FOUR-DIMENSIONAL FANO QUIVER FLAG ZERO LOCI

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Abstract. Quiver flag zero loci are subvarieties of quiver flag varieties cut out by sections of homogeneous vector bundles. We prove the Abelian/non-Abelian Correspondence in this context: this allows us to compute genus zero Gromov–Witten invariants of quiver flag zero loci. We determine the ample cone of a quiver flag variety, disproving a conjecture of Craw. In the Appendix, which is joint work with Tom Coates and Alexander Kasprzyk, we use these results to find four-dimensional Fano manifolds that occur as quiver flag zero loci in ambient spaces of dimension up to 8, and compute their quantum periods. In this way we find at least 139 new four-dimensional Fano manifolds.

1. Introduction

In this paper, we consider quiver flag zero loci: smooth projective algebraic varieties which occur as zero loci of sections of homogeneous vector bundles on quiver flag varieties. We prove the Abelian/non-Abelian Correspondence of Ciocan-Fontanine–Kim–Sabbah in this context, which allows us to compute genus zero Gromov–Witten invariants of quiver flag zero loci[1]. We also determine the ample cone of a quiver flag variety, disproving a conjecture of Craw. Our primary motivation for these results is as follows. There has been much recent interest in the possibility of classifying Fano manifolds using Mirror Symmetry. It is conjectured that, under Mirror Symmetry, n-dimensional Fano manifolds should correspond to certain very special Laurent polynomials in n variables [6]. This conjecture has been established in dimensions up to three [7], where the classification of Fano manifolds is known [13, 14, 15, 23, 24, 25, 26, 27]. Little is known about the classification of four-dimensional Fano manifolds, but there is strong evidence that the conjecture holds for four-dimensional toric complete intersections [8]. Not every Fano manifold is a toric complete intersection, but the constructions in [7] show that every Fano manifold of dimension at most three is either a toric complete intersection or a quiver flag zero locus. One might hope, therefore, that most four-dimensional Fano manifolds are either toric complete intersections or quiver flag zero loci.

In the Appendix, which is joint work with Tom Coates and Alexander Kasprzyk, we use the structure theory developed here to find four-dimensional Fano manifolds that occur as quiver flag zero loci in ambient spaces of dimension up to 8, and compute their quantum periods. We find 139 were previously unknown quantum periods. Thus we find at least 139 new four-dimensional Fano manifolds. The new period sequences are recorded in Table 2. A quiver flag zero locus is given for each quantum period in Table 1. Recent conjectures by Coates, Corti, Galkin, Golyshev, Kasprzyk and Prince say that under mirror symmetry, Fano manifolds of dimension n correspond under mirror symmetry to maximally mutable Laurent polynomials [1, 16] in n variables. So far, methods to produce Laurent polynomial mirrors for Fano varieties only apply when the Fano variety is a toric complete intersection or a complete intersection in a flag variety. In 7.2, we find Laurent polynomial mirrors for Picard rank 1 quiver flag varieties when the methods apply, otherwise we use the quantum periods found to compute the quantum local systems and find candidate Laurent polynomial mirrors.

2. Quiver flag varieties

Quiver flag varieties, which were introduced by Craw [9], are generalizations of Grassmannians and flag varieties. Like flag varieties, they are GIT quotients and fine moduli spaces. We begin by recalling Craw's construction. A quiver flag variety $M(Q, \mathbf{r})$ is determined by a quiver Q and

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[1] Another proof of this, using different methods, has recently been given by Rachel Webb [30].

a dimension vector $\mathbf{r}$. The quiver Q is assumed to be finite and acyclic, with a unique source. Let $Q_0 = \{0, 1, \ldots, \rho\}$ denote the set of vertices of Q and let Q_1 denote the set of arrows. Without loss of generality, after reordering the vertices if necessary, we may assume that $0 \in Q_0$ is the unique source and that the number n_{ij} of arrows from vertex i to vertex j is zero unless $i < j$. Write $s, t : Q_1 \to Q_0$ for the source and target maps, so that an arrow $a \in Q_1$ goes from $s(a)$ to $t(a)$. The dimension vector $\mathbf{r} = (r_0, \ldots, r_\rho)$ lies in $\mathbb{N}^{\rho+1}$, and we insist that $r_0 = 1$. $M(Q, \mathbf{r})$ is defined to be the moduli space of representations of the quiver Q with dimension vector $\mathbf{r}$.

2.1. **Quiver flag varieties as GIT quotients.** Consider the vector space

$$\operatorname{Rep}(Q, \mathbf{r}) = \bigoplus_{a \in Q_1} \operatorname{Hom}(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}})$$

and the action of $\operatorname{GL}(\mathbf{r}) := \prod_{i=0}^{\rho} \operatorname{GL}(r_i)$ on $\operatorname{Rep}(Q, \mathbf{r})$ by change of basis. The diagonal copy of $\operatorname{GL}(1)$ in $\operatorname{GL}(\mathbf{r})$ acts trivially, but the quotient $G := \operatorname{GL}(\mathbf{r})/\operatorname{GL}(1)$ acts effectively; since $r_0 = 1$, we may identify G with $\prod_{i=1}^{\rho} \operatorname{GL}(r_i)$. The quiver flag variety $M(Q, \mathbf{r})$ is the GIT quotient $\operatorname{Rep}(Q, \mathbf{r}) /\!\!/_\theta G$, where the stability condition θ is the character of G given by

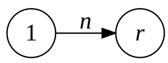
$$\theta(g) = \prod_{i=1}^{\rho} \det(g_i), \qquad\qquad g = (g_1, \ldots, g_\rho) \in \prod_{i=1}^{\rho} \operatorname{GL}(r_i).$$

For the stability condition θ, all semistable points are stable. To identify the θ-stable points in $\operatorname{Rep}(Q, \mathbf{r})$, set $s_i = \sum_{a \in Q_1, t(a)=i} r_{s(a)}$ and write

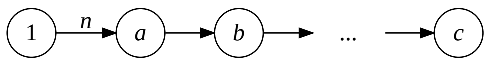
$$\operatorname{Rep}(Q, \mathbf{r}) = \bigoplus_{i=1}^{\rho} \operatorname{Hom}(\mathbb{C}^{s_i}, \mathbb{C}^{r_i}).$$

Then $w = (w_i)_{i=1}^{\rho}$ is θ-stable if and only if w_i is surjective for all i.

Example 2.1. *Consider the quiver Q given by*



so that $\rho = 1$, $n_{01} = n$, and the dimension vector $\mathbf{r} = (1, r)$. Then $\operatorname{Rep}(Q, \mathbf{r}) = \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^r)$, and the θ-stable points are surjections $\mathbb{C}^n \to \mathbb{C}^r$. The group G acts by change of basis, and therefore $M(Q, \mathbf{r}) = \operatorname{Gr}(n, r)$, the Grassmannian of r-dimensional quotients of $\mathbb{C}^n$. More generally, the quiver



gives the flag of quotients $\operatorname{Fl}(n, a, b, \ldots, c)$.

Quiver flag varieties are non-abelian GIT quotients unless the dimension vector $\mathbf{r} = (1, 1, \ldots, 1)$. In this case $G \cong \prod_{i=1}^{\rho} \operatorname{GL}_1(\mathbb{C})$ is abelian, and $M(Q; \mathbf{r})$ is a toric variety. We call such $M(Q, \mathbf{r})$ toric quiver flag varieties.

2.2. **Quiver flag varieties as moduli spaces.** To give a morphism to $M(Q, \mathbf{r})$ from a scheme B is the same as to give:

- globally generated vector bundles $W_i \to B$, $i \in Q_0$, of rank r_i such that $W_0 = \mathcal{O}_B$; and
- morphisms $W_{s(a)} \to W_{t(a)}$, $a \in Q_1$ satisfying the θ-stability condition

up to isomorphism. Thus $M(Q, \mathbf{r})$ carries universal bundles W_i, $i \in Q_0$. It is also a Mori Dream Space, and therefore there is an isomorphism between the Picard group of $M(Q, \mathbf{r})$ and the character group $\chi(G) \cong \mathbb{Z}^\rho$ of G. When tensored with $\mathbb{Q}$ this isomorphism induces an isomorphism of wall and chamber structures given by the Mori structure (on the effective cone) and the GIT structure (on $\chi(G) \otimes \mathbb{Q}$); in particular, the GIT chamber containing θ is the ample cone of $M(Q, \mathbf{r})$. The Picard group is generated by the determinant line bundles $\det(W_i)$, $i \in Q_0$.

2.3. **Quiver flag varieties as towers of Grassmannian bundles.** Generalizing Example 2.1, all quiver flag varieties are towers of Grassmannian bundles [9, Theorem 3.3]. For $0 \leq i \leq \rho$, let $Q(i)$ be the subquiver of Q obtained by removing the vertices $j \in Q_0$, $j > i$, and all arrows attached to them. Let $\mathbf{r}(i) = (1, r_1, \ldots, r_i)$, and write $Y_i = M(Q(i), \mathbf{r}(i))$. Denote the universal bundle $W_j \to Y_i$ by $W_j^{(i)}$. Then there are maps

$$M(Q, \mathbf{r}) = Y_\rho \to Y_{\rho-1} \to \cdots \to Y_1 \to Y_0 = \operatorname{Spec} \mathbb{C},$$

induced by isomorphisms $Y_i \cong \operatorname{Gr}(\mathcal{F}_i, r_i)$, where $\mathcal{F}_i$ is the locally free sheaf

$$\mathcal{F}_i = \bigoplus_{a \in Q_1, t(a)=i} W_{s(a)}^{(i-1)}$$

of rank s_i on Y_{i-1}. This makes clear that $M(Q, \mathbf{r})$ is a smooth projective variety of dimension $\sum_{i=1}^{\rho} r_i(s_i - r_i)$, and that W_i is the pullback to Y_ρ of the tautological quotient bundle over $\operatorname{Gr}(\mathcal{F}_i, r_i)$. Thus W_i is globally generated, and $\det(W_i)$ is nef. Furthermore the anticanonical line bundle of $M(Q, \mathbf{r})$ is

$$\bigotimes_{a \in Q_1} \det(W_{t(a)})^{r_{s(a)}} \otimes \det(W_{s(a)})^{-r_{t(a)}}.$$

In particular, $M(Q, \mathbf{r})$ is Fano if $s_i > s_i' := \sum_{a \in Q_1, s(a)=i} r_{t(a)}$. This condition is not if and only if.

2.4. **Quiver flag zero loci.** We have expressed the quiver flag variety $M(Q, \mathbf{r})$ as the quotient by G of the semistable locus $\operatorname{Rep}(Q, \mathbf{r})^{ss} \subset \operatorname{Rep}(Q, \mathbf{r})$. A representation E of G, therefore, defines a vector bundle $E_G \to M(Q, \mathbf{r})$ with fiber E; here $E_G = E \times_G \operatorname{Rep}(Q, \mathbf{r})^{ss}$. In the second half of this paper, we will study subvarieties of quiver flag varieties cut out by regular sections of such bundles. We refer to such subvarieties as *quiver flag zero loci*, and such bundles as homogeneous bundles.

The representation theory of $G = \prod_{i=1}^{\rho} \operatorname{GL}(r_i)$ is well-understood, and we can use this to write down the bundles E_G explicitly. Irreducible polynomial representations of $\operatorname{GL}(r)$ are indexed by partitions (or Young diagrams) of length at most r. The irreducible representation corresponding to a partition α is the Schur power $S^\alpha \mathbb{C}^r$ of the standard representation of $\operatorname{GL}(r)$ [11, chapter 8]. For example, if α is the partition (k) then $S^\alpha \mathbb{C}^r = \operatorname{Sym}^k \mathbb{C}^r$, the kth symmetric power, and if α is the partition $(1, 1, \ldots, 1)$ of length k then $S^\alpha \mathbb{C}^r = \bigwedge^k \mathbb{C}^r$, the kth exterior power. Irreducible polynomial representations of G are therefore indexed by tuples $(\alpha_1, \ldots, \alpha_\rho)$ of partitions, where α_i has length at most r_i. The tautological bundles on a quiver flag variety are homogenous: if $E = \mathbb{C}^{r_i}$ is the standard representation of the i^{th} factor of G, then $W_i = E_G$. More generally, the representation indexed by $(\alpha_1, \ldots, \alpha_\rho)$ is $\bigotimes_{i=1}^{\rho} S^{\alpha_i} \mathbb{C}^{r_i}$, and the corresponding vector bundle on $M(Q, \mathbf{r})$ is $\bigotimes_{i=1}^{\rho} S^{\alpha_i} W_i$.

Example 2.2. *Consider the vector bundle* $\operatorname{Sym}^2 W_1$ *on* $\operatorname{Gr}(8, 3)$*. By duality – which sends a quotient* $\mathbb{C}^8 \to V \to 0$ *to a subspace* $0 \to V^* \to (\mathbb{C}^8)^*$ *– this is equivalent to considering the vector bundle* $\operatorname{Sym}^2 S_1^*$ *on the Grassmannian of 3-dimensional subspaces of* $(\mathbb{C}^8)^*$*, where* S_1 *is the tautological sub-bundle. A generic symmetric 2-form* ω *on* $(\mathbb{C}^8)^*$ *determines a regular section of* $\operatorname{Sym}^2 S_1^*$*, which vanishes at a point* V^* *if and only if the restriction of* ω *to* V^* *is identically zero. So the associated quiver flag zero locus is the orthogonal Grassmannian* $\operatorname{OGr}(3, 8)$*.*

2.5. **The Euler sequence.** Quiver flag varieties, like both Grassmannians and toric varieties, have an Euler sequence.

Proposition 2.3. *Let* $X = M(Q, \mathbf{r})$ *be a quiver flag variety. There is a short exact sequence*

$$0 \to \bigoplus_{i=1}^{\rho} W_i \otimes W_i^* \to \bigoplus_{a \in Q_1} W_a \to T_X \to 0.$$

Proof. We proceed by induction on the Picard rank ρ of X. If $\rho = 1$ then this is the usual Euler sequence for the Grassmannian. Suppose that the proposition holds for quiver flag varieties of Picard rank $\rho - 1$, $\rho > 1$. Then the fibration $\pi \colon \operatorname{Gr}(\mathcal{F}_\rho, r_\rho) \to Y_{\rho-1}$ from §22.3 above gives a short exact sequence

$$0 \to W_\rho \otimes W_\rho^* \to \mathcal{F}_\rho^* \otimes W_\rho \to S^* \otimes W_\rho \to 0$$

where S is the kernel of the projection $\mathcal{F}_\rho \to W_\rho$. Note that

$$\mathcal{F}_\rho^* \otimes W_\rho = \bigoplus_{a \in Q_1, t(a)=\rho} W_a \qquad \text{and that} \qquad T_X = T_{Y_{\rho-1}} \oplus S^* \otimes W_\rho.$$

As $S^* \otimes W_\rho$ is the relative tangent bundle to π, the proposition follows by induction. □

If X is a quiver flag zero locus cut out of the quiver flag variety $M(Q, \mathbf{r})$ by a regular section of the homogeneous vector bundle E then there is an exact sequence

$$0 \to T_X \to T_{M(Q,\mathbf{r})}|_X \to E \to 0.$$

Thus T_X is the K-theoretic difference of homogeneous vector bundles.

3. Quiver flag varieties as subvarieties

There are three well-known constructions of flag varieties: as GIT quotients, as homogenous spaces, and as subvarieties of products of Grassmannians. Craw's construction gives quiver flag varieties as GIT quotients. General quiver flag varieties are not homogenous spaces, so the second construction does not generalize. In this section we generalize the third construction of flag varieties, exhibiting quiver flag varieties as subvarieties of products of Grassmannians. It is this description that will allow us to prove the Abelian/non-Abelian correspondence for quiver flag varieties.

Proposition 3.1. *Let $M(Q, \mathbf{r})$ be a quiver flag variety with $\rho > 1$. Let $\tilde{s}_i = \dim H^0(M(Q, \mathbf{r}), W_i)$. Then $M(Q, \mathbf{r})$ is cut out of $Y = \prod_{i=1}^{\rho} \mathrm{Gr}(\tilde{s}_i, r_i)$ by a canonical section of*

$$E = \bigoplus_{a \in Q_1, s(a) \neq 0} S^*_{s(a)} \otimes Q_{t(a)}$$

where S_i and Q_i are the tautological sub-bundle and quotient bundle on the i^{th} factor of Y.

Proof. Recall that $\tilde{s}_i$ is the number of paths from 0 to vertex i [9]. Thus

$$\mathbb{C}^{\tilde{s}_i} = \bigoplus_{a \in Q_1, t(a)=i, s(a)\neq 0} \mathbb{C}^{\tilde{s}_{s(a)}} \oplus \mathbb{C}^{n_{0i}}.$$

Let $F_i = \bigoplus_{t(a)=i} Q_{s(a)}$. Combining the canonical surjections $\mathbb{C}^{\tilde{s}_{s(a)}} \otimes \mathcal{O} \to Q_{s(a)}$ gives a surjection $\mathbb{C}^{\tilde{s}_i} \otimes \mathcal{O} \to F_i$ that fits into the exact sequence

$$0 \to \bigoplus_{t(a)=i, s(a)\neq 0} S_{s(a)} \to \mathbb{C}^{\tilde{s}_i} \otimes \mathcal{O} \to F_i \to 0.$$

Thus

$$(\mathbb{C}^{\tilde{s}_i *} \otimes \mathcal{O})/F_i^* \cong \bigoplus_{t(a)=i, s(a)\neq 0} S^*_{s(a)}$$

and it follows that $E = \bigoplus_{i=2}^{\rho} \mathrm{Hom}(Q_i^*, (\mathbb{C}^{\tilde{s}_i *} \otimes \mathcal{O})/F_i^*)$.

