Families of Gröbner degenerations

L. Bossinger (jt. F. Mohammadi and A. Nájera Chávez)

Universidad Nacional Autónoma de México, Oaxaca

September 2020
Overview

1 Motivation
Overview

1. Motivation
2. Review on Gröbner theory
Overview

1. Motivation
2. Review on Gröbner theory
3. Construction and Theorems
Overview

1. Motivation
2. Review on Gröbner theory
3. Construction and Theorems
4. Application: universal coefficients for cluster algebras
Motivation

Understand how different toric degenerations of a projective variety are related.
Motivation

Understand how different toric degenerations of a projective variety are related.

*Slogan: Knowing all possible toric degenerations of a variety is equivalent to knowing its mirror dual variety.*
Motivation

Understand how different toric degenerations of a projective variety are related.

_Slogan:_ Knowing all possible toric degenerations of a variety is equivalent to knowing its mirror dual variety.

_Today:_ understand those toric degenerations of a polarized projective variety that "share a common basis".
Initial ideals

Let \( f = \sum c_{\alpha} x^{\alpha} \in \mathbb{C}[x_1, \ldots, x_n] \)
Initial ideals

Let \( f = \sum c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \) with \( c_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}^n_{\geq 0} \) and \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

For \( w \in \mathbb{R}^n \) we define its initial form with respect to \( w \) as
\[
\text{in}_w(f) := \sum w \cdot \beta = \min_{c_\alpha \neq 0} \{ w \cdot \alpha \} c_\beta x^\beta.
\]
For \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) an ideal we define its initial ideal with respect to \( w \) as \( \text{in}_w(J) := \langle \text{in}_w(f) : f \in J \rangle \).
Initial ideals

Let \( f = \sum c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \) with \( c_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}^n_{\geq 0} \) and \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

For \( w \in \mathbb{R}^n \) we define its \textit{initial form with respect to} \( w \) as
Initial ideals

Let $f = \sum c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ with $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{Z}^n_{\geq 0}$ and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

For $w \in \mathbb{R}^n$ we define its *initial form with respect to $w$* as

$$\text{in}_w(f) := \sum_{w \cdot \beta = \min_{c_\alpha \neq 0} \{w \cdot \alpha\}} c_\beta x^\beta.$$


Initial ideals

Let \( f = \sum c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \) with \( c_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}^n_{\geq 0} \) and \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

For \( w \in \mathbb{R}^n \) we define its \textit{initial form with respect to} \( w \) as

\[
in_w(f) := \sum_{w \cdot \beta = \min \{w \cdot \alpha \mid c_\alpha \neq 0\}} c_\beta x^\beta.
\]

For \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) an ideal we define its \textit{initial ideal with respect to} \( w \) as \( \langle \text{in}_w(J) \rangle := \langle \text{in}_w(f) : f \in J \rangle \).
Initial ideals

Let \( f = \sum c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \) with \( c_\alpha \in \mathbb{C} \), \( \alpha \in \mathbb{Z}^n_{\geq 0} \) and \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

For \( w \in \mathbb{R}^n \) we define its initial form with respect to \( w \) as

\[
\text{in}_w(f) := \sum_{w \cdot \beta = \min_{c_\alpha \neq 0} \{w \cdot \alpha\}} c_\beta x^\beta.
\]

For \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) an ideal we define its initial ideal with respect to \( w \) as \( \text{in}_w(J) := \langle \text{in}_w(f) : f \in J \rangle \).

**Example**

For \( f = x_1 x_2^2 + x_1^2 + x_2 \in \mathbb{C}[x_1, x_2] \)...
Let \( f = \sum c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \) with \( c_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}_\geq 0^n \) and \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

For \( w \in \mathbb{R}^n \) we define its \textit{initial form with respect to} \( w \) as

\[
in_w(f) := \sum_{w \cdot \beta = \min_{c_\alpha \neq 0} \{w \cdot \alpha\}} c_\beta x^\beta.
\]

For \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) an ideal we define its \textit{initial ideal with respect to} \( w \) as \( \text{in}_w(J) := \langle \text{in}_w(f) : f \in J \rangle \).

\textbf{Example}

For \( f = x_1 x_2^2 + x_1^2 + x_2 \in \mathbb{C}[x_1, x_2] \) and \( w = (1, 0) \) we compute
Initial ideals

Let \( f = \sum c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \) with \( c_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}_{\geq 0}^n \) and \( x^\alpha := x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n} \).

For \( w \in \mathbb{R}^n \) we define its initial form with respect to \( w \) as

\[
\text{in}_w(f) := \sum_{w \cdot \beta = \min_{c_\alpha \neq 0} \{w \cdot \alpha\}} c_\beta x^\beta.
\]

For \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) an ideal we define its initial ideal with respect to \( w \) as \( \text{in}_w(J) := \langle \text{in}_w(f) : f \in J \rangle \).

Example

For \( f = x_1x_2^2 + x_1^2 + x_2 \in \mathbb{C}[x_1, x_2] \) and \( w = (1, 0) \) we compute

\[
\text{in}_w(f) = x_2.
\]
Gröbner fan and Gröbner degenerations

Definition

For a homogeneous ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$ its Gröbner fan $\text{GF}(J)$ is $\mathbb{R}^n$ with fan structure defined by $v, w \in \mathbb{C}^\circ \iff \text{in}_v(J) = \text{in}_w(J)$.

Notation: $	ext{in}_C(J) := \text{in}_w(J)$ for any $w \in \mathbb{C}^\circ$.

Every open cone $C^\circ \in \text{GF}(J)$ defines a Gröbner degeneration $\pi: V \to \mathbb{A}^1$ with $\pi^{-1}(0) = V(\text{in}_C(J))$. For $t \neq 0$ and $\pi^{-1}(0) = V(\text{in}_C(J))$. 

Gröbner fan and Gröbner degenerations
Gröbner fan and Gröbner degenerations

**Definition**

For a homogeneous ideal \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) its Gröbner fan \( GF(J) \) is \( \mathbb{R}^n \) with fan structure defined by

\[
in_C(J) := in_C(J)
\] for any \( w \in C^\circ \). Every open cone \( C^\circ \in GF(J) \) defines a Gröbner degeneration \( \pi : V \to A^1 \) with \( \pi^{-1}(t) \sim = V(J) \) for \( t \neq 0 \) and \( \pi^{-1}(0) = V(in_C(J)) \).
Definition

For a homogeneous ideal \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) its \textit{Gröbner fan} \( \text{GF}(J) \) is \( \mathbb{R}^n \) with fan structure defined by

\[
v, w \in C^\circ \iff \text{in}_v(J) = \text{in}_w(J).
\]

This means that for any open cone \( C^\circ \in \text{GF}(J) \), there exists a \textit{Gröbner degeneration} \( \pi: V \to \mathbb{A}^1 \) with \( \pi^{-1}(0) = V(\text{in}_w(J)) \) for \( t \neq 0 \) and \( \pi^{-1}(0) = V(\text{in}_w(J)) \).
Gröbner fan and Gröbner degenerations

