LAURENT INVERSION

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Abstract. We describe a practical and effective method for reconstructing the deformation class of a Fano manifold X from a Laurent polynomial f that corresponds to X under Mirror Symmetry. We explore connections to nef partitions, the smoothing of singular toric varieties, and the construction of embeddings of one (possibly-singular) toric variety in another. In particular, we construct degenerations from Fano manifolds to singular toric varieties; in the toric complete intersection case, these degenerations were constructed previously by Doran–Harder. We use our method to find models of orbifold del Pezzo surfaces as complete intersections and degeneracy loci, and to construct a new four-dimensional Fano manifold.

1. Introduction

The classification of Fano manifolds is an important open problem in geometry. As things stand the classification is understood only in dimensions one, two, and three [27–29, 33–37], but Golyshev et al. have announced a new approach to Fano classification [11, 22], using Mirror Symmetry, that could potentially work in all dimensions. Extensive computational experiments suggest that, under Mirror Symmetry, n-dimensional Fano manifolds correspond to certain Laurent polynomials in n variables with very special properties. We understand how to recover the known classifications in low dimensions from this perspective [1, 2, 12], but two essential questions remain:

(A) what is the class of Laurent polynomials f that correspond, under Mirror Symmetry, to Fano manifolds X?

(B) given such a Laurent polynomial f, how can we construct the corresponding X?

There has been significant recent progress on Question A: deformation families of Fano manifolds conjecturally correspond to mutation-equivalence classes of certain *rigid maximally mutable Laurent polynomials* [1, 31]. In this paper we make significant progress on Question B. There are well-understood methods, going back to Givental and Hori–Vafa, that to a Fano toric complete intersection X associate a Laurent polynomial f that corresponds to X under Mirror Symmetry. We describe a technique, *Laurent inversion*, for inverting this process, constructing the toric complete intersection X directly from its Laurent polynomial mirror f. In many cases this allows, given a Laurent polynomial f, the direct construction of a Fano manifold X that corresponds to f under Mirror Symmetry. Thus, in many cases, Laurent inversion answers Question B. In fact, as we explain in §9, when phrased appropriately, Laurent inversion is not limited to toric complete intersections: we can use it to construct Fano manifolds X as degeneracy loci (cut out by Pfaffian-type equations), and to give other classical constructions. As proof of concept, in §7 we construct a new four-dimensional Fano manifold by applying Laurent inversion to a rigid maximally-mutable Laurent polynomial in four variables.

The idea of reconstructing a Fano manifold X from its mirror f is not new. It is expected that, if a Fano manifold X is mirror to a Laurent polynomial f, then there is a degeneration from X to the (singular) toric variety X_f defined by the spanning fan of the Newton polytope of f; such a degeneration has been constructed for complete intersections in partial flag manifolds by Doran–Harder [20]. Thus one might hope to recover the Fano manifold X from f

by smoothing X_f, for instance using the Gross–Siebert program[1] [24], or via deformation theory [3, 4, 9, 10, 26]. Our new contribution here is to give an explicit construction of X, rather than a proof of its existence. Indeed, regardless of its context, Laurent inversion gives a powerful new method for constructing algebraic varieties. We illustrate this in §10 below, where we exhibit explicit models for del Pezzo surfaces with $1/3(1,1)$ singularities that played an essential role in the Corti–Heuberger classification [19], and which are hard to construct using more traditional methods.

As we will see in §5, in many cases Laurent inversion constructs, along with X, an embedded degeneration from X to the singular toric variety X_f – thus implementing the expected smoothing of X_f discussed above. We hope therefore that Laurent inversion will give a substantial hint as to the generalisations required to get a Gross–Siebert-style smoothing procedure working in higher dimensions. In the toric complete intersection case, such an embedded degeneration has been constructed by Doran–Harder [20]; we build an explicit link to their work in §12, where we describe how our main combinatorial construction, *scaffolding*, can be seen as a generalisation of their notion of amenable collection. We also discuss (in §11) how scaffolding gives a generalisation to the Fano case of Borisov's celebrated *nef partitions*, which have proved a powerful tool for constructing mirror partners to Calabi–Yau toric complete intersections [6, 7]. It will be very interesting to see how much of the theory survives to the Calabi–Yau case, and whether we can use Laurent inversion to construct and investigate Calabi–Yau manifolds that are not complete intersections.

2. Laurent Polynomial Mirrors for Toric Complete Intersections

We begin by recalling how to associate to a toric complete intersection X a Laurent polynomial that corresponds to X under Mirror Symmetry. This question has been considered by many authors [17, 20, 21, 25, 39, 40], and we will give a construction which generalises and unifies all these perspectives below (in §12). Consider first the ambient toric variety or toric stack Y. We consider the case where:

(1)
- (i) Y is a proper toric Deligne–Mumford stack;
- (ii) the coarse moduli space of Y is projective;
- (iii) the generic isotropy group of Y is trivial, that is, Y is a toric *orbifold*; and
- (iv) at least one torus-fixed point in Y is smooth.

Conditions (i)–(iii) here are essential; condition (iv) is less important and will be removed in §12. In the original work by Borisov–Chen–Smith [8], toric Deligne–Mumford stacks are defined in terms of stacky fans. In our context, since the generic isotropy is trivial, giving a stacky fan that defines Y amounts to giving a triple $(N;\Sigma;\rho_1,\ldots,\rho_R)$ where N is a lattice, Σ is a rational simplicial fan in $N\otimes\mathbb{Q}$, and $\rho_1,\ldots,\rho_R$ are elements of N that generate the rays of Σ. It will be more convenient for our purposes, however, to represent Y as a GIT quotient $\left[\mathbb{C}^R /\!\!/_\omega (\mathbb{C}^\times)^r\right]$. Any such Y can be realised this way, as we now explain.

Definition 2.1. We say that $(K;\mathbb{L};D_1,\ldots,D_R;\omega)$ are *GIT data* if $K\cong(\mathbb{C}^\times)^r$ is a connected torus of rank r; $\mathbb{L}=\mathrm{Hom}(\mathbb{C}^\times,K)$ is the lattice of subgroups of K; $D_1,\ldots,D_R\in\mathbb{L}^*$ are characters of K that span a strictly convex full-dimensional cone in $\mathbb{L}^*\otimes\mathbb{Q}$, and $\omega\in\mathbb{L}^*\otimes\mathbb{Q}$ lies in this cone.

GIT data $(K;\mathbb{L};D_1,\ldots,D_R;\omega)$ determine a quotient stack $\left[V_\omega/K\right]$ with $V_\omega\subset\mathbb{C}^R$, as follows. The characters $D_1,\ldots,D_R$ define an action of K on $\mathbb{C}^R$. For convenience write

[1] This works in dimension two [38], but the higher-dimensional case is significantly more involved.

$[R] := \{1, 2, \ldots, R\}$. We say that a subset $I \subset [R]$ *covers* ω if and only if $\omega = \sum_{i \in I} a_i D_i$ for some strictly positive rational numbers a_i. Set $\mathcal{A}_\omega = \{I \subset [R] \mid I \text{ covers } \omega\}$, and set

$$V_\omega = \bigcup_{I \in \mathcal{A}_\omega} (\mathbb{C}^\times)^I \times \mathbb{C}^{\bar{I}} \qquad \text{where} \qquad (\mathbb{C}^\times)^I \times \mathbb{C}^{\bar{I}} = \big\{(x_1, \ldots, x_R) \in \mathbb{C}^R \mid x_i \neq 0 \text{ if } i \in I\big\}.$$

The subset $V_\omega \subset \mathbb{C}^R$ is K-invariant, and $\big[V_\omega/K\big]$ is the GIT quotient stack given by the action of K on $\mathbb{C}^R$ and the stability condition ω. The convexity hypothesis in Definition 2.1 ensures that $\big[V_\omega/K\big]$ is proper.

Remark 2.2. Recall that the quotient $\big[V_\omega/K\big]$ depends on ω only via the minimal cone σ of the secondary fan such that $\omega \in \sigma$. The *secondary fan* for $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ is the fan defined by the wall-and-chamber decomposition of the cone in $\mathbb{L}^* \otimes \mathbb{Q}$ spanned by $D_1, \ldots, D_R$, where the walls are given by all $(r-1)$-dimensional cones of the form $\{D_i \mid i \in I\}$ with $I \subset [R]$.

Definition 2.3. *Orbifold GIT data* are those such that the quotient $\big[V_\omega/K\big]$ is a toric orbifold.

The quotient $\big[V_\omega/K\big]$ is a toric Deligne–Mumford stack if and only if ω lies in the strict interior of a maximal cone in the secondary fan. A toric orbifold Y satisfying conditions (1) above arises as the quotient $\big[V_\omega/K\big]$ for GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ as follows. Suppose that Y is defined by the stacky fan data $(N; \Sigma; \rho_1, \ldots, \rho_R)$. There is an exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^R \xrightarrow{\ \rho\ } N \longrightarrow 0 \tag{2}$$

where ρ maps the ith element of the standard basis for $\mathbb{Z}^R$ to ρ_i; this defines $\mathbb{L}$ and $K = \mathbb{L} \otimes \mathbb{C}^\times$. Dualising gives

$$0 \longleftarrow \mathbb{L}^* \xleftarrow{\ D\ } (\mathbb{Z}^*)^R \longleftarrow M \longleftarrow 0 \tag{3}$$

where $M := \operatorname{Hom}(N, \mathbb{Z})$, and we set $D_i \in \mathbb{L}^*$ to be the image under D of the ith standard basis element for $(\mathbb{Z}^*)^R$. The stability condition ω is taken to lie in the strict interior of

$$C := \bigcap_{\text{maximal cones } \sigma \text{ of } \Sigma} C_\sigma$$

where C_σ is the cone in $\mathbb{L}^* \otimes \mathbb{Q}$ spanned by $\{D_i \mid i \notin \sigma\}$; projectivity of the coarse moduli space of Y implies that C is a maximal cone of the secondary fan, and in particular that C has non-empty interior.

We can reverse this construction, defining a stacky fan $(N; \Sigma; \rho_1, \ldots, \rho_n)$ from GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ such that $D_1, \ldots, D_R$ span $\mathbb{L}^*$. The lattice $\mathbb{L}$ and elements $D_1, \ldots, D_R \in \mathbb{L}^*$ define the exact sequence (3), and dualising gives (2). This defines the lattice N and $\rho_1, \ldots, \rho_R$. The fan Σ consists of the cones spanned by $\{\rho_i \mid i \in I\}$ where $I \subset [R]$ satisfies $[R] \setminus I \in \mathcal{A}_\omega$.

Remark 2.4. Once K, $\mathbb{L}$, and $D_1, \ldots, D_R$ have been fixed, choosing ω such that the GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ define a toric Deligne–Mumford stack amounts to choosing a maximal cone in the secondary fan.

Remark 2.5. A character $\chi \in \mathbb{L}^*$ determines a line bundle on Y, which we denote also by χ.

Definition 2.6. Let $\Theta = (K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ be orbifold GIT data, and let Y denote the corresponding toric orbifold. A *convex partition with basis* for Θ is a partition $B, S_1, \ldots, S_k, U$ of $[R]$ such that:

(i) $\{D_b \mid b \in B\}$ is a basis for $\mathbb{L}^*$;
(ii) ω is a non-negative linear combination of $\{D_b \mid b \in B\}$;
(iii) each S_i is non-empty;

(iv) for each $i \in [k]$, the line bundle $L_i := \sum_{j \in S_i} D_j$ on Y is convex[2]; and
(v) for each $i \in [k]$, L_i is a non-negative linear combination of $\{D_b \mid b \in B\}$.

We allow $k = 0$, and we allow $U = \varnothing$.

Remark 2.7. Since ω here is taken to lie in the strict interior of a maximal cone in the secondary fan, it is given by a positive linear combination of $\{D_b \mid b \in B\}$. This positivity guarantees that the maximal cone spanned by $\{\rho_i \mid i \in [R] \setminus B\}$ defines a smooth torus-fixed point in Y.

Remark 2.8. It would be more natural to replace the condition that L_i be convex here with the weaker condition that L_i be nef. But, since we currently lack a Mirror Theorem that applies to toric complete intersections beyond the convex case, we will require convexity. If the ambient space Y is a manifold, rather than an orbifold, then a line bundle on Y is convex if and only if it is nef.

