SINGULARITY CONTENT

MOHAMMAD AKHTAR AND ALEXANDER KASPRZYK

Abstract. We show that a cyclic quotient surface singularity σ can be decomposed, in a precise sense, into a number of elementary T-singularities together with a cyclic quotient surface singularity called the residue of σ. A normal surface X with isolated cyclic quotient singularities $\{\sigma_i\}$ admits a $\mathbb{Q}$-Gorenstein partial smoothing to a surface with singularities given by the residues of the σ_i. We define the singularity content of a Fano lattice polygon P: this records the total number of elementary T-singularities and the residues of the corresponding toric Fano surface X_P. We express the degree of X_P in terms of the singularity content of P; give a formula for the Hilbert series of X_P in terms of singularity content; and show that singularity content is an invariant of P under mutation.

1. Introduction

Fano manifolds and mirror symmetry. This paper is motivated by a programme to classify Fano manifolds and orbifolds via mirror symmetry [10,11]. That programme suggests, roughly speaking, a correspondence between Fano manifolds and mutation-equivalence classes of lattice polytopes. Thus classifying Fano manifolds is closely related to the problem of classifying lattice polytopes up to mutation (see [1] and page 6 below). To approach this problem, which is of significant combinatorial interest in its own right, one needs invariants of polytopes under mutation. In this paper we give a new such invariant in the two-dimensional case: the singularity content of a lattice polygon P. We express the Hilbert series of the toric surface X_P in terms of the singularity content of P, and construct a $\mathbb{Q}$-Gorenstein (qG) deformation of X_P where the singularities of the general fibre are controlled, in a precise sense, by the singularity content of P. This latter result provides an important consistency check on the philosophy behind the Fano classification programme, which suggests that if a Fano manifold X corresponds under mirror symmetry to a Laurent polynomial f then there exists a degeneration from X to the singular toric variety X_Δ, where Δ is the Newton polytope of f, and that different Laurent polynomial mirrors to the same Fano manifold differ by mutation.

Singularity content. Let C be a two-dimensional rational cone and let X_C denote the corresponding affine toric surface singularity. Let u, v be primitive lattice points on the rays of C. Let ℓ, the *local index* of C, denote the lattice height of the line segment uv above the origin and let w, the *width* of C, denote the lattice length of $v-u$. Write $w = n\ell + \rho$ for $n, \rho \in \mathbb{Z}_{\geq 0}$ with $0 \leq \rho < \ell$. Then X_C is a T-singularity [16] if and only if $\rho = 0$, and we say that X_C is an *elementary T-singularity* if $n = 1$ and $\rho = 0$ (so $w = \ell$); these correspond to singularities of the form $\frac{1}{n\ell^2}(1, n\ell c - 1)$ and $\frac{1}{\ell^2}(1, \ell c - 1)$, respectively. Choose a crepant decomposition of C into a cone R, of width ρ and local index ℓ, and n other cones, each of width and local index ℓ (Notation 2.3). Then, up to lattice isomorphism, R depends only on C and not on the decomposition chosen (Proposition 2.4) and we give an explicit formula for R in terms of C. There is a qG-deformation of X_C such that the general fibre is the affine toric surface singularity X_R (Proposition 2.8). We call X_R the *residue* of C, and write it as $\mathrm{res}(C)$. Given a normal surface X with isolated cyclic quotient singularities $\{X_{C_i} : i \in I\}$ there exists a qG-deformation of X such that the general fibre is a surface with isolated singularities $\{\mathrm{res}(C_i) : i \in I\}$.

Let P be a Fano polygon and let X_P denote the corresponding toric Fano surface defined by the spanning fan Σ of P. For a cone C_i of Σ with width w_i and local index ℓ_i, write $w_i = n_i\ell_i + \rho_i$ with $0 \leq \rho_i < \ell_i$. The *singularity content* of P is the pair $(n, \mathcal{B})$ where $n = \sum_i n_i$ and $\mathcal{B}$ is the cyclically-ordered list $\{\mathrm{res}(C_i)\}_i$ with empty residues omitted. We compute the degree of X_P in terms of the

---

2010 *Mathematics Subject Classification.* 14M25 (Primary); 14B05, 14J45 (Secondary).

Key words and phrases. Del Pezzo surface, T-singularity, $\mathbb{Q}$-Gorenstein deformation, mutation.

Supported by EPSRC grants EP/I008128/1 and EP/N022513/1, and by ERC Starting Investigator Grant number 240123.

singularity content of P (Theorem 3.3) and express the Hilbert series of X_P, in the style of [17], as the sum of a leading term controlling the order of growth followed by contributions from the elements of $\mathcal{B}$ (Corollary 3.5). In Theorem 3.8 we show that the singularity content of P is invariant under mutation [1].

