta}$.

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Determine the Generic Newton Polytopes

Theorem 12 (F)

Let α be any dimension vector of Q. Each normal cone of $\mathsf{N}(\alpha)$ contains a cluster. Hence the Newton polytope $\mathsf{N}(\alpha)$ is completely determined by the Newton polytopes of real Schur representations.

Algorithm: For a fixed dimension vector α of Q, the (primitive) normal vectors of $\mathsf{N}(\alpha)$ are bounded. Let Δ_α be the set of all real δ-vectors within this bound. We can use the mutation algorithm (F-Z) to find the tropical F-polynomial of any $\delta \in \Delta_\alpha$. Since the exchange graph of acyclic quivers are connected [Happel], searching for all δ in Δ_α can be terminated in finite steps. Finally according to the above theorem, the generic Newton polytope $\mathsf{N}(\alpha)$ are determined by these tropical F-polynomials.

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A Conjecture about Bases and Newton Polytopes

A remarkable *universally positive* basis consisting of *theta functions* for all cluster algebras was introduced in [GHKK]. For each $\check{\delta}$-vector, there is a theta function $\vartheta_{\check{\delta}}$, which is of the form

$$\vartheta_{\check{\delta}} = \mathbf{x}^{-\check{\delta}} \varphi_{\check{\delta}}(\mathbf{y}).$$

Another very interesting positive (quantum) basis called *triangular basis* was introduced by [Qin] as a far-reaching generalization of [BZ]. It has a similar form

$$T_{\check{\delta},q} = \mathbf{x}^{-\check{\delta}} \psi_{\check{\delta},q}(\mathbf{y}).$$

In particular, $\varphi_{\check{\delta}}$ and $\psi_{\check{\delta},q}$ can be tropicalized and the tropicalization is determined by its Newton polytope.

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The Newton polytopes of $\varphi_{\check{\delta}}$ and $\psi_{\check{\delta},q}$ are the same as the generic Newton polytope $\mathrm{N}(\check{\delta})$. Moreover, the coefficients in $F_{\check{\delta}}$, $\varphi_{\check{\delta}}$, and $\psi_{\check{\delta},q}$ corresponding to the vertices of $\mathrm{N}(\check{\delta})$ are all 1's.

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The Fock-Goncharov Duality Pairing Conjecture

The duality conjecture further asserts that we can require the parametrized bases to be universally positive and satisfy several interesting properties. One of them concerns the pairing

$$\mathcal{A}(\mathbb{Z}^t) \times \mathcal{X}^\vee(\mathbb{Z}^t) \to \mathbb{Z}.$$

There are two canonical ways to define this pairing:

$$I_{\mathcal{A}}(a)^{\text{trop}}(x) \quad \text{and} \quad I_{\mathcal{X}^\vee}(x)^{\text{trop}}(a) \qquad \text{for } a \in \mathcal{A}(\mathbb{Z}^t),\ x \in \mathcal{X}^\vee(\mathbb{Z}^t).$$

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The Fock-Goncharov Duality Pairing in special cases

Theorem 14 (Fock-Goncharov duality pairing, F)

The pairings $\mathcal{A}(\mathbb{Z}^t) \times \mathcal{X}^\vee(\mathbb{Z}^t) \to \mathbb{Z}$ *given by* $I_{\mathcal{A}}(a)^{\widetilde{\text{trop}}}(\check{\delta})$ *and* $I_{\mathcal{X}^\vee}(\check{\delta})^{\widetilde{\text{trop}}}(a)$ *are both equal to* $\hom(aB^T, \check{\delta}) - a \cdot \check{\delta}$ *in the following two situations*

1. *The quiver is mutation-equivalent to an acyclic quiver.*
2. *Either* $I_{\mathcal{X}^\vee}(\check{\delta})$ *or* $I_{\mathcal{A}}(aB^T)$ *is a cluster variable, or equivalently either* $\check{\delta}$ *or* aB^T *is negative reachable.*

It is easy to see that Conjecture 11 ($f_{\check{\delta}}(\delta) = \check{f}_{\delta}(\check{\delta})$) implies the equality of the two pairings in all skew-symmetric cases. If B is invertible, we can set $\delta = aB^T$ and write $\hom(aB^T, \check{\delta}) - a \cdot \check{\delta}$ in a more symmetric form

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The Vertex Coefficients

Theorem 15 (F)

If γ is a vertex of $\mathrm{N}(M)$ then $\mathrm{Gr}_\gamma(M)$ must be a point. Conversely for a general representation M of an acyclic quiver, if $\mathrm{Gr}_\gamma(M)$ is a point, then γ is a vertex of $\mathrm{N}(M)$.

