# QUASI-PERIOD COLLAPSE FOR DUALS TO FANO POLYGONS: AN EXPLANATION ARISING FROM ALGEBRAIC GEOMETRY

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Abstract. The Ehrhart quasi-polynomial of a rational polytope $P$ is a fundamental invariant counting lattice points in integer dilates of $P$. The quasi-period of this quasi-polynomial divides the denominator of $P$ but is not always equal to it: this is called quasi-period collapse. Polytopes experiencing quasi-period collapse appear widely across algebra and geometry, and yet the phenomenon remains largely mysterious. By using techniques from algebraic geometry – $\mathbb{Q}$-Gorenstein deformations of orbifold del Pezzo surfaces – we explain quasi-period collapse for rational polygons dual to Fano polygons and describe explicitly the discrepancy between the quasi-period and the denominator.

## 1. Introduction

Let $P \subset \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{Q}$ be a convex lattice polytope of dimension $d$. Let $L_P(k) := \left|kP \cap \mathbb{Z}^d\right|$ count the number of lattice points in dilations $kP$ of $P$, $k \in \mathbb{Z}_{\geq 0}$. Ehrhart [9] showed that $L_P$ can be written as a degree $d$ polynomial

$$L_P(k) = c_d k^d + \ldots + c_1 k + c_0$$

which we call the *Ehrhart polynomial* of $P$. The leading coefficient $c_d$ is given by $\mathrm{Vol}(P)/d!$, $c_{d-1}$ is equal to $\mathrm{Vol}(\partial P)/2(d-1)!$, and $c_0 = 1$. Here $\mathrm{Vol}(\,\cdot\,)$ denotes the normalised volume, and $\partial P$ denotes the boundary of $P$. For example, if $P$ is two-dimensional (that is, $P$ is a lattice *polygon*) we obtain

$$L_P(k) = \frac{\mathrm{Vol}(P)}{2}k^2 + \frac{\left|\partial P \cap \mathbb{Z}^2\right|}{2}k + 1.$$

Setting $k = 1$ in this expression recovers Pick's Theorem [16]. The values of the Ehrhart polynomial of $P$ form a generating function $\mathrm{Ehr}_P(t) := \sum_{k \geq 0} L_P(k) t^k$ called the *Ehrhart series* of $P$.

When the vertices of $P$ are rational points the situation is more interesting. Recall that a *quasi-polynomial* with *period* $s \in \mathbb{Z}_{>0}$ is a function $q : \mathbb{Z} \to \mathbb{Q}$ defined by polynomials $q_0, q_1, \ldots, q_{s-1}$ such that

$$q(k) = q_i(k) \qquad \text{when } k \equiv i \pmod{s}.$$

The *degree* of $q$ is the largest degree of the $q_i$. The minimum period of $q$ is called the *quasi-period*, and necessarily divides any other period $s$. Ehrhart showed that $L_P$ is given by a quasi-polynomial of degree $d$, which we call the *Ehrhart quasi-polynomial* of $P$. Let $\pi_P$ denote the quasi-period of $P$. The smallest positive integer $r_P \in \mathbb{Z}_{>0}$ such that $r_P P$ is a lattice polytope is called the *denominator* of $P$. It is certainly the case that $L_P$ is $r_P$-periodic, however it is perhaps surprising that the quasi-period of $L_P$ does not always equal $r_P$; this phenomenon is called *quasi-period collapse*.

**Example 1.1** (Quasi-period collapse)**.** Consider the triangle $P := \mathrm{conv}\{(5,-1), (-1,-1), (-1,1/2)\}$ with denominator $r_P = 2$. This has $L_P(k) = 9/2k^2 + 9/2k + 1$, hence $\pi_P = 1$.

Quasi-period collapse is poorly understood, although it occurs in many contexts. For example, de Loera–McAllister [7,8] consider polytopes arising naturally in the study of Lie algebras (the Gel'fand–Tsetlin polytopes and the polytopes determined by the Clebsch–Gordan coefficients) that exhibit quasi-period collapse. In dimension two McAllister–Woods [15] show that there exist rational polygons with $r_P$ arbitrarily large but with $\pi_P = 1$ (see also Example 3.8). Haase–McAllister [10] give a constructive view of this phenomena in terms of $\mathrm{GL}_d(\mathbb{Z})$-*scissor congruence*; here a polytope is partitioned into pieces that are individually modified via $\mathrm{GL}_d(\mathbb{Z})$ transformation and lattice translation, then reassembled to give a new polytope which (by construction) has equal Ehrhart quasi-polynomial but different $r_P$.

