# Weierstrass sets on finite graphs

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Goal: tropical analogues of Weierstrass semigroups

$X$ smooth projective curve of genus $g$

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**Theorem (Weierstrass gap theorem)**

$$|\mathbb{N} \setminus H(P)| = g$$

**numerical semigroup** = cofinite submonoid of $\mathbb{N}$

**Question (Hurwitz 1893)**

*Which numerical semigroups are Weierstrass semigroups?*

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Recent work of Cotterill, Pflueger, Zhang (2022) certfies Weierstrass-realizability of some numerical semigroups.

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| | |
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**graph** := finite connected multigraph with no loops
**simple graph** := graph with no multiple edges

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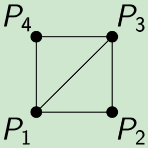
**Example**

$$D = 3P_1 + P_2 + P_3$$



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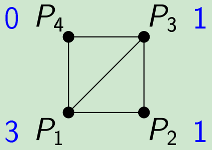

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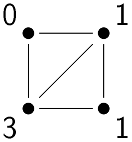


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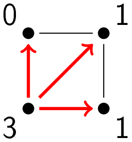


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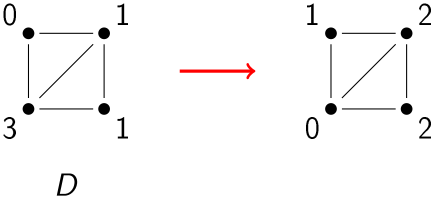

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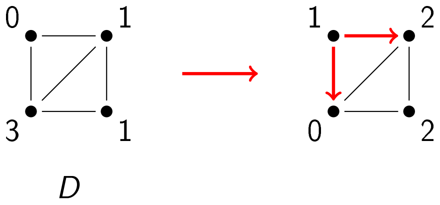

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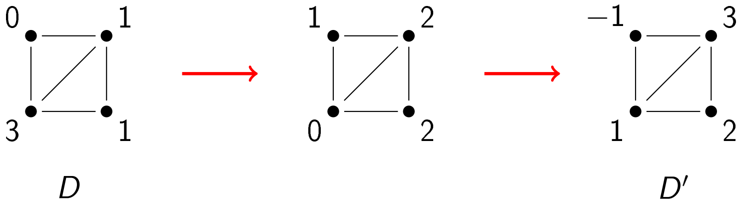

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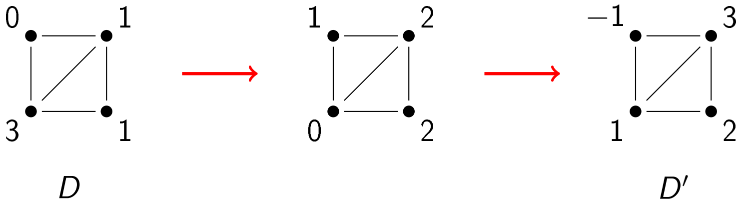
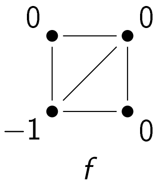



$$D \sim D'$$

# Principal divisors

Let $f : V(G) \to \mathbb{Z}$



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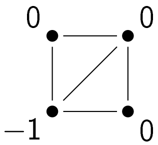


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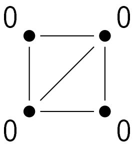


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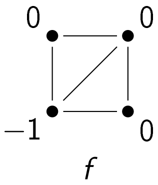


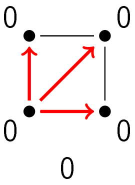

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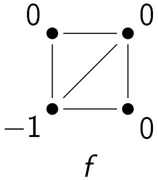


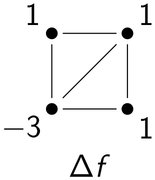

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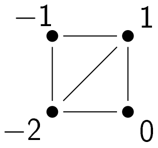


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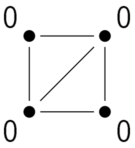


