A glimpse at the classification of Orbifold del Pezzo surfaces

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Del Pezzo Surfaces

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In particular, the associated graded ring

$$R(X, -K_X) = \bigoplus_{m \geq 0} H^0(X, -mK_X)$$

is finitely generated and gives X the structure of projective scheme

$$X \cong \mathrm{Proj}(R(X, -K_X)) \hookrightarrow \mathbb{P}^N$$

Del Pezzo Surfaces

The **degree** of a del Pezzo surface is $d = (-K_X)^2$.

Theorem (Castelnuovo)

Let X be a del Pezzo surface. Then X is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up in $9 - d$ general points (where $d \geq 1$).

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Remark

Every del Pezzo surface is realised as a complete intersection in Fano varieties.

Orbifold surfaces

Let X be a normal surface and $\varphi : Y \to X$ its minimal resolution, with exceptional locus given by a finite collection of curves $\{E_i\}$. Then there exist $d_i \in \mathbb{Q}$ such that

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$$K_Y = \varphi^*(K_X) + \sum d_i E_i$$

Then X is said to have

- **log canonical singularities** if $d_i \geq -1$;
- **log terminal singularities** if $d_i > -1$;
- **canonical singularities** if $d_i \geq 0$;
- **terminal singularities** if $d_i > 0$

Orbifold surfaces

Theorem (Kawamata)

A normal surface singularity $p \in X$ is log terminal $\Leftrightarrow$ p is a **quotient singularity**, *i.e. locally*

$$X \cong {}^{\mathbb{C}^2}\!/_{G}$$

where G is a finite group acting effectively on the open affine neighbourhood.

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If $G \cong \mu_r$ is a **cyclic** group of order r, then the action of G can always be rescaled to be

$$\mu_r \ : (x, y) \longmapsto (\zeta x, \zeta^a y)$$

where ζ is a primitive r-th root of unity, and $(a, r) = 1$.
The singularity $p \in X$ is denoted by $\frac{1}{r}(1, a)$.

Orbifold surfaces

Definition

A **del Pezzo orbifold** X is a complex projective surface with $\mathbb{Q}$–ample anticanonical class $-K_X$ and a finite number of cyclic quotient singularities.

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A **del Pezzo orbifold** X is a complex projective surface with $\mathbb{Q}$–ample anticanonical class $-K_X$ and a finite number of cyclic quotient singularities.

In particular, such surfaces are $\mathbb{Q}$–**Gorenstein**

$\Rightarrow$ the associated graded ring $R(X, -K_X)$ defines an embedding into a weighted projective space

$$X \cong \mathrm{Proj}(R(X,D)) \hookrightarrow \mathbb{P}(m_0, \ldots, m_N)$$

Cascades

Theorem (Reid–Suzuki)

- *Let $S = \mathbb{P}(1,1,3)$ be the surface having one $\frac{1}{3}(1,1)$ singularity, then there exists a cascade of blow ups $S^{(d)} \dashrightarrow S^{(d-1)}$ such that $S^{(d)} \dashrightarrow S$ is a blow up of S in $d \leq 8$ general points, and each $S^{(d)}$ has one $\frac{1}{3}(1,1)$ singularity only.*
- *Let $T_6 \subset \mathbb{P}(1,1,3,5)$ be a surface having one $\frac{1}{5}(1,2)$ singularity, then there exists a cascade of blow ups $T^{(d)} \dashrightarrow T^{(d-1)}$ such that $T^{(d)} \dashrightarrow T_6$ is a blow up of T_6 in $d \leq 6$ general points, and each $T^{(d)}$ has one $\frac{1}{5}(1,2)$ singularity only.*

Classifying del Pezzo orbifolds

Question

How many surfaces exist with such singularity type?

$\Rightarrow$ analyse **graded rings** to find numerical candidates

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How to find all possible base surfaces?

$\Rightarrow$ use **Mori Theory** to find birational models for minimal surfaces

Question

How to find all possible cascades for the given singularity type?