**Definition**

For a homogeneous ideal $J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ its *Gröbner fan* $\text{GF}(J)$ is $\mathbb{R}^n$ with fan structure defined by

$$\nu, w \in C^\circ \iff \text{in}_\nu(J) = \text{in}_w(J).$$

**Notation:** $\text{in}_C(J) := \text{in}_w(J)$ for any $w \in C^\circ$. 

Gröbner fan and Gröbner degenerations

Definition

For a homogeneous ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$ its **Gröbner fan** $GF(J)$ is $\mathbb{R}^n$ with fan structure defined by

$$\nu, \omega \in C^\circ \iff \text{in}_\nu(J) = \text{in}_\omega(J).$$

**Notation**: $\text{in}_C(J) := \text{in}_\omega(J)$ for any $\omega \in C^\circ$.

Every open cone $C^\circ \in GF(J)$ defines a **Gröbner degeneration**
**Definition**

For a homogeneous ideal \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) its **Gröbner fan** \( GF(J) \) is \( \mathbb{R}^n \) with fan structure defined by

\[ v, w \in C^\circ \iff \text{in}_v(J) = \text{in}_w(J). \]

**Notation:** \( \text{in}_C(J) := \text{in}_w(J) \) for any \( w \in C^\circ \).

Every open cone \( C^\circ \in GF(J) \) defines a **Gröbner degeneration**

\[ \pi : \mathbb{V} \to \mathbb{A}^1 \]
Gröbner fan and Gröbner degenerations

**Definition**

For a homogeneous ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$ its **Gröbner fan** $GF(J)$ is $\mathbb{R}^n$ with fan structure defined by

$$v, w \in C^\circ \iff \text{in}_v(J) = \text{in}_w(J).$$

**Notation:** $\text{in}_C(J) := \text{in}_w(J)$ for any $w \in C^\circ$.

Every open cone $C^\circ \in GF(J)$ defines a **Gröbner degeneration**

$$\pi : V \rightarrow \mathbb{A}^1$$

with $\pi^{-1}(t) \cong V(J)$ for $t \neq 0$ and $\pi^{-1}(0) = V(\text{in}_C(J))$. 
Example

Take \( I = \langle x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \rangle \subset \mathbb{C}[x_1, x_2, x_3]. \)
Example

Take \( I = \langle x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \rangle \subset \mathbb{C}[x_1, x_2, x_3] \). Then \( GF(I) \) is \( \mathbb{R}^3 \) with the fan structure:
Example

Take \( I = \langle x_1^2x_2^2 + x_1^4 + x_2x_3^3 \rangle \subset \mathbb{C}[x_1, x_2, x_3] \). Then \( GF(I) \) is \( \mathbb{R}^3 \) with the fan structure:

\[ w_3 = 0 \]

\[ \times \mathcal{L} = \langle (1, 1, 1) \rangle \]
Example

Take \( I = \langle x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \rangle \subset \mathbb{C}[x_1, x_2, x_3] \). Then \( GF(I) \) is \( \mathbb{R}^3 \) with the fan structure:

\[
\begin{aligned}
   w_3 &= 0 \\
   \times \mathcal{L} &= \langle (1, 1, 1) \rangle
\end{aligned}
\]
Example

Take $I = \langle x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$. Then $GF(I)$ is $\mathbb{R}^3$ with the fan structure:

$w_3 = 0$

$\langle x_1^4 \rangle \times \mathcal{L} = \langle (1, 1, 1) \rangle$
Example

Take $I = \langle x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$. Then $GF(I)$ is $\mathbb{R}^3$ with the fan structure:
Let $A := \mathbb{C}[x_1, \ldots, x_n]/J$ and $A_\tau := \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$ for $\tau \in GF(J)$. Then $B_{C,\tau}$ is a vector space basis for $A_\tau$, called the standard monomial basis. In particular, $B_{C,\{0\}}$ is a vector space basis for $A = A_{\{0\}}$. All degenerations $\{V(\text{in}_\tau(J)) : \tau \subseteq C\}$ share one standard monomial basis!
Let $A := \mathbb{C}[x_1, \ldots, x_n]/J$ and $A_{\tau} := \mathbb{C}[x_1, \ldots, x_n]/\text{in}_{\tau}(J)$ for $\tau \in GF(J)$.

Fix a maximal cone $C \in GF(J)$, then the ideal $\text{in}_C(J)$ is generated by monomials.

For every face $\tau \subseteq C$ we define $B_{C,\tau} := \{\bar{x}_\alpha \in A_\tau | x_\alpha \notin \text{in}_C(J)\}$.

Then $B_{C,\{0\}}$ is a vector space basis for $A = A_{\{0\}}$.

All degenerations $\{V(\text{in}_{\tau}(J)) : \tau \subseteq C\}$ share one standard monomial basis!
Standard monomial basis

Let $A := \mathbb{C}[x_1, \ldots, x_n]/J$ and $A_\tau := \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$ for $\tau \in \text{GF}(J)$.

Fix a maximal cone $C \in \text{GF}(J)$, then the ideal $\text{in}_C(J)$ is generated by monomials. For every face $\tau \subseteq C$ we define

$$B_{C,\tau} = \{ \bar{x}_\alpha \in A_\tau \mid x_\alpha \not\in \text{in}_C(J) \}.$$
Standard monomial basis

Let $A := \mathbb{C}[x_1, \ldots, x_n]/J$ and $A_\tau := \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$ for $\tau \in GF(J)$.

Fix a maximal cone $C \in GF(J)$, then the ideal $\text{in}_C(J)$ is generated by monomials. For every face $\tau \subseteq C$ we define

$$B_{C,\tau} := \{\bar{x}^\alpha \in A_\tau \mid x^\alpha \not\in \text{in}_C(J)\}.$$
Standard monomial basis

Let $A := \mathbb{C}[x_1, \ldots, x_n]/J$ and $A_\tau := \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$ for $\tau \in GF(J)$.

Fix a maximal cone $C \in GF(J)$, then the ideal $\text{in}_C(J)$ is generated by monomials. For every face $\tau \subseteq C$ we define

$$B_{C, \tau} := \{ \bar{x}^\alpha \in A_\tau \mid x^\alpha \notin \text{in}_C(J) \}.$$ 

Then $B_{C, \tau}$ is a vector space basis for $A_\tau$ called standard monomial basis$^1$.

---

$^1$Due to Lakshmibai–Seshadri, generalized by Sturmfels–White
Standard monomial basis

Let $A := \mathbb{C}[x_1, \ldots, x_n]/J$ and $A_\tau := \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$ for $\tau \in GF(J)$.