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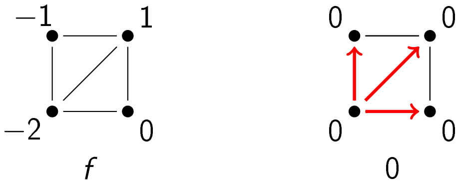

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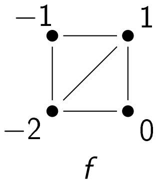


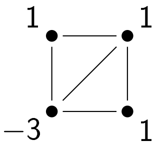

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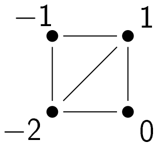

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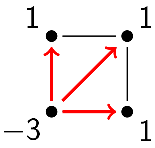

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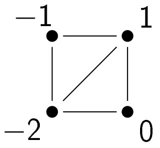


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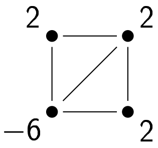

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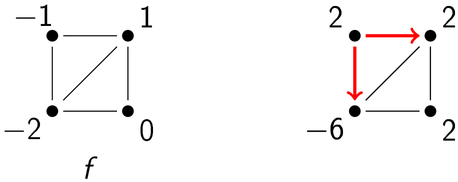

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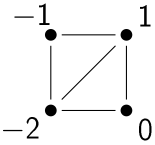


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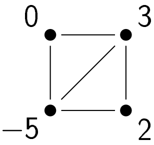

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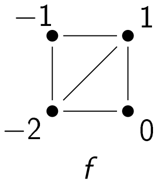


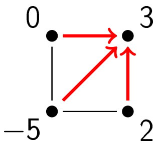

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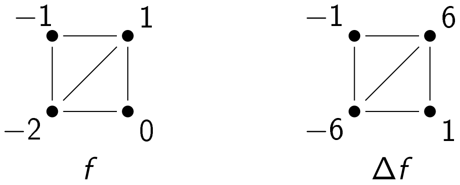

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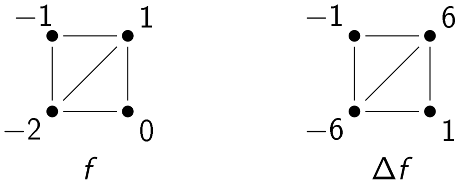


Two divisors $D, D' \in \mathrm{Div}(G)$ are **linearly equivalent** if

$$D - D' = \Delta f \quad \text{for some } f : V(G) \to \mathbb{Z}.$$

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Denote by $\mathrm{Div}_+^d(G)$ the set of effective of divisors of degree $d$.

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The **rank** of $D$ is $-1$ if $|D| = \emptyset$, otherwise

$$r(D) = \max\{d \in \mathbb{N} : |D - E| \neq \emptyset, \forall E \in \mathrm{Div}_+^d(G)\}.$$

# Weierstrass sets

Recall (for curves):

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**Definition (Kang, Matthews, Peachey 2020)**

Let $G$ be a graph and let $P \in V(G)$.

**Rank Weierstrass set**:

$$H_r(P) = \{n \in \mathbb{N} : r(nP) > r((n-1)P)\}$$

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For curves: $H_r(P) = H_f(P) = H(P)$,

For graphs: $H_f(P) \setminus H_r(P)$ can be arbitrarily large!

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The **genus** of a graph $G$ is $g = |E(G)| - |V(G)| + 1$.

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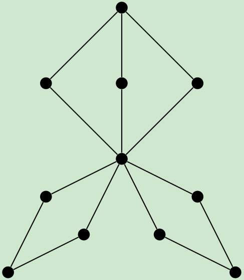
$H_f(P)$ is a semigroup, $H_r(P)$ is *not*.