Given:

(4)
(i) orbifold GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_R; \omega)$;
(ii) a convex partition with basis $B, S_1, \ldots, S_k, U$ for Θ; and
(iii) a choice of elements $s_i \in S_i$ for each $i \in [k]$;

we define a Laurent polynomial f as follows. This is the *Przyjalkowski method*; cf. [17, §5]. Without loss of generality we may assume that $B = [r]$. Writing $D_1, \ldots, D_R$ in terms of the basis $\{D_b \mid b \in B\}$ for $\mathbb{L}^*$ yields an $r \times R$ matrix $\mathbb{M} = (m_{i,j})$ of the form

$$\mathbb{M} = \left(\begin{array}{c|ccc} & m_{1,r+1} & \cdots & m_{1,R} \\ I_r & \vdots & & \vdots \\ & m_{r,r+1} & \cdots & m_{r,R} \end{array} \right) \tag{5}$$

where I_r is an $r \times r$ identity matrix. Consider the function

$$W = x_1 + x_2 + \cdots + x_R - k$$

subject to the constraints

$$\prod_{j=1}^{R} x_j^{m_{i,j}} = 1 \qquad\qquad 1 \leq i \leq r, \tag{6}$$

and

$$\sum_{j \in S_i} x_j = 1 \qquad\qquad 1 \leq i \leq k. \tag{7}$$

For each $i \in U$, introduce a new variable y_i. For each $i \in [k]$, introduce new variables y_j, where $j \in S_i \setminus \{s_i\}$, and set $y_{s_i} = 1$. Solve the constraints (7) by setting:

$$\begin{aligned} x_j &= \frac{y_j}{\sum_{l \in S_i} y_l} && j \in S_i, \\ x_j &= y_j && j \in U, \end{aligned}$$

and express the variables x_b, $b \in B$, in terms of the y_js using (6). The function W thus becomes a Laurent polynomial f in the variables y_j, $j \in [R] \setminus \{s_1, \ldots, s_k\}$. We refer to y_j, $j \in U$, as *uneliminated variables.*

Given data as in (4), let f be the Laurent polynomial just defined. Let Y denote the toric orbifold determined by Θ, let $L_1, \ldots, L_k$ denote the line bundles on Y from Definition 2.6, and

[2] A line bundle L on a Deligne–Mumford stack Y is convex if and only if L is nef and is the pullback of a line bundle on the coarse moduli space $|Y|$ of Y along the structure map $Y \to |Y|$. See [16].

let $X \subset Y$ be a complete intersection defined by a regular section of the vector bundle $\oplus_i L_i$. If X is Fano, then Mirror Theorems due to Givental, Hori–Vafa, and others [13, 14, 21, 25] imply that f corresponds to X under Mirror Symmetry; c.f. [17, §5]. We say that f *is a Laurent polynomial mirror* for X.

Remark 2.9. If f is a Laurent polynomial mirror for X then the Picard–Fuchs local system for $f\colon (\mathbb{C}^\times)^n \to \mathbb{C}$ coincides, after translation of the base if necessary, with the Fourier–Laplace transform of the quantum local system for X; see [11, 12]. Thus we regard f and $g := f - c$, where c is a constant, as Laurent polynomial mirrors for the same manifold Y, since the Picard–Fuchs local systems for f and g differ only by a translation of the base (by c).

Remark 2.10. If f and g are Laurent polynomials that differ by an invertible monomial change of variables then the Picard–Fuchs local systems for f and g coincide. Thus f is a Laurent polynomial mirror for X if and only if g is a Laurent polynomial mirror for X.

Example 2.11. Let X be a smooth cubic surface. The ambient toric variety $Y = \mathbb{P}^3$ is a GIT quotient $\mathbb{C}^4 /\!\!/ \mathbb{C}^\times$ where $\mathbb{C}^\times$ acts on $\mathbb{C}^4$ with weights $(1,1,1,1)$. Thus Y is given by GIT data $(K; \mathbb{L}; D_1, \ldots, D_4; \omega)$ with $K = \mathbb{C}^\times$, $\mathbb{L} = \mathbb{Z}$, $D_1 = D_2 = D_3 = D_4 = 1$, and $\omega = 1$. We consider the convex partition with basis B, S_1, $\varnothing$, where $B = \{1\}$ and $S_1 = \{2,3,4\}$, and take $s_1 = 4$. This yields

$$\mathbb{M} = \left(\begin{array}{c:ccc} 1 & 1 & 1 & 1 \end{array} \right)$$

and

$$W = x_1 + x_2 + x_3 + x_4 - 1$$

subject to

$$x_1 x_2 x_3 x_4 = 1 \qquad \text{and} \qquad x_2 + x_3 + x_4 = 1.$$

We set:

$$x_1 = \frac{1}{x_2 x_3 x_4}, \qquad x_2 = \frac{x}{1+x+y}, \qquad x_3 = \frac{y}{1+x+y}, \qquad x_4 = \frac{1}{1+x+y},$$

where, in the notation above, $x = y_2$ and $y = y_3$. Thus

$$f = \frac{(1+x+y)^3}{xy}$$

is a Laurent polynomial mirror to Y.

Example 2.12. Let Y be the projective bundle $\mathbb{P}\big(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)\big) \to \mathbb{P}^3$. This arises from the GIT data $(K; \mathbb{L}; D_1, \ldots, D_7; \omega)$ where $K = (\mathbb{C}^\times)^2$, $\mathbb{L} = \mathbb{Z}^2$,

$$D_1 = D_4 = D_6 = D_7 = (1,0), \qquad D_2 = D_3 = (0,1), \qquad D_5 = (-1,1),$$

and $\omega = (1,1)$. We consider the convex partition with basis B, S_1, S_2, U where $B = \{1,2\}$, $S_1 = \{3,4\}$, $S_2 = \{5,6\}$, $U = \{7\}$. This yields:

$$\mathbb{M} = \left(\begin{array}{cc:ccccc} 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array} \right)$$

Choosing $s_1 = 3$ and $s_2 = 5$, we find that

$$f = \frac{(1+x)}{xyz} + (1+x)(1+y) + z$$

Here, in the notation above, $x = y_4$, $y = y_6$, and $z = y_7$.

3. Scaffolding

In this section we give our central combinatorial construction: that of *scaffolding*. The output from the Przyjalkowski method is a Laurent polynomial f together with a decomposition of f as a sum of terms x_i, each of which is a Laurent polynomial in the variables y_j. The Newton polytope of each of the terms x_i is a product of translated dilates of standard simplices. Therefore each $\mathrm{Newt}(x_i)$ is the polyhedron P_D of sections of a nef divisor D on some (fixed) product of projective spaces. This motivates the following definition.

Definition 3.1. Fix the following data:

(i) a lattice N together with a splitting $N = \bar{N} \oplus N_U$;
(ii) the dual lattice $M := \mathrm{Hom}(N, \mathbb{Z})$, with the dual splitting $M = \bar{M} \oplus M_U$;
(iii) a Fano polytope $P \subset N_{\mathbb{Q}}$;
(iv) a projective toric variety Z given by a fan in $\bar{M}$ whose rays span the lattice $\bar{M}$.

Given such data, a *scaffolding* S of P is a set of pairs (D, χ) where D is a nef divisor on Z and χ is an element of N_U, such that

$$P = \mathrm{conv}\Big(P_D + \chi \,\Big|\, (D, \chi) \in S\Big).$$

We refer to Z as the *shape* of the scaffolding, and the elements $(D, \chi) \in S$ as *struts*.

Lemma 3.2. *Let f be a Laurent polynomial produced using the Przyjalkowksi method in §2. The polytopes $\mathrm{Newt}(x_i)$ determine a scaffolding of $P = \mathrm{Newt}(f)$ such that the shape Z is the product of projective spaces*

$$Z := \mathbb{P}^{|S_1|-1} \times \cdots \times \mathbb{P}^{|S_k|-1}$$

and S contains $r + |U|$ struts.

Proof. The polytope P is the convex hull of the union of the polytopes $\mathrm{Newt}(x_i)$ for x_i not appearing in any of the equations (7). There is a splitting of N into the sublattice N_U spanned by the exponents of uneliminated variables y_j, $j \in U$, and the sublattice $\bar{N}$ spanned by the exponents of variables y_i, $i \notin U$. If y_j is an uneliminated variable, add the strut $(\mathcal{O}, \mathrm{Newt}(y_j))$ to S. For $i \notin U$, $\mathrm{Newt}(x_i)$ is the polyhedron of sections of a nef divisor D on Z, translated by an element $\chi \in N_U$, and we add the strut (D, χ) to S. By construction P is the convex hull of this collection of struts. ∎

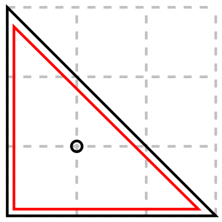
Remark 3.3. Note that any scaffolding generated by the proof of Lemma 3.2 contains a collection of struts $\{(\mathcal{O}, e_i) \mid i \in I\}$ for an index set I, corresponding to uneliminated variables, such that the collection $\{e_i \mid i \in I\}$ forms a basis of N_U. Although not the most general setting possible, we will assume from here onwards that this condition holds for every scaffolding.

Using the shape Z we can phrase the 'inversion' technique as a double application of Mirror Symmetry. Going forwards we start from a complete intersection $X \subset Y$ and form a Laurent polynomial f. The scaffolding obtained in the proof of Lemma 3.2 expresses f as a sum of sections of nef divisors on Z. Going backwards, the Givental/Hori–Vafa mirror of Z is a torus fibration $Z^\vee$ together with a regular function W on $Z^\vee$. The nef divisors we found to describe f determine the compactifying boundary divisors of $Z^\vee \subset Y$.

Example 3.4 (dP_3)**.** Consider the Laurent polynomial

$$f = \frac{(1 + x + y)^3}{xy}$$

from Example 2.11. A scaffolding for $\mathrm{Newt}(f)$ is given by a single standard 2-simplex, dilated by a factor of three:

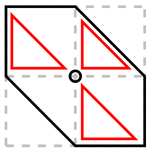


This gives a scaffolding of $\mathrm{Newt}(f)$ by single strut, with no uneliminated variables. The shape Z is $\mathbb{P}^2$ and the strut is given by choosing the entire toric boundary of $\mathbb{P}^2$.

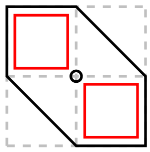
Example 3.5 (dP_6)**.** Consider the Laurent polynomial

$$f = x + y + \frac{1}{x} + \frac{1}{y} + \frac{x}{y} + \frac{y}{x}.$$

This is a mirror to the del Pezzo surface dP_6. We may scaffold $\mathrm{Newt}(f)$ in two different ways, using either three triangles or a pair of squares:



and



These choices correspond, respectively, to the decompositions

$$f = (1+x+y) + \frac{(1+x+y)}{x} + \frac{(1+x+y)}{y} - 3 \quad \text{and} \quad f = \frac{(1+x)(1+y)}{x} + \frac{(1+x)(1+y)}{y} - 2.$$

As discussed in Remark 2.9, we ignore the constant terms.

4. A Dual Perspective on Scaffolding

There is a dual characterisation of scaffolding which is often useful in applications. Instead of considering the polytope P, we consider the cone $C(P^*)$ over the dual polytope P^* embedded at height one in $M_\mathbb{Q} \oplus \mathbb{Q}$, and interpret the struts of a scaffolding as certain cones whose common intersection is exactly $C(P^*)$.

Definition 4.1. Given a Fano polytope P, let $C(P^*)$ be the cone obtained by embedding the rational polytope P^* in $M_\mathbb{Q} \oplus \{1\}$ and forming the cone over this affine polytope. Given a scaffolding S of P and a strut $s = (D, \chi)$ in S, define C_s to be the cone

$$C_s := \left\{ (\bar{m}, u, z) \in \left(\bar{M} \oplus M_U\right)_\mathbb{Q} \oplus \mathbb{Q} \mid z \geq \phi_D(\bar{m}) + \chi(u) \right\} \subset M_\mathbb{Q} \oplus \mathbb{Q}$$

where ϕ_D is the piecewise linear function on $\bar{M}$ determined by the $\mathbb{Q}$-Cartier divisor D on Z.

Remark 4.2. Recall that a torus invariant Weil divisor $D \in \mathrm{Div}_{T_{\bar{M}}}(Z)$ is, by definition, an integer-valued function on the set of rays of the fan Σ_Z determined by Z. The divisor D is $\mathbb{Q}$-Cartier if and only if this function is realised by a piecewise linear function ϕ_D on the fan of Z. Moreover the divisor D is nef if and only if the function ϕ_D is convex. The polyhedron of sections P_D of the divisor D is defined as the intersection of half-spaces $\langle \rho, - \rangle \geq -\phi_D(\rho)$ where ρ ranges over the integral generators of the rays of Σ_Z. Thus the rays of the cone C_s are generated by pairs (ρ, k) where $k = (\phi_D - \chi)(\rho)$ is the height of the supporting hyperplane of the strut $P_D + \chi$.

We can now interpret S as a collection of cones whose mutual intersection is equal to $C(P^*)$.

Lemma 4.3. *Given data as in (i)–(iv) of Definition 3.1 and a collection S of pairs $s = (D, \chi)$, where D is a nef divisor on Z and $\chi \in N_U$, then S is a scaffolding if and only if*

$$\bigcap_{s \in S} C_s = C(P^*).$$

Proof. Given a pair $s = (D, \chi) \in \overline{\mathrm{Amp}}(Z) \times N_U$ we prove that $C(P^*) \subseteq C_s$ if and only if the strut $P_D + \chi \subset P$. Since D is nef, C_s is a convex cone and so without loss of generality we can replace the condition that $C(P^*) \subset C_s$ with the condition that each of the rays of $C(P^*)$ is contained in C_s. Fixing a ray of $C(P^*)$ generated by an element $\rho \in M \oplus \mathbb{Z}$, recall that $\rho = (\rho', 1)$ where ρ' is a vertex of P^*. Considering the family of hyperplanes $H_{\rho',r} := \big\{ n \in N_{\mathbb{Q}} \mid \langle \rho', n \rangle = r \big\}$, $r \in \mathbb{Q}$, we see that -1 is the minimal r such that $H_{\rho',r}$ meets P^* and that the minimal value of r such that $H_{\rho',r}$ meets $P_D + \chi$ is $-(\phi_D - \chi)(\rho')$. Thus $P_D + \chi \subset P$ if and only if $-(\phi_D - \chi)(\rho') \geq -1$ for all ρ'.