T-singularities, Minkowski sums, and deformations. The deformation theory of two-dimensional cyclic quotient singularities is completely understood, following the work of Kollár and Shepherd-Barron [16]. There they establish, amongst other results, a one-to-one correspondence between components of the versal deformation space of the two-dimensional cyclic quotient singularity X_C and certain partial resolutions of X_C called P-resolutions. (The decompositions of the cone C considered above also give partial resolutions of X_C but, since these are crepant, they are not P-resolutions.) The P-resolutions of two-dimensional cyclic quotient singularities have been studied extensively by, amongst others, Christophersen [9], Stevens [18], Altmann [7], and Ilten [14]. The techniques of Altmann and Ilten, which make use of toric geometry, can also be used to construct the qG-deformation of X_C given in Proposition 2.8 below. Altmann has described a close relationship between Minkowski decompositions of cones and the deformation theory of the corresponding toric singularities [3–6], and as a very special case of this has proven Corollary 2.7 from this point of view [5, §7.3]. Deformations of affine toric varieties can also be understood using the theory of T-varieties and polyhedral divisors: see, for example, [8, §16]. Applying this theory to the case of the two-dimensional cyclic quotient singularity X_C, one finds a deformation of X_C with general fibre given by the residue X_R, and thus an alternative proof of Proposition 2.8.

2. Singularity content of a cone

Let N be a lattice of rank two, and consider a strictly convex (or 'pointed') two-dimensional cone $C \subset N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. This cone corresponds to a cyclic quotient singularity σ. There are several standard ways of describing σ; we give one here, along with the dictionary that allows us to move between the description of the singularity and the cone C. The cyclic singularity σ is given by a quotient of $\mathbb{A}^2$ by the group μ_r of r-th roots of unity, acting via $(x, y) \mapsto (\xi^a x, \xi^b y)$, where ξ is a generator of μ_r, $0 < a, b \leq r$, and $\gcd\{a, r\} = \gcd\{b, r\} = 1$. We denote this singularity by $\frac{1}{r}(a, b)$. From the combinatorial viewpoint, set $C' := \mathrm{conv}\{(1, 0), (0, 1)\}$ equal to the positive quadrant in the rank two lattice $N' := \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(a, b)$, where we interpret $\frac{1}{r}(a, b)$ as a rational vector. Then there exists a lattice isomorphism $N' \cong N$ sending C' to C.

Definition 2.1. Let u and v be the primitive lattice vectors in N defined by the rays of C. Define the *width* $w \in \mathbb{Z}_{>0}$ of C to be the lattice length of $v - u$, and the *local index* (or *Gorenstein index*) $\ell \in \mathbb{Z}_{>0}$ of C to be the lattice height of the line segment uv above the origin.

Lemma 2.2 ([2, Proof of Proposition 3.9])**.** *A cone $C \subset N_{\mathbb{Q}}$ with corresponding singularity $\frac{1}{r}(a, b)$ has width $w = \gcd\{r, a + b\}$ and local index $\ell = r / \gcd\{r, a + b\}$.*

Notation 2.3. Given C, u, and v as above, and a non-negative integer m such that $m \leq w/\ell$, we define a sequence of lattice points $v_0, v_1, \ldots, v_{n+1}$ on uv as follows:

(i) $v_0 = u$ and $v_{n+1} = v$;
(ii) $v_{i+1} - v_i$ is a non-negative scalar multiple of $v - u$, for $i \in \{0, 1, \ldots, n\}$;
(iii) $v_{i+1} - v_i$ has lattice length ℓ for $i \in \{0, \ldots, \widehat{m}, \ldots, n\}$ (here $\widehat{m}$ means that the element m is omitted from the set);
(iv) $v_{m+1} - v_m$ has lattice length ρ, with $0 \leq \rho < \ell$.

The sequence $v_0, \ldots, v_{n+1}$ is uniquely determined by m and the choice of u. Note that $w = n\ell + \rho$. We consider the partition of C into subcones $C_i := \mathrm{cone}\{v_i, v_{i+1}\}$, $0 \leq i \leq n$.

Proposition 2.4. *Let $C \subset N_{\mathbb{Q}}$ be a two-dimensional cone of singularity type $\frac{1}{r}(1, a - 1)$. Let u, v be the primitive lattice vectors defined by the rays of C, ordered such that u, v, and $\frac{a-1}{r}u + \frac{1}{r}v$ generate N. Let $v_0, \ldots, v_{n+1}$ be as in Notation 2.3. Then:*

(i) *The lattice points $v_0, \ldots, v_{n+1}$ are primitive;*

(ii) *The subcones C_i, $0 \leq i < m$, are of singularity type* $\frac{1}{\ell^2}(1, \frac{\ell a}{w} - 1)$;
(iii) *If $\rho \neq 0$ then the subcone C_m is of singularity type* $\frac{1}{\rho\ell}(1, \frac{\rho a}{w} - 1)$;
(iv) *The subcones C_i, $m < i \leq n$, are of singularity type* $\frac{1}{\ell^2}(1, \frac{\ell \bar{a}}{w} - 1)$.

