Families of Gröbner degenerations

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Overview

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2. Review on Gröbner theory
3. Construction and Theorems
4. Application: universal coefficients for cluster algebras

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Today: understand those toric degenerations of a polarized projective variety that *"share a common basis"*.

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with $\pi^{-1}(t) \cong V(J)$ for $t \neq 0$ and $\pi^{-1}(0) = V(\mathrm{in}_C(J))$.

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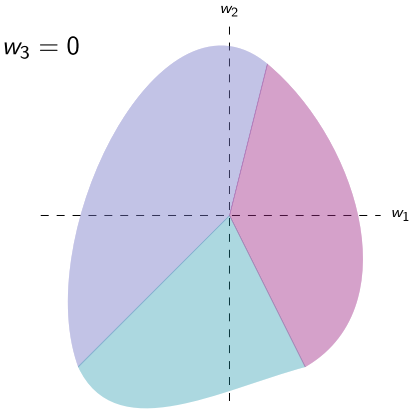

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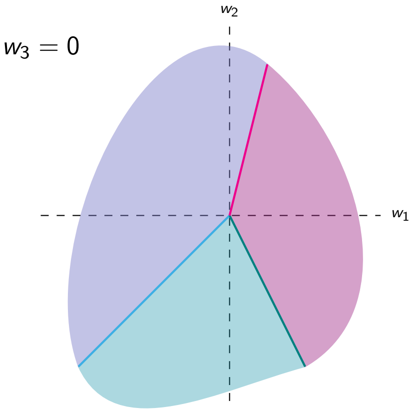

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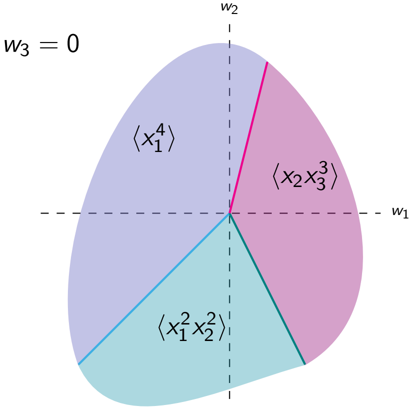

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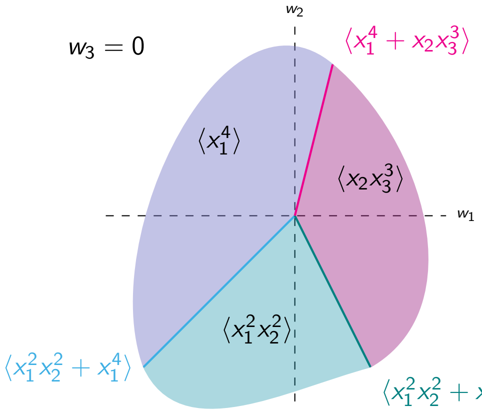

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Standard monomial basis

Let $A := \mathbb{C}[x_1, \dots, x_n]/J$ and $A_\tau := \mathbb{C}[x_1, \dots, x_n]/\mathrm{in}_\tau(J)$ for $\tau \in GF(J)$.

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In particular, $\mathbb{B}_C := \mathbb{B}_{C,\{0\}}$ is a vector space basis for $A = A_{\{0\}}$.

$\rightsquigarrow$ *All degenerations* $\{V(\mathrm{in}_\tau(J)) : \tau \subseteq C\}$ share one *standard monomial basis!*

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$$\mu(f) := (\min_{c_\alpha \neq 0}\{r_1 \cdot \alpha\}, \ldots, \min_{c_\alpha \neq 0}\{r_m \cdot \alpha\}) \in \mathbb{Z}^m.$$

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$$\tilde{f} := f(\mathbf{t}^{\mathbf{r}\cdot e_1} x_1, \dots, \mathbf{t}^{\mathbf{r}\cdot e_n} x_n)\mathbf{t}^{-\mu(f)} = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \mathbf{t}^{\mathbf{r}\cdot\alpha - \mu(f)}.$$

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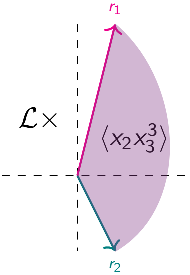

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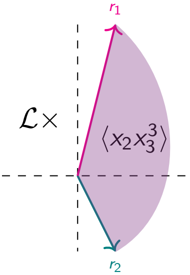

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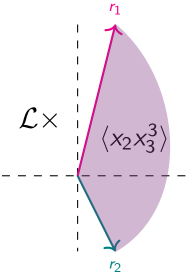

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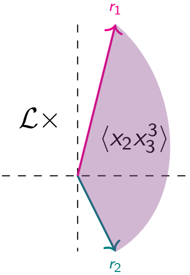

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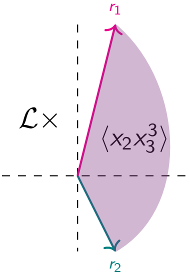