$\Rightarrow$ use **Toric Degenerations** to identify representatives for surfaces in the cascades

Graded rings and invariants

By analysing the structure of the graded ring of del Pezzo surfaces admitting a finite set $\mathrm{Sing}(X) = \{p_i = \frac{1}{r_i}(1, a_i)\}$ we can get a lot of information on a set of invariants

$$R(X, -K_X) = \bigoplus_{m \geq 0} H^0(X, -mK_X) \;\Rightarrow \begin{cases} h^0(-K_X) \\ (-K_X)^2 \\ \rho(X) \end{cases}$$

which strictly depend on the type of singularity.

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Example

For every $p = \frac{1}{r}(1, a)$ we can find the contributions d_i in

$$K_Y = \varphi^*(K_X) + \sum d_i E_i$$

which depend on the numbers r, a.

The case $\mathrm{Sing}(X) = \{k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2)\}$

If X is a del Pezzo orbifold with $\frac{1}{3}(1,1)$ or $\frac{1}{5}(1,2)$ points, we have for instance

$$K_Y = \varphi^*(K_X) - \frac{1}{3}\sum_{i=1}^{k_1} E_i - \frac{1}{5}\sum_{i=1}^{k_2}\left(2C_1^i + C_2^i\right) \tag{1}$$

where $E_i^2 = -3$, $(C_1^i)^2 = -3$ and $(C_2^i)^2 = -2$.

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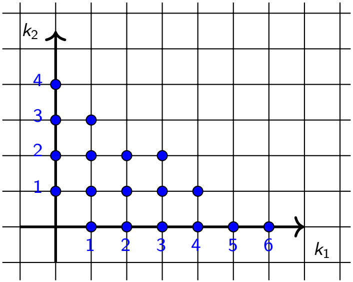
Then we have the following numerical candidates:



Mori Theory

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We start with a (smooth) surface X whose canonical divisor is not nef, and by subsequent birational maps we get to a surface $\overline{X}$ that we understand.

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We start with a (smooth) surface X whose canonical divisor is not nef, and by subsequent birational maps we get to a surface $\overline{X}$ that we understand.

For smooth surfaces, these birational maps are given by the following:

Theorem (Castelnuovo's contraction theorem)

Let X be a smooth surface and $C \subset X$ a rational curve such that $C^2 = -1$. Then there exists a morphism $X \dashrightarrow X'$ that contracts C to a point and it is an isomorphism outside of C.

Mori Theory

Theorem (Minimal Model Program for Surfaces)

Let X be a smooth surface, then after a finite number of extremal contractions, we have a map $X \dashrightarrow \overline{X}$ such that $\overline{X}$ is one of the following:

- **Minimal Model** *($K_{\overline{X}}$ is nef)*

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- $\mathbb{P}^2$

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Moreover, we have the following:

Theorem (Contraction Theorem)

Every proper birational morphism between smooth surfaces can be factored in a sequence of contractions of (-1)-curves.

Minimal Surfaces

As in the cascade the surfaces over the base surfaces are obtained by blow ups at smooth points, we introduce the notion of minimality in the following way:

Definition

A surface X is said to be **minimal** if it does not have any (-1)–curve passing through singularities.

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As in the cascade the surfaces over the base surfaces are obtained by blow ups at smooth points, we introduce the notion of minimality in the following way:

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A surface X is said to be **minimal** if it does not have any (-1)–curve passing through singularities.

Idea: construct an analogous MMP for orbifolds by analysing the possible extremal contractions and give a birational model for the base surfaces.

MMP for Orbifold surfaces

Since we know how the MMP works in the smooth case, we relate the birational morphisms in the singular case to the corresponding morphisms between the minimal resolutions.

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Let X be a minimal surface, $\psi: X \to \overline{X}$ be a birational morphism between orbifold surfaces, and let $Y, \overline{Y}$ be the respective minimal resolutions.

$$\begin{array}{ccc} Y & \xrightarrow{\bar{\psi}} & \bar{Y} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\bar{\varphi}} \\ X & \xrightarrow{\psi} & \bar{X} \end{array}$$

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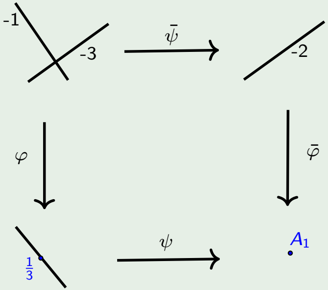
$$\begin{array}{ccc} Y & \xrightarrow{\bar{\psi}} & \bar{Y} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} \\ X & \xrightarrow{\psi} & \bar{X} \end{array}$$

Then $\overline{\psi} : Y \to \overline{Y}$ can be factored in a sequence of blow downs.