Consider the section s of E given by the compositions $Q_i^* \to \mathbb{C}^{\tilde{s}_i *} \otimes \mathcal{O} \to (\mathbb{C}^{\tilde{s}_i *} \otimes \mathcal{O})/F_i^*$. The section s vanishes at quotients $(V_1, \ldots, V_\rho)$ if and only if $V_i^* \subset \bigoplus_{t(a)=i} V^*_{s(a)}$; dually, the zero locus is where there is a surjection $F_i \to Q_i$ for each i. Now it isn't hard to see that $Z(s)$ is $M(Q, \mathbf{r})$. $M(Q, \mathbf{r})$ parametrizes surjections $\mathbb{C}^{\tilde{s}_i} \to V_i$ that factor as

$$\mathbb{C}^{\tilde{s}_i} \to \bigoplus_{t(a)=i} V_{s(a)} \to V_i.$$

Since the W_i are globally generated, there is a unique map

$$M(Q, \mathbf{r}) \to Y = \prod_{i=1}^{\rho} \mathrm{Gr}(\tilde{s}_i, r_i)$$

such that $Q_i \to Y$ pulls back to $W_i \to M(Q, \mathbf{r})$. The discussion above shows that the image of this map lies in the zero locus of s. Similarly, the universal property of $M(Q, \mathbf{r})$ gives rise to a unique map $Z(s) \to M(Q, \mathbf{r})$ such that $W_i \to M(Q, \mathbf{r})$ pulls back to $Q_i \to Z(s)$. Because $M(Q, \mathbf{r})$ is a fine moduli space, the composition of these maps $M(Q, \mathbf{r}) \to Z(s) \to M(Q, \mathbf{r})$ must be the identity. Similarly, the composition $Z(s) \to M(Q, \mathbf{r}) \to Z(s)$ is the identity, and we conclude that $Z(s)$ and $M(Q, \mathbf{r})$ are canonically isomorphic. □

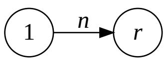
Suppose that X is a quiver flag zero locus cut out of $M(Q,\mathbf{r})$ by a regular section of a homogeneous vector bundle E. Since $M(Q,\mathbf{r})$ and the product of Grassmannians described above are both GIT quotients by the same group G, the representation of G that determines E also determines a vector bundle E' on $\prod_{i=1}^{\rho} \mathrm{Gr}(\tilde{s_i}, r_i)$. We see that X is deformation equivalent to the zero locus of a generic section of the vector bundle $F := E' \oplus \bigoplus_{a \in Q_1, s(a)\neq 0} S^*_{s(a)} \otimes Q_{t(a)}$. Although the product of Grassmannians is a quiver flag variety, this is not generally an additional model of X as a quiver flag zero locus, as the summand $S^*_{s(a)} \otimes Q_{t(a)}$ in F does not in general come from a representation of G.

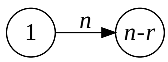
Remark 3.2. *Suppose α is a non-negative Schur partition. Then [29] shows that $S^\alpha(Q)$ is globally generated on $\mathrm{Gr}(n,r)$. The above construction allows us to generalise this to quiver flag varieties: $S^\alpha(W_i)$ is globally generated on $M(Q,\mathbf{r})$.*

4. Equivalences of quiver flag zero loci

The representation of a given variety X as a quiver flag zero locus, if it exists, is far from unique. In this section we describe various methods of passing between different representations of the same quiver flag zero locus. This is important in practice, because our systematic search for four-dimensional quiver flag zero loci described in the Appendix finds a given variety in many different representations. Furthermore, geometric invariants of a quiver flag zero locus X can be much easier to compute in some representations than in others. The observations in this section allow us to compute invariants of four-dimensional Fano quiver flag zero loci using only a few representations, where the computation is relatively cheap, rather than doing the same computation many times and using representations where the computation is expensive.

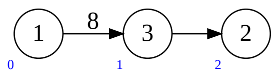
4.1. **Dualising.** As we saw in the previous section, a quiver flag zero locus X given by $(M(Q,\mathbf{r}), E)$ can be thought of as a zero locus in a product of Grassmannians Y. Unlike general quiver flag varieties, Grassmannians come in canonically isomorphic dual pairs:



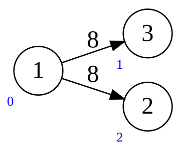


The isomorphism interchanges the tautological quotient bundle Q with S^*, where S is the tautological sub-bundle. One can then dualize some or none of the Grassmannian factors in Y, to get different models of X. Depending on the representations in E, after dualizing, E may still be a homogenous vector bundle, or the direct sum of a homogeneous vector bundle with bundles of the form $S_i^* \otimes W_j$. If this is the case, one can then undo the product representation process to obtain another model $(M(Q',\mathbf{r}'), E'_G)$ of X.

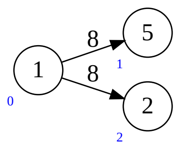
Example 4.1. *Consider X given by the quiver and bundle*



and bundle $\wedge^2 W_2$; here and below the vertex numbering is indicated in blue. Then writing it as a product:



with bundle $\wedge^2 W_2 \oplus S_1^* \otimes W_2$ and dualizing the first factor, we get



with bundle $W_1 \otimes W_2 \oplus \wedge^2 W_2$, which is a quiver flag zero locus.

4.2. **Removing arrows.**

Example 4.2. *Recall that* $\mathrm{Gr}(n, r)$ *is the quiver flag zero locus given by* $(\mathrm{Gr}(n+1, r), W_1)$*. This is because the space of sections of* W_1 *is* $\mathbb{C}^{n+1}$*, where the image of the section corresponding to* $v \in \mathbb{C}^{n+1}$ *at the point* $\phi\colon \mathbb{C}^{n+1} \to W$ *in* $\mathrm{Gr}(n+1, r)$ *is* $\phi(v)$*. This section vanishes precisely when* $v \in \ker \phi$*, so we can consider its zero locus to be* $\mathrm{Gr}(\mathbb{C}^{n+1}/\langle v \rangle, r) \cong \mathrm{Gr}(n, r)$*. The restriction of* W_1 *to this zero locus* $\mathrm{Gr}(n, r)$ *is* W_1*, and the restriction of the tautological sub-bundle* S *is* $S \oplus \mathcal{O}_{\mathrm{Gr}(n,r)}$*.*

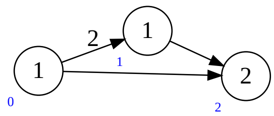
This example generalises. Let $M(Q, \mathbf{r})$ be a quiver flag variety. A choice of arrow $i \to j$ in Q determines a canonical section of $W_i^* \otimes W_j$, and the zero locus of this section is $M(Q', \mathbf{r})$, where Q' is the quiver obtained from Q by removing one arrow from $i \to j$.

Example 4.3. *Similarly,* $\mathrm{Gr}(n, r)$ *is the zero locus of a section of* S^**, the dual of the tautological sub-bundle, on* $\mathrm{Gr}(n+1, r+1)$*. The exact sequence* $0 \to W_1^* \to (\mathbb{C}^{n+1})^* \to S^* \to 0$ *shows that a global section of* S^* *is given by a linear map* $\psi : \mathbb{C}^{n+1} \to \mathbb{C}$*. The image of the section corresponding to* ψ *at the point* $s \in S$ *is* $\psi(s)$*, where we evaluate* ψ *on* s *via the tautological inclusion* $S \to \mathbb{C}^{n+1}$*. Splitting* $\mathbb{C}^{n+1} = \mathbb{C}^n \oplus \mathbb{C}$ *and choosing* ψ *to be projection to the second factor shows that* ψ *vanishes precisely when* $S \subset \mathbb{C}^n$*, that is, precisely along* $\mathrm{Gr}(n, r)$*. The restriction of* S *to this zero locus* $\mathrm{Gr}(n, r)$ *is* S*, and the restriction of* W_1 *is* $W_1 \oplus \mathcal{O}_{\mathrm{Gr}(n,r)}$*.*

4.3. **Grafting.** Let Q be a quiver and $S \subset \{1, \ldots, \rho\}$. We say that a vertex i is *graftable for* S if:

- $r_i = 1$ and $0 < i < \rho$;
- there is a path between vertex i and vertex j for all $j \in S$;
- if we remove all of the arrows from i to vertices in S, we get a disconnected quiver.

Example 4.4. *In the quiver below, vertex* 1 *is not graftable for* $\{2\}$*.*



If we removed the arrow from vertex 0 *to vertex* 2*, then vertex* 1 *would be graftable for* $\{2\}$*.*

Proposition 4.5. *Let* Q *be a quiver and let* i *be a vertex of* Q *that is graftable for* S*. Let* Q' *be the quiver obtained from* Q *by replacing each arrow* $i \to j$*, where* $j \in S$*, by an arrow* $0 \to j$*. Then*

$$M(Q, \mathbf{r}) = M(Q', \mathbf{r})$$

for any dimension vector $\mathbf{r}$*.*

Proof. Recall from §22.2 the moduli problem represented by $M(Q, \mathbf{r})$ and note, since i is graftable for S, that W_i is a line bundle. Setting

$$W_j \mapsto \begin{cases} W_i^* \otimes W_j & j \in S \\ W_j & j \notin S \end{cases}$$

transforms the moduli problem for $M(Q, \mathbf{r})$ into that for $M(Q', \mathbf{r})$. □

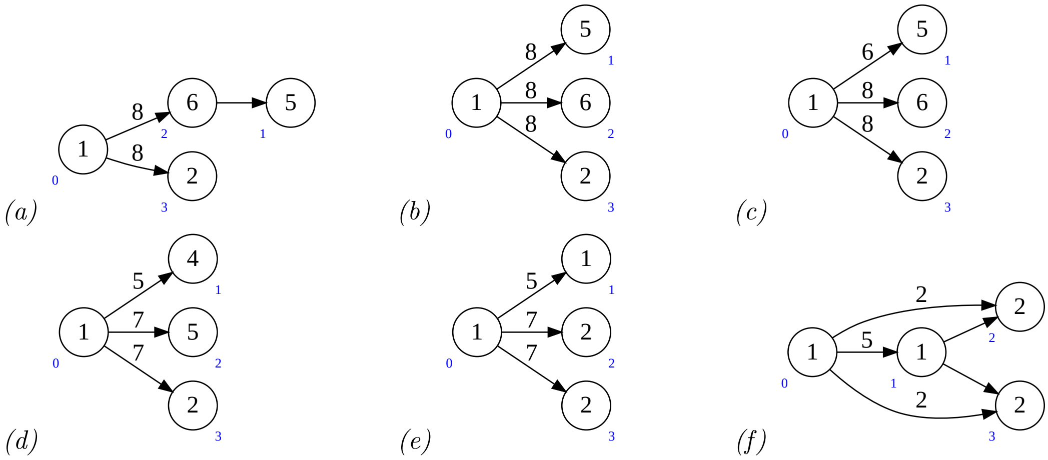
Example 4.6. *Consider the quiver flag zero locus* X *given by the quiver in (a) below, with bundle*

$$W_1 \otimes W_3 \oplus W_1^{\oplus 2} \oplus \det W_1.$$

Writing X *inside a product of Grassmannians gives* $W_1 \otimes W_3 \oplus W_1^{\oplus 2} \oplus \det W_1$ *on the quiver in (b), with arrow bundle* $S_2^* \otimes W_1$*. Removing the two copies of* W_1 *using Example 4.2 gives*

$$W_1 \otimes W_3 \oplus \det W_1$$

on the quiver in (c), with arrow bundle $S_2^* \otimes W_1$*. Applying Example 4.3 to remove* $\det W_1 = \det S_1^* = S_1^*$*, and taking care to absorb the extra factors of* W_3 *and* S_2^* *which arise from* $W_1 \otimes W_3$ *and the arrow bundle, we see that* X *is given by* $W_1 \otimes W_3$ *on the quiver in (d), with arrow bundle* $S_2^* \otimes W_1$*. Dualising at vertices* 1 *and* 2 *now gives the quiver in (e), with arrow bundle* $S_1^* \otimes W_2 \oplus S_1^* \otimes W_3$*. Finally, undoing the product representation exhibits* X *as the quiver flag variety for the quiver in (f).*



5. The ample cone

We now discuss how to compute the ample cone of a quiver flag variety. This is essential if one wants to search systematically for quiver flag zero loci that are Fano. In [9], Craw gives a conjecture that would in particular solve this problem, by relating a quiver flag variety X to a toric quiver flag variety. We give a counterexample to this conjecture, and determine the ample cone of X in terms of the combinatorics of the quiver: this is Theorem 5.12 below. Our method also involves a toric quiver flag variety: the abelianization of X.

5.1. **The multi-graded Plücker embedding.** Given a quiver flag variety $M(Q, \mathbf{r})$, Craw defines a multi-graded analogue of the Plücker embedding:

$$p : M(Q, \mathbf{r}) \hookrightarrow M(Q', \mathbf{1}) \qquad \text{with } \mathbf{1} = (1, \ldots, 1).$$

Here Q' is the quiver with the same vertices as Q but with the number of arrows $i \to j$ given by

$$\dim H^0 \left(\mathrm{Hom} \left(\det(W_i), \det(W_j) \right) \right) / S$$

where S is spanned by maps which factor through maps to $\det(W_k)$ with $i < k < j$. This induces an isomorphism p^* :
$Pic(X) \otimes \mathbb{R} \to \mathrm{Pic}(X) \otimes \mathbb{R}$ that sends $\det(W_i') \mapsto \det(W_i)$. In [9], it is conjectured that this induces a surjection of Cox rings $\mathrm{Cox}(M(Q', \mathbf{1})) \to \mathrm{Cox}(M(Q, \mathbf{r}))$. This would give information about the Mori wall and chamber structure of $M(Q, \mathbf{r})$. In particular, by the proof of Theorem 2.8 of [21], a surjection of Cox rings together with an isomorphism of Picard groups (which we have here) implies an isomorphism of effective cones.

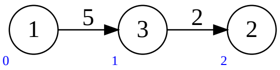
We provide a counterexample to the conjecture. To do this, we exploit the fact that quiver flag varieties are Mori Dream Spaces, and so the Mori wall and chamber structure on $\mathrm{NE}^1(M(Q, \mathbf{r}) \subset \mathrm{Pic}(M(Q, \mathbf{r})$ coincides with the GIT wall and chamber structure. This gives GIT characterizations for effective divisors, ample divisors, nef divisors, and the walls.

Theorem 5.1. [10] *Let X be a Mori Dream Space obtained as a GIT quotient of G acting on $Y = \mathbb{C}^N$ with stability condition $\omega \in \chi(G) = \mathrm{Hom}(G, \mathbb{C}^*)$. Identifying $\mathrm{Pic}(X) \cong \chi(G)$, we have that:*

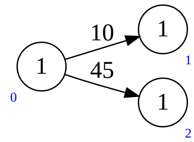
- *$v \in \chi(G)$ is ample if $Y^s(v) = Y^{ss}(v) = Y^s(\omega)$.*
- *v is on a wall if $Y^{ss}(v) \neq Y^s(v)$.*
- *$v \in \mathrm{NE}^1(X)$ if $Y^{ss} \neq \varnothing$.*

When combined with King's characterisation [18] of the stable and semistable points for the GIT problem defining $M(Q, \mathbf{r})$, this determines the ample cone of any given quiver flag variety. In Theorem 5.12 below we make this effective, characterising the ample cone in terms of the combinatorics of Q, but this is already enough to see a counterexample to the conjecture.

Example 5.2. *Consider the quiver Q and dimension vector $\mathbf{r}$ as in (a). The target $M(Q', \mathbf{1})$ of the multi-graded Plücker embedding has the quiver Q' shown in (b).*



(a)



(b)

In this case, $M(Q', \mathbf{1})$ is a product of Grassmannians and so the effective cone coincides with the nef cone, which is just the closure of the positive orthant. The ample cone of $M(Q, \mathbf{r})$ is indeed the positive orthant, as we will see later. However, the effective cone is strictly larger. We will use King's characterisation of semi-stable points with respect to a character χ of $\prod_{i=0}^{\rho} Gl(r_i)$: a representation $V = (V_i)_{i \in Q_0}$ is semi-stable with respect to $\chi = (\chi_i)_{i=0}^{\rho}$ if and only if

- $\sum_{i=0}^{\rho} \chi_i \dim_{\mathbb{C}}(V_i) = 0$; *and*
- *for any subrepresentation V' of V,* $\sum_{i=0}^{\rho} \chi_i \dim_{\mathbb{C}}(V'_i) \geq 0$.

Consider the character $\chi = (-1, 3)$ of G, which we lift to a character of $\prod_{i=0}^{\rho} Gl(r_i)$ by taking $\chi = (-3, -1, 3)$. We will show that there exists a representation $V = (V_0, V_1, V_2)$ which is semi-stable with respect to χ. The maps in the representation are given by a triple $(A, B, C) \in Mat(3 \times 5) \times Mat(2 \times 3) \times Mat(2 \times 3)$. Suppose that

$$A \text{ has full rank}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and that V' is a subrepresentation with dimensions a, b, c. We want to show that $-3a - b + 3c \geq 0$. If $a = 1$ then $b = 3$, as otherwise the image of A is not contained in V'_1. Similarly, this implies that $c = 2$. So suppose that $a = 0$. The maps B and C have no common kernel, so $b > 0$ implies $c > 0$, and $-b + 3c \geq 0$ as $b \leq 3$. Therefore V is a semi-stable point for χ, and χ is in the effective cone.

5.2. **Abelianization.** We consider now the toric quiver flag variety associated to a given quiver flag variety $M(Q, \mathbf{r})$ which arises from the corresponding abelian quotient. Let $T \subset G$ be the diagonal maximal torus. Then the action of G on $\mathrm{Rep}(Q, \mathbf{r})$ induces an action of T on $\mathrm{Rep}(Q, \mathbf{r})$, and the inclusion $i : \chi(G) \hookrightarrow \chi(T)$ allows us to interpret v as a stability condition for the action of T on $\mathrm{Rep}(Q, \mathbf{r})$. The abelian quotient is then $\mathrm{Rep}(Q, \mathbf{r}) /\!\!/_v T$. Let us see that $\mathrm{Rep}(Q, \mathbf{r}) /\!\!/_v T$ is a toric quiver flag variety. Let $\lambda = (\lambda_1, \ldots, \lambda_\rho)$ denote an element of $T = \prod_{i=1}^{\rho} (\mathbb{C}^*)^{r_i}$, where $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jr_j})$. The action of λ on the (i, j) entry x of the $r_{t(a)} \times r_{s(a)}$ matrix which is the $a \in Q_1$ component of an element in $\mathrm{Rep}(Q, \mathbf{r})$ is

$$x \mapsto \lambda_{s(a)i}^{-1} x \lambda_{t(a)j}.$$

Hence this is the same as the group action on the quiver Q^{ab} with vertices

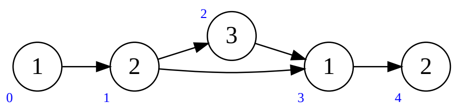
$$Q_0^{\mathrm{ab}} = \{v_{ij} : 0 \leq i \leq \rho, 1 \leq j \leq r_i\}$$

and $n_{v_{ij} v_{kl}} = n_{ik}$, the number of arrows in the original quiver. Hence

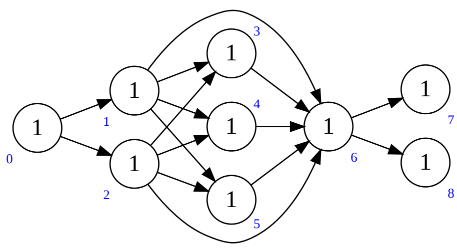
$$\mathrm{Rep}(Q, \mathbf{r}) /\!\!/_v T = M(Q^{\mathrm{ab}}, \mathbf{1}).$$

We call Q^{ab} the *abelianized* quiver.