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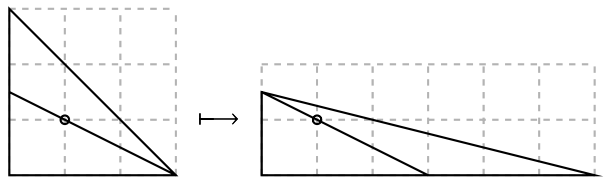
**Example 1.2** ($\mathrm{GL}_2(\mathbb{Z})$-scissor congruence)**.** The lattice triangle $Q := \mathrm{conv}\{(2,-1),(-1,-1),(-1,2)\}$ with Ehrhart polynomial $L_Q(k) = 9/2k^2+9/2k+1$ can be partitioned into two rational triangles as depicted on the left below. Fix the bottom-most triangle, and transform the top-most triangle via the lattice automorphism $e_1 \mapsto (3,-1)$, $e_2 \mapsto (4,-1)$. This gives the rational triangle $P$ (depicted on the right) from Example 1.1.



We give an explanation for quasi-period collapse in two dimensions for a certain class of polygons in terms of recent results in algebraic geometry arising from Mirror Symmetry. In §2 we explain how *mutation* – a combinatorial operation arising from the theory of cluster algebras – gives an explanation of this phenomenon, and explain how this is related to $\mathbb{Q}$*-Gorenstein (qG-) deformations* of del Pezzo surfaces as studied by Wahl [17], Kollár–Shepherd-Barron [14], Hacking–Prokhorov [11], and others. Finally, in Corollary 3.6 we completely characterise the discrepancy between the denominator and the quasi-period for this class of polygons.

## 2. Mutation

In [10] Haase–McAllister propose the open problem of finding a systematic and useful technique that implements $\mathrm{GL}_d(\mathbb{Z})$-scissor congruence for rational polytopes. In the case when the dual polyhedron is a lattice polytope it was observed in [2] that one such technique is given by *mutation*.

2.1. **The combinatorics of mutation.** Let $N \cong \mathbb{Z}^d$ be a rank $d$ lattice and set $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$. Let $P \subset N_\mathbb{Q}$ be a lattice polytope. We require – and will assume from here onwards – that $P$ satisfies the following two conditions:

(a) $P$ is of maximum dimension in $N$, $\dim(P) = d$;
(b) the origin is contained in the strict interior of $P$, $\mathbf{0} \in P^\circ$.

Condition (b) is not especially stringent, and can be satisfied by any polytope with $P^\circ \cap N \neq \varnothing$ by lattice translation. It is, however, an essential requirement in what follows.

Let $M := \mathrm{Hom}(N,\mathbb{Z}) \cong \mathbb{Z}^d$ denote the dual lattice. Given a polytope $P \subset N_\mathbb{Q}$, the dual polyhedron is defined by

$$P^* := \{u \in M_\mathbb{Q} \mid u(v) \geq -1 \text{ for all } v \in P\} \subset M_\mathbb{Q}.$$

Condition (b) gives that $P^*$ is a (typically rational) polytope. It is on rational polytopes dual to lattice polytopes that we focus. In this section we will explain how mutation corresponds to a piecewise-$\mathrm{GL}_d(\mathbb{Z})$ transformation of $P^*$, and hence is an instance of $\mathrm{GL}_d(\mathbb{Z})$-scissor congruence for $P^*$.

Following [2, §3], let $w \in M$ be a primitive lattice vector. Then $w : N \to \mathbb{Z}$ determines a height function (or grading) which naturally extends to $N_\mathbb{Q} \to \mathbb{Q}$. We call $w(v)$ the *height* of $v \in N_\mathbb{Q}$. We denote the set of all points of height $h$ by $H_{w,h}$, and write

$$w_h(P) := \mathrm{conv}\big(H_{w,h} \cap P \cap N\big) \subset N_\mathbb{Q}$$

for the (possibly empty) convex hull of lattice points in $P$ at height $h$.

**Definition 2.1.** A *factor* of $P \subset N_\mathbb{Q}$ with respect to $w \in M$ is a lattice polytope $F \subset w^\perp$ such that for every negative integer $h \in \mathbb{Z}_{<0}$ there exists a (possibly empty) lattice polytope $R_h \subset N_\mathbb{Q}$ such that

$$H_{w,h} \cap \mathrm{vert}(P) \subseteq R_h + |h|\, F \subseteq w_h(P).$$

Here '+' denotes Minkowski sum, and we define $\varnothing + Q = \varnothing$ for every lattice polytope $Q$.