Here $\bar{a}$ is an integer satisfying $(a-1)(\bar{a}-1) \equiv 1 \pmod{r}$, and so exchanging the roles of u and v exchanges a and $\bar{a}$. Note that the singularity type of C_m depends only on C.

Proof. Without loss of generality we may assume that $u = (0,1)$, that $v = (r, 1-a)$, and that $m \neq 0$. The primitive vector in the direction $v - u$ is $(\alpha, \beta) := (\ell, -a/w)$. Thus $v_1 = (\alpha^2, 1 + \alpha\beta)$, and so v_1 is primitive. There exists a change of basis sending v_1 to $(0,1)$ and leaving (α, β) unchanged. This change of basis sends v_i to v_{i-1} for each $1 \leq i \leq m$. It follows that the lattice points v_i, $1 \leq i \leq m$, are primitive, and that the cones C_i, $1 \leq i \leq m$, are isomorphic. Since

$$\tfrac{1}{\alpha^2}(\alpha^2, 1 + \alpha\beta) - \tfrac{1+\alpha\beta}{\alpha^2}(0,1) = (1,0),$$

we have that C_1 has singularity type $\frac{1}{\alpha^2}(1, -1-\alpha\beta) = \frac{1}{\ell^2}(1, \frac{\ell a}{w} - 1)$. Since $r = w\ell$ (Lemma 2.2) we have that $\frac{\ell(a+kr)}{w} - 1 \equiv \frac{\ell a}{w} - 1 \pmod{\ell^2}$ for any integer k, so that the singularity only depends on the equivalence class of a modulo r. This proves (ii). Switching the roles of u and v proves (i) and (iv).

It remains to prove (iii). As before, we may assume that $u = (0,1)$ and $v = (r, 1-a)$. Consider the change of basis described above. After applying this m times, the cone C_{m+1} has primitive generators $(0,1)$ and $(\rho\alpha, 1 + \rho\beta)$. Since

$$\tfrac{1}{\rho\alpha}(\rho\alpha, 1 + \rho\beta) - \tfrac{1+\rho\beta}{\rho\alpha}(0,1) = (1,0),$$

we see that C_{m+1} has singularity type $\frac{1}{\rho\alpha}(1, -1-\rho\beta) = \frac{1}{\rho\ell}(1, \frac{\rho a}{w} - 1)$. Since $r/w = \ell$ (again by Lemma 2.2) we see that $\frac{\rho(a+kr)}{w} - 1 \equiv \frac{\rho a}{w} - 1 \pmod{\rho\ell}$ for any integer k, hence the singularity only depends on a modulo r.

Next, we need to show that (iii) is well-defined: that is, that the quotient singularities $\frac{1}{\rho\ell}(1, \frac{\rho a}{w} - 1)$ and $\frac{1}{\rho\ell}(1, \frac{\rho\bar{a}}{w} - 1)$ are equivalent. It is sufficient to show that

$$\left(\frac{\rho a}{w} - 1\right)\left(\frac{\rho\bar{a}}{w} - 1\right) \equiv 1 \pmod{\rho\ell}.$$

Let $k, c \in \mathbb{Z}_{\geq 0}$, $0 \leq c < \rho\ell$ be such that

$$\left(\frac{\rho a}{w} - 1\right)\left(\frac{\rho\bar{a}}{w} - 1\right) = k\rho\ell + c. \tag{2.1}$$

From Lemma 2.2 we see that $0 \leq c < r$, and (2.1) becomes

$$\left(a - 1 - \frac{nra}{d}\right)\left(\bar{a} - 1 - \frac{nr\bar{a}}{d}\right) = kr - \frac{knr^2}{d} + c, \qquad \text{where } d := \gcd\{r,a\} \cdot \gcd\{r,\bar{a}\}.$$

Multiplying through by d and reducing modulo r we obtain $d(a-1)(\bar{a}-1) \equiv dc \pmod{r}$. Suppose that $d \not\equiv 0 \pmod{r}$. Since $(a-1)(\bar{a}-1) \equiv 1 \pmod{r}$, we conclude that $c = 1$.

Finally, suppose that $d \equiv 0 \pmod{r}$. Writing $a = a' \cdot \gcd\{r,a\}$ and $\bar{a} = \bar{a}' \cdot \gcd\{r,\bar{a}\}$, we obtain

$$1 \equiv (a' \cdot \gcd\{r,a\} - 1)(\bar{a}' \cdot \gcd\{r,\bar{a}\} - 1) \equiv 1 - a - \bar{a} \pmod{r},$$

and hence $a \equiv -\bar{a} \pmod{r}$. But this implies that $1 \equiv (a-1)(-a-1) \equiv 1 - a^2 \pmod{r}$ and so $a \mid r$. Hence $w = r$, $\ell = 1$, and the singularity in (iii) is equivalent to $\frac{1}{\rho}(1, \rho - 1)$. □

Notice that the quantities a/w and $\bar{a}/w$ appearing in Proposition 2.4 are integers by Lemma 2.2.