Fix a maximal cone $C \in GF(J)$, then the ideal $\text{in}_C(J)$ is generated by monomials. For every face $\tau \subseteq C$ we define

$$B_{C,\tau} := \{x^\alpha \in A_\tau \mid x^\alpha \notin \text{in}_C(J)\}.$$ 

Then $B_{C,\tau}$ is a vector space basis for $A_\tau$ called standard monomial basis\(^1\).

In particular, $B_C := B_{C,\{0\}}$ is a vector space basis for $A = A_{\{0\}}$.

$\leadsto$ All degenerations $\{V(\text{in}_\tau(J)) : \tau \subseteq C\}$ share one standard monomial basis!

\[^1\text{Due to Lakshmibai–Seshadri, generalized by Sturmfels–White}\]
Let $C \in GF(J)$ be a maximal cone and choose $r_1, \ldots, r_m$ representatives of primitive ray generators of $\overline{C} \in GF(J)/\mathcal{L}$.
Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \ldots, r_m$ representatives of primitive ray generators of $\overline{C} \in GF(J)/\mathcal{L}$. Let $r$ be the matrix with rows $r_1, \ldots, r_m$. 

Definition/Proposition

The lifted ideal $\tilde{J} := \langle \tilde{f} : f \in J \rangle \subset C[t_1, \ldots, t_m][x_1, \ldots, x_n]$ is generated by $\{\tilde{g} : g \in G\}$, where $G$ is a Gröbner basis for $J$ and $C$. 

Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \ldots, r_m$ representatives of primitive ray generators of $\overline{C} \in GF(J)/\mathcal{L}$. Let $r$ be the matrix with rows $r_1, \ldots, r_m$. Define for $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_{\alpha} x^\alpha \in J$

$$\mu(f) := (\min_{c_{\alpha} \neq 0} \{r_1 \cdot \alpha\}, \ldots, \min_{c_{\alpha} \neq 0} \{r_m \cdot \alpha\}) \in \mathbb{Z}^m.$$ 

and in $\mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]$ the lift of $f$
Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \ldots, r_m$ representatives of primitive ray generators of $\overline{C} \in GF(J)/L$. Let $r$ be the matrix with rows $r_1, \ldots, r_m$. Define for

$$f = \sum_{\alpha \in \mathbb{Z}_n^+} c_\alpha x^\alpha \in J$$

$$\mu(f) := \left( \min_{c_\alpha \neq 0} \{ r_1 \cdot \alpha \}, \ldots, \min_{c_\alpha \neq 0} \{ r_m \cdot \alpha \} \right) \in \mathbb{Z}^m.$$ 

and in $\mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]$ the lift of $f$

$$\tilde{f} := f(t^{r \cdot e_1} x_1, \ldots, t^{r \cdot e_n} x_n) t^{-\mu(f)} = \sum_{\alpha \in \mathbb{Z}_n^+} c_\alpha x^{\alpha} t^{r \cdot \alpha - \mu(f)}.$$
Let $C \in GF(J)$ be a maximal cone and choose $r_1, \ldots, r_m$ representatives of primitive ray generators of $\overline{C} \in GF(J)/\mathcal{L}$. Let $r$ be the matrix with rows $r_1, \ldots, r_m$. Define for $f = \sum_{\alpha \in \mathbb{Z}_n^+} c_{\alpha} x^\alpha \in J$

$$\mu(f) := (\min_{c_{\alpha} \neq 0} \{r_1 \cdot \alpha\}, \ldots, \min_{c_{\alpha} \neq 0} \{r_m \cdot \alpha\}) \in \mathbb{Z}^m.$$ 

and in $\mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]$ the \textit{lift} of $f$

$$\tilde{f} := f(t^{r \cdot e_1} x_1, \ldots, t^{r \cdot e_n} x_n) t^{-\mu(f)} = \sum_{\alpha \in \mathbb{Z}_n^+} c_{\alpha} x^\alpha t^{r \cdot \alpha - \mu(f)}.$$ 

\section*{Definition/Proposition}

The \textit{lifted ideal} $\tilde{J} := \langle \tilde{f} : f \in J \rangle \subset \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]$
Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \ldots, r_m$ representatives of primitive ray generators of $\overline{C} \in GF(J)/L$. Let $r$ be the matrix with rows $r_1, \ldots, r_m$. Define for $f = \sum_{\alpha \in \mathbb{Z}_n^\geq 0} c_\alpha x^\alpha \in J$

$$\mu(f) := \left( \min_{c_\alpha \neq 0} \{ r_1 \cdot \alpha \}, \ldots, \min_{c_\alpha \neq 0} \{ r_m \cdot \alpha \} \right) \in \mathbb{Z}^m.$$ 

and in $\mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]$ the lift of $f$

$$\tilde{f} := f(t^{r \cdot e_1} x_1, \ldots, t^{r \cdot e_n} x_n) t^{-\mu(f)} = \sum_{\alpha \in \mathbb{Z}_n^\geq 0} c_\alpha x^\alpha t^{r \cdot \alpha - \mu(f)}.$$ 

Definition/Proposition

The **lifted ideal** $\tilde{J} := \langle \tilde{f} : f \in J \rangle \subset \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]$ is generated by $\{ \tilde{g} : g \in G \}$, where $G$ is a **Gröbner basis** for $J$ and $C$. 

Example

Take \( f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3] \)
Example

Take $f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF(\langle f \rangle)$ the maximal cone $C$ spanned by $r_1 := (1, 4, 0), r_2 := (1, -2, 0)$ and $\mathcal{L}$.
Example

Take \( f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3] \) and consider in \( GF(\langle f \rangle) \) the maximal cone \( C \) spanned by \( r_1 := (1 \ 4 \ 0), r_2 := (1 \ -2 \ 0) \) and \( \mathcal{L} \). We compute

\[
\tilde{f}(t_1, t_2) = f(t_1 t_2 x_1, t_1^4 t_2^{-2} x_2, x_3) t_1^{-4} t_2^2
\]
Take $f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF(\langle f \rangle)$ the maximal cone $C$ spanned by $r_1 := (140), r_2 := (1-20)$ and $\mathcal{L}$. We compute

$$\tilde{f}(t_1, t_2) = f(t_1 t_2 x_1, t_1^4 t_2^{-2} x_2, x_3) t_1^{-4} t_2^2$$

$$= t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3$$
Example

Take \( f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3] \) and consider in \( GF(\langle f \rangle) \) the maximal cone \( C \) spanned by \( r_1 := (1 \ 4 \ 0) \), \( r_2 := (1 \ -2 \ 0) \) and \( \mathcal{L} \). We compute

\[
\tilde{f}(t_1, t_2) = f(t_1 t_2 x_1, t_1^4 t_2^{-2} x_2, x_3) t_1^{-4} t_2^2 \\
= t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3
\]