**Definition 2.2.** Let $P \subset N_\mathbb{Q}$ be a lattice polytope with $w \in M$ and $F \subset N_\mathbb{Q}$ as above. The *mutation* of $P$ with respect to the data $(w,F)$ is the lattice polytope

$$\mu_{(w,F)}(P) := \mathrm{conv}\left( \bigcup_{h \in \mathbb{Z}_{<0}} R_h \cup \bigcup_{h \in \mathbb{Z}_{\geq 0}} (w_h(P) + hF) \right) \subset N_\mathbb{Q}.$$

It is shown in [2, Proposition 1] that, for fixed data $(w, F)$, any choice of $\{R_h\}$ satisfying Definition 2.1 gives $\mathrm{GL}_d(\mathbb{Z})$-equivalent mutations. Since we regard lattice polytopes as being defined only up to $\mathrm{GL}_d(\mathbb{Z})$-equivalence, this means that mutation is well-defined. One can readily see that translating the factor $F$ by some lattice point $v \in w^\perp \cap N$ gives isomorphic mutations: $\mu_{(w,F+v)}(P) \cong \mu_{(w,F)}(P)$. In particular if $\dim(F) = 0$ then $\mu_{(w,F)}(P) \cong P$. Finally, we note that mutation is always invertible [2, Lemma 2]: if $Q := \mu_{(w,F)}(P)$ then $P = \mu_{(-w,F)}(Q)$.

**Remark 2.3.** Informally, mutation corresponds to the following operation on slices $w_h(P)$ of $P$: at height $h$ one Minkowski adds or "subtracts" $|h|$ copies of $F$, depending on the sign of $h$. Definition 2.1 ensures that the concept of Minkowski subtraction makes sense.

Mutation has a natural description in terms of the dual polytope $P^*$ [2, Proposition 4 and pg. 12].

**Definition 2.4.** The *inner-normal fan* in $M_\mathbb{Q}$ of a polytope $F \subset N_\mathbb{Q}$ is generated by the cones

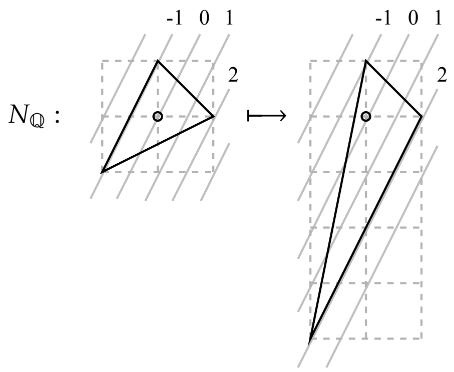
$$\sigma_{v_F} := \{u \in M_\mathbb{Q} \mid u(v_F) = \min\{u(v) \mid v \in F\}\}, \qquad \text{for each } v_F \in \mathrm{vert}(F).$$

A mutation $\mu_{(w,F)}$ induces a piecewise-$\mathrm{GL}_d(\mathbb{Z})$ transformation $\varphi_{(w,F)}$ on $M_\mathbb{Q}$ given by

$$\varphi_{(w,F)} : u \mapsto u - u_{\min} w, \qquad \text{where } u_{\min} := \min\{u(v_F) \mid v_F \in \mathrm{vert}(F)\}.$$

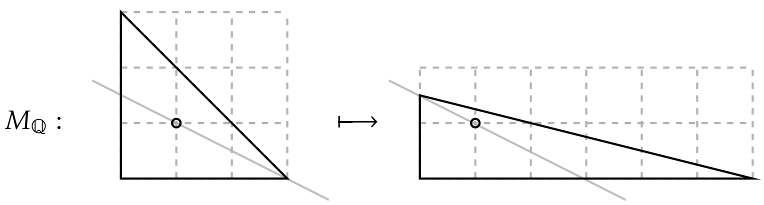
The inner-normal fan of $F$ determines a chamber decomposition of $M_\mathbb{Q}$, and $\varphi_{(w,F)}$ acts linearly within each chamber. Let $Q := \mu_{(w,F)}(P)$. Then $\varphi_{(w,F)}(P^*) = Q^*$. It is clear that the Ehrhart quasi-polynomials $L_{P^*}$ and $L_{Q^*}$ for the dual polytopes are equal, since the map $\varphi_{(w,F)}$ is piecewise-linear. Hence mutation gives a systematic way to produce examples of $\mathrm{GL}_d(\mathbb{Z})$-scissor congruence.