Example 5.3. *Let Q be the quiver*



Then Q^{ab} is



Martin [22] has studied the relationship between the cohomology of Abelian and non-Abelian quotients. We state his result specialized to quiver flag varieties, then extend this to a comparison of the ample cones. To simplify notation, denote $M_Q = M(Q, \mathbf{r})$, $M_{Q^{\mathrm{ab}}} = M(Q^{\mathrm{ab}}, (1, \ldots, 1))$ and $V = \mathrm{Rep}(Q, \mathbf{r}) = \mathrm{Rep}(Q^{\mathrm{ab}}, (1, \ldots, 1))$. For $v \in \chi(G)$, let $V^s_v(T)$ denote the T-stable points of V and $V^s_v(G)$ denote the G-stable points, dropping the subscript if it is clear from context. It is easy to see that $V^s(G) \subset V^s(T)$. The Weyl group W of (G, T) is $\prod_{i=1}^{\rho} S_{r_i}$. Let $\pi : V^s(G)/T \to V^s(G)/G$ be the projection. The Weyl group acts on the cohomology of $M(Q^{\mathrm{ab}}, \mathbf{1})$, and also on the Picard group, by permuting the $W_{v_{i1}}, \ldots, W_{v_{ir_i}}$. It is well-known (see e.g. Atiyah–Bott [3]) that

$$\pi^* : H^*(V^s(G)/T)^W \cong H^*(M_Q).$$

Theorem 5.4. [22] *There is a graded surjective ring homomorphism*

$$\phi : H^*(M_{Q^{\mathrm{ab}}}, \mathbb{C})^W \to H^*(V^s(G)/T, \mathbb{C}) \xrightarrow{\pi^*} H^*(M_Q, \mathbb{C})$$

where the first map is given by the restriction $V^s(T)/T \to V^s(G)/T$. The kernel is the annihilator of $e = \prod_{i=1}^{\rho} \prod_{1 \le j,k \le r_i} c_1(W^*_{v_{ij}} \otimes W_{v_{ik}})$.

Remark 5.5. *This means that any class $\sigma \in H^*(M_Q)$ can be lifted (non-uniquely) to a class $\tilde{\sigma} \in H^*(M_{Q^{\mathrm{ab}}})$. Moreover, $e \cap \tilde{\sigma}$ is uniquely determined by σ.*

Corollary 5.6. *Let E be a representation of G and hence T. We can form the vector bundles*

$$\begin{aligned} E_G &= (V^s(G) \times E)/G \to M_Q, \\ E'_G &= (V^s(G) \times E)/T \to V^s(G)/T, \\ E_T &= (V^s(T) \times E)/T \to M_{Q^{\mathrm{ab}}}, \end{aligned}$$

where in all cases the group acts diagonally. Then $\phi(c_i(E_T)) = c_i(E_G)$.

Proof. Let f be the inclusion $V^s(G)/T \to V^s(T)/T$. Clearly $f^*(E_T) = E'_G$ as E'_G is just the restriction of E_T. Considering the square

$$\begin{array}{ccc} E'_G = (V^s(G) \times E)/T & \longrightarrow & E_G = (V^s(G) \times E)/G \\ \downarrow & & \downarrow \\ V^s(G)/T & \xrightarrow{\quad\pi\quad} & V^s(G)/G, \end{array}$$

we see that $\pi^*(E_G) = E'_G$. Then we have that $f^*(E_T) = \pi^*(E_G)$, and so in particular $f^*(c_i(E_T) = \pi^*(c_i(E_G))$. The result now follows from Martin's theorem (Theorem 5.4). □

The corollary shows that the restriction of Martin's isomorphism to degree 2 is just

$$i : c_1(W_i) \mapsto \sum_{j=1}^{r_i} c_1(W_{v_{ij}}).$$

In particular we have that $i(\omega_{M_Q}) = \omega_{M_{Q^{\mathrm{ab}}}}$.

Proposition 5.7. *Let $\mathrm{Amp}(Q)$, $\mathrm{Amp}(Q^{\mathrm{ab}})$ denote the ample cones of M_Q and $M_{Q^{\mathrm{ab}}}$ respectively. Then*

$$i(\mathrm{Amp}(Q)) = \mathrm{Amp}(Q^{\mathrm{ab}})^W.$$

Proof. Let θ be a character for G, denoting its image under $i : \chi(G) \hookrightarrow \chi(T)$ as θ as well. First note that $V^{ss}_\theta(G) \subset V^{ss}_\theta(T)$. To see this, suppose $v \in V$ is semi-stable for G, θ. Let $\lambda : \mathbb{C}^* \to T$ be a one-parameter subgroup of T such that $\lim_{t \to 0} \lambda(t) \cdot v$ exists. By inclusion, λ is a one-parameter subgroup of G, and so $\langle \theta, \lambda \rangle \ge 0$ by semi-stability of v. Hence $v \in V^{ss}_\theta(T)$. It follows that, if $\theta \in NE^1(M_Q)$, then $V^{ss}_\theta(G) \ne \varnothing$, so $V^{ss}_\theta(T) \ne \varnothing$, and hence $\theta \in NE^1(M_{Q^{\mathrm{ab}}})^W$.

Ciocan-Fontanine–Kim–Sabbah use duality to construct a projection [4]

$$p : NE_1(M_{Q^{\mathrm{ab}}}) \to NE_1(M_Q).$$

Suppose that $\alpha \in \mathrm{Amp}(Q)$. Then for any $C \in NE_1(M_{Q^{\mathrm{ab}}})$, $i(\alpha) \cdot C = \alpha \cdot p(C) \ge 0$. So $i(\alpha) \in \mathrm{Amp}(Q^{\mathrm{ab}})^W$.

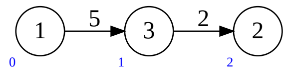
Let $\mathrm{Wall}(G) \subset \mathrm{Pic}(M_Q)$ denote the union of all GIT walls given by the G action, and similarly for $\mathrm{Wall}(T)$. Recall that $\theta \in \mathrm{Wall}(G)$ if and only if it has a non-empty strictly semi-stable locus.

Suppose $\theta \in \mathrm{Wall}(G)$, with v in the strictly semi-stable locus. That is, there exists a non-trivial $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot v$ exists and $\langle \theta, \lambda\rangle = 0$. Now we don't necessarily have $\mathrm{Im}(\lambda) \subset T$, but the image is in some maximal torus, and hence there exists $g \in G$ such that $\mathrm{Im}(\lambda) \subset g^{-1}Tg$. Consider $\lambda' = g\lambda g^{-1}$. Then $\lambda'(\mathbb{C}^*) \subset T$. Since $g \cdot v$ is in the orbit of v under G, it is semi-stable with respect to G, and hence with respect to T. In fact, it is strictly semi-stable with respect to T, since $\lim_{t\to 0} \lambda'(t)g \cdot v = \lim_{t\to 0} g\lambda(t) \cdot v$ exists, and $\langle \theta, \lambda'\rangle = \langle \theta, \lambda\rangle = 0$. So as a character of T, θ has a non-empty strictly semi-stable locus, and we have shown that

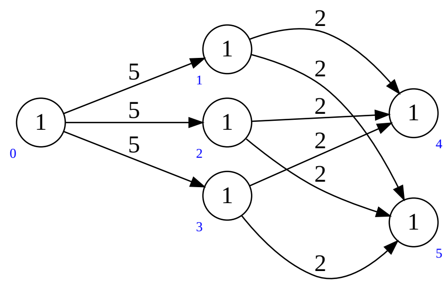
$$i(\mathrm{Wall}(G)) \subset \mathrm{Wall}(T)^W.$$

This means that the boundary of $i(\mathrm{Amp}(Q))$ is not contained in $\mathrm{Amp}(Q^{\mathrm{ab}})^W$. Since both are full dimensional cones in the W invariant subspace, the inclusion $i(\mathrm{Amp}(Q)) \subset \mathrm{Amp}(Q^{\mathrm{ab}})^W$ is in fact an equality. □

Example 5.8. *Consider again the example*



The abelianization of this quiver is



Walls are generated by collections of divisors that generate cones of codimension 1. We then intersect them with the Weyl invariant subspace, generated by $(1, 1, 1, 0, 0)$ *and* $(0, 0, 0, 1, 1)$. *In this subspace, the walls are generated by*

$$(1, 1, 1, 0, 0), \quad (0, 0, 0, 1, 1), \quad (-2, -2, -2, 3, 3).$$

This is consistent with the previous example.

5.3. **The toric case.** As a prelude to determining the ample cone of a general quiver flag variety, we first consider the toric case. Recall that a smooth projective toric variety (or orbifold) can be obtained as a GIT quotient of $\mathbb{C}^N$ by an r-dimensional torus. Let us spell this out.

Definition 5.9. *The* GIT data *for a toric variety is an r-dimensional torus K with cocharacter lattice $L = \mathrm{Hom}(\mathbb{C}^*, K)$, and N characters $D_1, \ldots, D_N \in L^\vee$, together with a stability condition $w \in L^\vee \otimes \mathbb{R}$.*

These linear data give a toric variety (or Deligne–Mumford stack) as the quotient of an open subset $U_w \subset \mathbb{C}^N$ by K, where K acts on $\mathbb{C}^N$ via the map $K \to (\mathbb{C}^*)^N$ defined by the D_i. U_w is defined as

$$\left\{ (z_1, \ldots, z_N) \in \mathbb{C}^N \;\middle|\; w \in \mathrm{Cone}(D_i : z_i \neq 0) \right\},$$

that is, its elements can have zeroes at $z_i, i \in I$, only if w is in the cone generated by D_i, $i \notin I$. Assume that all cones given by subsets of the divisors that contain w are full dimensional, as is the case for toric quiver flag varieties. Then the ample cone is the intersection of all of these.

GIT data for a toric quiver flag variety $M(Q, \mathbf{1})$ can be given as follows. The torus is $K = (\mathbb{C}^*)^\rho$. Let $e_1, \ldots, e_\rho$ be standard basis of $L^\vee = \mathbb{Z}^\rho$ and $e_0 = 0$. Then each $a \in Q_1$ gives a weight $D_a = -e_{s(a)} + e_{t(a)}$. The stability condition is $\mathbf{1} = (1, 1, \ldots, 1)$. Identify $L^\vee \cong \mathrm{Pic}\, M(Q, \mathbf{1})$. Then $D_a = W_a := W^*_{s(a)} \otimes W_{t(a)}$.

A minimal full dimensional cone for a toric quiver flag variety is given by ρ linearly independent D_{a_i}, $a_i \in Q_1$. Therefore for each vertex i with $1 \leq i \leq \rho$, we need an arrow a_i with either $s(a) = i$ or $t(a) = i$, and these arrows should be distinct. For the positive span of these divisors to contain $\mathbf{1}$ requires that D_{a_i} has $t(a_i) = i$. Fix such a set $S = \{a_1, \ldots, a_\rho\}$, and denote the corresponding cone by C_S. As mentioned, the ample cone is the intersection of such cones C_S. The set S determines a path from 0 to i for each i, given by concatenating (backwards) a_i with $a_{s(a_i)}$ and so on; let us write $f_{ij} = 1$ if a_j is in the path from 0 to i, and 0 otherwise. Then

$$e_i = \sum_{j=1}^{\rho} f_{ij} D_{a_j}.$$

This gives us a straightforward way to compute the cone C_S. Let B_S be the matrix with columns given by the D_{a_i}, and let $A_S = B_S^{-1}$. The columns of A_S are given by the aforementioned paths: the jth column of A_S is $\sum_{i=1}^{\rho} f_{ij} e_i$. If $c \in \operatorname{Amp}(Q)$, then $A_S c \in A_S \operatorname{Amp}(Q) \subset A_S C_S$. Since $A_S D_{a_i} = e_i$, this means that $A_S c$ is in the positive orthant.

Proposition 5.10. *Let $M(Q, \mathbf{1})$ be a toric quiver flag variety. Let $c \in \operatorname{Amp}(Q)$, $c = (c_1, \ldots, c_\rho)$, be an ample class, and suppose that vertex i of the quiver Q satisfies the following condition: for all $j \in Q_0$ such that $j > i$, there is a path from 0 to j not passing through i. Then $c_i > 0$.*

Proof. Choose a collection S of arrows $a_j \in Q_1$ such that the span of the associated divisors D_{a_j} contains the stability condition $\mathbf{1}$, and such that the associated path from 0 to j for any $j > i$ does not pass through i. Then the (i, i) entry of A_S is 1 and all other entries of the i^{th} row are zero. As $A_S c$ is in the positive orthant, $c_i > 0$. □

Corollary 5.11. *Let $M(Q, \mathbf{r})$ be a quiver flag variety, not necessarily toric. Then if $c = (c_1, \ldots, c_\rho) \in \operatorname{Amp}(Q)$ and $r_j > 1$, then $c_j > 0$.*

Proof. Consider the abelianized quiver. For any vertex $v \in Q_0^{\text{ab}}$, we can always choose a path from the origin to v that does not pass through v_{j1}: if there is an arrow between v_{j1} and v, then there is an arrow between v_{j2} and v, so any path through v_{j1} can be rerouted through v_{j2}. Then we obtain that the $j1$ entry of $i(c)$ is positive – but this is just c_j. □

5.4. **The ample cone of a quiver flag variety.** Let $M(Q, \mathbf{r})$ be a quiver flag variety and Q' be the associated abelianized quiver. For each $i \in \{1, \ldots, \rho\}$, define

$$T_i := \{j \in Q_0 \mid \text{all paths from 0 to } j \text{ pass through } i\},$$

but pretending that a path can always circumvent a vertex j when $r_j > 1$. This is motivated by the path structure of the abelianized quiver. Note that $i \in T_i$, and that if $r_i = 1$ then $T_i = \{i\}$.

Theorem 5.12. *The nef cone of $M(Q, \mathbf{r})$ is given by the following inequalities. Suppose that $a = (a_1, \ldots, a_\rho) \in \operatorname{Pic}(M_Q)$. Then a is nef if and only if*

$$\sum_{j \in T_i} r_j a_j \geq 0 \qquad i = 1, 2, \ldots, \rho. \tag{1}$$

Proof. We have already shown that the Weyl invariant part of the nef cone of $M_{Q'} := M(Q', \mathbf{1})$ is the image of the nef cone of $M_Q := M(Q, \mathbf{r})$ under the natural map $\pi : \operatorname{Pic}(M_Q) \to \operatorname{Pic}(M_{Q'})$. Label the vertices of Q' as v_{ij}, $i \in \{0, \ldots, \rho\}$, $j \in \{1, \ldots, r_i\}$, and index elements of $\operatorname{Pic}(M_{Q'})$ as (b_{ij}). The inequalities defining the ample cone of $M_{Q'}$ are given by a choice of arrow $A_{ij} \in Q'_1, t(A_{ij}) = v_{ij}$ for each v_{ij}. This determines a path P_{ij} from $0 \to v_{ij}$ for each vertex v_{ij}. For each v_{ij} the associated inequality is:

$$\sum_{v_{ij} \in P_{kl}} b_{kl} \geq 0. \tag{2}$$

Suppose that a is nef. First we show that a satisfies the inequalities (1). We have shown that $a_j \geq 0$ if $r_j > 1$, so it suffices to consider i such that $r_i = 1$. For each such i, take a choice of arrows such that if $v_{kl} \in P_{i1}, k \in T_i$. Moreover, choose arrows such that $s(A_{ik}) = s(A_{il})$ for all k, l. Then this gives the inequalities (1), after restricting to the Weyl invariant locus. Therefore, if C is the cone defined by (1), we have shown that $\operatorname{Nef}(M_Q) \subset C$.

Suppose now that $a \in C$ and take a choice of arrows A_{kl}. Write $\pi(a) = (a_{ij})$. We prove that the inequalities 2 are satisfied starting at $v_{\rho\rho}$. For ρ, the inequality is $a_{\rho\rho} \geq 0$, which is certainly satisfied. Suppose the $(ij+1), (ij+2), \ldots, (\rho\rho)$ inequalities are satisfied. The inequality we want to establish for (ij) is

$$a_i + \sum_{k \in T_i - \{i\}} r_k a_k + X = a_{ij} + \sum_{k \in T_i - \{i\}} \sum_{l=1}^{r_l} a_{kl} + X \geq 0,$$

where

$$X = \sum_{s(A_{kl})=v_{ij}, k\notin T_i} \left(a_{kl} + \sum_{v_{kl}\in P_{st}} a_{st} \right).$$

As $a \in C$ it suffices to show that $X \geq 0$. By the induction hypothesis $a_{kl} + \sum_{v_{kl}\in P_{st}} a_{st} \geq 0$, and therefore $X \geq 0$. This shows that $\pi(a)$ satisfies (2). □

5.5. **Nef line bundles are globally generated.** We conclude this section by proving that nef line bundles on quiver flag varieties are globally generated. This is well-known for toric varieties. This result will be important for us because in order to use the Abelian/non-Abelian Correspondence to compute the quantum periods of quiver flag zero loci, we need to know that the bundles involved are convex. Convexity is a difficult condition to understand geometrically, but it is implied by global generation.

Let $M(Q, \mathbf{r})$ be a quiver flag variety and Q' be the associated abelianized quiver. For each $i \in \{1, \ldots, \rho\}$, let T_i be as above, and define:

- $H_i := \{j \in Q_0 \mid j \neq 0 \text{ and all paths from } 0 \text{ to } i \text{ pass through } j\}$;
- $h(i) := \max H_i$; $h'(i) := \min H_i$; and
- $R_i := \{j \mid h'(j) = i\}$;

again pretending that a path can always circumvent a vertex j when $r_j > 1$. Note that $i \in H_i$. Observe too that the R_i give a partition of the vertices of Q, and that if $j, k \in T_i$, then $j, k \in R_{h'(i)}$.

Proposition 5.13. *Let L be a nef line bundle on $M(Q, r)$. Then L is globally generated.*

Proof. Let L be a nef line bundle on $M(Q, \mathbf{r})$ given by $(a_1, \ldots, a_\rho)$. A section of L is a G-equivariant section of the trivial line bundle on $\mathrm{Rep}(Q, \mathbf{r})$, where the action of G on the line bundle is given by the character $\prod \chi_i^{a_i}$. A point of $\mathrm{Rep}(Q, \mathbf{r})$ is given by $(\phi_a)_{a\in Q_1}, \phi_a : \mathbb{C}^{r_{s(a)}} \to \mathbb{C}^{r_{t(a)}}$, where G acts by change of basis. A choice of path $i \to j$ on the quiver gives an equivariant map $\mathrm{Rep}(Q, \mathbf{r}) \to \mathrm{Hom}(\mathbb{C}^{r_i}, \mathbb{C}^{r_j})$ where G acts on the image by $g \cdot \phi = g_j \phi g_i^{-1}$. If $r_i = r_j = 1$, such maps can be composed.