It remains to show that equality holds for the inclusion

$$C(P^*) \subseteq \bigcap_{s \in S} C_s$$

precisely when S is a scaffolding. In other words we need to show that the equality $C(P^*) = \bigcap_{s \in S} C_s$ is equivalent to the condition that

$$P = \mathrm{conv}\Big(P_D + \chi \,\Big|\, (D, \chi) \in S \Big).$$

If P is the convex hull of the polytopes $P_D + \chi$ then every vertex of P meets a strut $P_D + \chi$. In that case every facet of $C(P^*)$ is contained in a facet of some C_s and so, in particular, the intersection of the cones C_s is contained in the cone $C(P^*)$. Conversely if the intersection of cones C_s is equal to $C(P^*)$ then every ray $\langle (\rho', 1) \rangle$ of $C(P^*)$ is contained in some C_s, and therefore the minimal $r \in \mathbb{Q}$ such that $H_{\rho',r}$ meets *some* polytope $P_D + \chi$ is equal to -1. □

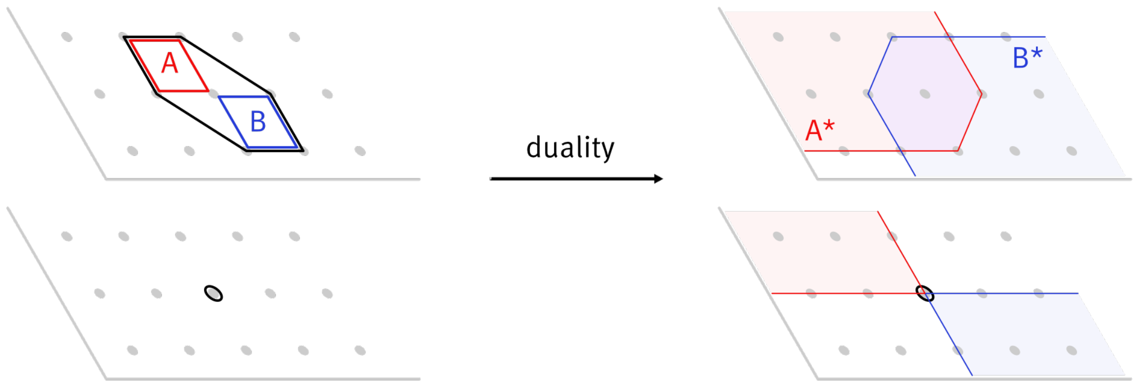


FIGURE 1. The dual picture of one of the scaffoldings from Example 3.5.

Example 4.4. Consider the right-hand scaffolding in Example 3.5. This is shown again on the left-hand side of Figure 1, placed at height 1 in $N_{\mathbb{Q}} \oplus \mathbb{Q}$ with the struts labelled as A and B. The corresponding cones C_A and C_B in $M_{\mathbb{Q}} \oplus \mathbb{Q}$ are shown on the right-hand side of Figure 1: C_A is the cone over the dual polyhedron A^*, placed at height 1 in $M_{\mathbb{Q}} \oplus \mathbb{Q}$, and similarly for C_B. The tail cones T_{A^*} of A^* and T_{B^*} of B^* are shown at height zero: these are faces of $C_A = C(A^*) = C(A)^\vee$ and $C_B = C(B^*) = C(B)^\vee$ respectively. The shape Z can be recovered by projecting the facets of C_A and C_B onto the height-zero slice in $M_{\mathbb{Q}} \oplus \mathbb{Q}$; this gives the fan of $Z = \mathbb{P}^1 \times \mathbb{P}^1$. The heights of the rays of C_A (respectively C_B) determine a divisor $D_A = D_1 + D_2$ (respectively $D_B = D_3 + D_4$) on Z. The strut A can be recovered as the polytope of sections of $\mathcal{O}(D_A)$, and similarly for B.

Note that in this dual perspective it makes sense to relax the condition that the divisors D of struts $s = (D, \chi)$ be nef on Z. Indeed, the new definition of scaffolding makes sense so long as D is $\mathbb{Q}$-Cartier, the cost of which is that the cones C_s cease to be convex. (Recall that the convexity of C_s is equivalent to D being a nef divisor.) Whilst we do not explore this further here, we hope that this notion will prove useful in the study of polytopes up to mutation.

5. Laurent Inversion

We have seen that if X is a Fano toric complete intersection defined by convex line bundles $L_1, \ldots, L_k$ on a toric orbifold Y, then there is a Laurent polynomial mirror f for X and a decomposition

$$f = f_1 + \cdots + f_r + \sum_{u \in U} x_u \tag{8}$$

where

$$f_a = \prod_{i=1}^{k} \prod_{j \in S_i} \left(\frac{\sum_{l \in S_i} y_l}{y_j} \right)^{m_{a,j}} \times \prod_{u \in U} x_u^{-m_{a,u}}.$$

This decomposition of f determines GIT data $(K; \mathbb{L}; D_1, \ldots, D_R)$ for Y, except for the stability condition, and also the line bundles $L_1, \ldots, L_k$. Indeed all of this data can be recovered from the scaffolding S of $\mathrm{Newt}(f)$ given by Lemma 3.2. In this section we generalise this observation, describing how to pass from a scaffolding S of a Fano polytope P to a toric variety Y and a toric embedding $X_P \to Y$.

Algorithm 5.1. Let S be a scaffolding of a Fano polytope P with shape Z. Let $u = \dim N_U$ and let $r = |S| - u$, so that S contains r struts that do not correspond to uneliminated variables and u struts that do correspond to uneliminated variables (see Remark 3.3). Let R be the sum of $|S|$ and the number z of rays of Z. We determine an $r \times R$ matrix $\mathbb{M}$, which will be the weight matrix for our toric variety Y, as follows. Let $m_{i,j}$ denote the (i, j) entry of $\mathbb{M}$. Fix an identification of the rows of $\mathbb{M}$ with the r elements (D_i, χ_i) of S which do not correspond to uneliminated variables, and an ordering $\Delta_1, \ldots, \Delta_z$ of the toric divisors in Z. Let $e_1, \ldots, e_u$ be the basis of N_U given by Remark 3.3.

(i) For $1 \leq j \leq r$ and any i, let $m_{i,j} = \delta_{i,j}$.

(ii) For $1 \leq j \leq u$ and any i, let $m_{i,r+j}$ be determined by the expansion
$$\chi_i = \sum_{j=1}^{u} m_{i,r+j} e_j.$$

(iii) For $1 \leq j \leq z$, let $m_{i,|S|+j}$ be determined by the expansion
$$D_i = \sum_{j=1}^{z} m_{i,|S|+j} \Delta_j.$$

The weight matrix $\mathbb{M}$ alone does not determine a unique toric variety – we also need to choose a stability condition ω. Let Y_ω denote the toric variety determined by this choice. Unless otherwise stated, we will take ω to be the sum of the first $|S|$ columns in $\mathbb{M}$.

Remark 5.2. In terms of the dual perspective on scaffoldings in §4, the entry $m_{i,|S|+j}$ in the matrix $\mathbb{M}$ is the height in $M_{\mathbb{Q}} \oplus \mathbb{Q}$ of the jth ray in the ith cone C_s.

Remark 5.3. In the case where the scaffolding S arises from a toric complete intersection X via Lemma 3.2, the choice of ω given above is equal to $-K_X - \sum_{i\in[k]} L_i$. The corresponding convex partition with basis B, S_1, $\ldots$, S_k, U can be recovered by setting $B = \{1, 2, \ldots, r\}$, $U = \{r + 1, \ldots, r + u\}$, and S_j equal to the subset of $\{|S| + 1, \ldots, |S| + z\}$ given by the toric divisors on the jth factor $\mathbb{P}^{a_j}$ of $Z = \prod_{i=1}^{k} \mathbb{P}^{a_i}$.

Remark 5.4. The ray lattice $\widetilde{N}$ of Y_ω, that is, the lattice of one-parameter subgroups of the dense torus in Y_ω, is equal to $\mathrm{Div}_{T_{\bar{M}}}(Z) \oplus N_U$.

In favourable cases, a suitable choice of stability condition ω gives a smooth toric orbifold Y_ω and convex line bundles $L_1, \ldots, L_k$ on Y_ω such that the complete intersection $X \subset Y_\omega$ defined by a regular section of the vector bundle $\oplus_i L_i$ is Fano. This can be very useful, and we use it in §7 to exhibit a new four dimensional Fano manifold. However our construction is not restricted to the case where the scaffolding comes from a toric complete intersection via Givental/Hori–Vafa Mirror Symmetry; that is, we do not insist that the shape Z is a product of projective spaces. In the Appendix we prove:

Theorem 5.5. *A scaffolding S of a Fano polytope P such that the shape Z is smooth determines an embedding of toric varieties $X_P \to Y_\omega$.*

Thus *any* scaffolding of a Fano polytope P with smooth shape determines a toric embedding of the corresponding Fano toric variety X_P into an ambient toric variety. If the scaffolding S arises, via Lemma 3.2, from a Fano toric complete intersection X defined by convex line bundles $L_1, \ldots, L_k$ on a Fano toric orbifold Y, then Theorem 5.5 embeds X_P as a complete intersection in a toric variety Y_ω defined using the same GIT data as Y (but with a possibly-different stability condition ω); see §8. There is then often an embedded degeneration from X to X_P. In general, however, the embedding in Theorem 5.5 is not a complete intersection, and X_P may not have an embedded smoothing inside Y_ω. Example 10.3 is instructive here.

The map of tori in Theorem 5.5, of which the embedding $X_P \hookrightarrow Y_\omega$ is the closure in Y_ω, is as follows. The dense tori in X_P and Y_ω are respectively T_N and $T_{\widetilde{N}}$. There is a map

$$\bar{N} \oplus N_U = N \to \widetilde{N} = \mathrm{Div}_{T_{\bar{M}}}(Z) \oplus N_U$$

defined on each factor as:

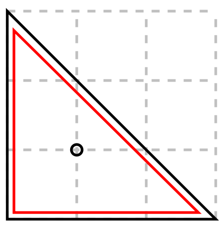
(i) $\bar{N} \to \mathrm{Div}_{T_{\bar{M}}}(Z) \oplus \{0\}$, the map taking characters of $T_{\bar{M}}$ to principal divisors;
(ii) $N_U \to \{0\} \oplus N_U$, the identity map.

For example, if Z is a product of projective spaces then the ray map dualises to give an inclusion of tori $T_N \hookrightarrow T_{\widetilde{N}}$ with ideal generated by binomials of the form $(\prod x_i = 1)$, where the product is taken over variables corresponding to divisors in the same projective space factor.

6. Examples

In this section we apply Algorithm 5.1 to several concrete examples.

Example 6.1 (dP_3)**.** Continuing Example 3.4, recall the scaffolding obtained from a mirror to dP_3 given by a single standard 2-simplex, dilated by a factor of three:



From this we read off $u=0$, $r=1$, $R=4$, $B=\{1\}$, $U=\varnothing$, $S_1=\{2,3,4\}$, and

$$\mathbb{M}=\left(\begin{array}{c:ccc} 1 & 1 & 1 & 1 \end{array}\right).$$

This gives GIT data $\Theta=(K;\mathbb{L};D_1,\ldots,D_4;\omega)$ with $K=\mathbb{C}^\times$, $\mathbb{L}=\mathbb{Z}$, $D_1=D_2=D_3=D_4=1$, and $\omega=1$; note that the secondary fan here has a unique maximal cone. The corresponding toric variety is $Y=\mathbb{P}^3$. The ideal defining X_P is principal in Cox co-ordinates on Y, generated by the equation $X_1X_2X_3-X_0^3$. This is a section of the nef line bundle $\mathcal{O}(3)$. Thus $B,S_1,\varnothing$ is a convex partition with basis for Θ, and we obtain the cubic hypersurface as in Example 2.11.

Example 6.2 (dP_6)**.** The projective plane blown up in three points, dP_6, is toric, but it has two famous models as a complete intersection:

(i) as a hypersurface of type $(1,1,1)$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$;
(ii) as the intersection of two bilinear equations in $\mathbb{P}^2\times\mathbb{P}^2$.

Recall the two scaffoldings from Example 3.5, which arose from the two decompositions

$$f=(1+x+y)+\frac{(1+x+y)}{x}+\frac{(1+x+y)}{y}-3 \quad\text{and}\quad f=\frac{(1+x)(1+y)}{x}+\frac{(1+x)(1+y)}{y}-2$$

of a Laurent polynomial mirror f for dP_6.