Definition 2.5. Let $C \subset N_{\mathbb{Q}}$ be a cone of singularity type $\frac{1}{r}(1, a-1)$. Let ℓ and w be as above, and write $w = n\ell + \rho$ with $0 \leq \rho < \ell$. The *residue* of C is given by

$$\operatorname{res}(C) := \begin{cases} \frac{1}{\rho\ell}(1, \frac{\rho a}{w} - 1) & \text{if } \rho \neq 0, \\ \varnothing & \text{if } \rho = 0. \end{cases}$$

The *singularity content* of C is the pair $\mathrm{SC}(C) := (n, \operatorname{res}(C))$.

Example 2.6. Let C be a cone corresponding to the singularity $\frac{1}{60}(1,23)$. Then $w = 12$, $\ell = 5$, and $\rho = 2$. Setting $m = 1$ we obtain a decomposition of C into three subcones: C_0 of singularity type $\frac{1}{25}(1,9)$, C_1 of singularity type $\frac{1}{10}(1,3)$, and C_2 of singularity type $\frac{1}{25}(1,4)$. In particular, $\mathrm{res}(C) = \frac{1}{10}(1,3)$.

Recall that a T-singularity is a quotient surface singularity which admits a qG-one-parameter smoothing; T-singularities correspond to cyclic quotient singularities of the form $\frac{1}{nd^2}(1, ndc-1)$, where $\gcd\{d,c\} = 1$ [16, Proposition 3.10]. We now show that T-singularities are precisely the cyclic quotient singularities with empty residue.

Corollary 2.7. *Let $C \subset N_{\mathbb{Q}}$ be a cone and let w, ℓ be as above. The following are equivalent:*

(i) $\mathrm{res}(C) = \varnothing$;
(ii) *There exists an integer n such that $w = n\ell$;*
(iii) *There exists a crepant subdivision of C into n cones of singularity type $\frac{1}{\ell^2}(1, \ell c - 1)$, $\gcd\{\ell, c\} = 1$;*
(iv) *C corresponds to a T-singularity of type $\frac{1}{n\ell^2}(1, n\ell c - 1)$, $\gcd\{\ell, c\} = 1$.*

Proof. (i) and (ii) are equivalent by definition. (iii) follows from (ii) by Proposition 2.4, and (i) follows from (iv) by Lemma 2.2. Assume (iii) and let the singularity type of C be $\frac{1}{R}(1, A-1)$. The width of C is n times the width of a given subcone. Since $\gcd\{\ell, c\} = 1$, Lemma 2.2 implies that

$$\gcd\{R, A\} = w = n \cdot \gcd\{\ell^2, \ell c\} = n\ell.$$

The local index of a given subcone coincides, by construction, with the local index of C. By Lemma 2.2 we see that

$$R = \ell \cdot \gcd\{R, A\} = n\ell^2.$$

Finally, Proposition 2.4 gives that $\ell A/w = \ell c$, hence $A = n\ell c$, and so (iii) implies (iv). □

2.1. **Residue and deformation.** Define the *residue* of a cyclic quotient singularity σ to be the residue of C, where C is any cone of singularity type σ. The residue encodes information about qG-deformations of σ.

Proposition 2.8. *A cyclic quotient singularity σ admits a qG-smoothing if and only if* $\mathrm{res}(\sigma) = \varnothing$. *Otherwise there exists a qG-deformation of σ such that the general fibre is a cyclic quotient singularity of type* $\mathrm{res}(\sigma)$.

Proof. By definition, σ admits a qG-smoothing if and only if it is a T-singularity. Thus the first statement follows from Corollary 2.7. Assume σ is not a T-singularity and let ω, ℓ, and ρ be as above. By Corollary 2.7 we must have $\rho > 0$. Now $\sigma = \frac{1}{r}(1, a-1)$ has index ℓ and canonical cover

$$\tfrac{1}{\omega}(1,-1) = (xy - z^{\omega}) \subset \mathbb{A}^3_{x,y,z}.$$

Taking the quotient by the cyclic group μ_ℓ, and noting that $\omega \equiv \rho \pmod{\ell}$, we have:

$$\tfrac{1}{r}(1, a-1) = (xy - z^{\omega}) \subset \tfrac{1}{\ell}(1, \tfrac{\rho a}{\omega} - 1, \tfrac{a}{w}).$$