We will show that L is globally generated. It suffices to show this in the case that $\{j \mid a_j \neq 0\} \subset R_i$ for some i. This is because if $j \in R_i$, the inequalities in Theorem 5.12 which involve j only involve a_k for $k \in R_i$. So if $L = L_1 \otimes \cdots \otimes L_\rho$ such that L_i only has non-zero powers of $\det(W_j)$, $j \in R_i$, then L is nef if and only if all the L_i are nef. So suppose that $\{j \mid a_j \neq 0\} \subset R_i$. If $r_i > 1$, then $R_i = \{i\}$ and $L = \det(W_i)^{\otimes a_i}$, which is known to be globally generated. So we further assume that $r_i = 1$.

Let f_i be any homogenous polynomial of degree $d_i = \sum_{k\in T_i} r_k a_k \geq 0$ in the maps given by paths $0 \to i$. That is, f_i is a G-equivariant map $\mathrm{Rep}(Q, \mathbf{r}) \to \mathbb{C}$, where G acts on the image with character $\chi_i^{d_i}$. For j in R_i with $r_j = 1$ and $j \neq i$, let f_j be any homogenous polynomial of degree $\sum_{k\in T_j} r_k a_k \geq 0$ in the maps given by paths $h(j) \to j$ (note that by definition, $r_{h(j)} = 1$). That is, f_j is a G-equivariant map $\mathrm{Rep}(Q, \mathbf{r}) \to \mathbb{C}$, where G acts on the image with character $\chi_{h(j)}^{-d_j} \chi_j^{d_j}$. For j in R_i with $r_j > 1$, let f_j be a homogenous polynomial of degree $a_k \geq 0$ in the minors of the matrix whose columns are given by the paths $h(j) \to j$. That is, f_j is a G-equivariant map $\mathrm{Rep}(Q, \mathbf{r}) \to \mathbb{C}$, where G acts on the image with character $\chi_{h(j)}^{-r_j a_j} \chi_j^{a_j}$. For any $x \in \mathrm{Rep}(Q, \mathbf{r})$ which is semi-stable, and for any $j \in R_i$, there exists such an f_j with $f_j(x) \neq 0$, by construction of $h(j)$. Define $s' := \prod_{j\in R_i} f_j : \mathrm{Rep}(Q, \mathbf{r}) \to \mathbb{C}$, and take the section s to be a sum of such s'. Then s is a G-equivariant map $\mathrm{Rep}(Q, \mathbf{r}) \to \mathbb{C}$, where G acts on the image with character

$$\prod_{j\in R_i} \chi_j^{b_j} = \chi_i^{d_i} \cdot \prod_{j\in R_i, j\neq i, r_j=1} \chi_{h(j)}^{-d_j} \chi_j^{d_j} \cdot \prod_{j\in R_i, j\neq i, r_j>1} \chi_{h(j)}^{-r_j a_j} \chi_j^{a_j}.$$

We need to check that $b_j = a_j$ for all j. This is obvious for $j \in R_i$ with $r_j > 1$. For $j = i$,

$$b_i = \sum_{j\in T_i} r_j a_j - \sum_{k\in R_i, k\neq i, h(k)=i} \sum_{j\in T_k} r_k a_k.$$

This simplifies to a_i in view of the fact that for all $j \in T_i$, there is a unique k such that $h(k) = i$ and $j \in T_k$. The check for $j \in R_i$ with $r_j = 1$ is similar. Therefore s gives a well-defined section of L. For any $x \in \mathrm{Rep}(Q, \mathbf{r})$ semi-stable, there exists an s such that $s(x) \neq 0$, so L is globally generated.

□

6. The Abelian/non-Abelian Correspondence

6.1. **A brief review of Gromov–Witten theory.** We very briefly review Gromov–Witten invariants and J-functions, in order to state the Abelian/non-Abelian correspondence. See [7] for more details and references.

Let X be a smooth projective variety. Given $g, n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X)$, let $M_{g,n}(X, \beta)$ be the moduli space of stable maps to X of genus g, class β, and with n marked points. We always consider $g = 0$. While this space may have components of different dimensions, it has a *virtual fundamental class* $[M_{0,n}(X, \beta)]^{virt}$ of the expected dimension. There are natural evaluation maps $ev_i : M_{0,n}(X, \beta) \to X$ taking a class of a stable map $f : C \to X$ to $f(x_i)$, where $x_i \in C$ is the i^{th} marked point. There is a universal curve $M_{0,n+1}(X, \beta) \to M_{0,n}(X, \beta)$, which has a line bundle whose fiber at $f : C \to X$ is the tangent space at x_i. The first Chern class of the push-forward of this line bundle to $M_{0,n}(X, \beta)$ is denoted ψ_i. Define:

$$\langle \tau_{a_1}(\alpha_1), \ldots, \tau_{a_n}(\alpha_n) \rangle_{n,\beta} = \int_{[M_{0,n}(X,\beta)]^{virt}} \prod_{i=1}^{n} ev_i^*(\alpha_i)\psi_i^{a_i}.$$

If $a_i = 0$ for all i, this is called a (genus 0) Gromov-Witten invariant and the τ notation is omitted; otherwise it is called a descendent invariant. It is deformation invariant.

Gromov-Witten invariants have been packaged in the theory in different ways: quantum cohomology, Frobenius manifolds, differential equations, and generating functions. We will use a generating function called the *J-function*.

Denote q^β for the element of $\mathbb{Q}[H_2(X)]$ representing β. Define $N(X)$ as the Novikov ring of X:

$$\{ \sum_{\beta \in NE_1(X)} c_\beta q^\beta; c_\beta \in \mathbb{C} \}.$$

The J-function assigns an element of $H^*(X) \otimes N(X)[[z^{-1}]]$ to every element of $H^*(X)$. The assignment is given as follows. Let $\phi_0 = 1, \phi_1, \ldots, \phi_N$ be a homogenous basis of $H^*(X)$, such that $\phi_1, \ldots, \phi_M \in H^2(X)$. Let ϕ^i be the Poincaré dual basis. Then the J-function maps

$$T \in H^*(X) \mapsto 1/z(z + T + \sum_i \langle\langle \phi_i/(z - \psi) \rangle\rangle \phi^i).$$

We divide by z to match with the notation of [7]. Here

$$\langle\langle \phi_i/(z - \psi) \rangle\rangle = \sum_{\beta \in NE_1(X)} q^\beta \sum_{n=0}^{\infty} \frac{1}{n!} \langle \frac{\phi_i}{z - \psi}, T, \ldots, T \rangle_{n+1,\beta}.$$

The *small* J-function is the restriction of the J-function to $H^0(X) \oplus H^2(X)$. The quantum period $G_X(t)$ is $J(\phi_1)$ under the map $q^\beta \to t^{\langle -K_X, \beta \rangle}$. For Fano varieties this is a power series in t. The quantum period satisfies an important differential equation called the quantum differential equation. The small J-function of toric complete intersections and the J-function of toric varieties is known. See [7] for the closed forms.

If $Y \subset X$ is the zero locus of a generic section of a convex vector bundle $E \to X$, then one can define the twisted J-function $J_{Y,X}$ which has the same target as J_X. It is defined using Gromov-Witten invariants *twisted* by the Euler class of the bundle $E_{0,n,\beta} = \pi_*(ev_{n+1}^*(E)) \to M_{0,n}(X, \beta)$. Here $\pi : M_{0,n+1}(X, \beta) \to M_{0,n}(X, \beta)$ is the universal curve, and $ev_{n+1} : M_{0,n}(X, \beta) \to X$ is the evaluation map. $E_{0,n,\beta}$ is a vector bundle because E is convex. That is, the invariants defining the twisted J-function are:

$$\langle \tau_{a_1}(\alpha_1), \ldots, \tau_{a_n}(\alpha_n) \rangle_{n,\beta}^{tw} = \int_{[M_{0,n}(X,\beta)]^{virt}} e(E_{0,n,\beta}) \prod_{i=1}^{n} ev_i^*(\alpha_i)\psi_i^{a_i}.$$

Using the embedding $j : Y \to X$, one can relate J_Y (defined as above) and $J_{Y,X}$; this is the statement of the conjecture of Cox (proved by [17]) that the virtual fundamental class is functorial. See [5] for the equivalent statement expressed in terms of J-functions. The upshot is that one can compute the quantum period of Y from the twisted J-function. Since the computational part of the paper is concerned with the quantum period of Fano fourfolds which are quiver zero loci, we will use the twisted J-functions.

The abelian/non-abelian correspondence is a conjecture [4] relating the J-functions (and more broadly, the Frobenius manifolds that arise from quantum cohomology) of GIT quotients $V/\!\!/G$ and $V/\!\!/T$, where $T \subset G$ is the maximal torus. It also extends to considering zero loci of representation theoretic bundles, by relating the associated twisted J-functions. In the next section, we give a closed form for the J-function of a Fano quiver flag zero locus by proving the conjecture for quiver flag varieties.

6.2. **Proof.** As the abelianization of a quiver flag variety is a toric quiver flag variety, the Abelian/non-Abelian correspondence conjectures a closed form for the J-functions of Fano quiver flag zero loci. [4] proved the conjecture in the case of flag manifolds. We can use this to prove the conjectures for quiver flag varieties.

We give the J-function in the way usual in the literature: first, by defining an I-function (which should be understood as a mirror object, but we omit this perspective here), then relating the J-function to the I-function. We follow the construction given by [4] in our special case.

Let X be a Fano quiver flag zero locus given by (Q, E_G), where the ambient quiver flag $M_Q = M(Q, \mathbf{r})$ variety is Fano. Let (Q^{ab}, E_T) be the associated abelianized quiver and bundle, $M_{Q^{\mathrm{ab}}} = M(Q^{\mathrm{ab}}, (1, \ldots, 1))$. Assume, moreover, that E_T splits into ample line bundles. To define the I-function, we need to relate the Novikov rings of M_Q and $M_{Q^{\mathrm{ab}}}$. Let $\mathrm{Pic}(Q)$ ($\mathrm{Pic}(Q^{\mathrm{ab}})$) denote the Picard group of M_Q ($M_{Q^{\mathrm{ab}}}$), and similarly for the cones of effective curves and effective divisors. The isomorphism $\mathrm{Pic}(Q) \to \mathrm{Pic}(Q^{\mathrm{ab}})^W$ gives a projection $p : NE_1(Q^{\mathrm{ab}}) \to NE_1(Q)$. In the bases dual to $\det(W_1), \ldots, \det(W_\rho)$ of $\mathrm{Pic}(Q)$ and $W_{ij}, 1 \leq i \leq \rho, 1 \leq j \leq r_i$ of $\mathrm{Pic}(Q^{\mathrm{ab}})$, the projection is given by

$$(d_{1,1}, \ldots, d_{1,r_1}, d_{2,1}, \ldots, d_{\rho,r_\rho}) \mapsto (\sum_{i=1}^{r_1} d_{1i}, \ldots, \sum_{i=1}^{r_\rho} d_{\rho i}).$$

For $\beta = (d_1, \ldots, d_\rho)$, define

$$\epsilon(\beta) = \sum_{i=1}^{\rho} d_i(r_i - 1).$$

Therefore, following (3.2.1) in [4], the map of Novikov rings $N(M_{Q^{\mathrm{ab}}}) \to N(M_Q)$ is

$$f(\sum_{\tilde{\beta}} c_{\tilde{\beta}} A^{\tilde{\beta}}) = \sum_{\beta} (-1)^{\epsilon(\beta)} (\sum_{p(\tilde{\beta})=\beta} c_{\tilde{\beta}}) q^{\beta}.$$

For a representation theoretic bundle V_G of rank r on M_Q, let $D_1, \ldots, D_r$ be the divisors of $M_{Q^{\mathrm{ab}}}$ giving the split bundle V_T. Given $\tilde{d} \in NE_1(M_{Q^{\mathrm{ab}}})$ define

$$I_{V_G}(\tilde{d}) = \frac{\prod_{i=1}^r \prod_{m \leq \langle \tilde{d}, D_i \rangle} (D_i + mz)}{\prod_{i=1}^r \prod_{m \leq 0} (D_i + mz)}.$$

Notice that the factors cancel to a finite product. If V is K-theoretically a representation theoretic bundle, in the sense that there exists A_G, B_G such that

$$0 \to A_G \to B_G \to E \to 0,$$

is an exact sequence, we can define

$$I_E = \frac{I_{B_G}}{I_{A_G}}.$$

For example, the Euler sequence from Proposition 2.3 shows that for the tangent bundle T_{M_Q}

$$I_{T_{M_Q}}(\tilde{d}) = \frac{\prod_{a \in Q_1^{ab}} \prod_{m \leq 0} (D_a + mz)}{\prod_{a \in Q_1^{ab}} \prod_{m \leq \langle \tilde{d}, D_a \rangle} (D_a + mz)} \frac{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq \langle \tilde{d}, D_{ij} - D_{ik} \rangle} (D_{ij} - D_{ik} + mz)}{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq 0} (D_{ij} - D_{ik} + mz)}.$$

Here D_a is the divisor on $M_{Q^{ab}}$ corresponding to the arrow $a \in Q_1^{ab}$, and D_{ij} is the divisor corresponding to the tautological bundle W_{ij} for vertex ij.

Similarly, if X is a quiver flag zero locus in M_Q defined by the bundle V_G, by the adjunction formula

$$I_{T_X}(\tilde{d}) = \frac{I_{T_{M_Q}}(\tilde{d})}{I_{V_G}(\tilde{d})}.$$

Define the small twisted I-function of $X \subset M_Q$ as

$$I_{X,M_Q}(\tau, z) = \sum_{d \in NE_1(Q)} e^{\int_\tau d} \Big(\sum_{\tilde{d} \in NE_1(Q^{\mathrm{ab}}), p(\tilde{d}) = d} f(q^{\tilde{d}}) I_{T_X}(\tilde{d}) \Big)$$

where (so $\tau \in H^0(M_Q) \oplus H^2(M_Q)$).

Theorem 6.1. *Let X be a Fano quiver flag zero locus given by (Q, E_G), where the ambient quiver flag $M_Q = M(Q, \mathbf{r})$ variety is Fano. If X has Fano index ≥ 2, then the I-function is the J-function. If X has Fano index 1, then*

$$J_{X,M_Q}(\tau, z) = e^{-C(\tau)} I_{X,M_Q}(\tau, z),$$

where:

$$I_{X,M_Q}(\tau, z) = 1 + \frac{1}{z}(\tau + C(\tau)) + O(1/z^2).$$

Proof. Notice that

$$(3) \qquad I_{T_X}(\tilde{d}) = \frac{\prod_{a \in Q_1^{ab}} \prod_{m \leq 0} (D_a + mz)}{\prod_{a \in Q_1^{ab}} \prod_{m \leq \langle \tilde{d}, D_a \rangle} (D_a + mz)} \frac{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq \langle \tilde{d}, D_{ij} - D_{ik} \rangle} (D_{ij} - D_{ik} + mz)}{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq 0} (D_{ij} - D_{ik} + mz)}$$

is Weyl anti-invariant; therefore, using the remark after Theorem 5.4, it can be interpreted as an element the Novikov ring $N(M_Q)$, the target of the twisted J-function of X.

In [4], the Abelian/non-Abelian correspondence is proved for Grassmannians and representation theoretic subvarieties of Grassmannians. We now find the I-function using the construction of quiver flag varieties as quiver flag zero loci.

Let $Y = \prod_{i=1}^{\rho} \mathrm{Gr}(\tilde{s}_i, r_i)$, and $\tilde{Y} = \prod_{i=1}^{\rho} (\mathbb{P}^{\tilde{s}_i - 1})^{r_i}$ its abelianization. We compare the I-function above with the I-function coming from $M_Q \subset Y$ using the canonical identification of the Picard group of M_Q with that of Y and the Picard group of $M_{Q^{ab}}$ with that of $\tilde{Y}$. Both Y and M_Q are GIT quotients by the same group; we can therefore canonically identify a representation theoretic vector bundle E'_G on Y such that $E'_G|_{M_Q}$ is E_G. Using the identification of Picard groups, it is clear that $I_{E'_G} = I_{E_G}$. It therefore suffices to compare I_{M_Q} with $I_{M_Q \subset Y}$.

If Z is a Fano quiver flag locus on Y cut out by a section of W_G, then [4] states that the I-function of $Z \subset Y$ is

$$I_{Z,Y}(\tau, z) = \sum_{d \in NE_1(Y)} e^{\int_\tau d} \Big(\sum_{\tilde{d} \in NE_1(\tilde{Y}), p(\tilde{d}) = d} f(q^{\tilde{d}}) I_{Ab}(\tilde{d}) I_{W_G}(\tilde{d}) \Big).$$

I_{W_G} is as above, and I_{Ab} is defined to be

$$(4) \quad I_{Ab}(\tilde{d}) = \frac{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq \langle \tilde{d}, D_{ij} - D_{ik} \rangle} (D_{ij} - D_{ik} + mz)}{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq 0} (D_{ij} - D_{ik} + mz)} \frac{\prod_{i=1}^{\rho} \prod_{j=1}^{r_i} \prod_{m \leq 0} (D_{ij} + mz)^{\tilde{s}_i}}{\prod_{i=1}^{\rho} \prod_{j=1}^{r_i} \prod_{m \leq \langle \tilde{d}, D_{ij} \rangle} (D_{ij} + mz)^{\tilde{s}_i}}.$$

We now determine $I_V(\tilde{d})$ for the vector bundle which V cuts out M_Q from Y:

$$V = \bigoplus_{i=2}^{\rho} Q_i \otimes \mathbb{C}^{\tilde{s}_i *} / F_i^*,$$

Recall from Section 3 that $\tilde{s}_i$ is the number of paths from 0 to i and $F_i = \bigoplus_{t(a)=i} Q_{s(a)}$. V is globally generated and hence convex. It is not representation theoretic, but it is K-theoretically:

$$0 \to F_i^* \otimes Q_i \to (\mathbb{C}^{\tilde{s_i}})^* \otimes Q_i \to (\mathbb{C}^{\tilde{s_i}})^* \otimes Q_i / F_i^* \to 0.$$

This is enough: the proof of Theorem 6.1.2 in [4] can be adjusted almost without change to this situation. Suppose A_G and B_G are homogenous vector bundles, and

$$0 \to A_G \to B_G \to V \to 0.$$

Then we can also consider

$$0 \to A_T \to B_T \to F \to 0$$

on the abelianization, and define $V_T := F$. Using the notation of the proof of the theorem, the point is that

$$\triangle(V) \triangle(A_G) = \triangle(B_G).$$

Here, $\triangle(V)$ is precisely the twist for the twisted Gromov–Witten invariants that appears in Quantum Lefschetz. We can then follow the same argument for

$$\triangle(B_G)/\triangle(A_G).$$

After abelianizing, we obtain $\triangle(B_T)/\triangle(A_T) = \triangle(F)$.