## Example

Consider the following graph $G$



It is the vertex gluing of $K_{2,3}$ and two copies of $K_{2,2}$.
Let $P \in V(G)$ be the vertex of degree 7. Then

$$H_r(P) = \{0, 3, 5, 7\} \cup (8 + \mathbb{N}).$$

Note that $H_r(P)$ is not a semigroup $6 = 3 + 3 \notin H_r(P)$.

This result was conjectured by Kang, Matthews and Peachey:

**Theorem (B. 2022)**

*Let $G$ be a simple graph. For every $P \in V(G)$*

$$H_r(P) \subseteq H_f(P)$$

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Let $K_{n+1}$ be the complete graph on $n+1$ vertices.

**Lemma (Kang, Matthews, Peachey 2020)**

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$H_r(P) \subseteq H_f(P)$ and $|\mathbb{N} \setminus H_r(P)| = g(K_{n+1})$ imply:

**Corollary**

*For every $P \in V(K_{n+1})$* $\quad H_r(P) = H_f(P) = \langle n, n+1 \rangle$.

Let $K_{m,n}$ be the complete bipartite graph.

### Proposition

*For every* $P \in V(K_{m,n})$

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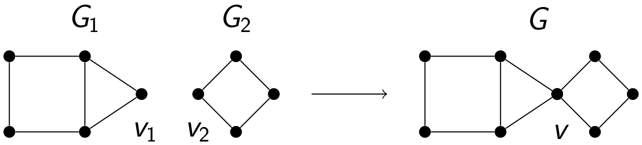
**Question**

*Under which conditions on $G$ we have $H_r(P) = H_f(P)$?*

# Vertex gluing

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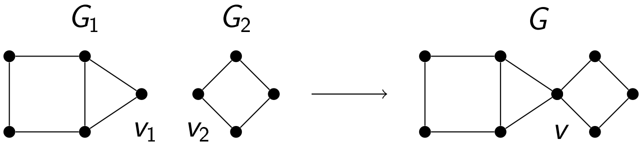
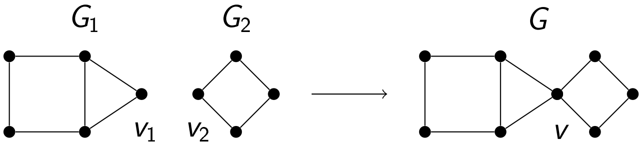



**Vertex gluing:** the graph $G$ obtained by identifying $v_1$ and $v_1$

**Proposition**

$$H_f^G(v) = H_f^{G_1}(v_1) + H_f^{G_2}(v_2)$$

# Vertex gluing




**Vertex gluing:** the graph $G$ obtained by identifying $v_1$ and $v_1$

**Proposition**

$$H_f^G(v) = H_f^{G_1}(v_1) + H_f^{G_2}(v_2)$$

**Theorem (B. 2022)**

| | | |
|---|---|---|
| *Functional Weierstrass sets of graphs* | $\longleftrightarrow$ | *submonoids of* $\mathbb{N}$ |
| *Functional Weierstrass sets of simple graphs* | $\longleftrightarrow$ | *numerical semigroups* |

Fix $P \in V(G)$, let $\lambda_P : \mathbb{N} \to \mathbb{N}$ defined by

$$\lambda_P(k) = \min\{n \in \mathbb{N} : r(nP) = k\}.$$

Note that $\lambda_P$ completely determines $H_r(P)$ and vice versa.

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**Proposition**

$$\lambda_v^G(k) = \max\{\lambda_{v_1}^{G_1}(k_1) + \lambda_{v_2}^{G_2}(k_2) : k_1 + k_2 = k\}$$

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### Proposition

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### Theorem (B. 2022)

*Let $e_1 \geq e_2 \geq \cdots \geq e_n \geq 0$ be integers, and set $s_i = \sum_{j=1}^{i} e_j$. There exists a simple graph $G$ with a vertex $P \in V(G)$ such that*

$$H_r(P) = \{0, s_1, \ldots, s_{n-2}\} \cup (s_{n-1} + \mathbb{N})$$

Thank you very much!