- \( \tilde{f}(0, 0) = x_2 x_3^3 \in \text{in}_C(f) \),
Take $f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF(\langle f \rangle)$ the maximal cone $C$ spanned by $r_1 := (1 4 0), r_2 := (1 -2 0)$ and $\mathcal{L}$. We compute

$$\tilde{f}(t_1, t_2) = f(t_1 t_2 x_1, t_1^4 t_2^{-2} x_2, x_3) t_1^{-4} t_2^2$$

$$= t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3$$

- $\tilde{f}(0, 0) = x_2 x_3^3 = \text{in}_C(f)$,
- $\tilde{f}(0, 1) = x_1^4 + x_2 x_3^3 = \text{in}_{r_1}(f)$,
Example

Take \( f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3] \) and consider in \( GF(\langle f \rangle) \) the maximal cone \( C \) spanned by \( r_1 := (1, 4, 0) \), \( r_2 := (1, -2, 0) \) and \( \mathcal{L} \). We compute

\[
\tilde{f}(t_1, t_2) = f(t_1 t_2 x_1, t_1 t_2^{-2} x_2, x_3) t_1^{-4} t_2^2
\]

\[
= t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3
\]

\[
\tilde{f}(0, 0) = x_2 x_3^3 = \text{in}_C(f),
\]

\[
\tilde{f}(0, 1) = x_1^4 + x_2 x_3^3 = \text{in}_{r_1}(f),
\]

\[
\tilde{f}(1, 0) = x_1^2 x_2^2 + x_2 x_3^3 = \text{in}_{r_2}(f),
\]
Take $f = x_1^2x_2^2 + x_1^4 + x_2x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF(\langle f \rangle)$ the maximal cone $C$ spanned by $r_1 := (1, 4, 0), r_2 := (1, -2, 0)$ and $\mathcal{L}$. We compute

$$
\tilde{f}(t_1, t_2) = f(t_1 t_2 x_1, t_1^4 t_2^{-2} x_2, x_3) t_1^{-4} t_2^2 \\
= t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3
$$

- $\tilde{f}(0, 0) = x_2 x_3^3 \in \text{in}_C(f)$,
- $\tilde{f}(0, 1) = x_1^4 + x_2 x_3^3 \in \text{in}_{r_1}(f)$,
- $\tilde{f}(1, 0) = x_1^2 x_2^2 + x_2 x_3^3 \in \text{in}_{r_2}(f)$,
- $\tilde{f}(1, 1) = f$. 

Theorem (B.–Mohammadi–Nájera Chávez)

\[ \tilde{A} \text{ is a free } C[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J}, \quad A_\tau = C[x_1, \ldots, x_n]/\text{in}_\tau(J). \]
First Theorem

Let $\tilde{A} := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J}$, $A_\tau = \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$.

Theorem (B.–Mohammadi–Nájera Chávez)
Let \( \tilde{A} := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J} \), \( A_\tau = \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J) \).

**Theorem (B.–Mohammadi–Nájera Chávez)**

\( \tilde{A} \) is a free \( \mathbb{C}[t_1, \ldots, t_m] \)-module with basis \( \mathbb{B}_C \).
Let $\tilde{A} := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J}$, $A_\tau = \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$.

**Theorem (B.–Mohammadi–Nájera Chávez)**

$\tilde{A}$ is a free $\mathbb{C}[t_1, \ldots, t_m]$-module with basis $\mathcal{B}_C$ and so the morphism

$$\pi : \text{Spec}(\tilde{A}) \to \mathbb{A}^m$$

is flat.
Let $\tilde{A} := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J}$, $A_\tau = \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$.

**Theorem (B.–Mohammadi–Nájera Chávez)**

$\tilde{A}$ is a free $\mathbb{C}[t_1, \ldots, t_m]$-module with basis $B_C$ and so the morphism

$$\pi : \text{Spec}(\tilde{A}) \rightarrow \mathbb{A}^m$$

is flat. In particular, $\pi$ defines a flat family with generic fiber $\text{Spec}(A)$.
Let $\tilde{A} := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J}$, $A_\tau = \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$.

**Theorem (B.–Mohammadi–Nájera Chávez)**

$\tilde{A}$ is a free $\mathbb{C}[t_1, \ldots, t_m]$-module with basis $B_C$ and so the morphism

$$\pi : \text{Spec}(\tilde{A}) \to \mathbb{A}^m$$

is flat. In particular, $\pi$ defines a flat family with generic fiber $\text{Spec}(A)$ and for every face $\tau \subseteq C$ there exists $a_\tau \in \mathbb{A}^m$ and a special fiber $\pi^{-1}(a_\tau) \cong \text{Spec}(A_\tau)$. 


Let $\tilde{A} := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J}$, $A_\tau = \mathbb{C}[x_1, \ldots, x_n]/\text{in}_\tau(J)$.

**Theorem (B.–Mohammadi–Nájera Chávez)**

$\tilde{A}$ is a free $\mathbb{C}[t_1, \ldots, t_m]$-module with basis $B_C$ and so the morphism

$$\pi : \text{Spec}(\tilde{A}) \rightarrow \mathbb{A}^m$$

is flat. In particular, $\pi$ defines a flat family with generic fiber $\text{Spec}(A)$ and for every face $\tau \subseteq \mathbb{C}$ there exists $a_\tau \in \mathbb{A}^m$ and a special fiber $\pi^{-1}(a_\tau) \cong \text{Spec}(A_\tau)$.

**Example**

$$\tilde{A} = \mathbb{C}[t_1, t_2][x_1, x_2, x_3]/\langle t_1^6 x_1^2 x_2^2 + t_1^6 x_1^4 + x_2 x_3^3 \rangle.$$
Toric degenerations

$GF(J)$ contains a subfan of dimension $\dim_{\text{Krull}} A$ called the tropicalization of $J$.
Toric degenerations

$GF(J)$ contains a subfan of dimension $\dim_{\text{Krull}} A$ called the
\textit{tropicalization of }$J$

\begin{align*}
Trop(J) &:= \{ w \in \mathbb{R}^n \mid \text{in}_w(J) \not\subseteq \text{monomials} \}.
\end{align*}
Toric degenerations

$GF(J)$ contains a subfan of dimension $\dim_{\operatorname{Krull}} A$ called the tropicalization of $J$

$$\operatorname{Trop}(J) := \{ w \in \mathbb{R}^n \mid \operatorname{in}_w(J) \not\approx \text{monomials} \}.$$
Toric degenerations

\( GF(J) \) contains a subfan of dimension \( \dim_{\text{Krull}} A \) called the tropicalization of \( J \)

\[
\text{Trop}(J) := \{ w \in \mathbb{R}^n \mid \text{in}_w(J) \not\in \text{monomials} \}.
\]

**Corollary (B.–Mohammadi–Nájera Chávez)**

Consider the fan \( \Sigma := C \cap \text{Trop}(J) \).
Toric degenerations

\( GF(J) \) contains a subfan of dimension \( \dim_{\text{Krull}} A \) called the tropicalization of \( J \)

\[
\text{Trop}(J) := \{ w \in \mathbb{R}^n \mid \text{in}_w(J) \not\equiv \text{monomials} \}.
\]