This shows in particular that the Newton polytope of M is the same as the usual Newton polytope of the polynomial F_M.

Conjecture 16

Let $F = \sum_\gamma c_\gamma \mathbf{y}^\gamma$ be the F-polynomial of a cluster variable (of any cluster algebra). Then γ is a vertex of the Newton polytope of F if and only if $c_\gamma = 1$.

The conjecture holds for acyclic cluster algebras.

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The Restriction to Faces

By the restriction of a polynomial $F = \sum_\gamma c_\gamma \mathbf{y}^\gamma$ to some face Λ of its Newton polytope, we mean

$$\sum_{\gamma | \gamma \in \Lambda} c_\gamma \mathbf{y}^\gamma.$$

Theorem 17 (F)

Let δ be the outer normal vector of some facet of the Newton polytope $\mathrm{N}(M)$. Then the restriction of F_M to this facet is given by

$$\mathbf{y}^{\underline{\dim} t_{\overline{\delta}}(M)} \iota_\delta \left(F_{\pi_\delta(M)} \right).$$

Here, $t_{\overline{\delta}}$ is the stabilization functor; $\pi_\delta(M)$ is a representation of another quiver, and ι_δ is a certain monomial change of variables. This result can be easily generalized to arbitrary faces of codimension greater than 1.

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The Saturation Property

We say the support of a polynomial $F = c_\gamma \mathbf{y}^\gamma$ *saturated* if $c_\gamma \neq 0$ for any lattice points in the Newton polytope of F.

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The support of the F-polynomial of a cluster monomial (of any cluster algebra) is saturated.

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The Dual Fan

Theorem 20 (F)

Let $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_m$ be finitely many clusters. Then there is some representation M such that each $\boldsymbol{\delta}_i$ spans a normal cone of $\mathrm{N}(M)$.

The normal cones of $\mathrm{N}(M)$ fit together into a complete fan $\mathrm{F}(M)$, the *normal fan* of $\mathrm{N}(M)$. The generalized cluster fan defined below refines the cluster fan introduced in [DF].

Definition 21

Let $\mathrm{F}(\operatorname{rep} A)$ be the set of all cones spanned by $\{\delta_1, \ldots, \delta_p\}$ such that each δ_i is normal and $\mathrm{e}(\delta_i, \delta_j) = 0$ for $i \neq j$. It turns out that $\mathrm{F}(\operatorname{rep} A)$ forms a simplicial fan. We call it *generalized cluster fan*.

Proposition 22 (F)

The fan $\mathrm{F}(M)$ *is a coarsening of the generalized cluster fan* $\mathrm{F}(\operatorname{rep} A)$.

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The Edge Quiver (1-Skelton)

Proposition 23 (F)

If $\overline{L_- L_+}$ is an edge in $\mathsf{N}(M)$, then either $L_- \subset L_+$ or $L_+ \subset L_-$. Say $L_- \subset L_+$, then

$$L_- = t_{\overline{\delta}}(M) \quad \text{and} \quad L_+ = \check{t}_{\overline{\delta}}(M) \quad \text{for any } \delta \text{ in the interior of } \mathsf{F}_{\overline{L_- L_+}}(M).$$

Moreover, we have the following

1. $\delta_+(L_+/L_-) \geq 0$ *for any* $\delta_+ \in \mathsf{F}_{L_+}(M)$ *and* $\delta_-(L_+/L_-) \leq 0$ *for any* $\delta_- \in \mathsf{F}_{L_-}(M)$.
2. *If* $\mathsf{F}_{L_-}(M)$ *is spanned by a cluster, then* L_+/L_- *is a direct sum of isomorphic real Schur representations.*

Definition 24

We assign the orientation $L_0 \to L_1$ for each edge $\overline{L_0 L_1}$ with $L_0 \subset L_1$. We call the resulting oriented graph the *edge quiver* of $\mathsf{N}(M)$, denoted by $\mathsf{N}_1(M)$.