A qG-deformation is given by

$$(xy - z^{\omega} + tz^{\rho}) \subset \tfrac{1}{\ell}\left(1, \tfrac{\rho a}{\omega} - 1, \tfrac{a}{\omega}\right) \times \mathbb{A}^1_t,$$

and the general fibre of this family is the cyclic quotient singularity $\frac{1}{\rho\ell}(1, \frac{\rho a}{\omega} - 1)$. □

3. Singularity content of a complete toric surface

Definition 3.1. Let Σ be a complete fan in $N_{\mathbb{Q}}$ with two-dimensional cones $C_1, \ldots, C_m$, numbered cyclically, with $\mathrm{SC}(C_i) = (n_i, \mathrm{res}(C_i))$. The *singularity content* of the corresponding toric surface X_Σ is

$$\mathrm{SC}(X_\Sigma) := (n, \mathcal{B}),$$

where $n := \sum_{i=0}^{m} n_i$ and $\mathcal{B}$ is the cyclically ordered list $\{\mathrm{res}(C_1), \ldots, \mathrm{res}(C_m)\}$, with the empty residues $\mathrm{res}(C_i) = \varnothing$ omitted. We call $\mathcal{B}$ the *residual basket* of X_Σ.

Notation 3.2. We recall some standard facts about toric surfaces; see for instance [12]. Let X be a toric surface with singularity $\frac{1}{r}(1, a-1)$. Let $[b_1, \dots, b_k]$ denote the Hirzebuch–Jung continued fraction expansion of $r/(a-1)$, having length $k \in \mathbb{Z}_{>0}$. For $i \in \{1, \dots, k\}$, define $\alpha_i, \beta_i \in \mathbb{Z}_{>0}$ as follows: Set $\alpha_1 = \beta_k = 1$ and set

$$\begin{aligned} \alpha_i/\alpha_{i-1} &:= [b_{i-1}, \dots, b_1], \quad 2 \le i \le k, \\ \beta_i/\beta_{i+1} &:= [b_{i+1}, \dots, b_k], \quad 1 \le i \le k-1. \end{aligned}$$

If $\pi : \widetilde{X} \to X$ is a minimal resolution then

$$K_{\widetilde{X}} = \pi^* K_X + \sum_{i=1}^{k} d_i E_i,$$

where $E_i^2 = -b_i$ and $d_i = -1 + (\alpha_i + \beta_i)/r$ is the discrepancy.

Theorem 3.3. *Let X be a complete toric surface with singularity content $(n, \mathcal{B})$. Then*

$$K_X^2 = 12 - n - \sum_{\sigma \in \mathcal{B}} A_\sigma, \quad \text{where } A_\sigma := k_\sigma + 1 - \sum_{i=1}^{k_\sigma} d_i^2 b_i + 2 \sum_{i=1}^{k_\sigma - 1} d_i d_{i+1}.$$

Proof. Let Σ be the fan in $N_{\mathbb{Q}}$ of X. If $C \in \Sigma$ is a two-dimensional cone whose rays are generated by the primitive lattice vectors u and v then, possibly by adding an extra ray through a primitive lattice vector on the line segment uv, we can partition C as $C = S \cup R_C$, where S is a (possibly smooth) T-singularity or $S = \varnothing$, and $R_C = \mathrm{res}(C)$. Repeating this construction for all two-dimensional cones of Σ gives a new fan $\widetilde{\Sigma}$ in $N_{\mathbb{Q}}$. If $\widetilde{X}$ is the toric variety corresponding to $\widetilde{\Sigma}$ then the natural morphism $\widetilde{X} \to X$ is crepant. In particular $K_{\widetilde{X}}^2 = K_X^2$. Notice that $\mathrm{SC}(X) = (n, \mathcal{B}) = \mathrm{SC}(\widetilde{X})$.

By resolving singularities on all the nonempty cones R_C, we obtain a morphism $Y \to \widetilde{X}$ where the toric surface Y (whose fan we denote Σ_Y) has only T-singularities. Thus by Noether's formula [13, Proposition 2.6]

$$K_Y^2 + \rho_Y + \sum_{\sigma \in \mathrm{Sing}(Y)} \mu_\sigma = 10, \tag{3.1}$$

where ρ_Y is the Picard rank of Y, and μ_σ denotes the Milnor number of σ. But $\rho_Y + 2$ is equal to the number of two-dimensional cones in Σ_Y, and the Milnor number of a T-singularity $\frac{1}{nd^2}(1, ndc-1)$ equals $n-1$, hence

$$\rho_Y + \sum_{\sigma \in \mathrm{Sing}(Y)} \mu_\sigma = -2 + n + \sum_{\sigma \in \mathcal{B}} (k_\sigma + 1), \tag{3.2}$$

where k_σ denotes the length of the Hirzebuch–Jung continued fraction expansion $[b_1, \dots, b_{k_\sigma}]$ of $\sigma \in \mathcal{B}$. With notation as in Notation 3.2,

$$K_Y^2 = K_X^2 + \sum_{\sigma \in \mathcal{B}} \left(- \sum_{i=1}^{k_\sigma} d_i^2 b_i + 2 \sum_{i=1}^{k_\sigma - 1} d_i d_{i+1} \right). \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1) gives the desired formula. □

Remark 3.4. If X has only T-singularities, or equivalently if $\mathcal{B} = \varnothing$, then Theorem 3.3 gives $K_X^2 = 12 - n$.