MMP for Orbifold surfaces

Example

If X contains a $\frac{1}{3}(1,1)$ point, and an extremal ray passes through it, then the contraction of the extremal ray is represented as follows:



$\Rightarrow$ obtain a list of (ordered) extremal contractions

MMP for Orbifold surfaces

Theorem (C.)

Directed MMP for del Pezzo Orbifolds: *Let X be a del Pezzo orbifold with*

$$\mathrm{Sing}(X) = \{k_1 \times \tfrac{1}{3}(1,1) + k_2 \times \tfrac{1}{5}(1,2)\}$$

Then there exists a finite sequence of extremal contractions $\psi_i : X_{i-1} \to X_i$ ($i = 0..m$ with $X_0 = X$), such that every X_i is a del Pezzo orbifold having at worst $\frac{1}{3}(1,1)$ or $\frac{1}{5}(1,2)$ singularities and for every ψ_i one of the following holds:

- *ψ_i is a* **divisorial contraction** *of a curve Γ, where $\Gamma^2 < 0$, and $\rho(X_i) = \rho(X_{i-1}) - 1$;*
- *ψ_i is a* **fibration** *and $X_i = \mathbb{P}^1$;*
- *$\psi_i = \psi_{m-1}$, and X_m is a surface with $\rho(X_m) = 1$ and at worst $\frac{1}{3}(1,1)$ and $\frac{1}{5}(1,2)$ singularities (e.g. $\mathbb{P}(1,2,5)$)*

The case $(0,1)$

$$
\begin{array}{ccccc}
\frac{1}{5}(1,2) & \longrightarrow & A_1 & \longrightarrow & \text{smooth} \\
 & \longrightarrow & A_2 & \longrightarrow & \text{smooth}
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$$

Thus, we can reconstruct a birational model for a minimal surface of type $(0,1)$ represented by a configuration of curves of its minimal resolution.

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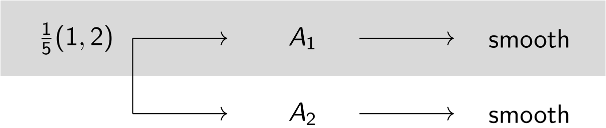

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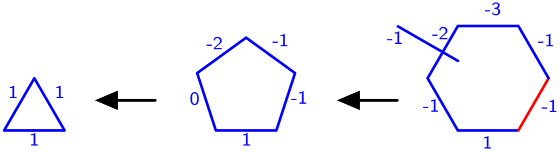

$X_2 = \mathbb{P}^2$ $\quad$ $X_1 \ni A_1$ $\quad$ $X_0 \ni \frac{1}{5}(1,2)$

$\Rightarrow$ not minimal as there is a (-1)–curve not passing through the singularities

The case $(0,1)$

$\frac{1}{5}(1,2) \longrightarrow A_1 \longrightarrow$ smooth

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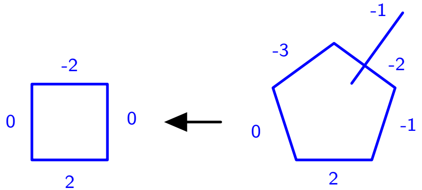

$X_1 = \mathbb{P}(1,1,2)$ $\quad$ $X_0 \ni \frac{1}{5}(1,5)$

$\Rightarrow$ This configuration represents a minimal surface with one $\frac{1}{5}(1,2)$ singularity.

Minimal surfaces with $\frac{1}{3}(1,1)$ and $\frac{1}{5}(1,2)$ singularities

Theorem (C.)

There are 41 isomorphism classes of minimal del Pezzo orbifolds with

$$\mathrm{Sing}(X) = \{k_1 \times \tfrac{1}{3}(1,1) + k_2 \times \tfrac{1}{5}(1,2)\}$$

and they are represented by a configuration of curves as described above.

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In particular, these surfaces represent the bases of the cascades with said singularity type.

qG–deformations

Definition

A flat family of surfaces $\mathcal{X} \to D$ over a base scheme D is called $\mathbb{Q}$–**Gorenstein** if the total space $\mathcal{X}$ is $\mathbb{Q}$–Gorenstein.