From the first scaffolding we read off $u=0$, $r=3$, $Z=\mathbb{P}^2$, $R=6$, $B=\{1,2,3\}$, $U=\varnothing$, $S_1=\{4,5,6\}$, and

$$\mathbb{M}=\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

This gives GIT data $\Theta=(K;\mathbb{L};D_1,\ldots,D_6;\omega)$ with $K=(\mathbb{C}^\times)^3$, $\mathbb{L}=\mathbb{Z}^3$, $D_1=D_4=(1,0,0)$, $D_2=D_5=(0,1,0)$, $D_3=D_6=(0,0,1)$, and $\omega=(1,1,1)$; the secondary fan here again has a unique maximal cone. The corresponding toric variety is $Y=\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. The line bundle $L_1=\sum_{j\in S_1}D_j$ is $\mathcal{O}(1,1,1)$, so we see that f is a Laurent polynomial mirror to a hypersurface of type $(1,1,1)$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

From the second scaffolding we read off $u=0$, $r=2$, $Z=\mathbb{P}^1\times\mathbb{P}^1$, $B=\{1,2\}$, $U=\varnothing$, $S_1=\{3,4\}$, $S_2=\{5,6\}$, and

$$\mathbb{M}=\left(\begin{array}{cc:cccc} 1 & 0 & 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

This gives GIT data $\Theta=(K;\mathbb{L};D_1,\ldots,D_6;\omega)$ with $K=(\mathbb{C}^\times)^2$, $\mathbb{L}=\mathbb{Z}^2$, $D_1=D_4=D_5=(1,0)$, $D_2=D_3=D_6=(0,1)$, and $\omega=(1,1)$; once again the secondary fan has a unique maximal cone. The corresponding toric variety Y is $\mathbb{P}^2\times\mathbb{P}^2$. The line bundles $L_1=D_3+D_4$ and $L_2=D_5+D_6$ are both equal to $\mathcal{O}(1,1)$, so we see that f is a Laurent polynomial mirror to the complete intersection of two hypersurfaces defined by bilinear equations in $\mathbb{P}^2\times\mathbb{P}^2$.

Example 6.3 ($\mathrm{MM}_{3\text{–}4}$)**.** Consider the rigid maximally-mutable Laurent polynomial

$$f=x+\frac{y^2}{z}+2y+\frac{3y}{z}+z+\frac{3}{z}+\frac{z}{y}+\frac{2}{y}+\frac{1}{yz}+\frac{y^2}{xz}+\frac{2y}{x}+\frac{2y}{xz}+\frac{z}{x}+\frac{2}{x}+\frac{1}{xz}.$$

The Newton polytope of f can be scaffolded as in Figure 2, and there is a corresponding decomposition of f:

$$f=x+\frac{(1+y+z)^2}{xz}+\frac{(1+y+z)^2}{z}+\frac{(1+y+z)^2}{yz}$$

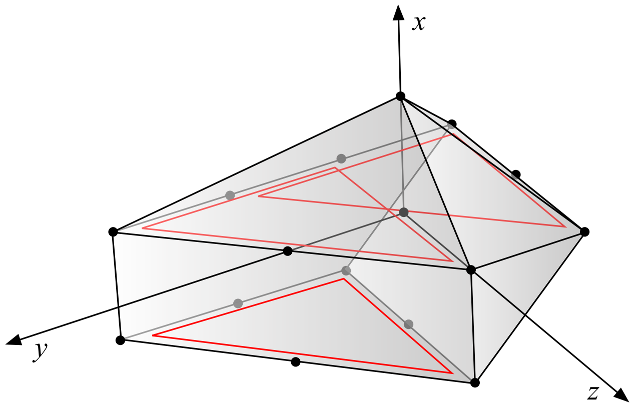


FIGURE 2. A scaffolding for Newt(f) in Example 6.3.

From this we read off $u = 1$, $r = 3$, $Z = \mathbb{P}^2$, $B = \{1, 2, 3\}$, $U = \{4\}$, $S_1 = \{5, 6, 7\}$, and

$$\mathbb{M} = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

This gives GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_6; \omega)$ with $K = (\mathbb{C}^\times)^3$, $\mathbb{L} = \mathbb{Z}^3$, $D_1 = D_4 = (1, 0, 0)$, $D_2 = (0, 1, 0)$, $D_3 = D_6 = (0, 0, 1)$, $D_4 = (1, 1, 0)$, and $D_7 = (1, 1, 1)$. The secondary fan is as shown in Figure 3. Choosing $\omega = (3, 2, 1)$ yields a weak Fano toric manifold Y_ω such that the line bundle $L_1 = \sum_{j \in S_1} D_j$ is convex. Let X denote the hypersurface in Y defined by a regular section of L_1. The class $-K_Y - L_1$ is nef but not ample on Y, but it becomes ample on restriction to X; thus X is Fano (cf. [12, §57]). We see that f is a Laurent polynomial mirror to X. This example shows that our Laurent inversion technique applies in cases where the ambient space Y is not Fano. In fact Y need not even be weak Fano.

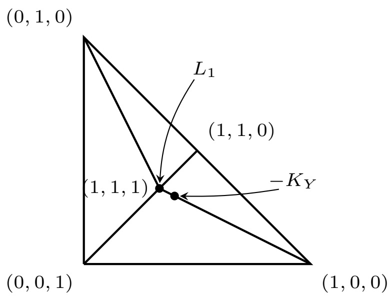


FIGURE 3. The secondary fan for Example 6.3, sliced by the plane $x + y + z = 1$.

7. Finding New Four-Dimensional Fano Manifolds

In this section we describe how Laurent inversion may be used to obtain previously unknown examples of Fano manifolds. We present two approaches. Firstly, given a Laurent polynomial f which is for some reason expected to correspond under Mirror Symmetry to a Fano manifold, one can search for decompositions of f of the form (8) and apply Algorithm 5.1 to construct Fano toric complete intersections X that correspond to f under Mirror Symmetry. An instance of this, with f given by a rigid maximally mutable Laurent polynomial in four variables, is Example 7.1 below. A second, more systematic, approach would be to fix a reflexive polytope P and search for deformations of X_P inside toric ambient spaces defined by scaffoldings which smooth X_P. For example, if one searches the Kreuzer–Skarke database of four-dimensional reflexive polytopes [32] for polytopes P that admit a scaffolding with the simplest possible shape $Z = \mathbb{P}^1$, such that the toric embedding given by Theorem 5.5 gives an embedded smoothing of X_P then one finds more than 450 such scaffoldings. One of these is Example 7.2.

Example 7.1. Consider the Laurent polynomial

$$f_1 = x + y + z + \frac{(1+w)^2}{xzw} + \frac{w}{y}$$

This is a rigid maximally-mutable Laurent polynomial in four variables. It is presented in scaffolded form, and we read off $r = 2$, $u = 3$, $B = \{1, 2\}$, $U = \{3, 4, 5\}$, $S_1 = \{6, 7\}$, and

$$\mathbb{M} = \left(\begin{array}{cc|ccccc} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 \end{array}\right).$$

This yields GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_6; \omega)$ with $K = (\mathbb{C}^\times)^2$, $\mathbb{L} = \mathbb{Z}^2$, $D_1 = D_3 = D_5 = (1, 0)$, $D_2 = D_4 = (0, 1)$, $D_6 = (1, 1)$, and $D_7 = (1, -1)$. We choose the stability condition $\omega = (3, 2)$, thus obtaining a Fano toric orbifold Y_1 such that the line bundle $L_1 = D_6 + D_7$ on Y is convex. Let X_1 denote the four-dimensional Fano manifold defined inside Y_1 by a regular section of L_1.

Example 7.2. Consider the Laurent polynomial

$$f_2 = x + y + z + \frac{1}{y} + \frac{(1+w)^2}{wxz} + \frac{(1+w)^2}{x^2yzw} + \frac{(1+w)^2}{xyzw}$$

This is presented in scaffolded form, and we read off $r = 4$, $u = 3$, $B = \{1, 2, 3, 4\}$, $U = \{5, 6, 7\}$, $S_1 = \{8, 9\}$, and

$$\mathbb{M} = \left(\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

This yields GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_9; \omega)$ with $K = (\mathbb{C}^\times)^4$, $\mathbb{L} = \mathbb{Z}^4$, $D_1 = (1, 0, 0, 0)$, $D_2 = (0, 1, 0, 0)$, $D_3 = (0, 0, 1, 0)$, $D_4 = (0, 0, 0, 1)$, $D_5 = (0, 1, 2, 1)$, $D_6 = (1, 0, 1, 1)$, and $D_7 = D_8 = D_9 = (0, 1, 1, 1)$. We choose the stability condition $\omega = (2, 3, 5, 4)$, thus obtaining a Fano toric orbifold Y_2 such that the line bundle $L_2 = D_8 + D_9$ on Y is convex. Let X_2 denote the four-dimensional Fano manifold defined inside Y_2 by a regular section of L_2.

To compare X_1 and X_2 with known four-dimensional Fano manifolds, we compute their regularised quantum periods. As is explained in detail in [11,12], since X_1 and X_2 correspond under Mirror Symmetry to f_1 and f_2, their regularised quantum periods $\widehat{G}_{X_1}$, $\widehat{G}_{X_2}$ coincide with the classical periods of f_1 and f_2. Here the classical period π_f of a Laurent polynomial f is

$$\pi_f(t) = \sum_{d=0}^{\infty} c_d t^d$$

where $c_d = \operatorname{coeff}_1 \left(f^d\right)$. Thus

$$\widehat{G}_{X_1} = \pi_f(t) = 1 + 12t^3 + 120t^5 + 540t^6 + 20160t^8 + 33600t^9 + \cdots$$

$$\widehat{G}_{X_2} = \pi_f(t) = 1 + 2t^2 + 12t^3 + 54t^4 + 360t^5 + 1280t^6 + 12600t^7 + 72310t^8 + 446880t^9 + \cdots$$

and in particular we see that neither $\widehat{G}_{X_1}$ nor $\widehat{G}_{X_2}$ is contained in the list of regularised quantum periods of known four-dimensional Fano manifolds [15, 17].

Remark 7.3. We did not find the Fano manifolds X_1 or X_2 in our systematic search for four-dimensional Fano toric complete intersections [17], because there we considered only ambient spaces that are Fano toric manifolds whereas the ambient spaces Y_1 and Y_2 here have non-trivial orbifold structure.

Although the Fano manifold X_1 does not occur in the list of four-dimensional Fano manifolds whose quantum periods are known, it is certainly not new. The ambient toric variety Y_1 can be obtained as the unique non-trivial flip of the projective bundle $\mathbb{P}\big(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1)\big)$ over $\mathbb{P}^1$ and, as was pointed out to us by Casagrande, the other extremal contraction of Y_1 exhibits X_1 as the blow-up of $\mathbb{P}^4$ in a plane conic[3]. On the other hand, we do not know of a classical construction of the Fano manifold X_2. We can analyse X_2 using its presentation as a toric complete intersection. Its ample cone coincides with that of the ambient space Y_2, which is the non-simplicial four-dimensional cone C with rays

$$(0,1,1,1), \qquad (0,1,2,1), \qquad (1,1,1,1), \qquad (1,1,2,2), \qquad (2,1,2,2).$$

Crossing each of the walls of C induces non-trivial birational transformations of X_2 and Y_2: two of these are flips and three of them are blow-downs. Indeed one of the cones C' of the secondary fan – that with rays $(0,0,1,0)$, $(0,1,0,0)$, $(1,0,0,0)$, and $(1,1,2,1)$ – gives the toric variety $\mathbb{P}^5$, and C' can be reached from C by crossing four walls. Following X_2 across these wall-crossings shows that X_2 can be obtained from $Q \subset \mathbb{P}^5$, the cone over a singular plane quadric Q', by taking the (weighted) blow-up of two points in the plane over the singularity of Q', followed by flipping a $\mathbb{P}^1$ and blowing up a surface S which is the crepant resolution of Q'. This construction of X_2 is in a sense classical, but it does not seem very natural. It is possible that the construction via scaffolding in Example 7.2 is the most meaningful available.

As mentioned above, Example 7.2 was discovered via a systematic search for four-dimensional reflexive polytopes P that admit a scaffolding with the simplest possible shape X_P, such that the toric embedding determined by the scaffolding gives an embedded smoothing of X_P. We imposed an additional condition – that singular cones of the normal fan to P lie in a unique hyperplane – that is not logically necessary but simplifies the search, as it determines the struts in the scaffolding. The search yields 450 such examples, which together give a total of 170 regularised quantum periods. Of these, 152 are the regularised quantum periods of known four-dimensional Fano manifolds; two are Examples 7.1 and 7.2; and 3 give complete intersection models that are beyond the reach of current Mirror Theorems. The remaining 13 examples give four-dimensional Fano manifolds with extremely beautiful complete intersection models. Mirror-theoretic analysis of these examples is delicate – we will discuss it elsewhere [18] – but the upshot is that these examples are proven to have previously-unknown regularised quantum periods. Since we know of only a few four-dimensional Fano manifolds in the literature with regularised quantum periods that have not yet been calculated, it is likely that at least some of these examples are new. In any case, relaxing the (restrictive) condition that the singularities lie in a unique hyperplane or the (very restrictive) condition that the scaffolding have shape $Z = \mathbb{P}^1$ will yield many more examples.

8. Scaffoldings and Embedded Degenerations of Complete Intersections

We next explain how, if P admits a scaffolding for which the shape Z is a product of projective spaces, X_P can be embedded as the common zero locus of a collection of sections of linear systems on Y. In this case X_P is a flat degeneration of the zero locus X of a

[3] This example suggests that restricting to smooth ambient spaces when searching for Fano toric complete intersections may omit many Fano manifolds with simple classical constructions.

general section. This embedded degeneration is often a smoothing of X_P. It was discovered independently by Doran–Harder [20]: see §12 for an alternative view on their construction.