The m-th Dedekind sum, $m \in \mathbb{Z}_{\ge 0}$, of the cyclic quotient singularity $\frac{1}{r}(a, b)$ is

$$\delta_m := \frac{1}{r} \sum \frac{\varepsilon^m}{(1 - \varepsilon^a)(1 - \varepsilon^b)},$$

where the summation is taken over those $\varepsilon \in \mu_r$ satisfying $\varepsilon^a \neq 1$ and $\varepsilon^b \neq 1$. By Theorem 3.3 and [17, §8] we obtain an expression for the Hilbert series of X in terms of its singularity content:

Corollary 3.5. *Let X be a complete toric surface with singularity content $(n, \mathcal{B})$. Then the Hilbert series of X admits a decomposition*

$$\mathrm{Hilb}(X, -K_X) = \frac{1 + (K_X^2 - 2)t + t^2}{(1-t)^3} + \sum_{\sigma \in \mathcal{B}} Q_\sigma(t), \quad \text{where } Q_{\frac{1}{r}(a,b)} := \frac{\sum_{i=1}^{r} (\delta_{(a+b)i} - \delta_0) t^{i-1}}{1 - t^r}.$$

3.1. **Singularity content and mutation.** A lattice polygon in $N_{\mathbb{Q}}$ is called *Fano* if $\mathbf{0}$ lies its strict interior, and all its vertices are primitive; see [15] for an overview. We define the *singularity content* of a Fano polygon P to be $\mathrm{SC}(P) := \mathrm{SC}(X_\Sigma)$, where Σ is the spanning fan of P; that is, Σ is the complete fan in $N_{\mathbb{Q}}$ with cones spanned by the faces of P. Under certain conditions, one can construct a new Fano polygon from P via a *(combinatorial) mutation*, which we describe below; see [1, §3] for details. We shall show in Theorem 3.8 that singularity content is invariant under mutation.

Let $h \in M$ be a primitive inner normal vector for an edge E of P. Then $h : N \to \mathbb{Z}$ induces a grading on $N_{\mathbb{Q}}$, and we call $h(v)$ the *height* of $v \in N_{\mathbb{Q}}$. Let $h_{\min}$ and $h_{\max}$ denote the minimum and maximum heights, respectively, of the points in P and let $h_i(P) := \mathrm{conv}\{v \in P \cap N \mid h(v) = i\}$. By definition $h_{\min}$ is equal to the height of the edge E, and $h_{h_{\min}}(P)$ is equal to E. Let $F \subset N_{\mathbb{Q}}$ be a polytope at height zero; that is, $h(v) = 0$ for all $v \in F$. Then F is either a line segment or a point. We call F a *factor* if for each negative height $h_{\min} \leq i < 0$ there exists a lattice polytope $G_i \subset N_{\mathbb{Q}}$ such that

$$\{v \in \mathrm{vert}(P) \mid h(v) = i\} \subseteq G_i + |i|\, F \subseteq h_i(P).$$

Here '+' denotes the Minkowski sum, and we allow the possibility that $G_i = \varnothing$ (with $\varnothing + Q = \varnothing$ for any polytope Q). The *mutation* of P with respect to h and factor F is:

$$\mathrm{mut}_h(P, F) := \mathrm{conv}\left(\bigcup_{i=h_{\min}}^{-1} G_i \cup \bigcup_{i=0}^{h_{\max}} (h_i(P) + iF) \right) \subset N_{\mathbb{Q}}.$$

Lemma 3.6 ([1, §3])**.** *Let $Q := \mathrm{mut}_h(P, F)$ be a mutation of P. Then:*

(i) *Q is independent of the choice of the $\{G_i\}$;*
(ii) *$Q \cong \mathrm{mut}_h(P, F + v)$ for any $v \in N$, $h(v) = 0$ and, in particular, if* $\dim F = 0$ *then $Q \cong P$;*
(iii) *Mutations are invertible, with* $\mathrm{mut}_{-h}(Q, F) = P$*;*
(iv) *P is a Fano polytope if and only if Q is a Fano polytope.*

Mutation acts on the dual lattice M via a piecewise linear transformation, which we now describe. Recall that the *inner normal fan* in M of a polytope $F \subset N_{\mathbb{Q}}$ is generated by the cones

$$\sigma_{v_F} := \{u \in M_{\mathbb{Q}} \mid u(v_F) = \min\{u(v') \mid v' \in F\}\}, \qquad \text{for each } v_F \in \mathrm{vert}(F).$$