Now consider the twisting for quiver flag varieties. Note that

$$(F_i^* \otimes Q_i)_T = \bigoplus_{a\in Q_1, t(a)=i} (\bigoplus_{j=1}^{r_{s(a)}} W_{s(a)j}) \bigotimes \bigoplus_{k=1}^{r_i} W_{ik}$$

and

$$(\mathbb{C}^{\tilde{s}_i *} \otimes Q_i)_T = \bigoplus_{k=1}^{r_i} W_{ik}^{\oplus \tilde{s}_i}.$$

Finally, we see that

$$I_V(\tilde{d}) = \frac{\prod_{a\in Q_1^{ab}} \prod_{m\leq 0}(D_a + mz)}{\prod_{a\in Q_1^{ab}} \prod_{m\leq \langle d, D_a\rangle}(D_a + mz)} \left(\frac{\prod_{i=1}^{\rho}\prod_{j=1}^{r_i}\prod_{m\leq 0}(D_{ij}+mz)^{\tilde{s}_i}}{\prod_{i=1}^{\rho}\prod_{j=1}^{r_i}\prod_{m\leq \langle \tilde{d}, D_{ij}\rangle}(D_{ij}+mz)^{\tilde{s}_i}}\right)^{-1} \tag{5}$$

The total correction term to the I-function of $M_Q \subset Y$ is the product of equation 5 with equation 4. This agrees with equation 3, which concludes the proof. □

With this theorem, the quantum period of homogenous Fano fourfolds in quiver flag varieties can be computed. In the appendix, joint work with T. Coates and A. Kasprzyk, all Fano fourfolds cut out of quiver flag varieties of dimension less than 8 by globally generated bundles are found, and their quantum periods computed. 139 of these are new.

7. Appendix

T. Coates, E. Kalashnikov, A. Kasprzyk

7.1. **Algorithms.** Our method is as follows. The first step is to find all Fano quiver flag varieties of dimension less than or equal to 8. We do this inductively on ρ (such a quiver must have $\rho \leq 8$). First, we find all Grassmannians ($\rho = 1$) of dimension less than or equal to 8. The induction step takes a given quiver with data $(Q, \mathbf{r})$, and finds all possible quivers which are 'grown' out of Q by adding a vertex and arrows in all possible ways, as well as an extra term of the dimension vector, such that the dimension of the quiver flag variety is less than or equal to 8. We then compute the ample cones, and discard any which are not Fano. We find 223044 Fano quiver flag varieties of dimension ≤ 8; 223017 of dimension $4 \leq d \leq 8$. Of these 50617 (respectively 50612) are non-toric quiver flag varieties.

The second step is to find all quiver flag zero loci. Using the equivalences of quiver flag varieties described above, we only consider quivers which are fully graft reduced (they have no branches which are graftable). To ensure that the zero locus of a generic section of a vector bundle is smooth, we need to assume that the vector bundle is globally generated. Globally generated also implies that the bundle is convex, which allows us to compute the quantum period. Given a quiver flag variety $M(Q, \mathbf{r})$, we search for vector bundles which are direct sums of bundles of the form

$$S^{\alpha_1}(W_1) \otimes \cdots \otimes S^{\alpha_\rho}(W_\rho).$$

Here $S^\alpha(W_i)$ is a positive Schur power of W_i. $S^\alpha(W_i)$ is globally generated by Remark 3.2.

Given a Fano quiver flag variety $M(Q, \mathbf{r})$ of dimension $4+c$, $c \leq 4$ with anti-canonical class $-K_Q$ and nef cone $\mathrm{Nef}(Q)$, we first find all suitable irreducible vector bundles. Let $\mathrm{Irr}(Q)$ be the set of all bundles of the above form such that $-K_Q - c_1(E)$ is ample. Write $\mathrm{Irr}(Q) = \mathrm{Irr}(Q)_1 \sqcup \mathrm{Irr}(Q)_2$, where $\mathrm{Irr}(Q)_1$ contains vector bundles of rank strictly larger than 1, and $\mathrm{Irr}(Q)_2$ contains only line bundles. We then search for bundles E on $M(Q, \mathbf{r})$ such that E is a direct sum of bundles from $\mathrm{Irr}(Q)$ and $-c_1(E) - K_Q \in \mathrm{Nef}(Q)$. The different combinatorial properties of line bundles and vector bundles mean that it is best to treat them separately. Therefore, we instead search for two vector bundles, E_1, E_2 such that $E = E_1 \oplus E_2$ and E_i is a direct sum of bundles from $\mathrm{Irr}(Q)_i$.

For each $x \in \mathrm{Nef}(Q)$ such that $-K_Q - x$ is ample, we find all possible ways to write x as

$$x = \sum_{i=1}^{l} a_i$$

where a_i are (possibly repeated) elements of a Hilbert basis. This is a knapsacking type problem. For each partition of the a_i into at most $c/2$ subsets $S_1, \dots, S_s$, we find all possible choices of $F_1, \dots, F_s \in \mathrm{Irr}_1$ such that

$$c_1(F_i) = \sum_{j \in S_i} a_j, \mathrm{rank}(F_1) + \cdots + \mathrm{rank}(F_s) = \tilde{c} \leq c.$$

Set $E_1 = \oplus_j F_j$. Then for each $y \in \mathrm{Nef}(Q)$ such that $-K_Q - x - y$ is ample, we again find all ways of writing

$$y = \sum_{i=1}^{p} b_i$$

as a sum of Hilbert basis elements. Each partition of the b_i into $c - \tilde{c}$ subsets gives a choice of line bundles $L_1, \dots, L_{c-\tilde{c}} \in \mathrm{Irr}_2(Q)$, and $E_2 = \oplus L_j$.

In this we find all possible Fano four dimensional quiver flag zero loci which have vector bundles of the given form. We check that they are non-empty and connected. Taking into account the equivalences of quiver flag varieties given by dualising and removing arrow bundles, we compute their quantum periods, Euler characteristic, Euler number $\chi(T_X)$, degree $(-K_X)^4$, and Hilbert series. We find 139new quantum periods, which means we find at least 139new Fano varieties. In 1 and 2 we describe one construction of a Fano fourfold for each new quantum period. In Figure 1 we plot their Euler numbers against their degrees.

7.2. **Picard rank one quiver zero loci.** As mentioned in the Introduction, recent conjectures of Coates, Corti, Galkin, Golyshev, Kasprzyk and Tveiten relate the classification of Fano manifolds to mirror symmetry. Specifically, it is expected that Fano n-manifolds up to deformation correspond to maximally mutable Laurent polynomials in n variables up to mutation [2] [12]. In this section we test these conjectures for quiver flag zero loci in Grassmannians (which were classified by [19], and are sometimes called *Küchle varieties*).

Throughout this section, let $\mathrm{Gr}(n, r)$ denote the Grassmannian of r-dimensional quotients of $\mathbb{C}^n$, and let $Q \to \mathrm{Gr}(n, r)$ denote the universal bundle of quotients. For some complete intersections in in flags (that is, quiver flag zero loci in flag varieties where the bundles are direct sums of line bundles), a method of Prince [28] allows one to find a mirror Laurent polynomial. In general, however, no methods are known for either more general quiver flag varieties or more general bundles.

For each Küchle variety, if Prince's method [28] applies, we use it to find a Laurent polynomial mirror and then compute the regularized quantum differential operator L_X using mirror symmetry and either Lairez's algorithm [20] or brute-force linear algebra. Otherwise, we compute the first 100 entries of the Picard sequence in order to determine the quantum differential operator, and then search for a maximally mutable Laurent polynomial which has the correct Hilbert series and Picard–Fuchs operator. We compute the regularized quantum differential operators and ramification data of all but one of the quiver flag zero loci found in Grassmannians.

7.2.1. $X_{3,1} \subset \mathrm{Gr}(5, 2)$. This is the subvariety X cut out of $\mathrm{Gr}(5, 2)$ by a regular section of $\det Q \oplus (\det Q)^{\otimes 3}$. It has degree 15. We can compute a Laurent polynomial mirror for X using Prince's method [28]:

$$f = (1 + x + y + z + w)^3 \left(\frac{1}{xy} + \frac{1}{xw} + \frac{1}{zw} \right)$$

This is rigid maximally mutable. We then determine the quantum differential operator using mirror symmetry and Lairez's algorithm. The quantum differential operator L_X is:

$$\begin{aligned}(18t+1)^4(5751t^2+261t-1)D^4 + 18t(18t+1)^3(57510t^2+3366t+23)D^3 +\\ 3t(18t+1)^2(21738780t^3+1821528t^2+37251t+88)D^2 +\\ 3t(18t+1)(558997200t^4+66379824t^3+2450412t^2+27342t+19)D +\\ 72t^2(201238992t^4+32516316t^3+1852146t^2+42579t+308)\end{aligned}$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = \infty$$

$$\begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \quad \text{at } s = -18$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } s^2 - 261s - 5751 = 0$$

The ramification defect of L_X is 1.

7.2.2. $X_{2,2} \subset \mathrm{Gr}(5, 2)$. This is the subvariety X cut out of $\mathrm{Gr}(5, 2)$ by a regular section of $(\det Q)^{\otimes 2} \oplus (\det Q)^{\otimes 2}$. It has degree 20. We can compute a Laurent polynomial mirror for X using Prince's method:

$$f = (1 + x + y + z)^2(1 + w)^2 \left(\frac{1}{xw} + \frac{1}{xzw} + \frac{1}{yzw} \right)$$

This is rigid maximally mutable. We then determine the quantum differential operator using mirror symmetry and Lairez's algorithm.

The quantum differential operator L_X is:

$$(12t + 1)^4(2224t^2 + 152t - 1)D^4 + 8t(12t + 1)^3(33360t^2 + 2936t + 29)D^3+$$
$$4t(12t + 1)^2(2802240t^3 + 352704t^2 + 10792t + 37)D^2+$$
$$32t(12t + 1)(6004800t^4 + 1070496t^3 + 59328t^2 + 993t + 1)D+$$
$$576t^2(1921536t^4 + 465984t^3 + 39856t^2 + 1377t + 15)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = \infty$$

$$\begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{at } s = -12$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } s^2 - 152s - 2224 = 0$$

The ramification defect of L_X is 1.

7.2.3. $X_{\mathrm{Sym}^2,\det^2} \subset \mathrm{Gr}(6, 2)$. This is the subvariety X cut out of $\mathrm{Gr}(6, 2)$ by a regular section of $(\mathrm{Sym}^2 Q) \oplus (\det Q)^{\otimes 2}$. It has degree 40. Computing 100 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence [4] suggests that the quantum differential operator L_X is:

$$(4t+1)^4(20t+1)(60t-1)D^4+8t(4t+1)^3(60t+7)(100t+1)D^3+4t(4t+1)^2(168000t^3+44480t^2+2368t+9)D^2+$$
$$8t(4t+1)(480000t^4+210560t^3+26064t^2+772t+1)D+64t^2(115200t^4+73920t^3+15904t^2+1247t+23)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = \infty$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = 60$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = -20$$

$$\begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \quad \text{at } s = -4$$

The ramification defect of L_X is 1. The rigid maximally mutable Laurent polynomial

$$f = \frac{xz^2}{w}+\frac{3xz}{w}+\frac{3x}{w}+\frac{x}{zw}+\frac{xz^2}{yw}+\frac{3xz}{yw}+\frac{3x}{yw}+\frac{x}{yzw}+2yz+4y+\frac{y}{w}+\frac{2y}{z}+\frac{y}{zw}+2z+\frac{2}{w}+\frac{2}{z}+\frac{2}{zw}+\frac{z^2w}{y}+\frac{3zw}{y}$$
$$+\frac{3w}{y}+\frac{1}{yw}+\frac{w}{yz}+\frac{1}{yzw}+\frac{y^2w}{x}+\frac{y^2w}{xz}+\frac{3yw}{x}+\frac{3yw}{xz}+\frac{3w}{x}+\frac{3w}{xz}+\frac{w}{xy}+\frac{w}{xyz}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.4. $X_{\det,\det,Q\otimes\det} \subset \mathrm{Gr}(6,2)$. This is the subvariety X cut out of $\mathrm{Gr}(6,2)$ by a regular section of $\det Q \oplus \det Q \oplus (Q \otimes \det Q)$. It has degree 33. Computing 100 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence suggests that the quantum differential operator L_X is:

$$(7t+1)^3(84t+11)^2(4887t^3+1188t^2+63t-1)D^4+6t(7t+1)^2(84t+11)(4789260t^4+1957662t^3+276525t^2+14788t+182)D^3+$$
$$t(7t+1)(59137782480t^6+35797617576t^5+8618367303t^4+1034016732t^3+62484336t^2+1613464t+7700)D^2+$$
$$t(591377824800t^7+411395772312t^6+117575488926t^5+17614041858t^4+1452921204t^3+62330565t^2+1099296t+1694)D+$$
$$6t^2(47310225984t^6+31376786616t^5+8478247932t^4+1186308228t^3+89634132t^2+3397911t+48400)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = \infty$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = -7$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } s^3 - 63s^2 - 1188s - 4887 = 0$$

The ramification defect of L_X is 1. The rigid maximally mutable Laurent polynomial

$$f = \frac{x^2w}{y}+3xw+2x+\frac{2xz}{y}+\frac{3x}{y}+\frac{x}{yz}+3yw+4y+\frac{y}{w}+5z+\frac{2z}{w}+\frac{3}{w}+\frac{2}{z}+\frac{1}{zw}+\frac{z^2}{y}+\frac{z^2}{yw}+\frac{3z}{y}+\frac{3z}{yw}+\frac{3}{y}+\frac{3}{yw}$$
$$+\frac{1}{yz}+\frac{1}{yzw}+\frac{y^2w}{x}+\frac{2y^2}{x}+\frac{y^2}{xw}+\frac{3yz}{x}+\frac{3yz}{xw}+\frac{4y}{x}+\frac{4y}{xw}+\frac{y}{xz}+\frac{y}{xzw}+\frac{z^2}{x}+\frac{3z^2}{xw}+\frac{3z}{x}$$
$$+\frac{8z}{xw}+\frac{3}{x}+\frac{7}{xw}+\frac{1}{xz}+\frac{2}{xzw}+\frac{z^3}{xyw}+\frac{4z^2}{xyw}+\frac{6z}{xyw}+\frac{4}{xyw}+\frac{1}{xyzw}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.5. $X_{1,1,1,2} \subset \mathrm{Gr}(6,2)$. This is the subvariety X cut out of $\mathrm{Gr}(6,2)$ by a regular section of $\det Q \oplus \det Q \oplus \det Q \oplus (\det Q)^{\otimes 2}$. It has degree 28. We can compute a Laurent polynomial mirror for X using Prince's method:

$$f = (1+x)^2(1+y)(1+z)(1+w)\left(\tfrac{1}{x} + \tfrac{1}{xw} + \tfrac{1}{xzw} + \tfrac{1}{xyzw}\right) - 8$$

This is rigid maximally mutable. We then determine the quantum differential operator using mirror symmetry and Lairez's algorithm. The quantum differential operator L_X is:

$$(8t+1)^4(12t+1)(100t-1)D^4+64t(8t+1)^3(1500t^2+163t+2)D^3+2t(8t+1)^2(1344000t^3+227968t^2+9258t+41)D^2+ 2t(8t+1)(15360000t^4+3856384t^3+296112t^2+6692t+9)D+32t^2(3686400t^4+1287168t^3+156816t^2+7583t+112)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=\infty$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=100$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=-12$$

$$\begin{pmatrix} \frac{2}{3}&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{2}&0\\0&0&0&\frac{1}{3} \end{pmatrix} \quad \text{at } s=-8$$

The ramification defect of L_X is 1.

7.2.6. $X_{\mathrm{Sym}^2,\mathrm{Sym}^2} \subset \mathrm{Gr}(7,2)$. This is the subvariety X cut out of $\mathrm{Gr}(7,2)$ by a regular section of $\mathrm{Sym}^2 Q \oplus \mathrm{Sym}^2 Q$. It has degree 80. Computing 100 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence suggests that the quantum differential operator L_X is:

$$((4t-1)^3(12t+1)^2(20t-1)(36t-5)^2)D^4+8t(4t-1)^2(12t+1)(36t-5)(43200t^3-11088t^2+836t+1)D^3+ 4t(4t-1)(522547200t^6-256877568t^5+42133248t^4-1762944t^3-158896t^2+13356t+5)D^2+ 32t^2(36t-5)(10368000t^5-5750784t^4+1053504t^3-56976t^2-2712t+245)D+ 64t^2(89579520t^6-59470848t^5+14637312t^4-1531344t^3+37512t^2+4095t-175)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=\infty$$

$$\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=4$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=20$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&\frac{1}{2}&1\\0&0&0&\frac{1}{2} \end{pmatrix} \quad \text{at } s=-12$$

The ramification defect of L_X is 1. The rigid maximally mutable Laurent polynomial

$$f = xz^3w + 2xz + \frac{x}{zw} + \frac{xzw}{y} + \frac{2xw}{y} + \frac{xw}{yz} + 2zw + \frac{2}{zw} + \frac{yz}{xw} + \frac{2y}{xw} + \frac{y}{xzw} + \frac{w}{xz} + \frac{2}{xz} + \frac{1}{xzw}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.7. $X_{\mathrm{Sym}^2,\det,\det,\det} \subset \mathrm{Gr}(7,2)$. This is the subvariety X cut out of $\mathrm{Gr}(7,2)$ by a regular section of $\mathrm{Sym}^2 Q \oplus \det Q \oplus \det Q \oplus \det Q$. It has degree 56. Computing 100 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence suggests that the quantum differential operator L_X is:

$$(4t{+}1)^3(8t{+}1)(34t{+}7)^2(176t^2{+}36t{-}1)D^4{+}4t(4t{+}1)^2(34t{+}7)(478720t^4{+}292352t^3{+}60864t^2{+}4680t{+}73)D^3{+}$$
$$4t(4t+1)(227870720t^6+214601472t^5+80083824t^4+14810972t^3+1366436t^2+52729t+329)D^2+$$
$$2t(2604236800t^7+2882625536t^6+1306737792t^5+309231424t^4+40050680t^3+2670444t^2+71610t+147)D+$$
$$192t^2(13021184t^6+13730560t^5+5882760t^4+1300344t^3+154330t^2+9093t+196)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=\infty$$
$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=-8$$
$$\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } s=-4$$
$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at the roots of } s^2-36s-176=0$$

The ramification defect of L_X is 1. The rigid maximally mutable Laurent polynomial

$$f = w{+}\frac{wz}{y}{+}\frac{w}{y}{+}x{+}\frac{x}{y}{+}y{+}\frac{y}{z}{+}2z{+}\frac{2}{z}{+}\frac{2z}{y}{+}\frac{3}{y}{+}\frac{1}{yz}{+}\frac{y}{x}{+}\frac{y}{xz}{+}\frac{z}{x}{+}\frac{3}{x}{+}\frac{2}{xz}{+}\frac{z}{xy}{+}\frac{2}{xy}{+}\frac{1}{xyz}{+}\frac{xy}{w}{+}\frac{xy}{wz}{+}\frac{2x}{w}$$
$$+\frac{2x}{wz}+\frac{x}{wy}+\frac{x}{wyz}+\frac{yz}{w}+\frac{3y}{w}+\frac{2y}{wz}+\frac{2z}{w}+\frac{6}{w}+\frac{4}{wz}+\frac{z}{wy}+\frac{3}{wy}+\frac{2}{wyz}+\frac{yz}{wx}+\frac{2y}{wx}$$
$$+\frac{y}{wxz}+\frac{2z}{wx}+\frac{4}{wx}+\frac{2}{wxz}+\frac{z}{wxy}+\frac{2}{wxy}+\frac{1}{wxyz}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.8. $X_{1,1,1,1,1,1} \subset \mathrm{Gr}(7,2)$. This is the subvariety X cut out of $\mathrm{Gr}(7,2)$ by a regular section of $(\det Q)^{\oplus 6}$. It has degree 42. We can compute a Laurent polynomial mirror for X using Prince's method:

$$f = (1+x)(1+y)(1+z)(1+w)\Big(1+\tfrac{1}{w}+\tfrac{1}{zw}+\tfrac{1}{yzw}+\tfrac{1}{xyzw}\Big)-5$$

This is rigid maximally mutable. We then determine the quantum differential operator using mirror symmetry and Lairez's algorithm. The quantum differential operator L_X is:

$$(5t{+}1)^3(14t{+}3)^2(2744t^3{+}784t^2{+}42t{-}1)D^4{+}2t(5t{+}1)^2(14t{+}3)(960400t^4{+}543312t^3{+}103292t^2{+}7084t{+}95)D^3{+}$$
$$t(5t+1)(470596000t^6+416045280t^5+144323424t^4+24490788t^3+2036174t^2+68939t+366)D^2+$$
$$t(3361400000t^7+3424786400t^6+1418291280t^5+303579696t^4+35086464t^3+2047680t^2+46656t+81)D+$$
$$84t^2(19208000t^6+18796400t^5+7402920t^4+1485036t^3+157114t^2+8037t+144)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = \infty$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = -5$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } s^3 - 42s^2 - 784s - 2744 = 0$$

The ramification defect of L_X is 1.