**Example 2.5** (Mutation)**.** Let $P = \mathrm{conv}\{(1,0),(0,1),(-1,-1)\} \subset N_\mathbb{Q}$ and $w = (2,-1) \in M$. Then $F = \mathrm{conv}\{(0,0),(-1,-2)\} \subset w^\perp$ is a factor. We see that $Q := \mu_{(w,F)}(P) = \mathrm{conv}\{(1,0),(0,1),(-1,-4)\}$.




On the dual side we have that $M_\mathbb{Q}$ is divided into two chambers whose boundary is given by $\mathbb{Q} \cdot w$, and

$$\varphi_{(w,F)} : (u_1, u_2) \mapsto \begin{cases} (u_1, u_2), & \text{if } u_1 + 2u_2 \leq 0; \\ (3u_1 + 4u_2, -u_1 - u_2), & \text{otherwise.} \end{cases}$$




Thus we recover Example 1.2 from the view-point of mutation.

From here onwards we assume that $P \subset N_\mathbb{Q}$ is *Fano*. That is, in addition to conditions (a) and (b) above, $P$ satisfies:

(c) the vertices $\mathrm{vert}(P)$ of $P$ are primitive lattice points.

The property of being Fano is preserved under mutation [2, Proposition 2]. A Fano polytope $P$ corresponds to a toric Fano variety $X_P$ via the *spanning fan* (that is, the fan whose cones are spanned by the faces of $P$). See [6] for the theory of toric varieties and [13] for a survey of Fano polytopes. When $P$ is a Fano polygon, $X_P$ corresponds to a toric del Pezzo surface with at worst log terminal singularities. The *singularity content* of $P$, which we recall in Definition 2.10 below, is a mutation-invariant of $P$ introduced in [3]. In §2.4 we remark briefly on the connection between singularity content and the qG-deformation theory of $X_P$, and how this gives a geometric explanation for the quasi-period collapse of $P^*$.

2.2. **Quotient singularities.** In order to state the definition of singularity content we first recall some of the theory of quotient or orbifold surface singularities. A cyclic quotient singularity is a surface singularity isomorphic to a quotient $\mathbb{A}^2/G$, where $G$ is a finite cyclic group acting diagonally on $\mathbb{A}^2$. Assuming that $G$ acts faithfully means that it can be expressed as a subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by

$$\begin{pmatrix} \varepsilon & \\ & \varepsilon^a \end{pmatrix}$$

where $\varepsilon$ is a root of unity and $a \in \mathbb{Z}$. Suppose that $G$ has order $r$; all possible representations are obtained (non-uniquely) by letting $a$ range over $0, \dots, r-1$. If $G$ is generated by the matrix above for $\varepsilon$ a primitive $r$-th root of unity, then denote by $\frac{1}{r}(1,a)$ the singularity $\mathbb{A}^2/G$. As a quotient of affine space by an abelian group, $\frac{1}{r}(1,a)$ is an affine toric variety whose fan we now describe.

Let $N \cong \mathbb{Z}^2$ and $M = \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ be the cocharacter and character lattices respectively of an algebraic two-torus $(\mathbb{C}^\times)^2$. A cone $\sigma \subset N_{\mathbb{Q}}$ whose rays are generated by lattice points in $N$ describes an affine toric variety $X_\sigma$. More generally, a collection of cones given by a fan $\Sigma$ describes a non-affine toric variety $X_\Sigma$. The singularity $\frac{1}{r}(1,a)$ is the affine toric variety associated to the cone

$$\sigma = \mathrm{cone}\{e_2, re_1 - ae_2\} \subset N_{\mathbb{Q}}.$$

The lattice height of such a cone – that is, the lattice distance between the origin and the line segment joining the two primitive ray generators of the cone (the *edge* of the cone) – is called the *local index*, and can be calculated to be

$$\ell_\sigma = \frac{r}{\gcd\{r, a+1\}}.$$

The *width* of the cone is the number of unit-length lattice line segments along the edge of the cone or, equivalently, one less than the number of lattice points along the edge. The width is equal to $\gcd\{r, a+1\}$. We will often conflate a singularity and its corresponding cone in $N_{\mathbb{Q}}$. An isolated cyclic quotient singularity is a *T-singularity* if it is smoothable by a qG-deformation.