That is, σ_{v_F} consists of those linear functions which are minimal on the vertex v_F of F. Notice that the cones σ_{v_F} are not strictly convex. A mutation of $P \subset N_{\mathbb{Q}}$ induces a piecewise linear transformation φ of $M_{\mathbb{Q}}$ given by

$$\varphi : u \mapsto u - u_{\min} h, \qquad \text{where } u_{\min} := \min\{u(v_F) \mid v_F \in \mathrm{vert}(F)\}.$$

We have that $(\varphi(P^\vee))^\vee = \mathrm{mut}_h(P, F)$. The inner normal fan of F determines a chamber decomposition of $M_{\mathbb{Q}}$, and φ acts as a $\mathrm{GL}_2(\mathbb{Z})$-transformation within each chamber σ_{v_F}. As a consequence, we have:

Corollary 3.7 ([1, Proposition 4])**.** *Let $Q := \mathrm{mut}_h(P, F)$ be a mutation of P, and let X_P and X_Q be the toric varieties defined by the spanning fans of P and Q, respectively. Then:*

$$\mathrm{Hilb}(X_P, -K_{X_P}) = \mathrm{Hilb}(X_Q, -K_{X_Q}).$$

Theorem 3.8. *Let $Q := \mathrm{mut}_h(P, F)$. Then* $\mathrm{SC}(P) = \mathrm{SC}(Q)$*. In particular, singularity content is an invariant of Fano polygons under mutation.*

Proof. The dual polygon $P^\vee \subset M_{\mathbb{Q}}$ is an intersection of cones

$$P^\vee = \bigcap \left(C_L^\vee - v_L \right),$$

where the intersection ranges over all facets L of P. Here $C_L \subset N_{\mathbb{Q}}$ is the cone over the facet L and v_L is the vertex of $P^\vee$ corresponding to L.

If F is a point then $P \cong Q$ and we are done. Let F be a line segment and let $P_{\max}$ and $P_{\min}$ (resp. $Q_{\max}$ and $Q_{\min}$) denote the faces of P (resp. Q) at heights $h_{\min}$ and $h_{\max}$ with respect to h. By assumption the mutation Q exists, hence $P_{\min}$ must be a facet, and so there exists a corresponding vertex $v_0 \in M$ of $P^\vee$. $P_{\max}$ can be either facet or a vertex. The argument is similar in either case, so we will assume that $P_{\max}$ is a facet with corresponding vertex $v_1 \in M$ of $P^\vee$.

The inner normal fan Σ of F defines a decomposition of $M_{\mathbb{Q}}$ into half-spaces Σ^+ and Σ^-, whose boundaries are given by the line H generated by h. The vertices v_0 and v_1 of $P^\vee$ lie in H; any other vertex lies in the interior of exactly one of Σ^+ or Σ^-. Mutation acts as an automorphism in both half-spaces. Thus the contribution to $\mathrm{SC}(Q)$ from cones over all facets excluding $Q_{\max}$ and $Q_{\min}$ is equal to the contribution to $\mathrm{SC}(P)$ from cones over all facets excluding $P_{\max}$ and $P_{\min}$. Finally, mutation acts by exchanging T-singular subcones between the facets $P_{\max}$ and $P_{\min}$, leaving the residue unchanged. Hence the contribution to $\mathrm{SC}(Q)$ from $Q_{\max}$ and $Q_{\min}$ is equal to the contribution to $\mathrm{SC}(P)$ from $P_{\max}$ and $P_{\min}$. □

Example 3.9. If two Fano polygons are related by a sequence of mutations then the corresponding toric surfaces have the same anti-canonical degree [1, Proposition 4]. The Fano polygons $P_1 := \mathrm{conv}\{(0,1),(5,4),(-7,-8)\}$ and $P_2 := \mathrm{conv}\{(0,1),(3,1),(-112,-79)\}$ correspond to $\mathbb{P}(5,7,12)$ and $\mathbb{P}(3,112,125)$, respectively. These both have degree $48/35$, however their singularity contents differ:

$$\mathrm{SC}(P_1) = \left(12, \left\{\tfrac{1}{5}(1,1), \tfrac{1}{7}(1,1)\right\}\right), \qquad \mathrm{SC}(P_2) = \left(5, \left\{\tfrac{1}{14}(1,9), \tfrac{1}{125}(1,79)\right\}\right).$$

Hence they are not related by a sequence of mutations.

Lemma 3.10. *Let P be a Fano polygon with $\mathrm{SC}(P) = (n, \mathcal{B})$, and let ρ_X denote the Picard rank of the corresponding toric surface. Then $\rho_X \leq n + |\mathcal{B}| - 2$.*

Proof. The cone over any facet of P admits a subdivision (in the sense of Notation 2.3) into at least one subcone. Therefore we must have that $|\mathrm{vert}(P)| \leq n + |\mathcal{B}|$. Recalling that $\rho_X = |\mathrm{vert}(P)| - 2$ we obtain the result. □

Since singularity content is preserved under mutation, Lemma 3.10 gives an upper bound on the rank of the resulting toric varieties.