**Corollary (B.–Mohammadi–Nájera Chávez)**

Consider the fan \( \Sigma := C \cap \text{Trop}(J) \). If there exists \( \tau \in \Sigma \) with \( \text{in}_\tau(J) \) binomial and prime,
Toric degenerations

$GF(J)$ contains a subfan of dimension $\dim_{\text{Krull}} A$ called the tropicalization of $J$

$$\text{Trop}(J) := \{ w \in \mathbb{R}^n \mid \text{in}_w(J) \not\in \text{monomials} \}.$$  

**Corollary (B.–Mohammadi–Nájera Chávez)**

Consider the fan $\Sigma := C \cap \text{Trop}(J)$. If there exists $\tau \in \Sigma$ with $\text{in}_\tau(J)$ binomial and prime, then the family

$$\pi : \text{Spec}(\tilde{A}) \to \mathbb{A}^m$$
$GF(J)$ contains a subfan of dimension $\dim_K A$ called the \textit{tropicalization of $J$}

\[
\text{Trop}(J) := \{ w \in \mathbb{R}^n \mid \text{in}_w(J) \not\subseteq \text{monomials} \}.
\]

\textbf{Corollary (B.–Mohammadi–Nájera Chávez)}

Consider the fan $\Sigma := C \cap \text{Trop}(J)$. If there exists $\tau \in \Sigma$ with $\text{in}_\tau(J)$ binomial and prime, then the family

\[
\pi : \text{Spec}(\tilde{A}) \to \mathbb{A}^m
\]

contains toric fibers isomorphic to $\text{Spec}(A_{\tau})$ (affine toric scheme).
Let $C \in \text{GF}(J)$ be a maximal cone, $X_C$ the affine toric variety and $p_C : \mathbb{A}^m \to X_C$ the universal torsor of $X_C$ (by Cox construction)
Toric families

Let $C \in \text{GF}(J)$ be a maximal cone, $X_C$ the affine toric variety and $p_C : \mathbb{A}^m \to X_C$ the universal torsor of $X_C$ (by Cox construction).

By work of Kaveh–Manon there exists a (piecewise linear) valuation on $A$ defined by $C$. Denote its Rees algebra by $\mathcal{R}_C$. By Theorem (Kaveh–Manon) the Rees algebra $\mathcal{R}_C$ is a direct sum of line bundles over $X_C$ and $\psi : \text{Spec}(\mathcal{R}_C) \to X_C$ defines a toric family with generic fiber $\text{Spec}(A)$, and $\psi^{-1}(p) \sim \text{Spec}(A_\tau)$ for all $p \in O_{\tau}$ torus orbit of a face $\tau \subseteq C$. 

Toric families

Let $C \in \text{GF}(J)$ be a maximal cone, $X_C$ the affine toric variety and $p_C : \mathbb{A}^m \to X_C$ the universal torsor of $X_C$ (by Cox construction).

By work of Kaveh–Manon there exists a (piecewise linear) valuation on $A$ defined by $C$. Denote its Rees algebra by $\mathcal{R}_C$.

**Theorem (Kaveh–Manon)**

The Rees algebra $\mathcal{R}_C$ is a direct sum of line bundles over $X_C$ and $\psi : \text{Spec}(\mathcal{R}_C) \to X_C$ defines a toric family with
Let $C \in GF(J)$ be a maximal cone, $X_C$ the affine toric variety and $p_C : \mathbb{A}^m \to X_C$ the universal torsor of $X_C$ (by Cox construction).

By work of Kaveh–Manon there exists a (piecewise linear) valuation on $A$ defined by $C$. Denote its Rees algebra by $R_C$.

**Theorem (Kaveh–Manon)**

The Rees algebra $R_C$ is a direct sum of line bundles over $X_C$ and $\psi : \text{Spec}(R_C) \to X_C$ defines a **toric family** with

- generic fiber $\text{Spec}(A)$, and
- $\psi^{-1}(p) \cong \text{Spec}(A_\tau)$ for all $p \in O_\tau$ torus orbit of a face $\tau \subseteq C$. 

Let \( C \in \text{GF}(J) \) be a maximal cone, \( X_C \) the affine toric variety and \( p_C : \mathbb{A}^m \to X_C \) the universal torsor of \( X_C \) (by Cox construction).
Let \( C \in \text{GF}(J) \) be a maximal cone, \( X_C \) the affine toric variety and \( p_C : \mathbb{A}^m \to X_C \) the universal torsor of \( X_C \) (by Cox construction).

**Theorem (B–Mohammadi–Nájera Chávez)**

*The flat family defined by \( \pi : \text{Spec}(\tilde{A}) \to \mathbb{A}^m \) fits into the following pullback diagram:*
Let \( C \in GF(J) \) be a maximal cone, \( X_C \) the affine toric variety and \( p_C : \mathbb{A}^m \to X_C \) the universal torsor of \( X_C \) (by Cox construction).

**Theorem (B–Mohammadi–Nájera Chávez)**

The flat family defined by \( \pi : \text{Spec}(\tilde{A}) \to \mathbb{A}^m \) fits into the following pullback diagram:

\[
\begin{array}{ccc}
\text{Spec}(\mathcal{R}_C) & \xleftarrow{\psi} & \text{Spec}(\tilde{A}) \\
\downarrow \psi & & \downarrow \pi \\
X_C & \xleftarrow{p_C} & \mathbb{A}^m
\end{array}
\]

where \( \psi : \text{Spec}(\mathcal{R}_C) \to X_C \) is Kaveh–Manon’s toric family.
A cluster algebra\(^2\) \(A \subset \mathbb{C}(x_1, \ldots, x_n)\) is a commutative ring defined recursively by
A *cluster algebra*\(^2\) \(A \subset \mathbb{C}(x_1, \ldots, x_n)\) is a commutative ring defined recursively by

1. **seeds**: maximal sets of algebraically independent algebra generators,

\(^2\text{Defined by Fomin–Zelevinsky.}\)
A \textit{cluster algebra} \(^2\) \(A \subset \mathbb{C}(x_1, \ldots, x_n)\) is a commutative ring defined recursively by

\begin{itemize}
  \item \textit{seeds}: maximal sets of algebraically independent algebra generators, its elements are called \textit{cluster variables};
\end{itemize}

\(^2\)Defined by Fomin–Zelevinsky.
Application: universal coefficients for cluster algebras

A *cluster algebra*\(^2\) \(A \subset \mathbb{C}(x_1, \ldots, x_n)\) is a commutative ring defined recursively by

1. **seeds**: maximal sets of algebraically independent algebra generators, its elements are called *cluster variables*;
2. **mutation**: an operation to create a new seed from a given one by replacing one element.