**Lemma 2.6** ([14, Proposition 3.11])**.** *An isolated cyclic quotient singularity is a $T$-singularity if and only if it takes the form*

$$\frac{1}{dn^2}(1, dnc-1)$$

*for some $c$ with* $\gcd\{n,c\} = 1$.

The cone $\sigma \subset N_{\mathbb{Q}}$ associated to a $T$-singularity $\frac{1}{dn^2}(1, dnc-1)$ has local index $\ell = n$ and width $dn$; it is easily seen that $T$-singularities are characterised by having the width divisible by the local index. Suppose that $P \subset N_{\mathbb{Q}}$ is a Fano polygon with edge $E$ spanning $\sigma$. Let $w \in M$ be the primitive inner-normal such that $w(E) = -\ell$, and choose $F \subset w^\perp$ of lattice length $d$. The mutation $\mu_{(w,F)}(P)$ collapses the edge $E$ to a vertex, removing the cone $\sigma$. This is equivalent to a local qG-smoothing of the $T$-singularity.

**Example 2.7.** Consider the polytope $Q := \mathrm{conv}\{(1,0),(0,1),(-1,-4)\}$ appearing in Example 2.5. The corresponding spanning fan has three two-dimensional cones, two of which are smooth and one of which, $\mathrm{cone}\{(1,0),(-1,-4)\}$, corresponds to a $\frac{1}{4}(1,1)$ $T$-singularity.

The other relevant class of quotient singularities are the *R-singularities* introduced in [3].

**Definition 2.8.** A cyclic quotient singularity of local index $\ell$ and width $k$ is an *R-singularity* if $k < \ell$.

Let $\sigma \subset N_{\mathbb{Q}}$ be a cone of local index $\ell$ and width $k$. Write $k = d\ell + r$, where $d, r \in \mathbb{Z}_{\geq 0}$, $0 \leq r < \ell$. If $r = 0$ then $\sigma$ is a $T$-singularity. Assume that $r \neq 0$ and, as before, suppose that $P \subset N_{\mathbb{Q}}$ is a Fano polygon with edge $E$ spanning $\sigma$. Let $w \in M$ be the corresponding inner-normal, and pick $F \subset w^\perp$ of lattice length $d$. The mutation $\mu_{(w,F)}(P)$ transforms $\sigma$ to a cone $\tau$ of width $r$ corresponding to a $\frac{1}{r\ell}(1, rc/k - 1)$ singularity. Crucially, $\tau$ has width strictly less than the local index, and so cannot be simplified via further mutation. This is equivalent to a partial qG-smoothing of the original singularity $\sigma$, resulting in a singularity $\tau$ that is rigid under qG-deformation. The $R$-singularity $\tau$ is independent of the choices made [3, Proposition 2.4].

**Definition 2.9.** Let $\sigma \subset N_{\mathbb{Q}}$ be a cone corresponding to a $\frac{1}{r}(1, c-1)$ singularity. Let $\ell$ be the local index and let $k$ be the width of the cone. Write $k = d\ell + r$, where $d, r \in \mathbb{Z}_{\geq 0}$, $0 \leq r < \ell$. The *residue* of $\sigma$ is

$$\operatorname{res}(\sigma) = \begin{cases} \frac{1}{r\ell}(1, rc/k - 1), & \text{if } r \neq 0; \\ \varnothing, & \text{otherwise.} \end{cases}$$

The *singularity content* of $\sigma$ is the pair $(d, \operatorname{res}(\sigma))$. The singularity content is local qG-deformation-theoretic data about $\sigma$.

**Definition 2.10.** Let $P \subset N_{\mathbb{Q}}$ be a Fano polygon with cones $\sigma_1, \ldots, \sigma_n$. The *basket* of $P$ is the multiset

$$\mathcal{B} := \{\operatorname{res}(\sigma_i) \mid 1 \leq i \leq n\},$$

where the empty residues are omitted[1]. The *singularity content* of $P$ is the pair

$$(d_1 + \cdots + d_n, \mathcal{B}),$$

where the $d_i$ are the integers appearing in the singularity content of the $\sigma_i$. Singularity content is a qG-deformation-invariant of $X_P$.