In particular, for surfaces with cyclic quotient singularities have the following characterisation:

Theorem (Kollár, Shepherd–Barron)

If $\mathcal{X} \to D$ is a one–paramenter deformation of a cyclic quotient singularity (X_0, p), then, up to base change, there exists a birational morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ such that over a general point $\varphi : Y_t \to X_t$ is the minimal resolution, and the special fibre Y_0 is normal with quotient singularities. In particular, $K^2_{X_t}$ is locally constant on the base.

qG–deformations

Cyclic quotient singularities are thus divided in two classes:

- **T–singularties** if they admit a qG–smoothing;
- **R–singularities** if they admit some "residual content", thus they are not smoothable.

Example

- The singularities $A_n = \frac{1}{n+1}(1, n)$ are smoothable for every $n \geq 1$.
- The singularities $\frac{1}{3}(1, 1)$ and $\frac{1}{5}(1, 2)$ are rigid.

Toric log del Pezzo

Toric varieties arise as Zariski closures of torus embeddings and have a very combinatorial structure related to lattice polyhedra.

Definition

A lattice polygon $P \in \mathbb{Z}^2$ is called **LDP–polygon** if the following hold:

- P is convex
- $0 \in \mathbb{Z}^2$ is a strict interior point of P
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- $0 \in \mathbb{Z}^2$ is a strict interior point of P
- all of the vertices of P are primitive

There exists a $1:1$ correspondence between

$$\begin{pmatrix} \text{Toric del Pezzo with} \\ \text{quotient singularities} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{LDP–polygons} \\ \text{up to isomorphism} \end{pmatrix}$$

Mutations

For toric surfaces the notion of qG–deformation is linked to the one of **mutation**, which describes a transformation of the LDP–polygon associated to the toric variety that does not change the rigid content of the variety and preserves a set of invariants.

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Theorem (Ilten)

Let P, P' two LDP–polygons associated to the toric varieties $X_P, X_{P'}$, and suppose that there exists a mutation between the two polygons. Then there exists a qG-pencil $g : \mathcal{X} \to \mathbb{P}^1$ with scheme-theoretic fibres $g^*(0) = X_P$ and $g^*(\infty) = X'_P$.

Toric Degenerations

Via computer algebra we can find all possible mutation classes for LDP–polygons with given singularity type, thus we can choose the relative toric variety as a representative for a qG–class.

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Theorem (C., Kutas)

There are 95 mutation classes of toric del Pezzo orbifolds with $\frac{1}{3}(1,1)$ *and* $\frac{1}{5}(1,2)$ *points.*

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Theorem (Corti, Heuberger)

There are 29 qG–deformation families of del Pezzo orbifolds with $\frac{1}{3}(1,1)$ *points. Exactly 26 of them admit a qG–degeneration to a toric surface.*

Toric Degenerations

So far we have built two sets of surfaces and we still have to figure out how they are related:

$$\begin{pmatrix}\text{Minimal del Pezzo orbifolds}\\ \text{w/ } \frac{1}{3}(1,1) \text{ and } \frac{1}{5}(1,2) \text{ pts}\end{pmatrix} \quad \begin{pmatrix}\text{Mutation classes of del Pezzo}\\ \text{orbifolds w/ } \frac{1}{3}(1,1) \text{ and } \frac{1}{5}(1,2) \text{ pts}\end{pmatrix}$$

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Thus, we want to find a correspondence between the birational models of the minimal del Pezzo orbifols and the minimal toric surfaces representing mutation classes.