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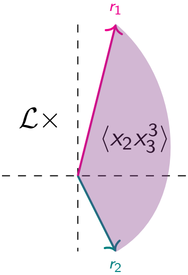

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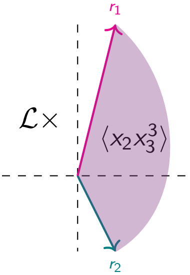

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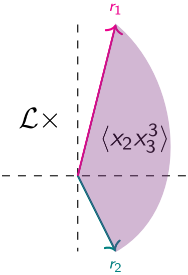
Example

Take $f = x_1^2x_2^2 + x_1^4 + x_2x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF(\langle f\rangle)$ the maximal cone C spanned by $r_1 := (\,1\ 4\ 0\,)$, $r_2 := (\,1\ {-2}\ 0\,)$ and $\mathcal{L}$. We compute

$$\begin{aligned}\tilde{f}(t_1, t_2) &= f(t_1t_2x_1, t_1^4t_2^{-2}x_2, x_3)t_1^{-4}t_2^2\\ &= t_1^6x_1^2x_2^2 + t_2^6x_1^4 + x_2x_3^3\end{aligned}$$



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contains toric fibers isomorphic to $Spec(A_\tau)$ *(affine toric scheme).*

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Let $C \in \mathrm{GF}(J)$ be a maximal cone, X_C the affine toric variety and $p_C : \mathbb{A}^m \to X_C$ the universal torsor of X_C (by Cox construction)

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$\rightsquigarrow$ the cluster structure is encoded in a simplicial complex called the *cluster complex* (seeds $\leftrightarrow$ maximal simplices).

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The homogeneous coordinate ring of $\mathrm{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding is a cluster algebra:

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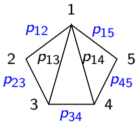

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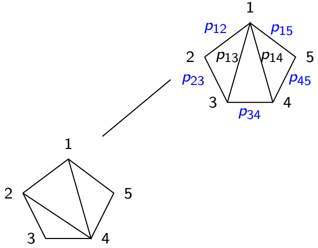

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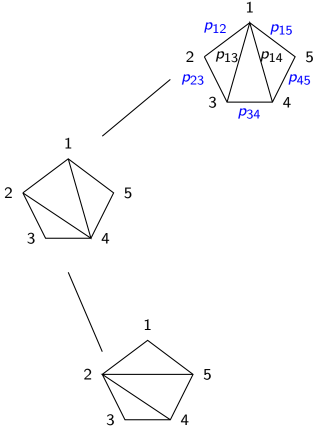

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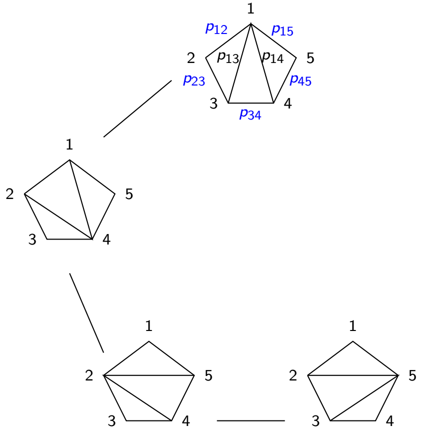

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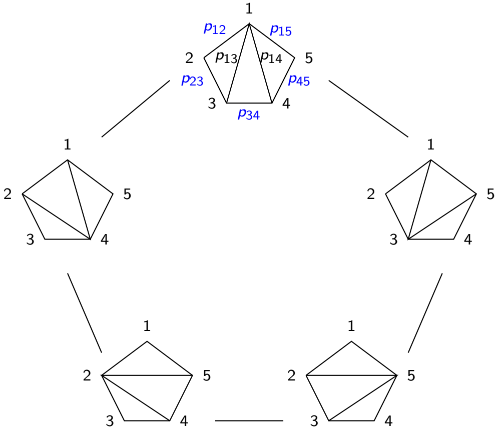

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$\rightsquigarrow$ All these degenerations *share the ϑ-basis*, i.e. $A_s^{\mathrm{prin}} = \bigoplus_{\vartheta \in \Theta} \vartheta$ for all s.

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4. *for every seed s there exists a face $\tau_s \subset C \cap \mathrm{Trop}(J_{k,n})$ such that the toric variety $X_{s,0}$ is isomorphic to $\mathrm{Spec}(A_{\tau_s})$.*

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Corollary

$Gr_3(\mathbb{C}^6)$, a cone over $\mathbb{P}(\mathtt{D}_4)$ (namely $Proj(A_C)$) and the toric schemes $Proj(A_{s,0})$ for all seeds s all lie on the same component of the Hilbert scheme.

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Question: Can we obtain similar results for arbitrary Grassmannians?

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