By assumption we have, as in §2, an $r \times R$ matrix $\mathbb{M} = (m_{i,j})$ of the form:

$$\mathbb{M} = \left(\begin{array}{c|ccc} & m_{1,r+1} & \cdots & m_{1,R} \\ I_r & \vdots & & \vdots \\ & m_{r,r+1} & \cdots & m_{r,R} \end{array} \right)$$

such that $l_{b,i} := \sum_{j \in S_i} m_{b,j}$ is non-negative for all $b \in [r]$ and $i \in [k]$. The exact sequence (2) becomes

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{\mathbb{M}^T} \mathbb{Z}^R \xrightarrow{\rho} \widetilde{N} \longrightarrow 0$$

and, writing $\rho_i \in \widetilde{N}$ for the image under ρ of the ith standard basis vector in $\mathbb{Z}^R$, we find that $\{\rho_i \mid r < i \leq R\}$ is a distinguished basis for $\widetilde{N}$ and that

$$\rho_i = -\sum_{j=r+1}^{R} m_{i,j}\rho_j \qquad\qquad \text{for all } 1 \leq i \leq r.$$

Let $\widetilde{M} = \operatorname{Hom}\left(\widetilde{N}, \mathbb{Z}\right)$ and define $u_j \in \widetilde{M}$, $1 \leq j \leq k$, by

$$u_j(\rho_i) = \begin{cases} 0, & \text{if } r < i \leq R \text{ and } i \notin S_j; \\ 1, & \text{if } r < i \leq R \text{ and } i \in S_j. \end{cases}$$

Let $N' := \widetilde{N} \cap H_{u_1} \cap \ldots \cap H_{u_k}$ be the sublattice of $\widetilde{N}$ given by restricting to the intersection of the hyperplanes $H_{u_i} := \{v \in N \mid u_i(v) = 0\}$. Let Σ' denote the fan defined by intersecting Σ with $N'_{\mathbb{Q}}$, and let X' be the toric variety defined by Σ'.

Lemma 8.1. *The lattice N' is the image of N under the map dual to the ray map of Z.*

Proof. The lattice N' is defined as the vanishing of a collection of elements of the dual lattice $\widetilde{M}$. Since these intersect transversely we have that $\dim N' = \dim N$. To check that $N \subset N'$ we check that each u_i vanishes on N. But the vectors u_i form a basis of the kernel of the ray map of Z dual to the inclusion of $\bar{N} \hookrightarrow \operatorname{Div}_{T_{\bar{M}}}(Z)$. □

Thus $X' = X_P$, and we have embedded X_P in Y as the common zero locus of sections of linear systems defined by the hyperplanes H_{u_i}.

9. Beyond Complete Intersections

Any Laurent polynomial obtained from the Givental/Hori–Vafa model gives a scaffolding with shape Z equal to a product of projective spaces (Lemma 3.2) but the definition of scaffolding allows for much more general choices of Z. We now show how certain classical constructions appear via scaffolding. For example, for any reflexive polytope P there is a distinguished choice of scaffolding S_{can} with shape Z given by a toric crepant terminal $\mathbb{Q}$-factorialisation of the toric variety defined by the normal fan of P, and a single strut covering all of P.

Proposition 9.1. *The embedding $X_P \hookrightarrow Y \cong \mathbb{P}^{\rho-1}$ determined by the scaffolding S_{can} of P is the anticanonical embedding of X_P, where ρ is the number of integral points of P^*.*

Proof. That $Y \cong \mathbb{P}^{\rho-1}$ follows from the definition of polar polytope: the nef divisor of Z used to cover P as a single strut is precisely the toric boundary of Z. Indeed every torus invariant section of $-K_{X_P}$ defines a character of T_N which in turn generates a ray of Z. The map of tori $T_N \hookrightarrow \mathbb{C}^{\star\rho}$ defining the embedding in Theorem 5.5 is precisely the map of tori defined by these characters of T_N. □

Remark 9.2. Note that Proposition 9.1 does not reply on Theorem 5.5. Indeed, the hypotheses of Theorem 5.5 require that the shape be a smooth toric variety, and in general it will not be possible to choose Z to be smooth in dimensions higher than three.

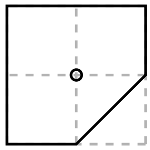


FIGURE 4. Polygon for dP_7

Example 9.3 (dP_7)**.** Let P be the polytope shown in Figure 4 and let Z be the toric variety associated to the normal fan of P, that is, the blow up of $\mathbb{P}^2$ in two points. The image of the anticanonical embedding of X_P is the closure in the projective space $\mathbb{P}^5$ of the variety X_0 defined via the following five equations in $\mathbb{C}^5$:

$$x_1x_3 = 1, \qquad x_2x_4 = x_3, \qquad x_3x_5 = x_4, \qquad x_4x_1 = x_5, \qquad x_5x_2 = 1.$$

The variety X_0 admits a flat deformation to the variety X_t defined by the 4×4 Pfaffians of the following skew-symmetric matrix:

$$\begin{pmatrix} & 1 & x_1 & x_2 & 1 \\ & & t & x_3 & x_4 \\ & & & 1 & x_5 \\ & & & & t \\ & & & & \end{pmatrix} \tag{9}$$

Scaffoldings of a Fano polygon P using this shape Z produce ambient toric varieties Y which exhibit X_P as the closure in Y of the affine variety defined by these five binomial equations, homogenising each equation to an equation in Cox co-ordinates. In forthcoming work we will show that the existence of the flat deformation of X_P in Y given by these Pfaffians exists if and only if the following 'mutability condition' holds.

Proposition 9.4. *Given a scaffolding S of X_P with shape Z, X_P deforms in the ambient space Y to a variety defined by the homogenisation of the 4×4 Pfaffians of (9) if and only if each strut in S, regarded as a polyhedron in N, admits mutations, in the sense of [2], with weight vectors equal to the elements x_1, x_2 (regarded as elements of the dual lattice M).*

This condition ensures that we can homogenise the Pfaffian equations, replacing the entries on the superdiagonal and in the upper-right corner of (9) with monomials in Cox co-ordinates with *non-negative* exponents.

It follows that Y contains five toric degenerations of the variety defined by these Pfaffians, since cyclically permuting the positions of the variables $x_1, \ldots, x_5$ shown in the matrix (9) gives rise to five distinct toric degenerations.

10. Models of Orbifold del Pezzo Surfaces

Scaffolding has a practical advantage even in the surface case. In this section we show how to find models of del Pezzo surfaces with $1/3(1,1)$ singularities that were used by Corti–Heuberger in their classification [19]. Two of these models are toric complete intersections; the third is a degeneracy locus cut out by Pfaffian equations. The Fano polygons we use for these models were classified in [30]. Following [1,19,30] we refer to the del Pezzo surface with $n \times 1/3(1,1)$ singular points and degree d as $X_{n,d}$.

Example 10.1 ($X_{2,5/3}$)**.** Consider the Fano polygon P with scaffolding shown in Figure 5. This scaffolding defines the weight matrix:

y_1	y_2	x_1	x_2	x_3
1	0	2	1	1
0	1	1	2	−1

Fixing the stability condition $\omega = (1,1)$ defines a toric variety Y. The toric variety X_P is a hypersurface in Y defined by the vanishing of the binomial section $y_1^4y_2^2 - x_1x_2x_3$ of the bundle $L = (4,2)$. A general section of L is a del Pezzo surface with $2 \times 1/3(1,1)$ singularities and no other singular points.

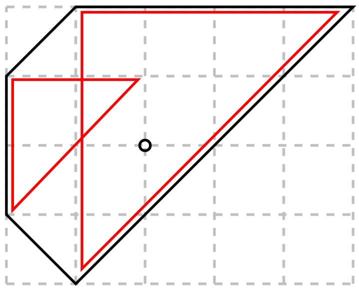


FIGURE 5. Polygon for a degeneration of the surface $X_{2,5/3}$

Example 10.2 ($X_{3,1}$)**.** Now consider the Fano polygon P with the scaffolding shown in Figure 6. This scaffolding defines the weight matrix:

y_1	y_2	y_3	x_1	x_2	x_3
1	0	0	2	1	1
0	1	0	1	2	1
0	0	1	1	1	2

Fixing the stability condition $\omega = (1,1,1)$ defines a toric variety Y. The toric variety X_P is a hypersurface in Y defined by the vanishing of the binomial section $y_1^4y_2^4y_3^4 - x_1x_2x_3$ of the bundle $L = (4,4,4)$. A general section of L is a del Pezzo surface with $3 \times 1/3(1,1)$ singularities and no other singular points.

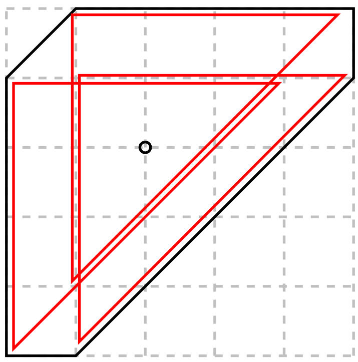


FIGURE 6. Polygon for a degeneration of the surface $X_{3,1}$

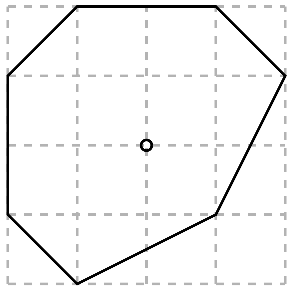


FIGURE 7. Polygon for a degeneration of the surface $X_{5,5/3}$

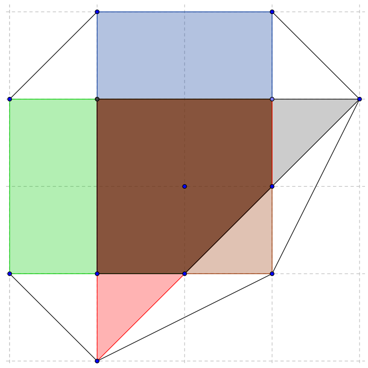


FIGURE 8. A scaffolding of the polygon in Figure 7

Example 10.3 ($X_{5,5/3}$)**.** The surface $X_{5,5/3}$ in [19] is found as a degeneracy locus defined by five 4×4 Pfaffian equations. We show how this appears as an instance of Laurent inversion.

Consider the polygon P shown in Figure 7 and let Z be the toric variety with fan given by the normal fan of the polygon in Figure 4. Figure 8 exhibits a scaffolding of P with 5 struts and shape Z. To write out the corresponding weight matrix $\mathbb{M}$ we first have a 5×5 identity block, identifying each row with a strut; the remaining columns are found by expanding the nef divisors making up the scaffolding in the basis of torus invariant divisors. In this way we obtain:

$$
\begin{array}{cccccccccc}
y_1 & y_2 & y_3 & y_4 & y_5 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2
\end{array}.
$$

This scaffolding satisfies the mutability condition in Proposition 9.4. Taking stability condition $(1,\ldots,1)$ and homogenising the Pfaffian equations given in (9) we obtain a flat deformation of X_P given by the 4×4 Pfaffians of the skew-symmetric matrix:

$$\begin{pmatrix} y_1^2y_2y_3y_4^2 & x_1 & x_2 & y_1y_2^2y_4^2y_5 \\ & y_1^2y_3^2y_4y_5 & x_3 & x_4 \\ & & y_1^2y_2^2y_3y_5 & x_5 \\ & & & y_2^2y_3y_4y_5^2 \end{pmatrix} \tag{10}$$

Hence we realise the surface $X_{5,5/3}$ as a degeneracy locus in a rank 5 toric variety Y. In this example all five toric degenerations of the surface are isomorphic. This is not typical, but a consequence of the symmetries of the Fano polygon P.

11. Nef Partitions

We now consider the connection between Laurent inversion and the *nef partitions* studied by Batyrev and Borisov [6,7]. We begin with a motivating example. The notion of mutation of polytopes [2] extends naturally to scaffoldings, and we illustrate this by mutating one of the scaffoldings considered in Example 6.2.