2.3. **Hilbert series.** Any projective toric variety $X_P$ arising from a polytope $P$ comes with a natural ample divisor $D$ given by its toric boundary $D = X_P \setminus \mathbb{T}$, where $\mathbb{T}$ is the big torus inside $X_P$. When $P$ is Fano, $D = -K$, the anti-canonical divisor on $X_P$. In this case, due to the standard toric dictionary allowing one to move between lattice points in $M$ and sections of line bundles on $X_P$, one has that the Hilbert function of $(X_P, -K)$ equals the Ehrhart quasi-polynomial $L_{P^*}(k)$ of the rational polytope $P^*$. Hence the generating function $\operatorname{Hilb}_{(X_P,-K)}(t)$ for the Hilbert function of $(X_P, -K)$ – the *Hilbert series* of $(X_P, -K)$ – is equal to the Ehrhart series of $P^*$. From here onwards we supress $-K$ from the notation.

The Hilbert series of an orbifold del Pezzo surface $X$ with basket $\mathcal{B}$ can be written in the form [3, Corollary 3.5]:

$$\operatorname{Hilb}_X(t) = \frac{1 + (K^2 - 2)t + t^2}{(1-t)^3} + \sum_{\sigma \in \mathcal{B}} Q_\sigma,$$

where $Q_\sigma$ are *orbifold correction terms* given by certain rational functions with denominators $1 - t^{\ell_\sigma}$. For example, the orbifold correction term for the $R$-singularity $\frac{1}{3}(1,1)$ is

$$Q_{\frac{1}{3}(1,1)} = \frac{-t}{3(1-t^3)} = -\frac{1}{3}(t + t^4 + t^7 + \ldots)$$

which contributes $-1/3$ to the coefficient of $t^d$ when $d \equiv 1 \pmod 3$.

The Hilbert function is a quasi-polynomial when $X$ is an orbifold (because the anti-canonical divisor is $\mathbb{Q}$-Cartier rather than Cartier). The anti-canonical divisor does not correspond to a line bundle, but some integer multiple of it does. The smallest integer $d$ such that $-dK$ is Cartier is called the *Gorenstein index* of $X$ and denoted by $\ell_X$. In the toric setting, $-dK$ is Cartier if and only if $dP^*$ is a lattice polytope. Hence the Gorenstein index $\ell_{X_P}$ of $X_P$ equals the denominator $r_{P^*}$ of $P^*$.

2.4. **Algebraic geometry and the quasi-period.** Mutations were introduced in [2] as part of an ongoing program investigating Mirror Symmetry for Fano manifolds [5]. In two dimensions the picture is very well understood: see [1] for the details. In summary, if two Fano polygons $P$ and $Q \subset N_{\mathbb{Q}}$ are related by a sequence of mutations then there exists a qG-deformation between the corresponding toric del Pezzo surfaces $X_P$ and $X_Q$. Such a qG-deformation preserves the anti-canonical Hilbert series, hence $L_{P^*} = L_{Q^*}$ and so the quasi-periods of $P^*$ and $Q^*$ agree. However it does not in general preserve the Gorenstein index, and hence the denominators $r_{P^*}$ and $r_{Q^*}$ need not be equal. The cones over the edges of $P$ correspond to the singularities of $X_P$, and these admit partial qG-smoothings to the qG-rigid singularities given by the basket $\mathcal{B}$ of residues.

Suppose that the singularity content of $P$ is $(d, \mathcal{B})$. Then, by the absence of global obstructions to qG-deformations on Fano varieties, $X_P$ is qG-deformation-equivalent to a (*not necessarily toric*) del Pezzo surface $X$ with singularities $\mathcal{B}$ and whose non-singular locus has topological Euler number $d$. Since $\operatorname{Hilb}_{X_P}(t) = \operatorname{Hilb}_X(t)$, we have an explanation for quasi-period collapse of the dual polytope $P^*$. Specifically, the Gorenstein index of $X$ is equal to the quasi-period of $P^*$.

[1] In [3] the basket is cyclically ordered. Although important from the viewpoint of classification, it is not required here.

## 3. Studying quasi-period collapse

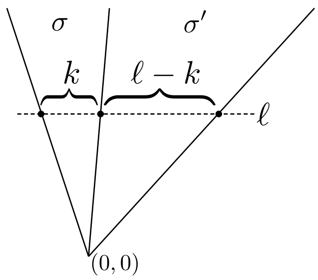
The Hilbert series of orbifold del Pezzo surfaces were studied in [18] with the aim of describing the structure of the set of possible baskets $\mathcal{B}$ of $R$-singularities on orbifold del Pezzo surfaces with a fixed Hilbert series. This is achieved by partitioning $\mathcal{B}$ into two pieces: a *reduced basket* and an *invisible basket*. The latter, along with the $T$-singularities, is not detectable by the Hilbert series, and from our viewpoint it is this invisibility that causes quasi-period collapse.