\(^2\)Defined by Fomin–Zelevinsky.
A cluster algebra\(^2\) \(A \subset \mathbb{C}(x_1, \ldots, x_n)\) is a commutative ring defined recursively by

1. **seeds**: maximal sets of algebraically independent algebra generators, its elements are called cluster variables;

2. **mutation**: an operation to create a new seed from a given one by replacing one element.

\[\mapsto\] the cluster structure is encoded in a simplicial complex called the cluster complex (seeds \(\leftrightarrow\) maximal simplices).

\(^2\)Defined by Fomin–Zelevinsky.
Example: Grassmannian $\text{Gr}_2(\mathbb{C}^5)$

The homogeneous coordinate ring of $\text{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding is a cluster algebra:
Example: Grassmannian $\text{Gr}_2(\mathbb{C}^5)$

The homogeneous coordinate ring of $\text{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding is a cluster algebra: seeds $\leftrightarrow$ triangulations of a 5-gon
The homogeneous coordinate ring of $\text{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding is a cluster algebra: seeds $\leftrightarrow$ triangulations of a 5-gon
Example: Grassmannian $\text{Gr}_2(\mathbb{C}^5)$

The homogeneous coordinate ring of $\text{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding is a cluster algebra: seeds $\leftrightarrow$ triangulations of a 5-gon
Example: Grassmannian $\text{Gr}_2(\mathbb{C}^5)$

The homogeneous coordinate ring of $\text{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding is a cluster algebra: seeds $\leftrightarrow$ triangulations of a 5-gon
Example: Grassmannian $\text{Gr}_2(\mathbb{C}^5)$

The homogeneous coordinate ring of $\text{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding is a cluster algebra: seeds $\leftrightarrow$ triangulations of a 5-gon
Example: Grassmannian $\text{Gr}_2(\mathbb{C}^5)$

The homogeneous coordinate ring of $\text{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding is a cluster algebra: seeds $\leftrightarrow$ triangulations of a 5-gon
Consider the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$ with Plücker embedding.  

\[ 3 \text{ Assume } k \leq \left\lfloor \frac{n}{2} \right\rfloor. \]
Consider the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$ with Plücker embedding$^3$. Then its homogeneous coordinate ring

$$A_{k,n} = \mathbb{C} [p_J \mid J = \{j_1, \ldots, j_k\} \subset [n]] / I_{k,n}$$

is a cluster algebra [Scott06].

---

$^3$Assume $k \leq \lfloor \frac{n}{2} \rfloor$. 
Consider the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$ with Plücker embedding. Then its homogeneous coordinate ring

$$A_{k,n} = \mathbb{C} [p_J \mid J = \{j_1, \ldots, j_k\} \subset [n]] / I_{k,n}$$

is a cluster algebra \cite{Scott06}. $k \leq 2$ Plücker coordinates = cluster variables.
Consider the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$ with Plücker embedding. Then its homogeneous coordinate ring

$$A_{k,n} = \mathbb{C}[p_J \mid J = \{j_1, \ldots, j_k\} \subset [n]] / I_{k,n}$$

is a cluster algebra [Scott06].

$k \leq 2$ Plücker coordinates $= \text{cluster variables}$.  
$k \geq 3$ Plücker coordinates $\subsetneq \text{cluster variables}$.

---

3 Assume $k \leq \lfloor \frac{n}{2} \rfloor$. 
Consider the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$ with Plücker embedding. Then its homogeneous coordinate ring

$$A_{k,n} = \mathbb{C} [p_J | J = \{j_1, \ldots, j_k\} \subset [n]] / I_{k,n}$$

is a cluster algebra \cite{Scott06}.

- $k \leq 2$ Plücker coordinates $=$ cluster variables.
- $k \geq 3$ Plücker coordinates $\subsetneq$ cluster variables.
- $k = 2$ or $k = 3$, $n \in \{6, 7, 8\}$ finitely many seeds;

---

$^3$Assume $k \leq \left\lfloor \frac{n}{2} \right\rfloor$. 
Consider the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$ with Plücker embedding\(^3\). Then its homogeneous coordinate ring

$$A_{k,n} = \mathbb{C} \left[ p_J \mid J = \{j_1, \ldots, j_k\} \subset [n] \right] / I_{k,n}$$

is a cluster algebra [Scott06].

- $k \leq 2$ Plücker coordinates = cluster variables.
- $k \geq 3$ Plücker coordinates $\subsetneq$ cluster variables.
- $k = 2$ or $k = 3$, $n \in \{6, 7, 8\}$ finitely many seeds; otherwise infinitely many seeds.

\(^3\)Assume $k \leq \lfloor \frac{n}{2} \rfloor$. 

Fix a seed $s$, then $A$ can be endowed with principal coefficients at the seed $s$.\footnote{Due to Gross–Hacking–Keel–Kontsevich.}
Fix a seed $s$, then $A$ can be endowed with \textit{principal coefficients at the seed $s$}

$$A_s^{\text{prin}} \subset \mathbb{C}[t_1, \ldots, t_n](x_1, \ldots, x_n).$$

\footnote{Due to Gross–Hacking–Keel–Kontsevich.}
Fix a seed $s$, then $A$ can be endowed with *principal coefficients at the seed $s*

$$A_s^{\text{prin}} \subset \mathbb{C}[t_1, \ldots, t_n](x_1, \ldots, x_n).$$

Under some technical assumptions:

1. $A_s^{\text{prin}}$ has a $\mathbb{C}[t_1, \ldots, t_n]$-basis called $\vartheta$-basis$^4$, which is independent of $s$;

---

$^4$Due to Gross–Hacking–Keel–Kontsevich.
Fix a seed $s$, then $A$ can be endowed with principal coefficients at the seed $s$

$$A_{s}^{\text{prin}} \subset \mathbb{C}[t_1, \ldots, t_n](x_1, \ldots, x_n).$$

Under some technical assumptions:

1. $A_{s}^{\text{prin}}$ has a $\mathbb{C}[t_1, \ldots, t_n]$-basis called $\vartheta$-basis\(^4\), which is independent of $s$;

2. if $A$ is the homogeneous coordinate ring of a projective variety $X$ then $A_{s}^{\text{prin}}$ defines a toric degeneration of $X$ to $X_{s,0}$.

\(^4\)Due to Gross–Hacking–Keel–Kontsevich.
Fix a seed $s$, then $A$ can be endowed with \textit{principal coefficients at the seed} $s$

$$A_s^{\text{prin}} \subset \mathbb{C}[t_1, \ldots, t_n](x_1, \ldots, x_n).$$

Under some technical assumptions:

1. $A_s^{\text{prin}}$ has a $\mathbb{C}[t_1, \ldots, t_n]$-basis called $\vartheta$-\textit{basis}$^4$, which is independent of $s$;

2. if $A$ is the homogeneous coordinate ring of a projective variety $X$ then $A_s^{\text{prin}}$ defines a toric degeneration of $X$ to $X_{s,0}$.

$\rightsquigarrow$ All these degenerations \textit{share the} $\vartheta$-\textit{basis}, i.e. $A_s^{\text{prin}} = \bigoplus_{\vartheta \in \Theta} \vartheta$ for all $s$.