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We assign the orientation $L_0 \to L_1$ for each edge $\overline{L_0L_1}$ with $L_0 \subset L_1$. We call the resulting oriented graph the *edge quiver* of $\mathsf{N}(M)$, denoted by $\mathsf{N}_1(M)$.

The Edge Quiver (1-Skelton)

Proposition 23 (F)

If $\overline{L_- L_+}$ is an edge in $\mathsf{N}(M)$, then either $L_- \subset L_+$ or $L_+ \subset L_-$. Say $L_- \subset L_+$, then

$$L_- = t_{\overline{\delta}}(M) \ \ \text{and} \ \ L_+ = \check{t}_{\overline{\delta}}(M) \ \ \text{for any } \delta \text{ in the interior of } \mathsf{F}_{\overline{L_- L_+}}(M).$$

Moreover, we have the following

1. $\delta_+(L_+/L_-) \geq 0$ *for any* $\delta_+ \in \mathsf{F}_{L_+}(M)$ *and* $\delta_-(L_+/L_-) \leq 0$ *for any* $\delta_- \in \mathsf{F}_{L_-}(M)$.
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Examples

Proposition 25 (F)

Suppose that A is cluster-finite. Let M be the direct sum of all E*-rigid representations. Then the normal fan* $\mathsf{F}(M)$ *is the cluster fan of A, and the edge quiver* $\mathsf{N}_1(M)$ *is the exchange quiver of A.*

In view of Proposition 22 and 25, the generalized cluster fan $\mathsf{F}(\operatorname{rep} A)$ can be viewed heuristically as the normal fan of the infinite dimensional representation $\bigoplus_{M \in \operatorname{rep} A} M$.

Proposition 26 (F)

Suppose that A is a preprojective algebra of Dynkin type. The vertices of $\mathsf{N}(A)$ *are labelled by the ideals I_w, and* $\mathsf{F}_{I_w}(A)$ *is the cluster corresponding to I_w. So* $\mathsf{F}(A)$ *is the cluster fan* $\mathsf{F}(\operatorname{rep} A)$*, which is a Weyl fan.*

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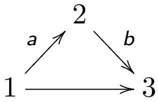
Examples

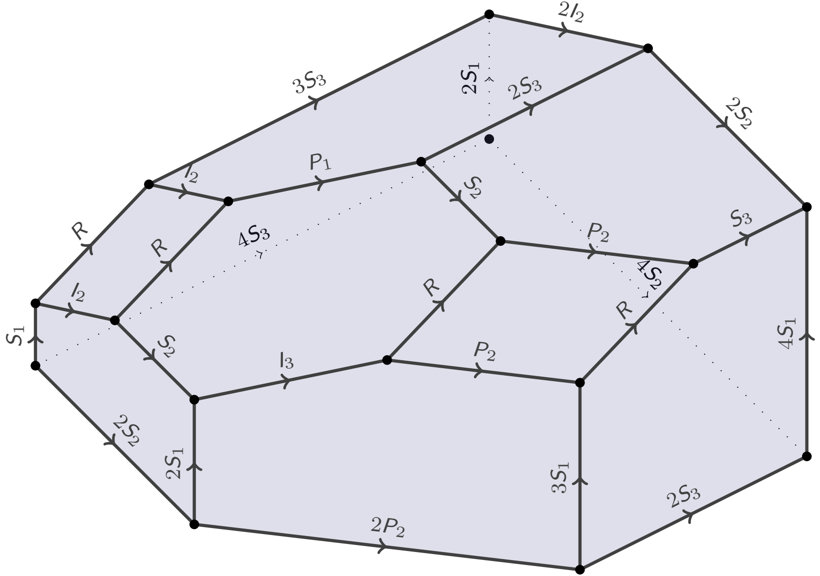
Example 27

There are 9 indecomposable representations for the quiver



with relation *ab*.

Except for indecomposable projective, injective, and simple representations, they are $R = \text{Coker}(1, -1, 0)$ and $T = \text{Coker}(1, 1, -1)$. It turns out that to get the cluster fan of A, we do not need all of them as in Proposition 25. One minimal choice is P_2, P_3, I_1, I_2, R, T. The 18 vertices correspond to the 18 clusters.


2I2
3S3
2S1
2S3
2S2
I2
P1
S2
R
R
4S3
P2
S3
4S2
R
R
I2
S1
S2
I3
P2
4S1
2S2
2S1
3S1
2S3
2P2

Thank You

Time for comments and questions ☺