7.2.9. $X_{\bigwedge^2,\det,\det^2} \subset \mathrm{Gr}(6,3)$. This is the subvariety X cut out of $\mathrm{Gr}(6,3)$ by a regular section of $\bigwedge^2 Q \oplus \det Q \oplus (\det Q)^2$. It has degree 32. Computing 60 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence suggests that the quantum differential operator L_X is:

$$(8t+1)^3(448t^2+80t-1)D^4+16t(8t+1)^2(40t+7)(56t+1)D^3+8t(8t+1)(125440t^3+29952t^2+1640t+9)D^2+ \\ 16t(716800t^4 + 222208t^3 + 21408t^2 + 624t + 1)D + 256t^2(21504t^3 + 5952t^2 + 492t + 11)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = \infty$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{at } s = -8$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } s^2 - 80s - 448 = 0$$

The operator L_X is extremal. The rigid maximally mutable Laurent polynomial

$$f = \frac{wyz^2}{x}+\frac{3wyz}{x}+\frac{3wy}{x}+\frac{wy}{xz}+\frac{x^2}{y}+\frac{x^2}{yz}+3x+\frac{3x}{z}+\frac{3xz}{y}+\frac{6x}{y}+\frac{3x}{yz}+3y+\frac{3y}{z}+4z+\frac{4}{z}+\frac{3z^2}{y}+\frac{9z}{y}+\frac{9}{y} \\ + \frac{3}{yz} + \frac{y^2}{x} + \frac{y^2}{xz} + \frac{yz}{x} + \frac{2y}{x} + \frac{y}{xz} + \frac{z^3}{xy} + \frac{4z^2}{xy} + \frac{6z}{xy} + \frac{4}{xy} + \frac{1}{xyz} + \frac{z}{wy} + \frac{2}{wy} \\ + \frac{1}{wyz} + \frac{z}{wx} + \frac{2}{wx} + \frac{1}{wxz} + \frac{z^2}{wxy} + \frac{3z}{wxy} + \frac{3}{wxy} + \frac{1}{wxyz}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.10. $X_{1,1,1,1,1} \subset \mathrm{Gr}(6,3)$. This is the subvariety X cut out of $\mathrm{Gr}(6,3)$ by a regular section of $(\det Q)^{\oplus 5}$. It has degree 42. Computing 60 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence suggests that the quantum differential operator L_X is:

$$(5t+1)(6t+1)^3(58t-1)D^4+2t(6t+1)^2(8700t^2+1852t+35)D^3+t(6t+1)(365400t^3+103044t^2+6850t+45)D^2+ \\ 2t(1566000t^4 + 598752t^3 + 71388t^2 + 2576t + 5)D + 144t^2(10440t^3 + 3564t^2 + 363t + 10)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=\infty$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=58$$

$$\begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&\frac{3}{4}&0\\ 0&0&0&\frac{1}{4} \end{pmatrix} \quad \text{at } s=-6$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=-5$$

The operator L_X is extremal. The rigid maximally mutable Laurent polynomial

$$f = \frac{x^2}{yw}+\frac{x^2}{yzw}+\frac{3xz}{w}+x+\frac{6x}{w}+\frac{x}{z}+\frac{3x}{zw}+\frac{3x}{yw}+\frac{x}{yz}+\frac{3x}{yzw}+\frac{3yz^2}{w}+2yz+\frac{9yz}{w}+5y+\frac{9y}{w}+\frac{3y}{z}+\frac{3y}{zw}+\frac{6z}{w}$$
$$+\frac{12}{w}+\frac{6}{z}+\frac{6}{zw}+\frac{1}{y}+\frac{3}{yw}+\frac{3}{yz}+\frac{3}{yzw}+\frac{y^2z^3}{xw}+\frac{y^2z^2}{x}+\frac{4y^2z^2}{xw}+\frac{4y^2z}{x}+\frac{6y^2z}{xw}+\frac{y^2w}{x}+\frac{5y^2}{x}$$
$$+\frac{4y^2}{xw}+\frac{y^2w}{xz}+\frac{2y^2}{xz}+\frac{y^2}{xzw}+\frac{3yz^2}{xw}+\frac{5yz}{x}+\frac{9yz}{xw}+\frac{2yw}{x}+\frac{11y}{x}+\frac{9y}{xw}+\frac{3yw}{xz}+\frac{6y}{xz}+\frac{3y}{xzw}+\frac{z}{x}$$
$$+\frac{3z}{xw}+\frac{w}{x}+\frac{7}{x}+\frac{6}{xw}+\frac{3w}{xz}+\frac{6}{xz}+\frac{3}{xzw}+\frac{1}{xy}+\frac{1}{xyw}+\frac{w}{xyz}+\frac{2}{xyz}+\frac{1}{xyzw}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.11. $X_{\bigwedge^2,\bigwedge^2,\det,\det} \subset \mathrm{Gr}(7,3)$. This is the subvariety X cut out of $\mathrm{Gr}(7,3)$ by a regular section of $\bigwedge^2 Q \oplus \bigwedge^2 Q \oplus \det Q \oplus \det Q$. It has degree 61. Computing 100 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence suggests that the quantum differential operator L_X is:

$$(5t+1)^3(224t+61)^2(293t^3+195t^2+32t-1)D^4+2t(5t+1)^2(224t+61)(1640800t^4+1541054t^3+502194t^2+60590t+1261)D^3+$$
$$t(5t+1)(12863872000t^6+16332532160t^5+8226951733t^4+2053006369t^3+254400487t^2+12966797t+99247)D^2+$$
$$t(91884800000t^7+126172580800t^6+71012714910t^5+20886378172t^4+3367189424t^3+279910761t^2+9384240t+22326)D+$$
$$2t^2(22052352000t^6+28967265600t^5+15476500980t^4+4274001704t^3+636131976t^2+47310441t+1309792)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=\infty$$

$$\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=-5$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at the roots of } s^3-32s^2-195s-293=0$$

The ramification defect of L_X is 1. The rigid maximally mutable Laurent polynomial

$$f = \frac{xyz^2}{w}+\frac{3xyz}{w}+\frac{3xy}{w}+\frac{xy}{zw}+xz+\frac{2xz}{w}+2x+\frac{4x}{w}+\frac{x}{z}+\frac{2x}{zw}+\frac{x}{yw}+\frac{x}{yzw}+\frac{2yz^2}{w}+2yz+\frac{6yz}{w}+4y+\frac{6y}{w}+\frac{2y}{z}+$$
$$\frac{2y}{zw}+z+\frac{4z}{w}+\frac{8}{w}+\frac{w}{z}+\frac{4}{z}+\frac{4}{zw}+\frac{2}{yw}+\frac{1}{yz}+\frac{2}{yzw}+\frac{yz^2}{xw}+\frac{2yz}{x}+\frac{3yz}{xw}+\frac{yw}{x}+\frac{4y}{x}+$$
$$\frac{3y}{xw}+\frac{yw}{xz}+\frac{2y}{xz}+\frac{y}{xzw}+\frac{2z}{xw}+\frac{3}{x}+\frac{4}{xw}+\frac{w}{xz}+\frac{3}{xz}+\frac{2}{xzw}+\frac{1}{xyw}+\frac{1}{xyz}+\frac{1}{xyzw}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.12. $X_{Q\otimes\det Q} \subset \mathrm{Gr}(6,4)$. This is the subvariety X cut out of $\mathrm{Gr}(6,4)$ by a regular section of $Q\otimes\det Q$. It has degree 42. Computing 50 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence suggests that the quantum differential operator L_X is:

$$\begin{gathered}(5t+1)^3(14t+3)^2(2744t^3+784t^2+42t-1)D^4+\\ 2t(5t+1)^2(14t+3)(960400t^4+543312t^3+103292t^2+7084t+95)D^3+\\ t(5t+1)(470596000t^6+416045280t^5+144323424t^4+24490788t^3+2036174t^2+68939t+366)D^2+\\ t(3361400000t^7+3424786400t^6+1418291280t^5+303579696t^4+35086464t^3+2047680t^2+46656t+81)D+\\ 84t^2(19208000t^6+18796400t^5+7402920t^4+1485036t^3+157114t^2+8037t+144)\end{gathered}$$

The local log-monodromies for the quantum local system, with $s=1/t$:

$$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \quad \text{at } s=\infty$$

$$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \quad \text{at } s=-5$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \quad \text{at the roots of } s^3-42s^2-784s-2744=0$$

The ramification defect of L_X is 1. The rigid maximally mutable Laurent polynomial

$$\begin{gathered}f=\frac{xy^2z}{w}+\frac{xy^2}{w}+\frac{3xyz}{w}+xy+\frac{4xy}{w}+\frac{xy}{zw}+\frac{3xz}{w}+x+\frac{5x}{w}+\frac{2x}{zw}+\frac{xz}{yw}+\frac{2x}{yw}+\frac{x}{yzw}+\frac{2y^2z^2}{w}+\frac{4y^2z}{w}+\frac{2y^2}{w}+\frac{6yz^2}{w}+\\ 3yz+\frac{14yz}{w}+3y+\frac{10y}{w}+\frac{2y}{zw}+\frac{6z^2}{w}+4z+\frac{16z}{w}+w+\frac{14}{w}+\frac{1}{z}+\frac{4}{zw}+\frac{2z^2}{yw}+\frac{z}{y}+\frac{6z}{yw}+\frac{2}{y}+\\ \frac{6}{yw}+\frac{1}{yz}+\frac{2}{yzw}+\frac{y^2z^3}{xw}+\frac{3y^2z^2}{xw}+\frac{3y^2z}{xw}+\frac{y^2}{xw}+\frac{3yz^3}{xw}+\frac{2yz^2}{x}+\frac{10yz^2}{xw}+\frac{4yz}{x}+\frac{12yz}{xw}+\frac{2y}{x}+\\ \frac{6y}{xw}+\frac{y}{xzw}+\frac{3z^3}{xw}+\frac{3z^2}{x}+\frac{11z^2}{xw}+\frac{zw}{x}+\frac{7z}{x}+\frac{15z}{xw}+\frac{w}{x}+\frac{5}{x}+\frac{9}{xw}+\frac{1}{xz}+\\ \frac{2}{xzw}+\frac{z^3}{xyw}+\frac{z^2}{xy}+\frac{4z^2}{xyw}+\frac{3z}{xy}+\frac{6z}{xyw}+\frac{3}{xy}+\frac{4}{xyw}+\frac{1}{xyz}+\frac{1}{xyzw}\end{gathered}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.13. $X_{\bigwedge^3\oplus\bigwedge^3} \subset \mathrm{Gr}(7,4)$. This is the subvariety X cut out of $\mathrm{Gr}(7,4)$ by a regular section of $\bigwedge^3 Q\oplus\bigwedge^3 Q$. It has degree 72. Computing 55 terms of the quantum period sequence using the Abelian/non-Abelian Correspondence suggests that the quantum differential operator L_X is:

$$\begin{gathered}(3t+1)(4t+1)(8t+9)^2(1408t^4+1024t^3+256t^2+20t-1)D^4+\\ 2t(8t+9)(675840t^6+1620992t^5+1276032t^4+468224t^3+85584t^2+6912t+125)D^3+\\ t(37847040t^7+132349952t^6+165777920t^5+91994496t^4+25327744t^3+3434832t^2+190448t+1449)D^2+\\ 4t(13516800t^7+47024128t^6+57428736t^5+29291456t^4+7077184t^3+798552t^2+33300t+81)D+\\ 64t^2(405504t^6+1406208t^5+1691072t^4+811952t^3+178024t^2+17388t+567)\end{gathered}$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=\infty$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=-4$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=-3$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at the roots of } s^4-20s^3-256s^2-1024s-1408=0$$

The ramification defect of L_X is 1. The rigid maximally mutable Laurent polynomial

$$f = \frac{x^2z}{w}+\frac{x^2}{w}+\frac{xyz}{w}+\frac{xy}{w}+\frac{3xz}{w}+2x+\frac{3x}{w}+\frac{xz}{yw}+\frac{2y^2z}{w}+\frac{6yz}{w}+\frac{2y}{w}+\frac{6z}{w}+\frac{3}{w}+\frac{w}{z}+\frac{2z}{yw}+\frac{2}{y}+\frac{y^3z^2}{xw} + \frac{4y^2z^2}{xw}+\frac{2y^2z}{xw}+\frac{6yz^2}{xw}+\frac{5yz}{xw}+\frac{y}{xw}+\frac{4z^2}{xw}+\frac{4z}{xw}+\frac{2}{x}+\frac{1}{xw}+\frac{w}{xz}+\frac{z^2}{xyw}+\frac{z}{xyw}+\frac{2}{xy}+\frac{w}{xyz}$$

has the correct Hilbert series and has Picard–Fuchs operator equal to the quantum differential operator L_X above; thus f is a good candidate for a mirror to X.

7.2.14. $X_{\bigwedge^4 \oplus \det^2} \subset \mathrm{Gr}(7,5)$. This is the subvariety X cut out of $\mathrm{Gr}(7,5)$ by a regular section of $\bigwedge^4 Q \oplus (\det Q)^{\otimes 2}$. It has degree 36. Computing 6 terms of the quantum period sequence, finding (numerically) a rigid maximally mutable Laurent polynomial f with the correct Hilbert series and correct initial terms of the period sequence, and finding the Picard–Fuchs operator for f using Lairez's algorithm suggests that the quantum differential operator L_X is:

$$(6t+1)^3(828t^2+60t-1)D^4+12t(6t+1)^2(4140t^2+516t+7)D^3+54t(6t+1)(19320t^3+4048t^2+198t+1)D^2+ 12t(745200t^4+238896t^3+23058t^2+636t+1)D+144t^2(29808t^3+8532t^2+693t+14)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } s=\infty$$

$$\begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&\frac{1}{2}&0\\ 0&0&0&\frac{1}{2} \end{pmatrix} \quad \text{at } s=-6$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at the roots of } s^2-60s-828=0$$

The operator L_X is extremal. The Laurent polynomial f, which is a good candidate for the mirror to X, is:

$$f = xyz+xyw+2xy+\frac{xyw}{z}+\frac{xy}{z}+\frac{yz}{w}+2y+\frac{2y}{w}+\frac{2y}{z}+\frac{y}{zw}+\frac{2z^2}{w}+3z+\frac{6z}{w}+2w+\frac{6}{w}+\frac{2w}{z}+\frac{3}{z}+\frac{2}{zw}+\frac{z^3}{yw} +\frac{z^2}{y}+\frac{4z^2}{yw}+\frac{3z}{y}+\frac{6z}{yw}+\frac{3}{y}+\frac{4}{yw}+\frac{1}{yz}+\frac{1}{yzw}+\frac{y}{xw}+\frac{y}{xzw}+\frac{2z}{xw}+\frac{2}{x}+\frac{4}{xw}+\frac{2}{xz}+\frac{2}{xzw} +\frac{z^2}{xyw}+\frac{2z}{xy}+\frac{3z}{xyw}+\frac{w}{xy}+\frac{4}{xy}+\frac{3}{xyw}+\frac{w}{xyz}+\frac{2}{xyz}+\frac{1}{xyzw}$$

7.2.15. $X_{\bigwedge^3 \oplus \det} \subset \mathrm{Gr}(8,5)$. This is the subvariety X cut out of $\mathrm{Gr}(8,5)$ by a regular section of $\bigwedge^3 Q \oplus \det Q$. It has degree 102. Computing 6 terms of the quantum period sequence, finding (numerically) a rigid maximally mutable Laurent polynomial f with the correct Hilbert series and correct initial terms of the period sequence, and finding the Picard–Fuchs operator for f using Lairez's algorithm suggests that the quantum differential operator L_X is:

$$(24t-17)^2(t^2-11t-1)(64t^2+16t-1)D^4+2t(24t-17)(4608t^4-37376t^3+29304t^2+8194t+61)D^3+$$
$$t(479232t^5-3246336t^4+4925056t^3-2002124t^2-423285t-1326)D^2+$$
$$t(442368t^5-2434560t^4+3712032t^3-2131188t^2-287674t-289)D+8t^2(18432t^4-86304t^3+125748t^2-93381t-9248)$$