$^4$Due to Gross–Hacking–Keel–Kontsevich.
Now assume $A$ has finitely many seeds.
Now assume $A$ has finitely many seeds.

Algebraically, we can endow $A$ with *universal coefficients*: 

$$A_{\text{univ}} \subset \mathbb{C}[t_1, \ldots, t_N](x_1, \ldots, x_n),$$

where $N$ is the number of cluster variables.
Now assume $A$ has finitely many seeds.

Algebraically, we can endow $A$ with \textit{universal coefficients}:

$$A_{\text{univ}} \subset \mathbb{C}[t_1, \ldots, t_N](x_1, \ldots, x_n),$$

where $N$ is the number of cluster variables.
Application: Universal coefficients for cluster algebras

Now assume $A$ has finitely many seeds.

Algebraically, we can endow $A$ with \textit{universal coefficients}:

$$A^{\text{univ}} \subset \mathbb{C}[t_1, \ldots, t_N](x_1, \ldots, x_n),$$

where $N$ is the number of cluster variables. Moreover, we have a unique \textit{specialization map} for every seed $s$: 
Now assume $A$ has finitely many seeds. Algebraically, we can endow $A$ with \textit{universal coefficients}:

$$A^{\text{univ}} \subset \mathbb{C}[t_1, \ldots, t_N](x_1, \ldots, x_n),$$

where $N$ is the number of cluster variables. Moreover, we have a unique \textit{specialization map} for every seed $s$:

$$A^{\text{univ}} \rightarrow A_s^{\text{prin}}.$$
Now assume $A$ has finitely many seeds.

Algebraically, we can endow $A$ with universal coefficients:

$$A^{\text{univ}} \subset \mathbb{C}[t_1, \ldots, t_N](x_1, \ldots, x_n),$$

where $N$ is the number of cluster variables. Moreover, we have a unique specialization map for every seed $s$:

$$A^{\text{univ}} \to A^{\text{prin}}_s.$$

$A^{\text{univ}}$ knows all toric degenerations $X_{s,0}$.
Now assume $A$ has finitely many seeds.

Algebraically, we can endow $A$ with *universal coefficients*:

$$A^\text{univ} \subset \mathbb{C}[t_1, \ldots, t_N](x_1, \ldots, x_n),$$

where $N$ is the number of cluster variables. Moreover, we have a unique *specialization map* for every seed $s$:

$$A^\text{univ} \rightarrow A_s^\text{prin}.$$

- $A^\text{univ}$ knows all toric degenerations $X_{s,0}$.
- $A^\text{univ}$ is defined only recursively.
Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$

Consider $\text{Gr}_k(\mathbb{C}^n)$ for $(k, n) \in \{(2, n), (3, 6)\}$. 
Consider $\text{Gr}_k(\mathbb{C}^n)$ for $(k, n) \in \{(2, n), (3, 6)\}$. Then there exists a presentation

$$A_{k,n} \cong \mathbb{C}[\text{cluster variables}]/J_{k,n}.$$
Consider $\text{Gr}_k(\mathbb{C}^n)$ for $(k, n) \in \{(2, n), (3, 6)\}$. Then there exists a presentation

$$A_{k,n} \cong \mathbb{C}[\text{cluster variables}]/J_{k,n}.$$ 

$\text{Gr}_2(\mathbb{C}^n)$: \{cluster variables\} = \{Plücker coordinates\} and $J_{2,n} = l_{2,n}$.
Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$

Consider $\text{Gr}_k(\mathbb{C}^n)$ for $(k, n) \in \{(2, n), (3, 6)\}$. Then there exists a presentation

$$A_{k,n} \cong \mathbb{C}[\text{cluster variables}]/J_{k,n}.$$ 

$\text{Gr}_2(\mathbb{C}^n)$: \{cluster variables\} = \{Plücker coordinates\} and $J_{2,n} = I_{2,n}$;

$\text{Gr}_3(\mathbb{C}^6)$: \{cluster variables\} = \{Plücker coordinates, $x$, $y$\},

$J_{3,6}$ is homogeneous w.r.t. non-standard grading $\deg(x) = \deg(y) = 2$ and $\deg(p_{ijk}) = 1$, eliminating $x$ and $y$ from $J_{3,6}$ gives $I_{3,6}$. 

Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$

Consider $\text{Gr}_k(\mathbb{C}^n)$ for $(k, n) \in \{(2, n), (3, 6)\}$. Then there exists a presentation

$$A_{k,n} \cong \mathbb{C}[\text{cluster variables}]/J_{k,n}.$$ 

**$\text{Gr}_2(\mathbb{C}^n)$**: $\{\text{cluster variables}\} = \{\text{Plücker coordinates}\}$ and $J_{2,n} = I_{2,n}$;

**$\text{Gr}_3(\mathbb{C}^6)$**: $\{\text{cluster variables}\} = \{\text{Plücker coordinates}, x, y\}$, $J_{3,6}$ is homogeneous w.r.t. non-standard grading $\deg(x) = \deg(y) = 2$ and $\deg(p_{ijk}) = 1$,
Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$

Consider $\text{Gr}_k(\mathbb{C}^n)$ for $(k, n) \in \{(2, n), (3, 6)\}$. Then there exists a presentation

$$A_{k,n} \cong \mathbb{C}[\text{cluster variables}]/J_{k,n}.$$ 

**Gr$_2(\mathbb{C}^n)$:** \{cluster variables\} = \{Plücker coordinates\} and $J_{2,n} = l_{2,n}$.

**Gr$_3(\mathbb{C}^6)$:** \{cluster variables\} = \{Plücker coordinates, $x, y$\}, $J_{3,6}$ is homogeneous w.r.t. non-standard grading $\deg(x) = \deg(y) = 2$ and $\deg(p_{ijk}) = 1$, eliminating $x$ and $y$ from $J_{3,6}$ gives $l_{3,6}$. 
Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$

Theorem (B.–Mohammadi–Nájera Chávez)

There exists a unique maximal cone $C$ in the Gröbner fan of $J_{k,n}$ for $(k,n) \in \{(2,n), (3,6)\}$ such that

1. we have a canonical isomorphism $\tilde{A}_{k,n} \cong A_{\text{univ}}^{k,n}$ identifying universal coefficients with rays of $C$;
2. the standard monomial basis $B_C$ for $A_{k,n}$ (and $\tilde{A}_{k,n}$) coincides with the $\vartheta$-basis for $A_{k,n}$ (and $A_{\text{univ}}^{k,n}$);
3. the Stanley–Reisner ideal of the cluster complex is the initial ideal in $C(J_{k,n})$;
4. for every seed $s$ there exists a face $\tau_s \subset C \cap \text{Trop}(J_{k,n})$ such that the toric variety $X_s,0$ is isomorphic to $\text{Spec}(A_{\tau_s})$. 

Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$

Theorem (B.–Mohammadi–Nájera Chávez)

There exists a unique maximal cone $C$ in the Gröbner fan of $J_{k,n}$ for $(k, n) \in \{(2, n), (3, 6)\}$ such that

1. we have a canonical isomorphism $\tilde{A}_{k,n} \sim A_{\text{univ}}$ identifying universal coefficients with rays of $C$;
2. the standard monomial basis $B_{\mathbb{C}}$ for $A_{k,n}$ (and $\tilde{A}_{k,n}$) coincides with the $\vartheta$-basis for $A_{k,n}$ (and $A_{\text{univ}}$);
3. the Stanley–Reisner ideal of the cluster complex is the initial ideal in $\mathbb{C}(J_{k,n})$;
4. for every seed $s$ there exists a face $\tau_s \subset C \cap \text{Trop}(J_{k,n})$ such that the toric variety $X_s, 0$ is isomorphic to $\text{Spec}(A_{\tau_s})$. 
Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$

**Theorem (B.–Mohammadi–Nájera Chávez)**

There exists a unique maximal cone $C$ in the Gröbner fan of $J_{k,n}$ for $(k, n) \in \{(2, n), (3, 6)\}$ such that

1. we have a canonical isomorphism $\tilde{A}_{k,n} \cong A_{k,n}^{\text{univ}}$ identifying universal coefficients with rays of $C$;

2. the standard monomial basis $B$ for $A_{k,n}$ (and $\tilde{A}_{k,n}$) coincides with the $\vartheta$-basis for $A_{k,n}$ (and $A_{k,n}^{\text{univ}}$);

3. the Stanley–Reisner ideal of the cluster complex is the initial ideal in $\mathbb{C}(J_{k,n})$;

4. for every seed $s$ there exists a face $\tau_s \subset C \cap \text{Trop}(J_{k,n})$ such that the toric variety $X_s, 0$ is isomorphic to $\text{Spec}(A_{\tau_s})$. 

Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$

Theorem (B.–Mohammadi–Nájera Chávez)

There exists a unique maximal cone $C$ in the Gröbner fan of $J_{k,n}$ for $(k, n) \in \{(2, n), (3, 6)\}$ such that

1. we have a canonical isomorphism $\tilde{A}_{k,n} \cong A_{k,n}^{\text{univ}}$ identifying universal coefficients with rays of $C$;

2. the standard monomial basis $\mathbb{B}_C$ for $A_{k,n}$ (and $\tilde{A}$) coincides with the $\vartheta$-basis for $A_{k,n}$ (and $A_{k,n}^{\text{univ}}$);
Theorem (B.–Mohammadi–Nájera Chávez)

There exists a unique maximal cone $C$ in the Gröbner fan of $J_{k,n}$ for $(k, n) \in \{(2, n), (3, 6)\}$ such that

1. we have a canonical isomorphism $\tilde{A}_{k,n} \cong A_{k,n}^{\text{univ}}$ identifying universal coefficients with rays of $C$;
2. the standard monomial basis $\mathcal{B}_C$ for $A_{k,n}$ (and $\tilde{A}$) coincides with the $\vartheta$-basis for $A_{k,n}$ (and $A_{k,n}^{\text{univ}}$);
3. the Stanley–Reisner ideal of the cluster complex is the initial ideal $\text{in}_C(J_{k,n})$;
**Application: Grassmannians $\text{Gr}_2(\mathbb{C}^n)$ and $\text{Gr}_3(\mathbb{C}^6)$**

**Theorem (B.–Mohammadi–Nájera Chávez)**

There exists a unique maximal cone $C$ in the Gröbner fan of $J_{k,n}$ for $(k, n) \in \{(2, n), (3, 6)\}$ such that

1. we have a canonical isomorphism $\tilde{A}_{k,n} \cong A^\text{univ}_{k,n}$ identifying universal coefficients with rays of $C$;

2. the standard monomial basis $\mathbb{B}_C$ for $A_{k,n}$ (and $\tilde{A}$) coincides with the $\vartheta$-basis for $A_{k,n}$ (and $A^\text{univ}_{k,n}$);

3. the Stanley–Reisner ideal of the cluster complex is the initial ideal $\text{in}_C(J_{k,n})$;

4. for every seed $s$ there exists a face $\tau_s \subset C \cap \text{Trop}(J_{k,n})$ such that the toric variety $X_{s,0}$ is isomorphic to $\text{Spec}(A_{\tau_s})$. 

Further directions

(i) Expect to extend the application to all (graded) cluster algebras of finite type (with frozen directions).

N. Ilten showed that the type $D_n$ associahedron is unobstructed $\Rightarrow$ the Stanley–Reisner scheme $P(D_n)$ of the cluster complex is a smooth point in its Hilbert scheme.

Corollary $Gr_3(C_6)$, a cone over $P(D_4)$ (namely Proj($A_{C_6}$)) and the toric schemes Proj($A_{s, 0}$) for all seeds $s$ all lie on the same component of the Hilbert scheme.

Question: Can we obtain similar results for arbitrary Grassmannians?
Further directions

(i) Expect to extend the application to all (graded) cluster algebras of finite type (with frozen directions).

(ii) N. Ilten showed that the type $D_n$ associahedron is unobstructed.
Further directions

(i) Expect to extend the application to all (graded) cluster algebras of finite type (with frozen directions).

(ii) N. Ilten showed that the type $\mathcal{D}_n$ associahedron is *unobstructed* ⇒ the Stanley–Reisner scheme $\mathbb{P}(\mathcal{D}_n)$ of the cluster complex is a smooth point in its Hilbert scheme.
Further directions

(i) Expect to extend the application to all (graded) cluster algebras of finite type (with frozen directions).

(ii) N. Ilten showed that the type $D_n$ associahedron is unobstructed $\Rightarrow$ the Stanley–Reisner scheme $\mathbb{P}(D_n)$ of the cluster complex is a smooth point in its Hilbert scheme.

Corollary

$Gr_3(\mathbb{C}^6)$, a cone over $\mathbb{P}(D_4)$ (namely $\text{Proj}(A_C)$) and the toric schemes $\text{Proj}(A_{s,0})$ for all seeds $s$ all lie on the same component of the Hilbert scheme.
Further directions

(i) Expect to extend the application to all (graded) cluster algebras of finite type (with frozen directions).

(ii) N. Ilten showed that the type $D_n$ associahedron is unobstructed $\Rightarrow$ the Stanley–Reisner scheme $\mathbb{P}(D_n)$ of the cluster complex is a smooth point in its Hilbert scheme.

Corollary

Gr$_3(\mathbb{C}^6)$, a cone over $\mathbb{P}(D_4)$ (namely $\text{Proj}(A_C)$) and the toric schemes $\text{Proj}(A_{s,0})$ for all seeds $s$ all lie on the same component of the Hilbert scheme.

Question: Can we obtain similar results for arbitrary Grassmannians?
References