Figure 9. Mutating a scaffolding

Example 11.1. The mutation that takes the left-hand polygon in Figure 9 (previously seen in Example 6.2) to the right-hand polygon transforms the scaffolding as shown. In Example 4.4 we analysed the dual picture of the left-hand scaffolding in Figure 9, obtaining Figure 10 (which is a copy of Figure 1). Repeating this analysis for the right-hand scaffolding in Figure 9

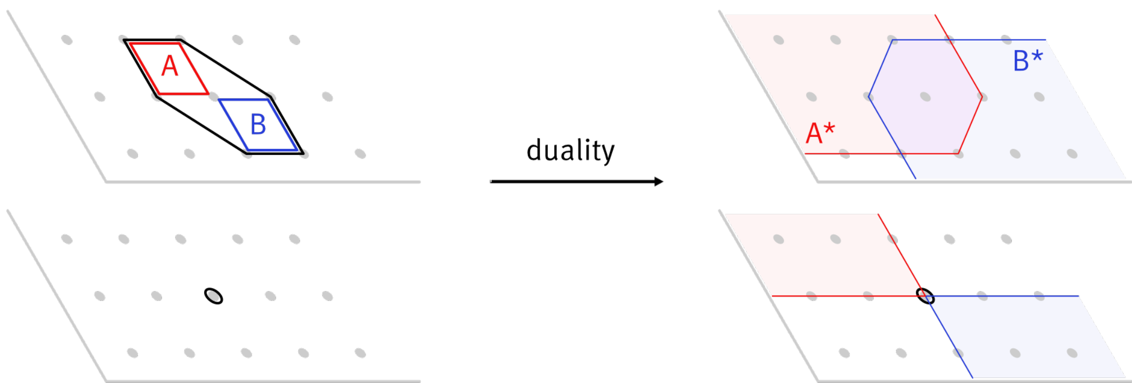


Figure 10. The dual picture of the left-hand scaffolding in Figure 9.

yields Figure 11. As before, on the left-hand side of Figure 11 the scaffolding is placed at height 1 in $N_{\mathbb{Q}}\oplus\mathbb{Q}$, with the struts labelled as A and B. The corresponding cones C_A and C_B in $M_{\mathbb{Q}}\oplus\mathbb{Q}$ are shown on the right-hand side of Figure 11: C_A is the cone over the dual

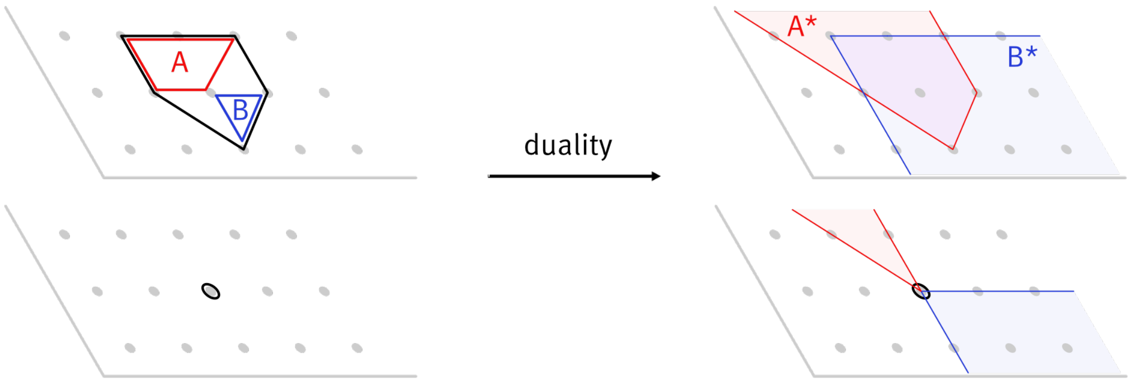


FIGURE 11. The dual picture of the right-hand scaffolding in Figure 9.

polyhedron A^*, placed at height 1 in $M_{\mathbb{Q}} \oplus \mathbb{Q}$, and similarly for C_B. The tail cones T_{A^*} of A^* and T_{B^*} of B^* are shown at height zero: these are faces of C_A and C_B respectively. The shape Z can be found by projecting the facets of C_A and C_B onto the height-zero slice in $M_{\mathbb{Q}} \oplus \mathbb{Q}$, where we see the fan of the Hirzebruch surface $Z = \mathbb{F}_1$.

Mutation here is the piecewise-linear transformation of $M_{\mathbb{Q}} \oplus \mathbb{Q}$ given by

$$(x, y, z) \mapsto \begin{cases} (x, y - x, z), & \text{if } x < 0; \\ (x, y, z), & \text{if } x \geq 0. \end{cases} \tag{11}$$

This maps the right-hand side of Figure 10 to the right-hand side of Figure 11. One could also apply the definition of N-side mutation from [2] directly to the struts in the left-hand side of Figure 10; note that in loc. cit. the polytope being mutated is not required to be Fano, or even to contain the origin in its interior. This yields the struts shown in the left-hand side of Figure 11.

Since in this example the shape $Z = \mathbb{P}^1 \times \mathbb{P}^1$ is a toric surface there is an alternative, and more geometric, description of its mutations which makes contact with the work of Gross–Hacking–Keel [23]. A mutation of such a Z is given by fixing a morphism $\pi \colon Z \to \mathbb{P}^1$ and making an *elementary transformation*[4] of this $\mathbb{P}^1$ bundle. In this case the mutation takes Z to the Hirzebruch surface $\mathbb{F}_1$. In general the fan determined by Z undergoes a piecewise linear transformation T which fixes the rays corresponding to the torus invariant sections of π. In this case T is the restriction of (11) to the height-zero slice $z = 0$.

Turning now to nef partitions, we first extend the definition of nef partition to the setting of Fano toric complete intersections and then show that scaffolding offers a substantial generalisation of this new notion. We begin by recalling the basic definition and main results [5,7].

Definition 11.2. Given a lattice N and a reflexive polytope $\Delta \subset N_{\mathbb{Q}}$, a *nef partition of length* r is a partition $E_1 \cup \ldots \cup E_r$ of the set verts(Δ) of vertices of Δ such that there are $\Sigma[\Delta]$-piecewise linear functions ϕ_i satisfying $\phi_i(v) = 1$ if $v \in E_i$ and $\phi_i(v) = 0$ otherwise. We write $\phi := \phi_1 + \ldots + \phi_r$.

A nef partition defines a set of nef divisors $D_i = \sum_{\rho \in E_i} D_\rho$ on X_Δ such that $\sum_{i=1}^r D_i = -K_{X_\Delta}$; thus a general section of the bundle $\bigoplus_{i=1}^r \mathcal{O}(D_i)$ is a Calabi–Yau variety.

[4]That is, blow up a point on one of the two torus invariant sections and contract the strict transform of the fibre containing this point.

From the dual perspective, a nef partition is a Minkowski decomposition

$$\Delta^* = \nabla_1 + \ldots + \nabla_r$$

where the polytopes ∇_i are the polyhedra of sections of the line bundles $\mathcal{O}(D_i)$, together with points $p_i \in \nabla_i$ for each $1 \le i \le r$ such that $\sum_i p_i = 0$. The points p_i themselves may be interpreted as the torus invariant divisors D_i, which determine unique sections of the bundles $\mathcal{O}(D_i)$. More explicitly, the polytopes ∇_i are

$$\nabla_i := \{n \in N_{\mathbb{Q}} \mid \langle n, m\rangle \ge \phi_i(m) \text{ for any } m \in M_{\mathbb{Q}}\}, \qquad 1 \le i \le r.$$

In the case of a *Fano* complete intersection we can make a directly analogous definition:

Definition 11.3. Let Y be the toric variety defined by a fan Σ_Y, and consider a partition of the rays $\Sigma_Y(1)$ into subsets E_i, $1 \le i \le r$, and F. Let D_i be the torus invariant divisor corresponding to the set E_i and let D_F be the torus invariant divisor corresponding to F. The partition is a *Fano nef partition* if:

(i) the divisor D_F is ample; and
(ii) each of the divisors D_i is nef.

Note that since D_F is ample and the divisor $\sum_{i=1}^r D_i$ is nef, the divisor $-K_Y = D_F + \sum_{i=1}^r D_i$ is ample, that is, Y is a Fano toric variety.

Lemma 11.4. *The rays of Σ_Y in the set $\bigcup_{i=1}^r E_i$ generate a Gorenstein cone of the fan Σ_Y.*

Proof. Since D_F is ample the stability condition defining Y is covered by the divisor classes in F, and so the complement of these rays define a cone σ in the fan Σ_Y. Note that $\Sigma_Y(1)\backslash F$ is precisely the set $\bigcup_{i=1}^r E_i$. Moreover, since each divisor D_i is nef there is a function ϕ_i which is linear on σ and evaluates to one on each of the ray generators of E_i and to zero on all other ray generators of the cone σ. The sum ϕ of the ϕ_i defines a linear function on σ evaluating to one on every generator, which implies that σ is a Gorenstein cone. □

Consider a Fano polytope $P \subset N_{\mathbb{Q}}$ and a scaffolding S of P with shape $Z = \prod_{i=1}^k \mathbb{P}^{a_i}$. Lemma 3.2 and Theorem 5.5 imply that these data determine a toric variety Y_S, divisors $D_1, \ldots, D_r$, on Y_S whose linear systems define a Fano toric complete intersection, and a Laurent polynomial f_S with $P = \mathrm{Newt}(f_S)$. Write Σ_{Y_S} for the fan of Y_S, E_i for the subset of the rays of Σ_{Y_S} determined by D_i, and F for the set $\Sigma_{Y_S}(1) \setminus \bigcup_{i=1}^r D_i$. If the divisors D_i of Y_S are nef, then $F \cup E_1 \cup \cdots \cup E_r$ is a Fano nef partition. Furthermore if Y_S is $\mathbb{Q}$-factorial, then the Laurent polynomial f_S is mirror dual to the complete intersection defined by the vanishing of a general section of $\bigoplus_{i=1}^r \mathcal{O}(D_i)$. Conversely, a Fano nef partition for which the rays in $\bigcup_{i=1}^r E_i$ span a smooth cone determines a scaffolding of a Fano polytope with shape Z equal to a product of projective spaces.

Remark 11.5. The condition that the D_i are nef is much stronger than it appears. In general Y_S is far from being $\mathbb{Q}$-factorial, in which case there is no reason for the D_i to lie in $\mathbb{Q}$-Cartier divisor classes. After making a small resolution of Y_S it is reasonable to then expect the D_i to be nef divisors, but we then usually lose the conclusion of Theorem 5.5.

Remark 11.6. Recall that the ray generators of the fan Σ_{Y_S} lie in $\mathrm{Div}_{T_{\bar{M}}}(Z) \oplus N_U$. The set E_i in the nef partition above is given by the $a_i + 1$ divisors of the ith factor $\mathbb{P}^{a_i}$ of the shape $Z = \prod_{i=1}^k \mathbb{P}^{a_i}$. In particular, therefore, E_i spans a smooth cone in Σ_{Y_S}. This suggests a further generalisation of the notion of scaffolding in which the cone generated by the standard basis in $\mathrm{Div}_{T_{\bar{M}}}(Z)$ is replaced by an arbitrary Gorenstein cone. This is the most natural setting from the point of view of nef partitions: it would allow us to treat a broader class of toric complete intersections. We chose here, however, to pursue the alternative generalisation where the shape Z need no longer be the product of projective spaces, as this allows us to describe embeddings of toric varieties that are very far from complete intersections. It would be very interesting to see if these ideas can be translated back to the Calabi–Yau setting, and whether they give access to more general embeddings of Calabi–Yau manifolds in toric varieties.

Batyrev–Nill have determined necessary and sufficent conditions for a polytope to admit a nef partition [5], based on certain *Cayley cones* associated to a Minkowski decomposition of Δ^*.

Definition 11.7. Given polytopes $\nabla_1, \ldots, \nabla_r$ in $N_{\mathbb{Q}}$ the *Cayley polytope* of length r is

$$\nabla_1 \star \cdots \star \nabla_r := \operatorname{conv}(\nabla_1 + e_1, \ldots, \nabla_r + e_r) \subset N_{\mathbb{Q}} \times \mathbb{Q}^r.$$

The *Cayley cone* is the cone

$$\mathbb{Q}_{\geq 0}(\nabla_1 \star \cdots \star \nabla_r) = \mathbb{Q}_{\geq 0}(\nabla_1 + e_1) + \ldots + \mathbb{Q}_{\geq 0}(\nabla_r + e_r).$$

Proposition 11.8 ([5, Proposition 3.6])**.** *Given a reflexive polytope Δ and a Minkowski decomposition*

$$\Delta^* = \nabla_1 + \ldots + \nabla_r$$

the following conditions are equivalent:

(i) *the dual of the Cayley cone is a reflexive Gorenstein cone of index r that can be realised as the Cayley cone of r polytopes;*
(ii) *the Cayley polytope $\nabla_1 \star \cdots \star \nabla_r$ is a Gorenstein polytope of index*[5] *r containing a special $(r-1)$-simplex (see [5]);*
(iii) *the given Minkowski decomposition is a nef partition, that is, there are points $p_i \in \nabla_i$ for each $1 \leq i \leq r$ such that $\sum_i p_i = 0$.*

Given any scaffolding S of a Fano polytope P, we can produce a large number of polytopes $\widetilde{P}$ which project to P using Cayley product-type constructions. For any lattice L and any set of lattice vectors $R = \{r_s \in L \mid s \in S\}$, the polytope

$$\widetilde{P}_R := \operatorname{conv}((P_D + \chi) + r_s \mid s = (D, \chi) \in S) \subset (N \oplus L) \otimes_{\mathbb{Z}} \mathbb{Q}$$

admits a projection to P, induced by the projection $N \oplus L \to N$. The scaffolding S determines a canonical such polytope, given by setting

$$L = \operatorname{Pic}(Z), \qquad R = \{\mathcal{O}(D) \in \operatorname{Pic}(Z) \mid (D, \chi) \in S\}.$$

We denote this polytope $\widetilde{P}_R$ by $\widetilde{P}$. In the case where the shape Z is a product of projective spaces, there is a natural choice of coefficents on the integral points of $\widetilde{P}_R$ (for any R) that defines a Laurent polynomial with Newton polytope $\widetilde{P}_R$ which projects to f_S.

Given a scaffolding which defines a Fano nef partition we can describe both the toric ambient space Y_S and the Laurent polynomial f_S determined by S in terms of Cayley products.

Definition 11.9. Fix a Fano polytope P and a scaffolding S of P with shape $Z = \prod_{i=1}^k \mathbb{P}^{a_i}$ which determines a Fano nef partition of the toric ambient space Y_S. Define the polytope

$$P_S := \operatorname{conv}(\{(e_i, 0) \mid i \in \Sigma_Z(1)\} \cup S) \subset \widetilde{N} = \operatorname{Div}_{T_{\bar{M}}}(Z) \oplus N_U.$$

[5] That is, a polytope P such that rP is reflexive, possibly after translation.