The local log-monodromies for the quantum local system, with $s=1/t$:

$$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \quad \text{at } s=\infty$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \quad \text{at the roots of } s^2+11s-1=0$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \quad \text{at the roots of } s^2-16s-64=0$$

$$\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \quad \text{at } s=0$$

The ramification defect of L_X is 1. The Laurent polynomial f that is a reasonable candidate for the mirror to X is

$$f=\frac{xy^2z}{w}+\frac{xy^2}{w}+\frac{xyz}{w}+\frac{2xy}{w}+\frac{xy}{zw}+\frac{2yz^2}{w}+\frac{5yz}{w}+\frac{4y}{w}+\frac{y}{zw}+\frac{2z^2}{w}+\frac{5z}{w}+\frac{4}{w}+\frac{1}{zw}+\frac{z}{yw}+\frac{2}{y}+\frac{2}{yw}+\frac{w}{yz}+\frac{2}{yz}$$
$$+\frac{1}{yzw}+\frac{z^3}{xw}+\frac{4z^2}{xw}+\frac{6z}{xw}+\frac{4}{xw}+\frac{1}{xzw}+\frac{z^3}{xyw}+\frac{4z^2}{xyw}+\frac{2z}{xy}+\frac{6z}{xyw}+\frac{w}{xy}+\frac{4}{xy}$$
$$+\frac{4}{xyw}+\frac{w}{xyz}+\frac{2}{xyz}+\frac{1}{xyzw}$$

7.2.16. $X_{\bigwedge^2,\bigwedge^2,\det}\subset \mathrm{Gr}(10,5)$. This is the subvariety X cut out of $\mathrm{Gr}(10,5)$ by a regular section of $\bigwedge^2 Q\oplus\bigwedge^2 Q\oplus\det Q$. It can also be expressed as a $(1,1,1,1,1)$ hypersurface in $(\mathbb{P}^1)^5$. In particular this makes it clear that X has degree 120. The Przyjalkowski method gives a mirror Laurent polynomial:

$$f=(1+x+y+z+w)\Big(1+\tfrac{1}{x}+\tfrac{1}{y}+\tfrac{1}{z}+\tfrac{1}{w}\Big)-5$$

which is rigid maximally mutable. Lairez's algorithm then shows that the quantum differential operator L_X is:

$$(4t-1)(4t+1)(5t+1)^2(20t-1)D^4+4t(5t+1)(4000t^3+320t^2-166t-5)D^3+$$
$$t(280000t^4+64400t^3-4820t^2-1275t-13)D^2+t(400000t^4+78400t^3-5280t^2-1022t-3)D+$$
$$80t^2(2400t^3+420t^2-27t-4)$$

The local log-monodromies for the quantum local system, with $s = 1/t$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = \infty$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = 4$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = 20$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = -5$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } s = -4$$

The operator L_X is extremal.

What is Missing? The calculation for $X_{\bigwedge^5 \oplus \det \oplus \det} \subset \mathrm{Gr}(8,6)$. This is the remaining quiver flag zero locus inside a Grassmannian; it is in principle amenable to the methods used here, although in practice the computations are quite involved.

7.3. **Tables.** We label the new period sequences from 1 to 139. For each, we give one quiver flag zero locus with that period. We describe it by the adjacency matrix of the quiver of the ambient quiver flag variety, $\mathbf{r}$, and the partitions of the irreducible bundles corresponding to the vector bundle determining the quiver flag zero locus.

The adjacency matrix is a $\rho+1 \times \rho+1$ matrix with with non-negative integer entries: the (i,j) entry records the number of arrows from vertex $i-1$ to vertex $j-1$ where $Q_0 = \{0, \ldots, \rho\}$. Note that we describe these using the *normal form* of the quiver flag variety, where we allow arrows from $i \to j$ even if $i > j$ (but still no cycles), and where we have sorted the entries of $\mathbf{r}$ and then the columns of A. The source is always the first vertex.

Each irreducible summand of the vector bundle is of the form $S^{\alpha_1}(W_1) \otimes \cdots \otimes S^{\alpha_\rho}(W_\rho)$. Here α_i is a length r_i decreasing sequence. Finally, we record the first 8 terms of each of the 139 new period sequences.

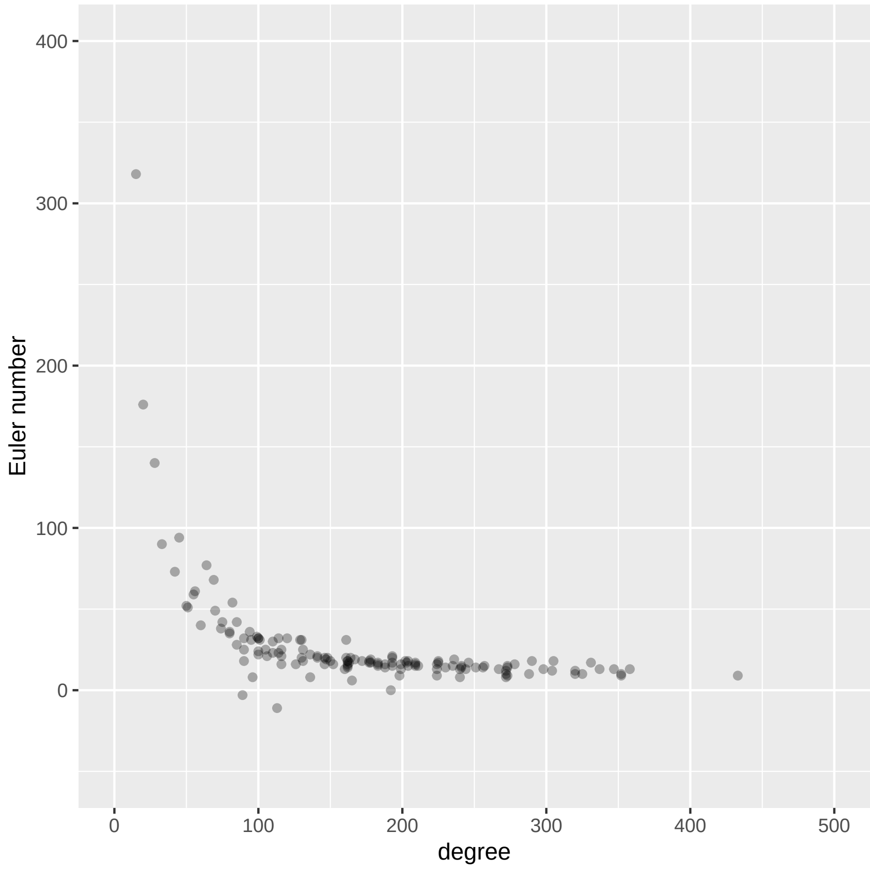


Figure 1. The distribution of degrees with Euler number

Table 1: Certain 4-dimensional Fano manifolds with Fano index 1 that arise as quiver flag zero loci

Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
1	0 5 2 0 0 1 0 0 0	1 1 2	((1),(1,0))	433	9
2	0 3 0 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 3	((0),(0,0),(1,1,1)),((1),(1,0),(0,0,0))	358	13
3	0 0 5 2 0 0 0 0 0 0 0 1 0 1 0 0	1 1 1 2	((0),(1),(1,0)),((1),(0),(0,0))	347	13
4	0 0 5 0 0 0 0 1 0	1 2 3	((0,0),(1,1,1)),((0,0),(1,1,1)),((1,0),(0,0,0))	325	10
5	0 0 0 0 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0	1 1 1 1 2	((0),(0),(1),(0,0)),((0),(1),(0),(0,0)),((1),(0),(0),(0,0))	290	18
6	0 0 4 0 0 0 0 2 0	1 1 2	((1),(0,0)),((1),(0,0)),((1),(0,0))	273	9
7	0 1 2 4 0 0 1 0 0 0 0 0 0 1 0 0	1 1 2 2	((0),(1,0),(1,0))	337	13
8	0 0 3 3 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0	1 1 1 1 2	((0),(1),(0),(1,0)),((1),(0),(0),(0,0))	331	17
9	0 0 5 0 0 0 0 1 0	1 2 3	((0,0),(1,1,0)),((1,1),(0,0,0))	352	9
10	0 0 5 0 0 0 0 1 0	1 1 2	((0),(1,1)),((0),(1,1)),((1),(0,0))	320	10
11	0 3 3 0 3 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0	1 1 1 2 2	((0),(0),(1,0),(0,0)),((1),(0),(1,0),(0,0))	305	18

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
12	0 0 5 0 0 0 0 1 0	1 1 2	((0),(1,1)),((0),(1,1)),((0),(1,1))	272	8
13	0 2 4 0 0 0 0 1 0	1 1 2	((0),(1,1)),((1),(1,0))	272	10
14	0 0 1 4 0 0 0 0 0 0 0 0 0 1 1 0	1 1 2 2	((0),(1,1),(0,0)),((0),(1,1),(0,0)),((1),(0,0),(0,0))	257	15
15	0 5 1 3 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 2	((1),(1,0),(0,0)),((1),(1,0),(0,0))	211	15
16	0 0 5 0 0 0 0 1 0	1 1 3	((0),(1,1,0)),((1),(1,1,1))	224	13
17	0 3 3 0 0 1 0 0 0	1 1 2	((0),(1,1)),((0),(1,1))	352	10
18	0 3 0 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 3	((0),(1,1),(0,0,0)),((1),(1,0),(0,0,0))	320	12
19	0 3 1 2 0 0 0 1 0 0 0 0 0 0 1 0	1 1 2 2	((0),(1,1),(0,0)),((0),(1,1),(0,0))	298	13
20	0 0 0 4 0 0 0 0 0 0 0 0 0 1 1 0	1 1 2 3	((0),(1,1),(0,0,0)),((1),(0,0),(0,0,0)),((1),(0,0),(0,0,0))	304	12
21	0 5 0 3 0 0 1 0 0 0 0 0 0 0 1 0	1 1 2 2	((0),(1,0),(0,0)),((1),(1,0),(0,0))	273	15
22	0 2 3 4 0 0 1 0 0 0 0 0 0 0 0 0	1 1 1 3	((0),(1),(1,1,0))	272	12

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
23	0 0 3 3 0 0 0 0 0 0 0 1 0 1 0 0	1 1 1 2	((0),(0),(1,1)),((0),(0),(1,1)),((1),(0),(0,0))	256	14
24	0 1 4 0 0 0 0 1 0	1 2 3	((0,0),(1,1,1)),((1,1),(0,0,0)),((1,1),(0,0,0))	288	10
25	0 0 3 0 4 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0	1 1 1 2 3	((0),(0),(0,0),(1,1,1)),((0),(1),(1,0),(0,0,0)),((1),(0),(0,0),(0,0,0))	278	16
26	0 0 4 1 0 0 0 0 0 0 0 1 0 1 0 0	1 1 2 2	((0),(0,0),(1,1)),((0),(1,1),(0,0)),((1),(0,0),(0,0))	267	13
27	0 0 0 5 0 0 0 0 0 0 0 0 0 1 1 0	1 1 1 2	((0),(0),(1,1)),((0),(0),(1,1)),((0),(1),(0,0)),((1),(0),(0,0))	241	15
28	0 1 3 2 0 0 0 0 0 0 0 1 0 1 0 0	1 1 1 2	((1),(0),(1,0))	235	15
29	0 1 4 0 0 0 0 1 0	1 2 3	((1,1),(0,0,0)),((1,1),(0,0,0)),((1,1),(0,0,0))	224	9
30	0 0 1 4 0 0 1 0 0 0 0 0 0 1 0 0	1 2 2 3	((0,0),(0,0),(1,1,1)),((0,0),(1,1),(0,0,0)),((0,0),(1,1),(0,0,0))	251	14
31	0 3 5 2 0 0 0 1 0 0 0 0 0 0 0 0	1 1 1 2	((0),(1),(1,0)),((0),(1),(1,0))	225	18
32	0 0 0 2 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0	1 1 1 1 2	((0),(0),(1),(1,1)),((0),(1),(0),(0,0)),((1),(0),(0),(0,0))	236	19
33	0 4 4 0 0 0 0 0 0	1 2 2	((0,0),(1,1)),((1,1),(0,0)),((1,0),(1,1))	240	8

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
34	0 2 5 2 0 0 0 0 0 0 0 1 0 0 0 0	1 1 1 2	((0),(1),(1,0)),((1),(0),(1,1))	225	17
35	0 0 2 4 0 0 1 0 0 0 0 0 0 1 0 0	1 1 1 2	((0),(0),(1,1)),((0),(1),(1,0))	230	14
36	0 1 3 4 0 0 0 0 0 0 0 0 0 1 0 0	1 1 2 2	((0),(1,0),(1,1)),((1),(1,0),(0,0))	199	16
37	0 4 4 0 0 0 0 0 0	1 2 3	((1,1),(1,1,0))	192	0
38	0 1 5 4 0 0 0 0 0 1 0 0 0 0 0 0	1 1 1 3	((0),(1),(1,1,0)),((1),(0),(1,1,1))	224	16
39	0 1 4 4 0 0 0 0 0 1 0 0 0 0 0 0	1 1 1 2	((0),(0),(1,1)),((0),(1),(1,0)),((1),(0),(1,1))	203	17
40	0 5 3 0 0 1 0 0 0	1 1 2	((0),(1,1)),((0),(1,1)),((1),(1,0))	177	17
41	0 3 0 4 0 0 1 0 0 0 0 0 0 0 1 0	1 1 2 2	((0),(0,0),(1,1)),((0),(1,1),(0,0)),((0),(1,0),(0,0))	273	14
42	0 2 5 0 0 0 0 0 0	1 1 2	((0),(1,1)),((0),(1,1)),((1),(1,1))	240	13
43	0 3 1 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 2	((0),(0,0),(1,1)),((0),(1,1),(0,0)),((1),(1,0),(0,0))	241	14
44	0 0 3 2 0 0 0 0 0 0 0 1 0 2 0 0	1 1 1 2	((1),(0),(0,0)),((1),(0),(0,0)),((1),(0),(0,0))	244	13

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
45	0 0 0 2 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0	1 1 1 1 2	((0),(0),(0),(1,1)),((0),(1),(0),(0,0)),((1),(0),(1),(0,0))	246	17
46	0 3 3 3 0 0 0 0 0 0 0 1 0 0 0 0	1 1 1 2	((0),(0),(1,1)),((0),(0),(1,1)),((1),(0),(1,0))	209	16
47	0 0 3 2 0 0 0 0 0 0 0 1 0 1 0 0	1 1 2 2	((0),(0,0),(1,1)),((0),(0,0),(1,1)),((1),(0,0),(0,0))	209	15
48	0 3 4 2 0 0 0 1 0 0 0 0 0 0 0 0	1 1 1 2	((0),(1),(1,1)),((0),(1),(1,0))	193	20
49	0 0 3 2 0 0 0 0 0 0 0 1 0 1 0 0	1 1 1 2	((1),(0),(1,1))	202	18
50	0 3 3 3 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 1 2	((0),(1),(0),(1,0)),((1),(0),(1),(1,0))	198	9
51	0 2 1 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 2	((0),(1,1),(0,0)),((0),(1,1),(0,0)),((1),(0,0),(1,1))	209	17
52	0 1 0 4 0 0 0 0 0 1 0 0 0 0 1 0	1 1 2 3	((0),(0,0),(1,1,1)),((1),(1,0),(0,0,0))	204	15
53	0 0 3 1 0 0 0 0 0 0 0 1 0 2 0 0	1 1 2 2	((1),(0,0),(0,0)),((1),(0,0),(0,0)),((1),(0,0),(0,0))	199	13
54	0 3 0 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 3	((0),(1,1),(1,1,1)),((1),(1,0),(0,0,0))	178	19

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
55	0 0 0 4 0 0 0 0 0 0 0 0 0 1 1 0	1 1 2 3	((0),(1,1),(0,0,0)),((1),(1,0),(0,0,0))	188	16
56	0 1 3 4 0 0 0 0 0 0 0 0 0 1 0 0	1 1 2 2	((0),(0,0),(1,1)),((0),(1,1),(1,1)),((1),(1,0),(0,0))	183	17
57	0 0 3 4 0 0 0 0 0 0 0 0 0 1 0 0	1 2 2 3	((0,0),(1,1),(1,1,1)),((1,1),(1,0),(0,0,0))	178	17
58	0 5 3 3 0 0 0 0 0 0 0 0 0 0 0 0	1 1 2 2	((1),(1,0),(1,0))	150	18
59	0 4 2 0 0 1 0 0 0	1 2 3	((0,0),(1,1,1)),((0,0),(1,1,1)),((0,0),(1,1,1))	177	18
60	0 4 3 0 0 1 0 0 0	1 1 2	((0),(1,1)),((0),(1,1)),((1),(1,1))	129	31
61	0 7 2 0 0 1 0 0 0	1 1 2	((1),(1,0)),((1),(1,0))	193	21
62	0 1 5 0 0 0 0 1 0	1 2 2	((0,0),(1,1)),((0,0),(1,1)),((1,1),(0,0)),((1,1),(0,0))	193	15
63	0 3 4 0 0 1 0 0 0	1 1 2	((0),(1,1)),((0),(2,0))	164	20
64	0 0 2 2 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0	1 1 1 1 2	((0),(0),(0),(1,1)),((0),(0),(1),(1,1)),((1),(1),(0),(0,0))	204	18
65	0 2 1 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 2	((0),(0,0),(1,1)),((0),(1,1),(0,0)),((1),(1,1),(0,0))	193	17

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
66	0 0 3 3 0 0 0 0 0 1 0 0 0 1 0 0	1 2 2 2	((1,1),(0,0),(0,0)),((1,1),(0,0),(0,0)),((1,0),(0,0),(0,0))	183	16
67	0 6 2 0 0 1 0 0 0	1 1 2	((1),(1,1)),((1),(1,0))	161	31
68	0 1 0 4 0 0 0 0 0 1 0 0 0 0 1 0	1 1 2 3	((0),(0,0),(1,1,1)),((0),(1,1),(0,0,0)),((2),(0,0),(0,0,0))	188	14
69	0 0 2 5 0 0 0 0 0 0 0 0 0 1 0 0	1 1 1 2	((0),(0),(1,1)),((0),(0),(1,1)),((0),(0),(1,1)),((1),(1),(0,0))	183	15
70	0 0 3 4 0 0 0 0 0 0 0 0 0 1 0 0	1 1 1 2	((0),(1),(1,0)),((1),(0),(1,1))	172	18
71	0 2 1 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 3 3	((0),(1,1,1),(0,0,0)),((0),(1,1,1),(0,0,0)),((1),(1,1,1),(0,0,0))	167	19
72	0 2 3 3 0 0 0 0 0 0 0 1 0 0 0 0	1 1 1 2	((0),(0),(1,1)),((0),(0),(1,1)),((1),(0),(1,1))	161	20
73	0 0 5 0 0 0 0 1 0	1 2 3	((1,0),(0,0,0)),((1,0),(1,1,1))	147	19
74	0 0 2 4 0 0 0 0 0 0 0 0 0 2 0 0	1 1 1 2	((0),(1),(1,1)),((1),(0),(0,0)),((1),(0),(0,0)),((1),(0),(0,0))	163	17
75	0 3 0 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 3	((0),(1,1),(0,0,0)),((1),(0,0),(1,1,1)),((1),(1,1),(0,0,0))	162	18
76	0 3 3 0 0 1 0 0 0	1 2 2	((0,0),(1,1)),((0,0),(2,0))	148	20

