Nottingham COW

char $k = 0$

Log geometry and Chow theory

Motivation: Chow theory and bivariant theory

Given a scheme X we have the Chow groups

$A_\bullet(X)$ graded Abelian group consisting of closed integral subschemes of X up to rational equivalence

these are:

- Covariant and grading preserving for proper maps ie given $f: X \to Y$ proper we have

$$f_*: A_\bullet(X) \to A_\bullet(Y).$$

- Contravariant and codimension preserving for flat maps ie given $f: X \to Y$ flat rel dim d we have

$$f^*: A_\bullet(Y) \to A_{\bullet+d}(X).$$

Fulton + Macpherson define bivariant theory.

For a map $f: X \to Y$ the group $A^k(X \xrightarrow{f} Y)$ is spanned by collections of maps $\alpha^!$ for every diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \qquad \alpha^!: A_\bullet(V) \to A_{\bullet-k}(W)$$

Ask that these be compatible with proper pushforward, flat pullback and intersecting with Cartier divisors (Gysin maps).

Note these are composable:

$$A^\bullet(X \xrightarrow{f} Y) \otimes A^\bullet(Y \xrightarrow{g} Z) \to A^\bullet(X \xrightarrow{gf} Z)$$

Constructing elements of $A^\bullet(X \xrightarrow{f} Y)$ uses (normally) the virtual fundamental class construction. We embed

$$\begin{tikzcd} \mathfrak{C}_f \arrow[r, hook] & E \arrow[d, "p_E"] \\ & X \end{tikzcd}$$

E vector bundle stack on X

We take $0_E^!([\mathfrak{C}_f])$

Manolache: This gives you a bivariant class.

In log geometry rather than using $f: X \to Y$ we use an intermediate stack.

$$\begin{tikzcd} X \arrow[r] & \mathcal{L}og_Y \arrow[r] & Y \\ & \text{Log ét} \arrow[u] & \end{tikzcd}$$

Thm (Fulton - Macpherson) Poincaré duality

Given $f: X \to Y$ a map of schemes and $g: Y \to Z$ a smooth of relative dim d then the map

$$A^\bullet(X \xrightarrow{f} Y) \xrightarrow{g^*} A^{\bullet-d}(X \xrightarrow{gf} Z) \text{ is an isom.}$$

§ So what is a log scheme?

Defn: A log scheme X consists of:

- Underlying scheme $\underline{X}$.
- A sheaf of monoids M_X on X.
- A map of sheaves of monoids $\alpha_X: M_X \to \mathcal{O}_X$ such that $\alpha_X: \alpha_X^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$ is an isom.

So we can talk about $\overline{M}_X = M_X / \mathcal{O}_X^*$

A map of log schemes $X \xrightarrow{f} Y$ is:

- $\underline{f}: \underline{X} \to \underline{Y}$
- a map $f^\#$ fitting into the following diagram

$$\begin{tikzcd} f^{-1}M_Y \arrow[r, dashed, "f^\#"] \arrow[d, "f^{-1}\alpha_Y"'] \arrow[dr, phantom, "\circ"] & M_X \arrow[d, "\alpha_X"] \\ f^{-1}\mathcal{O}_Y \arrow[r, "f^*"'] & \mathcal{O}_X \end{tikzcd}$$

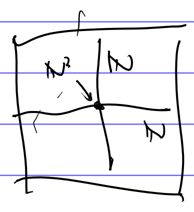
this gives a map $f^{-1}\bar{M}_y \to \bar{M}_x$ and we say f is strict if this is an isomorphism.

Toric varieties have a canonical log str given by:

P a fg integral sat monoid

$\underline{P} \xrightarrow{\alpha_P} \mathcal{O}_{\mathrm{Spec}\,k[P]}$ $\quad p \in \underline{P} \quad p \to z^p$

The log structure $\underline{P} \oplus_{\alpha^{-1}\mathcal{O}^*_{\mathrm{Spec}\,k[P]}} \mathcal{O}^*_{\mathrm{Spec}\,k[P]}$ is a log structure



Def: X is fs if ét locally we have charts:

$$\begin{array}{ccc} U & \xrightarrow{ch} & \mathrm{Spec}\,k[P] \\ {\scriptstyle ét}\downarrow & & \\ X & & \end{array}$$

P fg sat integral monoid, ch be strict.

Defn (K. Kato) $f: \tilde{X} \to X$ is a log blow up if locally on $\tilde{X}$ and X we have diagrams

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{ch_{\tilde{X}}} & Bl_I\, \mathrm{Spec}\,k[P] \\ f\downarrow & & \downarrow \\ X & \xrightarrow{ch_X} & \mathrm{Spec}\,k[P] \end{array}$$

I a toric ideal of $k[P]$.

Lemma: $\mathrm{Spec}\,k[P]$ smooth $\iff P \cong \mathbb{N}^r \iff \bar{M}_{P,p} \cong \mathbb{N}^r$

So define X is locally free if $\bar{M}_{X,x} \cong \mathbb{N}^{r_x}$

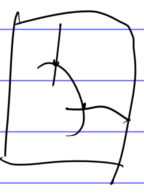
$f: \tilde{X} \to X$ is a locally free blow up if f is a log blow up and X and $\tilde{X}$ are locally free.

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & Bl_I\, \mathbb{A}^n \\ \downarrow & & \downarrow f_P \\ X & \longrightarrow & \mathbb{A}^n \end{array}$$

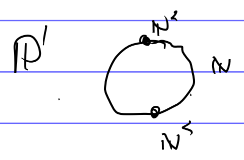
f_P has a bivariant $f_P^! \in A^0(Bl_I \mathbb{A}^n \to \mathbb{A}^n)$

glue to $f^! \in A^0(\tilde{X} \to X)$

Example



$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & Bl_0 \mathbb{A}^2 \\ \downarrow & & \downarrow \\ \underset{\mathbb{N}^2}{\bullet} & \longrightarrow & \mathbb{A}^2 \\ \uparrow & & \\ M_x \cong k^* \oplus \mathbb{N}^2 & & \end{array}$$



Defn: log Chow group $A_\bullet^+(X)$ is the lim over all $\tilde{X} \xrightarrow[\text{log blow}]{p} X$ with $\tilde{X}$ locally free, and all log blow ups between them

§ Take care with pushforward etc.

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\mathrm{II}} & \mathbb{P}^1 \\ \text{log blowup} \to \ \downarrow \mathrm{II} & & \downarrow \leftarrow \text{log blow up} \\ \mathbb{P}^1 & \longrightarrow & \bullet\, \mathbb{N}^2 \end{array}$$

This doesn't commute using the naive push forward definition, because the map $\mathbb{P}^1 \to \mathbb{N}^2$ is not integral

Thm (F. Kato) Given $f: X \to Y$ we can find a diagram

$$\begin{array}{ccccc} \bar{X} & \xrightarrow{\pi_{\hat{X}}} & \hat{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ & {\scriptstyle \pi_{\bar{X}}}\searrow & \downarrow \pi_{\tilde{X}} & & \downarrow \pi_{\tilde{Y}} \\ & & X & \xrightarrow{f} & Y \end{array}$$

$\hat{f}$ is integral

$\pi_{\hat{X}}$, $\pi_{\tilde{Y}}$ are log blow ups
$\pi_{\tilde{X}}$