The toric variety defined by the spanning fan of P_S is Y_S. Furthermore the polytopes $\widetilde{P}$ and P_S are related by mutation. To describe this mutation we fix a boundary divisor v_i of each projective space factor of Z. The divisors v_i generate the kernel of a projection $\pi\colon \widetilde{N} \to N$ and hence determine an isomorphism $\widetilde{N} \to N \oplus \mathrm{Pic}(Z)$. Let $\widetilde{P}_1$ denote the convex hull of $\widetilde{P}$ and the set

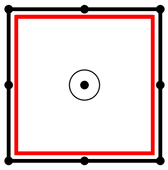
$$\{\pi_i^\star \mathcal{O}(1) \mid 1 \le i \le r\} \subset \{0\} \times \mathrm{Pic}(Z).$$

A mutation of $\widetilde{P}_1$ (or indeed of any other lattice polytope in $\widetilde{N}_\mathbb{Q}$) is determined by a *weight vector* $w \in \widetilde{M}$ and a polytope, the *factor*, $F \subset w^\perp$. We fix a sequence of mutations indexed by $[r]$ by specifying their weight vectors w_i and factors F_i, as follows:

(i) let $w_i \in \widetilde{M}$ be $-f_i^\star$, where $f_i^\star$ is the ith element of the basis dual to $\{v_1, \ldots, v_r\} \subset \widetilde{N}$;
(ii) let F_i be the the convex hull of the $(a_i + 1)$ elements of the standard basis of $\mathrm{Div}_{T_{\bar{M}}}(Z)$ corresponding to the ith projective space factor in Z.

The polytope obtained by applying the given sequence of mutations (in any order) to $\widetilde{P}_1$ is P_S.

Example 11.10. We verify this in a simple example. Let P and S be the Fano polygon and scaffolding shown:



The shape here is $Z = \mathbb{P}^1 \times \mathbb{P}^1$. The Laurent polynomial associated to this scaffolding is

$$f_S = \frac{(1+x)^2(1+y)^2}{xy}.$$

Applying Algorithm 5.1 to the scaffolding S we obtain the toric variety $Y_S = \mathbb{P}^4$ and an embedded toric degeneration of the del Pezzo surface dP_4 to the surface X_P. The polytope $\widetilde{P}_1$ is the Newton polytope of the polynomial

$$g_S = z_1 + z_2 + \frac{(1+x)^2(1+y)^2}{xyz_1^2z_2^2}.$$

Recall that the divisor D defining the (unique) element $(D, 0)$ of S is a section of the line bundle $\mathcal{O}(2,2) \in \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$. Mutating g_S as described, we obtain the Laurent polynomial

$$h_S = z_1(1+x) + z_2(1+y) + \frac{1}{xyz_1^2z_2^2}.$$

The Newton polytope of h_S is isomorphic to the Newton polytope of the polynomial

$$f_{\mathbb{P}^4} = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1x_2x_3x_4},$$

that is, to the polytope P_S.

Both of the scaffoldings described in Example 6.2 arise from Fano nef partitions. Example 11.1 shows that this property is not preserved under mutation of scaffoldings, whereas the Cayley polytope $\widetilde{P}$ always exists. Thus the polytope $\widetilde{P}$ associated to a scaffolding S of P is a natural generalisation of the notion of nef partition.

12. Amenable Collections and Towers of Projective Bundles

Theorem 5.5 asserts that any scaffolding of a polytope P determines an embedding of the toric variety X_P into an ambient toric variety Y. Lemma 3.2 tells us that the Laurent polynomials obtained via the Przyjalkowski method encode enough data to reconstruct X_P as a complete intersection, via a scaffolding on P with shape a product of projective spaces. In fact the Przyjalkowski method can be generalised via the use of *amenable collections subordinate to a nef partition*, introduced by Doran–Harder in [20]. These allow one to consider both more general toric complete intersection models for X_P and more general Laurent polynomial mirrors f. In this section we show that these embeddings and Laurent polynomials are determined by scaffoldings of P with a shape which is a tower of projective space bundles, rather than a product of projective spaces; in particular we see that our Laurent inversion construction (which allows the shape Z to be any toric variety) generalises the methods of [20].

Suppose, as before, that we have:

(12)
(i) orbifold GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_R; \omega)$;
(ii) a convex partition with basis $B, S_1, \ldots, S_k, U$ for Θ; and
(iii) a choice of elements $s_i \in S_i$ for each $1 \leq i \leq k$.

Let Y be the corresponding toric orbifold, let $X \subset Y$ denote the complete intersection defined by a regular section of the vector bundle $\bigoplus_i L_i$ and, following the notation used in §5, let $\widetilde{N}$ denote the ray lattice of Y. Following [20], an *amenable collection* subordinate to the partition $S_1, \ldots, S_k$ is a collection of vectors $w_1, \ldots, w_k$ that satisfies:

(13)
(i) $\langle w_i, \rho_j \rangle = -1$ for all $j \in S_i$ and all i;
(ii) $\langle w_i, \rho_j \rangle = 0$ for all $j \in S_l$ such that $l < i$ or $j \in U$ and all i;
(iii) $\langle w_i, \rho_j \rangle \geq 0$ for all $j \in S_l$ such that $l > i$ and all i.

Remark 12.1. The condition $\langle w_i, \rho_j \rangle = 0$ for $j \in U$ stems from the particular form of the algorithm used in §2. There is a more general form of this algorithm in which this condition may be dropped.

An amenable collection determines both a toric section of the bundle $\bigoplus_{1 \leq i \leq k} L_i$, and so a toric degeneration of X, and a Laurent polynomial mirror f. These constructions are both explained in detail in [20].

Proposition 12.2. *An amenable collection determines and is determined by a tower of projective space bundles Z. Furthermore, given an amenable collection subordinate to a nef partition, the toric degeneration of X to X_P constructed in [20] is equal to the toric embedding determined by Theorem 5.5 from a scaffolding of P with shape Z.*

Proof. The toric embedding $X_P \hookrightarrow Y$ determined by an amenable collection has the following straightforward description in terms of the Cox co-ordinates of Y [20, Proposition 2.7]. For each $1 \leq i \leq k$ consider the binomial equation in Cox co-ordinates

$$\prod_{j \in S_i} x_j - \prod_{j \notin S_i} x_j^{\langle w_i, \rho_j \rangle} = 0.$$

The toric variety cut out by all of these equations is a toric degeneration of X.

From an amenable collection we define Z inductively, starting from a point Z_0. For each $1 \leq j \leq k$ we define a toric variety Z_j and a $\mathbb{P}^{|S_j|-1}$ bundle $\pi_j \colon Z_j \to Z_{j-1}$. Each Z_j is the projectivisation of a split vector bundle, and so is determined by a collection of line bundles on Z_{j-1}. First we specify line bundles $L_{m,n}$ for all $n \in S_j$ and $m < j$ recursively by setting

$$L_{m,n} := \pi_m^\star(L_{m-1,n}) \otimes \mathcal{O}(-\langle w_m, \rho_n \rangle).$$

Here $\mathcal{O}(-1)$ is the tautological line bundle on the projective space fibration π_j and $L_{0,n} := \mathcal{O}$. Define π_j to be the projectivisation of

$$\mathbb{P}_{Z_{j-1}}\left(\bigoplus_{n\in S_j} L_{j-1,n}\right)$$

and define $Z := Z_k$. By construction the variety Z is toric, and we can easily write down a generating set for the relations between rays of the fan of Z. Indeed, writing z for the number of rays of Z, there is a partition of $[z]$ into k sets of sizes $|S_1|,\dots,|S_k|$ determined by the iterated bundle structure of Z. For each $1 \leq i \leq k$ there is a relation $\sum_{j=1}^{z}\alpha_{i,j}\rho_j$ where $\alpha_{i,j} = -\langle\rho_j, w_i\rangle$. Note that the value of $-\langle\rho_j, w_i\rangle$ is positive only if $j \in S_i$, in which case it is equal to 1.

Recall that a scaffolding with shape Z defines an embedding of lattices $N \to \widetilde{N} = \mathrm{Div}_{T_{\bar{M}}}(Z) \oplus N_U$. The relations described in the previous paragraph define hyperplanes in the lattice $\mathrm{Div}_{T_{\bar{M}}}(Z)$ and thus on $\widetilde{N}$. However any element w in the dual lattice to $\widetilde{N}$ defines a binomial in Cox co-ordinates:

$$\prod_{\rho \text{ s.t. } \langle w,\rho\rangle>0} x_\rho^{\langle w,\rho\rangle} - \prod_{\rho \text{ s.t. } \langle w,\rho\rangle<0} x_\rho^{-\langle w,\rho\rangle}.$$

Evidently these binomials are precisely those defining X_P as a subvariety of Y. Thus the system of binomials determined by an amenable collection is also determined by a scaffolding S with shape Z, obtained by fixing the struts of S (nef divisors on Z) via the projection $\widetilde{N} \to \mathrm{Div}_{T_{\bar{M}}}(Z)$. □

Remark 12.3. This result is compatible with Mirror Symmetry: an amenable collection defines a Laurent polynomial f much as in §2, so that f is the sum of terms x_i whose Newton polyhedra are nef divisors on a tower of projective space bundles Z. Thus we can determine Y and the toric embedding of X_P from this Laurent polynomial f and its scaffolding.

Example 12.4. A del Pezzo surface X_4 of degree 4 is a $(2,2)$ complete intersection in $\mathbb{P}^4$. Using the methods discussed in §2 one can construct a toric degeneration of this del Pezzo surface with central fibre

$$x_0^2 - x_1x_2 = 0, \qquad\qquad x_0^2 - x_3x_4 = 0,$$

where $x_0,\dots,x_4$ are the homogeneous co-ordinates on $\mathbb{P}^4$. Using amenable collections we now describe another toric degeneration of X_4. Let $\widetilde{N} \cong \mathbb{Z}^4$ be the ray lattice of $\mathbb{P}^4$, and fix a convex partition with basis by setting $B = \{1\}$, $S_1 = \{2,3\}$, $S_2 = \{4,5\}$, and $U = \varnothing$. Choose an amenable collection $\{w_1, w_2\}$ in M by setting

$$w_1 = (-1,-1,0,2), \qquad\qquad w_2 = (0,0,-1,-1).$$

The two equations defined by the w_i are

$$x_4^2 - x_1x_2 = 0, \qquad\qquad x_0^2 - x_3x_4 = 0. \tag{14}$$

To compute the corresponding scaffolding we first need to determine Z. Following the proof of Proposition 12.2 we see that $Z \cong \mathbb{F}_2 := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(-2))$. The scaffolding is determined by taking rays of Y not contained in the standard basis and viewing them as nef divisors on $\mathbb{F}_2$. In this case we have only the ray $(-1,-1,-1,-1)$, which corresponds to the toric boundary of $\mathbb{F}_2$. Consequently the scaffolding we obtain consists of a single triangle: see Figure 12. The rays shown on Figure 12 are obtained by pulling back the fan of $\mathbb{P}^4$ along the inclusion of the subspace of $\widetilde{N}$ annihilating both w_1 and w_2. In particular the toric variety defined by this fan

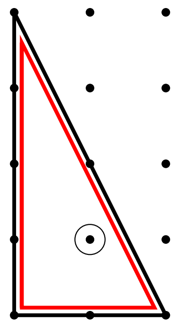


FIGURE 12. A scaffolding determined by an amenable collection.

is a quotient of the weighted projective plane $\mathbb{P}(1,1,2)$ defined by the binomial quadrics (14) in $\mathbb{P}^4$. This is an instance of Proposition 9.1, with shape $\mathbb{F}_2$.

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Appendix A. The Proof of Theorem 5.5

Throughout this section we fix a Fano polytope P together with a scaffolding S of P with shape Z, where Z is smooth. We show that X_P is a toric subvariety of the ambient space Y_S defined in §5, via the embedding of tori defined in the discussion following Theorem 5.5. We begin by constructing a polytope Q_S defined by a polarisation of the toric variety Y_S.

Definition A.1. Let $\widetilde{N}$ denote the lattice $\mathrm{Div}_{T_{\bar{M}}}(Z) \oplus N_U$ and let $\widetilde{M}$ denote the dual lattice. Denote the standard basis elements of $\mathrm{Div}_{T_{\bar{M}}}(Z) \cong \mathbb{Z}^k$ by e_i for $1 \leq i \leq k$. Define elements $\rho_s = (-D, \chi) \in \widetilde{N}$ for each $s = (D, \chi) \in S$. The polytope Q_S is defined by

$$Q_S := \left\{ u \in \widetilde{M}_{\mathbb{Q}} \mid \langle u, e_i \rangle \geq 0 \text{ and } \langle u, \rho_s \rangle \geq -1 \text{ for all } s \in S \text{ and } 1 \leq i \leq k \right\}.$$

We write Σ_S for the normal fan of Q_S.

Remark A.2. We assume throughout this section that, given an element $(D, \chi) \in S$, the strut $P_D + \chi$ contains a vertex of P. In fact X_P embeds in Y_S if and only if it embeds into $Y_{S'}$ where S' is the scaffolding obtained by removing all struts which do not meet a vertex of P.

Remark A.3. It is elementary to check that the toric variety defined by Σ_S is precisely Y_S. Indeed, the polarising class is precisely the one chosen in Algorithm 5.1.

We now study the faces of Q_S in more detail. Our first step is to introduce a polyhedral decomposition of $Q := P^*$.