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
77	0 3 5 3 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 2	((0),(1),(1,0)),((1),(1),(1,0))	131	25
78	0 0 5 0 0 0 0 1 0	1 2 3	((0,0),(1,1,1)),((0,0),(1,1,1)),((1,1),(0,0,0)),((1,1),(0,0,0))	162	14
79	0 1 2 4 0 0 0 0 0 0 0 0 0 1 0 0	1 1 1 2	((0),(1),(1,1)),((1),(0),(1,0))	162	18
80	0 1 4 0 0 0 0 1 0	1 2 2	((1,1),(1,0))	146	20
81	0 3 5 0 0 0 0 0 0	1 2 2	((0,0),(1,1)),((0,0),(1,1)),((1,0),(1,1))	165	6
82	0 2 1 4 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 3	((0),(1,1),(0,0,0)),((0),(1,1),(0,0,0)),((0),(1,1),(0,0,0)),((1),(0,0),(1,1,1))	162	16
83	0 2 3 3 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 1 2	((0),(1),(1),(1,0)),((1),(0),(0),(1,1))	162	15
84	0 2 4 0 0 0 0 1 0	1 2 2	((0,0),(1,1)),((1,1),(0,0)),((1,1),(0,0)),((1,1),(0,0))	146	16
85	0 3 5 0 0 0 0 0 0	1 1 2	((0),(1,1)),((0),(1,1)),((0),(1,1)),((1),(1,1))	160	13
86	0 0 5 0 0 0 0 1 0	1 2 3	((0,0),(1,1,1)),((1,0),(0,0,0)),((1,1),(1,1,1))	130	31
87	0 1 2 4 0 0 0 0 0 0 0 0 0 1 0 0	1 1 1 2	((0),(0),(1,1)),((0),(1),(1,1)),((2),(0),(0,0))	152	16
88	0 2 2 3 0 0 0 0 0 0 0 0 0 0 1 0	1 1 2 2	((0),(1,1),(0,0)),((0),(1,1),(0,0)),((1),(1,1),(0,0))	141	20

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
89	0 1 5 0 0 0 0 1 0	1 1 2	((0),(1,1)),((0),(1,1)),((1),(1,0))	131	18
90	0 1 2 4 0 0 0 0 0 0 0 0 0 1 0 0	1 1 1 2	((0),(0),(1,1)),((1),(0),(1,1)),((1),(1),(0,0))	141	21
91	0 0 2 4 0 0 0 0 0 1 0 0 0 1 0 0	1 1 1 3	((1),(0),(1,1,0))	136	22
92	0 0 0 4 0 0 0 0 0 0 0 0 0 1 1 0	1 1 2 3	((0),(2,2),(0,0,0)),((1),(0,0),(0,0,0)),((1),(0,0),(0,0,0))	136	8
93	0 3 4 3 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 2	((0),(1),(1,1)),((1),(1),(1,0))	116	25
94	0 3 4 0 0 1 0 0 0	1 1 2	((0),(1,1)),((0),(1,1)),((0),(1,1)),((0),(1,1))	114	23
95	0 1 4 0 0 0 0 1 0	1 2 3	((1,1),(1,0,0))	110	23
96	0 0 4 0 0 0 0 2 0	1 1 2	((1),(0,0)),((1),(1,0))	116	21
97	0 1 5 0 0 0 0 1 0	1 1 2	((0),(1,1)),((0),(1,1)),((0),(1,1)),((2),(0,0))	126	16
98	0 4 4 0 0 0 0 0 0	1 1 2	((1),(2,0))	90	18
99	0 2 5 4 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 3	((0),(1),(1,1,0)),((1),(1),(1,1,1))	120	32

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
100	0 0 5 0 0 0 0 1 0	1 2 3	((0,0),(2,2,2)),((1,1),(0,0,0)),((1,0),(0,0,0))	114	32
101	0 0 5 0 0 0 0 1 0	1 1 2	((0),(1,1)),((0),(1,1)),((1),(1,1))	110	30
102	0 3 3 0 0 1 0 0 0	1 1 2	((0),(2,1))	99	33
103	0 0 5 0 0 0 0 1 0	1 1 2	((0),(1,1)),((0),(2,2)),((1),(0,0))	94	36
104	0 0 4 0 0 0 0 2 0	1 1 2	((1),(0,0)),((1),(0,0)),((1),(1,1))	101	31
105	0 0 5 0 0 0 0 1 0	1 1 3	((0),(1,1,1)),((0),(1,1,1)),((0),(1,1,1)),((2),(0,0,0))	116	16
106	0 4 4 0 0 0 0 0 0	1 2 2	((1,0),(1,1)),((1,1),(1,0))	106	21
107	0 3 5 0 0 0 0 0 0	1 2 2	((1,0),(1,1)),((1,0),(1,1))	85	28
108	0 1 2 5 0 0 0 0 0 1 0 0 0 0 0 0	1 1 1 2	((0),(0),(1,1)),((0),(0),(1,1)),((0),(0),(1,1)),((1),(0),(1,1))	130	20
109	0 2 4 0 0 0 0 1 0	1 1 2	((1),(1,1)),((1),(1,0))	100	32
110	0 0 4 0 0 0 0 2 0	1 1 2	((0),(1,1)),((1),(0,0)),((2),(0,0))	100	24
111	0 3 3 0 0 1 0 0 0	1 2 2	((0,0),(1,1)),((0,0),(1,1)),((0,0),(1,1)),((0,0),(1,1))	100	22
112	0 1 4 0 0 0 0 1 0	1 2 3	((1,1),(0,0,0)),((1,1),(0,0,0)),((1,1),(1,1,1))	95	31

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
113	0 3 5 3 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 2	((0),(2),(1,0)),((1),(0),(1,0))	113	-11
114	0 4 4 0 0 0 0 0 0	1 2 2	((0,0),(1,1)),((1,0),(1,1)),((2,2),(0,0))	96	8
115	0 1 4 0 0 0 0 1 0	1 2 3	((0,0),(1,1,1)),((2,1),(0,0,0))	90	25
116	0 2 2 5 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 2	((0),(0),(1,1)),((0),(0),(1,1)),((0),(1),(1,1)),((1),(0),(1,1))	105	25
117	0 5 4 0 0 0 0 0 0	1 1 2	((1),(1,1)),((1),(1,1)),((1),(1,0))	85	42
118	0 2 4 0 0 0 0 1 0	1 1 2	((0),(1,1)),((1),(1,1)),((2),(0,0))	90	32
119	0 3 4 0 0 0 0 1 0	1 1 3	((2),(0,0,0)),((1),(1,1,0))	80	36
120	0 3 5 0 0 0 0 0 0	1 2 2	((0,0),(1,1)),((1,1),(1,1)),((1,0),(1,1))	80	35
121	0 2 2 5 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 2	((0),(0),(1,1)),((0),(0),(1,1)),((0),(0),(1,1)),((1),(1),(1,1))	100	32
122	0 3 3 0 0 1 0 0 0	1 1 2	((0),(1,1)),((0),(2,2))	82	54
123	0 3 5 0 0 0 0 0 0	1 1 2	((0),(1,1)),((0),(1,1)),((1),(1,1)),((1),(1,1))	75	42
124	0 1 4 0 0 0 0 1 0	1 2 3	((0,0),(1,1,1)),((1,1),(0,0,0)),((2,2),(0,0,0))	74	38
125	0 0 5 0 0 0 0 1 0	1 1 3	((0),(1,1,0)),((2),(1,1,1))	69	68

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Period ID	Adjacency matrix	Dimension vector	Generalized partitions	Degree	Euler number
126	0 4 4 0 0 0 0 0 0	1 1 2	((1),(2,2)),((1),(1,0))	64	77
127	0 2 5 3 0 0 0 0 0 0 0 0 0 0 0 0	1 1 1 2	((0),(2),(1,0)),((1),(0),(1,1))	89	-3
128	0 3 5 0 0 0 0 0 0	1 1 2	((0),(1,1)),((0),(1,1)),((0),(1,1)),((2),(1,1))	70	49
129	0 7 3 0 0 0 0 0 0	1 1 2	((1),(1,0)),((2),(1,0))	55	59
130	0 3 4 0 0 0 0 0 0	1 1 2	((1),(2,1))	56	61
131	0 6 3 0 0 0 0 0 0	1 1 2	((1),(1,1)),((2),(1,0))	51	51
132	0 2 5 0 0 0 0 0 0	1 1 2	((0),(2,1)),((1),(1,1))	60	40
133	0 6 0 0	1 4	((2,1,1,1))	42	73
134	0 2 5 0 0 0 0 0 0	1 1 2	((0),(1,1)),((0),(2,2)),((1),(1,1))	50	52
135	0 2 5 0 0 0 0 0 0	1 1 2	((0),(1,1)),((0),(1,1)),((1),(2,2))	45	94
136	0 6 0 0	1 2	((1,1)),((1,1)),((2,1))	33	90
137	0 6 0 0	1 2	((1,1)),((1,1)),((1,1)),((2,2))	28	140
138	0 5 0 0	1 2	((2,2)),((2,2))	20	176
139	0 5 0 0	1 2	((1,1)),((3,3))	15	318

Table 2: 139 regularized period sequences obtained from 4-dimensional Fano manifolds that arise as quiver flag zero loci.

Period ID	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
1	1	0	0	12	0	120	540	0	20160
2	1	0	0	12	48	0	900	7560	15120
3	1	0	0	12	48	120	540	7560	35280
4	1	0	0	18	48	0	1710	11340	15120
5	1	0	0	18	48	120	2430	11340	45360
6	1	0	0	24	48	120	3600	15120	55440
7	1	0	2	12	30	300	920	9240	42910
8	1	0	2	12	30	300	1640	7980	59710
9	1	0	2	18	6	300	1730	2940	62230
10	1	0	2	18	6	420	1730	5460	92470
11	1	0	2	18	30	360	2450	9660	91630
12	1	0	2	18	102	300	3170	26460	146230
13	1	0	2	24	54	360	4340	18480	129430
14	1	0	2	24	54	480	4700	21000	179830
15	1	0	2	30	78	960	7670	46200	483070
16	1	0	2	30	126	540	7670	56700	340270
17	1	0	4	6	36	360	490	12600	25060
18	1	0	4	6	84	300	2290	13020	87220
19	1	0	4	12	60	480	2740	18480	131740
20	1	0	4	12	84	420	2740	19320	117460
21	1	0	4	12	84	480	3100	23520	140980
22	1	0	4	12	108	600	3820	28560	207340
23	1	0	4	12	108	600	4900	32340	254380
24	1	0	4	18	36	720	2110	21000	159460
25	1	0	4	18	84	600	4270	26880	208180
26	1	0	4	18	84	720	3910	32340	224980
27	1	0	4	18	108	840	4990	44940	311500
28	1	0	4	18	108	960	6070	49980	395500
29	1	0	4	18	180	1020	7870	66780	577780
30	1	0	4	24	84	840	5800	38220	329140
31	1	0	4	24	84	1140	5800	51660	450100
32	1	0	4	24	108	960	6880	49560	429100
33	1	0	4	24	132	840	7240	53760	418180
34	1	0	4	24	132	840	7960	54600	451780

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Period ID	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
35	1	0	4	24	132	1020	7600	60480	502180
36	1	0	4	24	180	1440	11560	99120	903700
37	1	0	4	24	276	1680	13720	137760	1360660
38	1	0	4	30	132	1140	9490	70560	616420
39	1	0	4	30	204	1440	12730	113820	1013740
40	1	0	4	48	180	1920	22000	156240	1663060
41	1	0	6	6	138	420	4830	24360	216090
42	1	0	6	18	114	1140	4650	61740	334530
43	1	0	6	18	138	1020	6450	57960	410970
44	1	0	6	18	138	1080	7890	66780	538650
45	1	0	6	18	162	960	7890	58380	484050
46	1	0	6	18	210	1320	11850	96180	868770
47	1	0	6	24	210	1500	12660	110460	971250
48	1	0	6	24	210	1800	13380	133980	1145970
49	1	0	6	24	210	1800	15180	138600	1266930
50	1	0	6	24	234	1800	16620	146160	1390410
51	1	0	6	30	162	1680	11670	103320	944370
52	1	0	6	30	186	1800	13470	122220	1121610
53	1	0	6	36	210	2100	16800	154980	1468530
54	1	0	6	36	234	2520	19680	201600	1958250
55	1	0	6	36	258	2280	20400	191520	1870050
56	1	0	6	36	258	2340	21120	196980	1964130
57	1	0	6	36	282	2520	22920	224280	2235450
58	1	0	6	36	330	2640	27600	274680	2888970
59	1	0	6	42	162	2760	17610	178920	1942290
60	1	0	6	84	714	6840	96360	1211280	14830410
61	1	0	8	12	216	1440	8540	126000	544600
62	1	0	8	18	288	1560	16910	136500	1308160
63	1	0	8	24	360	2160	25640	218400	2401000
64	1	0	8	30	288	2100	19250	171360	1627360
65	1	0	8	30	288	2220	20330	185640	1795360
66	1	0	8	30	336	2520	25010	233940	2378320
67	1	0	8	36	216	3600	16100	294000	2224600
68	1	0	8	36	312	2760	25100	243600	2439640

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Period ID	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
69	1	0	8	42	312	3000	27350	263340	2715160
70	1	0	8	42	360	3540	32750	346080	3613960
71	1	0	8	42	408	3480	36350	368340	3978520
72	1	0	8	42	456	4320	44270	479220	5374600
73	1	0	8	48	336	4680	35000	467040	4747120
74	1	0	8	48	384	3960	37160	400680	4285120
75	1	0	8	48	432	4140	42560	454440	5029360
76	1	0	8	48	432	4320	44720	487200	5513200
77	1	0	8	60	672	7200	83060	1032360	13152160
78	1	0	10	36	486	3720	42400	420000	4761190
79	1	0	10	48	486	4680	47260	519960	5752390
80	1	0	10	48	486	5400	49420	631680	6706630
81	1	0	10	48	510	5160	56260	632100	7307230
82	1	0	10	54	534	5100	54550	594300	6783910
83	1	0	10	60	582	5760	61480	683760	7892710
84	1	0	10	60	582	6360	68680	807240	9730630
85	1	0	12	48	636	5940	68880	780780	9337020
86	1	0	12	54	732	7680	84810	1136520	13271580
87	1	0	12	60	684	6840	76620	893760	10773420
88	1	0	12	66	804	8400	100830	1237740	15842820
89	1	0	12	66	828	8880	108030	1369620	17787420
90	1	0	12	72	756	8580	97320	1209180	15283380
91	1	0	12	72	828	9000	108480	1354920	17558940
92	1	0	12	84	876	9960	122700	1540560	20335980
93	1	0	12	96	1140	14400	193080	2721600	39626580
94	1	0	12	96	1356	17640	245640	3609480	55232940
95	1	0	12	120	1284	17700	253200	3671640	56408100
96	1	0	14	78	1146	12780	174830	2377620	33993610
97	1	0	14	84	1074	12600	161600	2187360	30345490
98	1	0	14	96	1434	18600	276620	4253760	67316410
99	1	0	14	102	1242	15720	211910	2994600	43838410
100	1	0	14	108	1218	17400	224600	3334800	49589890
101	1	0	14	108	1314	18240	245120	3690960	55634530
102	1	0	14	138	2106	30120	474530	7913220	136419850

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Period ID	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
103	1	0	14	180	2082	33480	560480	9276960	164968930
104	1	0	16	96	1632	19320	302200	4447800	71407840
105	1	0	16	108	1488	18960	267460	3922800	59619280
106	1	0	16	108	1632	20640	309220	4640160	73470880
107	1	0	16	204	3264	52680	952180	18086880	352525600
108	1	0	18	108	1542	18180	249480	3486420	50929830
109	1	0	18	120	2022	26160	421020	6607440	110986470
110	1	0	18	144	2118	30960	481860	7971600	135188550
111	1	0	18	156	2358	34920	564120	9502920	165863670
112	1	0	18	174	2454	38880	636030	11007780	197230950
113	1	0	20	144	2148	31800	505280	8329440	141091300
114	1	0	20	168	2580	39600	648920	11239200	200541460
115	1	0	20	198	3228	52260	925130	17075100	326431420
116	1	0	22	168	2634	38040	613660	10263120	178335850
117	1	0	22	186	3354	52980	960970	17852100	345361450
118	1	0	22	192	3258	51720	914620	16742880	318415930
119	1	0	22	264	4122	77880	1476220	29789760	623443450
120	1	0	22	264	4554	82200	1613740	33027120	699596170
121	1	0	24	186	3144	47280	804390	14118720	257786760
122	1	0	24	264	5352	101040	2040360	43219680	953236200
123	1	0	28	342	6540	129540	2770570	61901700	1433166700
124	1	0	32	384	8112	167520	3766640	88438560	2152526320
125	1	0	34	390	8694	179520	4180750	100127580	2501049670
126	1	0	34	498	10278	245040	5923330	153543600	4097686390
127	1	0	36	396	7572	143160	2921580	62324640	1376946900
128	1	0	36	456	9876	214680	5072760	125137740	3200341620
129	1	0	36	768	18996	500640	14713200	450203040	14177280180
130	1	0	44	744	18396	492360	14028200	419215440	12984240220
131	1	0	52	1044	29124	874080	28285540	956113200	33495075460
132	1	0	58	888	23694	632400	18393340	559525680	17662680910
133	1	0	84	1932	69636	2622480	106446900	4526098920	199524037860
134	1	0	90	1788	59886	2032920	74950920	2894154480	116002941390
135	1	0	102	2274	84330	3207480	132223890	5710371660	255843458010
136	1	0	150	4866	241002	12623040	711272850	42024975300	2574748234170

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Period ID	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
137	1	0	224	9312	580704	38555520	2752140320	206084027520	16004302058080
138	1	0	540	37632	3836268	420664320	49565795760	6131551910400	786906243004140
139	1	0	1386	166284	28575342	5322513240	1065056580360	223880895211680	48835950883718670

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