Complexity 1 constructions

P, P' two mutation equivalent polygons $\Rightarrow X_P, X_{P'}$ are the special fibres of a qG–pencil

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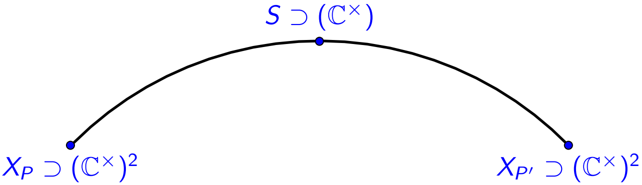
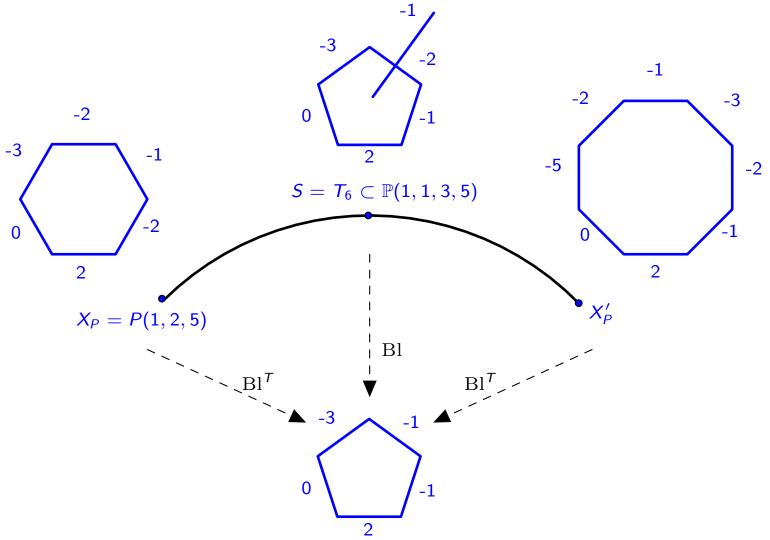
Moreover, the general element of the pencil is a variety that inherits a $\mathbb{C}^\times$ action (which is said to be of **complexity 1**).



Thus, to link the minimal del Pezzo orbifolds given by their birational models with the toric surfaces, we will use the structure of the complexity 1 surface to reconstruct the curve configurations.

Complexity 1 constructions



Complexity 1 constructions

Theorem (C.)

Let X_1, X_2 *be two toric orbifold del Pezzo surfaces corresponding to the two LDP–polygons* P_1, P_2. *Furthermore, assume that the two polygons are mutation equivalent , so there exists a qG–deformation family* $\pi : \mathcal{X} \to B$ *such that for* $\lambda_1, \lambda_2 \in B$ *then* $\pi^{-1}(\lambda_1) = X_1$ *and* $\pi^{-1}(\lambda_2) = X_2$. *Then the general element* S *of the family is a* T*–variety corresponding to an equivariant blow up of a toric surface* $\bar{X}$. *Moreover, the toric surfaces* X_1, X_2 *are obtained from* $\bar{X}$ *via toric blow ups.*

Complexity 1 constructions

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It turns out that the curve configurations of the minimal del Pezzo orbifolds can have three type of configurations: toric, complexity 1 or composition of non toric blow–ups.

Results

Theorem (C., Kutas)

There are 41 isomorphism classes of minimal del Pezzo orbifolds with

$$\mathrm{Sing}(X) = \{k_1 \times \tfrac{1}{3}(1,1) + k_2 \times \tfrac{1}{5}(1,2)\}$$

admitting a toric degeneration.

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admitting a toric degeneration.

Lastly, we can recontruct the (concurring) cascades by looking at the remaining toric candidates and finding the respective blow ups.

Results

$n = 3$ $\quad X^3_{(0,1)} = \mathbb{P}(1,2,5)$

$n = 4$ $\quad X^{4,1}_{(0,1)} = \mathrm{Bl}(X^3_{(0,1)}) \qquad X^{4,2}_{(0,1)} = \mathbb{P}(1,3,5)$

$n = 5$ $\quad X^{5,1}_{(0,1)} = \mathrm{Bl}(X^{4,1}_{(0,1)}) \qquad X^{5,2}_{(0,1)} = \mathrm{Bl}(X^{4,2}_{(0,1)})$

$n = 6$ $\quad X^6_{(0,1)} = \mathrm{Bl}(X^{5,1}_{(0,1)}) = \mathrm{Bl}(X^{5,2}_{(0,1)})$

$n = 7$ $\quad X^7_{(0,1)} = \mathrm{Bl}(X^6_{(0,1)})$

$n = 8$ $\quad X^8_{(0,1)} = \mathrm{Bl}(X^7_{(0,1)})$

$n = 9$ $\quad X^9_{(0,1)} = \mathrm{Bl}(X^8_{(0,1)})$

Thank you for your attention!

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