Definition A.4. Let $\mathrm{verts}(S)$ be the set of torus fixed points of Z, and observe there is a canonical bijection $\mathrm{verts}(S) \to \mathrm{verts}(P_{D'})$ for an ample divisor D', and a canonical surjection $\mathrm{verts}(S) \to \mathrm{verts}(P_D)$ for a nef divisor D which we denote by $u \mapsto u^D$. We refer to $\mathrm{verts}(S)$ as the set of vertices of the scaffolding S. Each element $u \in \mathrm{verts}(S)$ defines a function $S \to N$, which we also denote by u, defined by setting $u\left((D,\chi)\right) = u^D + \chi$.

Definition A.5. Given $u \in \mathrm{verts}(S)$ we define

$$\mathrm{verts}(P,u) := \{v \in \mathrm{verts}(P) \mid v = u(s) \text{ for some } s \in S\}.$$

Definition A.6. Define a polyhedral decomposition of Q by intersecting Q with the fan $\Sigma_Z \times (N_U \otimes \mathbb{Q})$ defining the toric variety $Z \times T_{N_U}$. Maximal cells C_u of this decomposition are indexed by elements $u \in \mathrm{verts}(S)$.

Remark A.7. If we identify $\mathrm{verts}(S)$ with the vertices of the polyhedron of sections of an ample divisor D on Z the chambers C_u are precisely the maximal domains of linearity of the convex piecewise linear function

$$\min_{u\in\mathrm{verts}(S)} \langle u^D, -\rangle \colon Q \to \mathbb{Q}$$

indexed by the vertex u^D on which this function attains its minimum. If instead D is nef then the analogous maximal domains of linearity are unions of chambers C_u.

We next identify certain faces of Q_S with images $\iota(C_u)$ of a piecewise linear function ι.

Definition A.8. Let $n = \dim M$. Define ι to be the inverse map to the restriction to $E \oplus N_U$ of the canonical projection $\widetilde{M}_{\mathbb{Q}} \to M_{\mathbb{Q}}$, where E is the union of n-dimensional faces of the standard coordinate cone in $\mathrm{Div}_{T_{\bar{M}}}(Z)^\vee$ which project onto maximal dimensional cones of the fan of Z.

The fact that Z is smooth ensures that ι is well defined and maps the integral points of C_u bijectively to the integral points of a face of Q_S. Note that ι is linear on each chamber $C_u \subset Q_{\mathbb{Q}}$.

Proposition A.9. *For each $u \in \mathrm{verts}(S)$, the polytope $\iota(C_u)$ is a face of Q_S.*

Proof. We first show that, for any $p \in C_u$:

(i) $\langle e_i, \iota(p)\rangle = 0$ for some $1 \le i \le k$;
(ii) $\langle e_j, \iota(p)\rangle \ge 0$ for all $1 \le j \le k$; and
(iii) $\langle \rho_s, \iota(p)\rangle \ge -1$ for all $s \in S$.

Fixing a point $p \in C_u$ the first two are obvious: $\iota(p)$ lies in the positive co-ordinate cone of $\widetilde{M}_{\mathbb{Q}}$ so the second condition is automatic, the first follows from the fact that $\iota(p)$ lies in the cone spanned by $n = \dim M$ of the standard coordinate vectors and hence in the hyperplane defined by $\langle e_i^\star, -\rangle$ for e_i not among these n vectors.

Next we consider $\langle \rho_s, \iota(p)\rangle$; the map ι is linear on C_u and let ι_u denote the linear extension of $\iota|_{C_u}$ to $M_{\mathbb{Q}}$. We now compute $\langle \iota_u^\star \rho_s, p\rangle$. It is clear that $\iota_u(M_{\mathbb{Q}})$ is the span of M_U together with co-ordinate vectors $e_i^\star$ in $\widetilde{M}$ such that the divisor in Z corresponding to e_i meets the vertex u. By definition $\langle \rho_s, e_i^\star\rangle$ is the height of the supporting hyperplane of the polyhedron of sections P_D where $s = (D,\chi)$. Thus $\iota_u^\star \rho_s$ is the vertex u of $P_D + \chi$ and therefore lies in P. Since $\iota_u^\star \rho_s \in P$ and $p \in Q$, $\langle \iota_u^\star \rho_s, p\rangle \ge -1$.

We have shown that $\iota(C_u)$ is contained in a face of Q_S, to show the reverse inclusion we need only to check that if $\langle \rho_s, m'\rangle \ge -1$ for $m' \in \iota_u(M)$ and m' in the standard positive cone then $m' \in C_u$. However this also follows from the fact that $\iota_u^\star \rho_s$ is the vertex u of $P_D + \chi$. □

The polytope Q_S determines its normal fan Σ_S, which in turn determines a toric variety Y_S. We now prove that the pullback of the fan Σ_S under the inclusion $M \to \widetilde{M}$ is the spanning fan of the Fano polytope P.

Lemma A.10. *The set of rays of Σ_S is $\{\rho_s \mid s \in S\} \cup \{e_i \mid 1 \leq i \leq k\}$. That is, all the rays used in Definition A.1 to define Q_S appear in the normal fan of Q_S.*

Proof. Finding facets of Q_S with normal direction e_i, $1 \leq i \leq k$, is straightforward: intersecting Q_S with a small ball B, so that $\langle \rho_s, p \rangle > -1$ for all $p \in B$, centered at the origin we obtain a smooth (not necessarily strictly convex) cone. The normal directions to the facets meeting the origin are precisely the co-ordinate vectors e_i.

Now fix an element $s = (D, \chi) \in S$ and a vertex $v \in P$ which meets the strut $P_D + \chi$. Let B' be a small ball around a point $\iota(p)$, where p is a point in the interior of the facet $v^\star$ dual to the vertex v. Now recall from the proof of Proposition A.9 that for any $s' \in S$ and $u \in \mathrm{verts}(S)$

$$\iota_u^\star \rho_{s'} = u(s')$$

where $\iota_u^\star$ is the dual map to the linear map defined by restricting ι to C_u. Consequently considering $\rho_{s'}$ as a function on $\iota(\partial Q)$ we see that $\rho_{s'}$ achieves its minimum, -1, precisely along facets $u(s)^\star$ for $u(s)$ a vertex of P. Therefore, taking a point p' in the intersection of B' with the hyperplane $\langle \rho_s, - \rangle = -1$ and the half spaces $\langle e_i, - \rangle > 0$ for $1 \leq i \leq k$, by construction p' lies on the facet with normal ρ_s. ☐

We require an explicit description of those cones in Σ_S which intersect the image of $N_\mathbb{Q}$ non-trivially. Fixing a face E of ∂P we first identify a cone in $\widetilde{N}_\mathbb{Q}$ which intersects $N_\mathbb{Q}$ precisely in the cone generated by E.

Definition A.11. Given a stratum $E \in \partial P$ let

$$\mathrm{verts}(S, E) := \{u \in \mathrm{verts}(S) \mid u(s) \in E \text{ for some } s \in S\}.$$

Let $C_E \subset \widetilde{N}_\mathbb{Q}$ be the cone generated by the following rays:

(i) ρ_s, for $s = (D, \chi) \in S$ such that the strut $P_D + \chi$ meets E;
(ii) e_i, the divisors of Z which miss some $u \in \mathrm{verts}(S, E)$.

Proposition A.12. *We have that $C_E \cap N_\mathbb{Q} = \mathrm{cone}(E)$.*

Proof. First we show that $\mathrm{cone}(E) \subset C_E \cap N_\mathbb{Q}$. In fact we prove that every ray generator of $\mathrm{cone}(E)$ appears in $C_E \cap N_\mathbb{Q}$. By definition the ray generators of $\mathrm{cone}(E)$ are precisely the vertices of the face $E \subset P$. Choose such a vertex v. Writing θ for the inclusion $N \to \widetilde{N}$, since $v = \iota_u^\star(\rho_s)$ for any u such that $v \in \mathrm{verts}(P, u)$ we see that $(\theta(v) - \rho_s) \in \ker(\iota_u^\star)$. However $\ker(\iota_u^\star)$ is the annihilator of $\iota_u(M_\mathbb{Q}) \subset \widetilde{M}$, which is generated by those divisors e_i which miss the torus fixed point $u \in \mathrm{verts}(S)$ which determined the linear map ι_u. That is, we can express v as a linear combination of ray generators of $\mathrm{cone}(E)$. Thus to prove that $v \in C_E \cap N_\mathbb{Q}$ we now only need to prove that the divisor $\theta(v) - \rho_s$ is effective on Z. However the polyhedron of sections of this divisor is easily seen to be that of $-\rho_s$ translated so that the origin is identified with the u vertex of P_D (for $s = (D, \chi)$). Since this polyhedron contains the origin it must be the polyhedron of sections of an effective divisor.

To prove the reverse inclusion observe that it suffices to consider the case where E is a vertex. Since the generators of C_E different from ρ_s generate the subspace $\ker(\iota_u^\star) \subset \widetilde{N}$ they span a complement to $\theta(N)$, and therefore $\dim(C_E \cap N_\mathbb{Q}) \leq 1$. Thus this intersection is contained in a ray which, by the first part, must be $\mathrm{cone}(E)$. ☐

We now conclude the proof of Theorem 5.5 by showing that the cones C_E appear in the fan Σ_S. We show that the following diagram commutes:

$$\begin{tikzcd} \widetilde{\Sigma}_S^{max} \arrow[r] \arrow[d, "-\cap N_{\mathbb{Q}}"] & \mathrm{verts}(Q_S) \\ \Sigma_P^{max} \arrow[r] & \mathrm{verts}(Q) \arrow[u, "\iota"'] \end{tikzcd} \tag{15}$$

Here the horizontal arrows are the usual bijections between k-strata of a polytope and codimension-k cones of its normal fan, and Σ_P is the spanning fan of the polytope P. The right-hand vertical arrow is the piecewise linear map ι and the left-hand vertical map is intersection with the subspace $N_{\mathbb{Q}}$, where we slightly abuse notation to make this map well-defined by restricting to those maximal cones in $\widetilde{N}_{\mathbb{Q}}$ which meet $N_{\mathbb{Q}}$ in a maximal dimensional cone. Choose a maximal cone in Σ_S which meets $N_{\mathbb{Q}}$ in maximal dimensional cone σ; let v be the corresponding vertex of Q. We need to check that the normal directions to the facets which meet $\iota(v) \in \mathrm{verts}(Q_S)$ generate C_σ.

First consider the collection of struts $\{\rho_s \mid s \in S, \langle \rho_s, \iota(v)\rangle = -1\}$. Recall from the proof of Proposition A.9 that given an element $u \in \mathrm{verts}(S)$ and $w \in \mathrm{verts}(v^\star)$ such that $w \in \mathrm{verts}(P, u)$ we have that $\iota_u^\star(\rho_s) = w$. Now $v \in C_u$ for some $u \in \mathrm{verts}(S)$ and observe that $\iota(v) = \iota_u(v)$ for any such $u \in \mathrm{verts}(S)$, and $\langle \iota_u^\star(\rho_s), v\rangle = \langle u(s), v\rangle$, so the condition that $\langle \rho_s, \iota(v)\rangle = -1$ is equivalent to the condition that

$$\langle u(s), v\rangle = -1$$

for some $u \in \mathrm{verts}(S)$ such that $v \in C_u$. That is, that $u(s) \in v^\star$ for some u. In other words, the set of struts $\{\rho_s \mid s \in S, \langle \rho_s, \iota(v)\rangle = -1\}$ is precisely the set of those struts which meet $v^\star$. Thus the collection of hyperplanes $\langle \rho_s, -\rangle = -1$ meeting $\iota(v)$ are precisely the generators ρ_s of $C_{v^\star}$. Furthermore, by Lemma A.10 these hyperplanes do all appear as facets of Q_S that meet v. It remains to check that the generators e_i of $C_{v^\star}$ appear as rays in the normal cone to $\iota(v)$.

Consider the collection $\{e_i \mid 1 \leq i \leq k, \langle e_i, \iota(v)\rangle = 0\}$, and observe that by Lemma A.10 each of these hyperplanes defines a facet of Q_S that meets v. Recall that the elements $u \in \mathrm{verts}(S)$ are in bijection with the maximal cones σ_u in the fan Σ_Z defining Z, and that by definition $\iota(\sigma_u)$ is the cone in $\widetilde{M}$ whose ray generators are standard basis vectors corresponding to rays of σ_u. Choosing any u such that $v \in C_u$ the functionals e_i which vanish on the cone $\iota(\sigma_u)$ are precisely those elementary torus invariant divisors which miss the torus fixed point $u \in \mathrm{verts}(S)$. Conversely $\iota(\mathrm{cone}(v))$ is the intersection of cones $\iota(\mathrm{cone}(\sigma_u))$ for $v \in C_u$, and so the annihilator of $\iota(\mathrm{cone}(v))$ is generated by the union of those standard basis vectors generating the annihilator of $\iota(\sigma_u)$. That is, if e_i vanishes on $\iota(v)$ it vanishes on some σ_u.

Thus the facets which meet $\iota(v)$ exactly generate the cone $C_{\mathrm{cone}(v^\star)}$. Proposition A.12 now implies that diagram (15) commutes. Consider now the fan Σ'_P obtained by pulling back the fan Σ_S along the inclusion $\theta\colon N_{\mathbb{Q}} \hookrightarrow \widetilde{N}_{\mathbb{Q}}$. Commutativity of diagram (15) guarantees that the cone over each face of P occurs in Σ'_P. Since the spanning fan of P is a complete fan, we have determined every cone of Σ'_P. This completes the proof of Theorem 5.5.

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