**Definition 3.1.** A collection $\sigma_1, \ldots, \sigma_n$ of $R$-singularities is a *cancelling tuple* if

$$Q_{\sigma_1} + \cdots + Q_{\sigma_n} = 0.$$

A collection of $R$-singularities is called *invisible* if it is a union of cancelling tuples.

**Example 3.2.** Let $\sigma$ be an $R$-singularity of local index $\ell$ and width $k$. Then there exists an $R$-singularity $\sigma'$ of local index $\ell$ and width $\ell - k$ such that $Q_\sigma + Q_{\sigma'} = 0$. Combinatorially, this is understood by the observation that the union of the two cones gives a $T$-singularity.




**Definition 3.3.** Let $X$ be an orbifold del Pezzo surface. A maximal invisible subcollection of the basket $\mathcal{B}$ of $X$ is called an *invisible basket* for $X$. Notice that such a maximal subcollection is not unique, since singularities can appear in many different cancelling tuples. Given a choice of invisible basket $\mathcal{IB} \subset \mathcal{B}$, the complement $\mathcal{RB} = \mathcal{B} \setminus \mathcal{IB}$ is called the *reduced basket* for $X$ corresponding to the choice of $\mathcal{IB}$.

Let $P \subset N_\mathbb{Q}$ be a Fano polygon with singularity content $(d, \mathcal{B})$. Let $\mathcal{IB}$ be an invisible basket of $\mathcal{B}$ with corresponding reduced basket $\mathcal{RB}$. Hence $\mathcal{B} = \mathcal{RB} \amalg \mathcal{IB}$. Denote the collection of $T$-singularities on $X_P$ by $\mathcal{T}$ (so $|\mathcal{T}| = d$).

**Theorem 3.4.** *Let $P \subset N_\mathbb{Q}$ be a Fano polygon. The quasi-period of $P^*$ is given by*[2]

$$\pi_{P^*} = \operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}.$$

*Furthermore, $P^*$ exhibits quasi-period collapse if and only if there exists some $\tau \in \mathcal{IB} \cup \mathcal{T}$ of local index not dividing $\operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}$. Moreover, the quasi-period collapse is measured by $\mathcal{IB}$:*

$$r_{P^*} = \operatorname{lcm}(\{\pi_{P^*}\} \cup \{\ell_\sigma \mid \sigma \in \mathcal{IB} \cup \mathcal{T}\}).$$

*Proof.* We have

$$\mathrm{Ehr}_{P^*}(t) = \mathrm{Hilb}_{X_P}(t) = \text{initial term} + \sum_{\sigma \in \mathcal{B}} Q_\sigma = \text{initial term} + \sum_{\sigma \in \mathcal{RB}} Q_\sigma.$$

As discussed, each orbifold correction term $Q_\sigma$ contributes to the coefficients of this series as a quasi-polynomial with quasi-period $\ell_\sigma$. When $\sigma \in \mathcal{RB}$ these terms are not cancelled and so make non-zero contributions to the coefficients of the Ehrhart series, hence its quasi-period is given by:

$$\pi_{P^*} = \operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}.$$

The Gorenstein index of $P$ is equal to $\ell_{X_P} = \operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{B} \cup \mathcal{T}\}$. Hence

$$r_{P^*} = \ell_{X_P} = \operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB} \cup \mathcal{IB} \cup \mathcal{T}\} = \operatorname{lcm}(\{\pi_{P^*}\} \cup \{\ell_\sigma \mid \sigma \in \mathcal{IB} \cup \mathcal{T}\}).$$

This is distinct from $\pi_{P^*}$ if and only if $\operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{IB} \cup \mathcal{T}\}$ does not divide $\pi_{P^*}$. □

**Remark 3.5.** It follows from [18, §4] that the choice of $\mathcal{IB}$ is irrelevant in the statement of Theorem 3.4.

As a corollary to Theorem 3.4 we immediately obtain:

[2] We adopt the convention that $\operatorname{lcm}\{\varnothing\} = 1$.