Example 3.11. In [2] we classified one-step mutations of (fake) weighted projective planes. It is natural to ask how much of the graph of mutations of a given (fake) weighted projective plane is captured by the graph of one-step mutations. Lemma 3.10 shows that the two graphs coincide if the singularity content of the (fake) weighted projective plane in question satisfies $n + |\mathcal{B}| = 3$. For example the full mutation graph of $\mathbb{P}^2$ is isomorphic to the graph of solutions of the Markov equation $3xyz = x^2 + y^2 + z^2$ [2, Example 3.14]. More interestingly, the weighted projective plane $\mathbb{P}(3,5,11)$ does not admit *any* mutations [2, Example 3.5].

Acknowledgements. We thank Tom Coates, Alessio Corti, and Diletta Martinelli for many useful conversations, and Klaus Altmann and Nathan Ilten for explaining the connections between the decompositions and deformations constructed here and their work on Minkowski decompositions, T-varieties, and polyhedral divisors.

References

[1] Mohammad Akhtar, Tom Coates, Sergey Galkin, and Alexander M. Kasprzyk. Minkowski polynomials and mutations. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 8:Paper 094, 17, 2012.

[2] Mohammad Akhtar and Alexander M. Kasprzyk. Mutations of fake weighted projective planes. *Proc. Edinb. Math. Soc. (2)*, 59(2):271–285, 2016.

[3] Klaus Altmann. Deformations of affine torus varieties. *Beiträge Algebra Geom.*, 34(1):119–150, 1993.

[4] Klaus Altmann. Computation of the vector space T^1 for affine toric varieties. *J. Pure Appl. Algebra*, 95(3):239–259, 1994.

[5] Klaus Altmann. Minkowski sums and homogeneous deformations of toric varieties. *Tohoku Math. J. (2)*, 47(2):151–184, 1995.

[6] Klaus Altmann. The versal deformation of an isolated toric Gorenstein singularity. *Invent. Math.*, 128(3):443–479, 1997.

[7] Klaus Altmann. P-resolutions of cyclic quotients from the toric viewpoint. In *Singularities (Oberwolfach, 1996)*, volume 162 of *Progr. Math.*, pages 241–250. Birkhäuser, Basel, 1998.

[8] Klaus Altmann, Nathan Owen Ilten, Lars Petersen, Hendrik Süß, and Robert Vollmert. The geometry of T-varieties. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 17–69. Eur. Math. Soc., Zürich, 2012.

[9] Jan Arthur Christophersen. On the components and discriminant of the versal base space of cyclic quotient singularities. In *Singularity theory and its applications, Part I (Coventry, 1988/1989)*, volume 1462 of *Lecture Notes in Math.*, pages 81–92. Springer, Berlin, 1991.

[10] Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander M. Kasprzyk. Mirror symmetry and Fano manifolds. In *Proceedings of the 6th European Congress of Mathematics (Kraków, 2-7 July, 2012)*, pages 285–300. European Mathematical Society, November 2013.

[11] Tom Coates, Alessio Corti, Sergey Galkin, and Alexander M. Kasprzyk. Quantum periods for 3-dimensional Fano manifolds. *Geom. Topol.*, 20(1):103–256, 2016.

[12] William Fulton. *Introduction to toric varieties*, volume 131 of *Ann. of Math. Stud.* Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[13] Paul Hacking and Yuri Prokhorov. Smoothable del Pezzo surfaces with quotient singularities. *Compos. Math.*, 146(1):169–192, 2010.

[14] Nathan Owen Ilten. One-parameter toric deformations of cyclic quotient singularities. *J. Pure Appl. Algebra*, 213(6):1086–1096, 2009.

[15] Alexander M. Kasprzyk and Benjamin Nill. Fano polytopes. In Anton Rebhan, Ludmil Katzarkov, Johanna Knapp, Radoslav Rashkov, and Emanuel Scheidegger, editors, *Strings, Gauge Fields, and the Geometry Behind – the Legacy of Maximilian Kreuzer*, pages 349–364. World Scientific, 2012.

[16] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Invent. Math.*, 91(2):299–338, 1988.

[17] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[18] Jan Stevens. On the versal deformation of cyclic quotient singularities. In *Singularity theory and its applications, Part I (Coventry, 1988/1989)*, volume 1462 of *Lecture Notes in Math.*, pages 302–319. Springer, Berlin, 1991.

Institut des Hautes Études Scientifiques, 35 Route de Chartres, 91440 Paris, France
E-mail address: akhtar@ihes.fr

School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, UK
E-mail address: a.m.kasprzyk@nottingham.ac.uk