**Corollary 3.6.** *Let $P \subset N_{\mathbb{Q}}$ be a Fano polygon. The discrepancy between the quasi-period and denominator of $P^*$ is*

$$\frac{r_{P^*}}{\pi_{P^*}} = \frac{\operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{IB} \cup \mathcal{T}\}}{\gcd\{\operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}, \operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{IB} \cup \mathcal{T}\}\}}.$$

**Example 3.7** (Detecting quasi-period collapse)**.** Consider the polytope $Q := \operatorname{conv}\{(1,0),(0,1),(-1,-4)\}$ appearing in Example 2.5. This has singularity content $(3, \varnothing)$, and $\mathcal{T} = \{2 \times \text{smooth}, \frac{1}{4}(1,1)\}$. Applying Corollary 3.6 we have that $r_{Q^*} = 2\pi_{Q^*}$.

We now give an example of an infinite family of Fano triangles, obtained via mutation, where the denominator $r_{P^*}$ can become arbitrarily large but where $\pi_{P^*} = 1$. Let $P \subset N_{\mathbb{Q}}$ be a Fano triangle. Recall that the corresponding toric variety $X_P$ is a *fake weighted projective plane* [12]: a quotient of a weighted projective plane by a finite group $N/N'$ acting free in codimension one, where $N'$ is the sublattice generated by the vertices of $P$.

**Example 3.8** (Mutations of $\mathbb{P}^2$)**.** In [4,11] the graph of mutations of $\mathbb{P}^2$ is constructed. The vertices of this graph are given by $\mathbb{P}(a^2,b^2,c^2)$, where $(a,b,c) \in \mathbb{Z}^3_{>0}$ is a *Markov triple* satisfying

$$a^2 + b^2 + c^2 = 3abc. \tag{3.1}$$

Let $X_P = \mathbb{P}(a^2,b^2,c^2)$ be such a weighted projective plane, with $P \subset N_{\mathbb{Q}}$ the corresponding Fano triangle. Since $X_P$ is qG-deformation-equivalent to $\mathbb{P}^2$, so $X_P$ is smoothable and its anti-canonical Hilbert function has quasi-period one. Hence $\pi_{P^*} = 1$. However, the denominator $r_{P^*}$ of $P^*$ can be arbitrarily large. To see this, note first that $a, b, c$ must be pairwise coprime: if $p \mid a$ and $p \mid b$ then $p^2 \mid 3abc = a^2 + b^2 + c^2$, and hence $p \mid c$; but then $p$ appears as a square on the left-hand side and as a cube on the right-hand side of (3.1). Let $\bar{b}$ be an inverse of $b \pmod{a^2}$. Note that $c^2\bar{b}^2 + 1 \equiv (3abc - b^2)\bar{b}^2 + 1 \equiv 3a\bar{b}c \pmod{a^2}$, and so the singularity $\frac{1}{a^2}(b^2,c^2)$ on $X_P$ has local index

$$\frac{a^2}{\gcd\left\{a^2, c^2\bar{b}^2 + 1\right\}} = \begin{cases} a, & \text{if } a \not\equiv 0 \pmod 3; \\ a/3, & \text{if } a \equiv 0 \pmod 3). \end{cases}$$

Considering equation (3.1) $\pmod 3$ shows that no Markov numbers are divisible by three. Hence the three local indices on $X_P$ are $a$, $b$, and $c$, and so $r_{P^*} = abc$. The two triangles $P$ and $Q$ in Example 2.5 are the simplest examples, arising from the Markov triples $(1,1,1)$ and $(1,1,2)$ respectively, and corresponding to $\mathbb{P}^2$ and $\mathbb{P}(1,1,4)$.

**Remark 3.9.** There exist Fano triangles of quasi-period one not arising from the construction in Example 3.8. For example, consider

$$P = \operatorname{conv}\{(3,2),(-1,2),(-1,-2)\} \subset N_{\mathbb{Q}}.$$

The corresponding fake weighted projective plane $X_P = \mathbb{P}(1,1,2)/(\mathbb{Z}/4)$ has $2 \times \frac{1}{4}(1,3)$ and $\frac{1}{8}(1,3)$ $T$-singularities. We see that $P^*$ has $r_{P^*} = 2$ and $\pi_{P^*} = 1$. In fact $X_P$ is qG-smoothable to the nonsingular del Pezzo surface of degree two, and hence $L_{P^*}(k) = k^2 + k + 1$.

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