Toric Fano Varieties and Convex Polytopes

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Summary

In this thesis we study toric Fano varieties. Toric varieties are a particular class of algebraic variety which can be described in terms of combinatorial data. Toric Fano varieties correspond to certain convex lattice polytopes whose boundary lattice points are dictated by the singularities involved.

Terminal toric Fano varieties correspond to convex lattice polytopes which contain only the origin as an internal lattice point, and whose boundary lattice points are precisely the vertices of the polytope. The situation is similar for canonical toric Fano varieties, with the exception that the condition on boundary lattice points is relaxed. We call these polytopes terminal (or canonical) Fano polytopes.

The heart of this thesis is the development of an approach to classifying Fano polytopes, and hence the associated varieties. This is achieved by ordering the polytopes with respect to inclusion. There exists a finite collection of polytopes which are minimal with respect to this ordering. It is then possible to "grow" these minimal polytopes in order to obtain a complete classification.

Critical to this method is the ability to find the minimal polytopes. Their description is inductive, requiring an understanding of the lower-dimensional minimal polytopes. A generalisation of weighted projective space plays a crucial role – the associated simplices form the building blocks of the minimal polytopes. A significant part of this thesis is dedicated to attempting to understand these building blocks.

A classification of all toric Fano threefolds with at worst terminal singularities is given. The three-dimensional minimal canonical polytopes are also found, making a complete classification possible.

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Contents

CHAPTER 1

Introduction

1.1 Background

Toric varieties form an important class of algebraic varieties. The structure of a toric variety is intimately connected with a corresponding combinatorial description, allowing one to easily illustrate such concepts as linear systems, invertible sheaves, cohomology, and resolution of singularities. This provides a powerful tool for explicit constructions, whilst still admitting a large enough class of varieties to reflect many general principles of algebraic geometry.

A substantial body of introductory literature exists on the subject of toric varieties. Fulton's book [Ful93] is the standard reference, along with [Ewa96] and [Oda78]. Of equal merit are [BB02] and [Dan78]. For a survey of the current state of research in this field, consult [Cox02]. The basic constructions in toric geometry are reviewed in Chapter 2. We concentrate on the tools we shall need, to the detriment of the well-developed theory of cohomology on toric varieties.

In Chapter 3 we review terminal and canonical singularities. The importance of terminal singularities lies chiefly in the fact that, together with $\mathbb{Q}$-factoriality, it characterises the most restrictive collection of singularities in which the Minimal Model Program (or Mori Program) operates (see Section 3.4).

Canonical singularities can be regarded as the limits of terminal singularities. They characterise the singularities of the canonical models of varieties of general type whose canonical rings are finitely generated $\mathbb{C}$-algebras – an important birational invariant. Technical details can be found in [Mat02] or [Deb01].

Section 3.5 introduces the concept of a Fano variety, named after the Italian mathematician Gino Fano (1871–1952). A Fano variety has restrictions on its singularities, and possesses an ample anticanonical bundle. In other words, it comes equipped with a natural projective embedding.

Fano varieties are one of the major classes of algebraic varieties studied in algebraic geometry. Their classification has been known in dimensions one and two for over a century, and there has been a lot of work more recently in dimension three after an initial classification by Fano himself in the 1930s.

A smooth Fano surface is usually called a del Pezzo surface, in honour of the German-born mathematician and politician Pasquale Del Pezzo (1859–1936). They consist of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^2$ blown up in at most eight points (in general position). Smooth Fano threefolds have also been classified. There are seventeen families with Picard number one, and eighty-nine other families (for references see Section 3.5). There are eighteen smooth toric Fano threefolds ([Bat81, WW82]) and 128 smooth toric Fano fourfolds ([Bat99]).

The singularities discussed above give rise to a particularly beautiful construction in the toric setting (Section 3.6). If, in addition, we require that our toric varieties are Fano, we see that an equivalence exists between classes of convex lattice polytopes and toric varieties with prescribed singularities. The polytopes in question are particularly elegant – perhaps philosophically similar to the Platonic solids. The remainder of this thesis is dedicated to the study of these polytopes, which are called Fano polytopes.

1.2 Fake Weighted Projective Space

We begin Chapter 4 by recalling the definition of weighted projective space. We then proceed to present a toric construction of these varieties. Many useful consequences can be derived from the toric construction. A classification of those weighted projective spaces of the form $\mathbb{P}(1, \lambda_1, \lambda_2, \lambda_3)$ is given in Section 4.3.

Section 4.4 introduces a generalisation of weighted projective space, called fake weighted projective space. A weighted projective space corresponds to a simplex. Fake weighted projective spaces are defined to be all those spaces whose corresponding polytope is a simplex. This generalisation of weighted projective space will prove to be particularly useful in Chapter 8.

Fake weighted projective spaces arise naturally in toric Mori theory (consult [Rei83a, Kry02, Fuj03] or [Mat02, Remark 14.2.4]). Loosely speaking, they appear as the fibres of an elementary contraction. For a precise statement, see Proposition 4.4.4.

Associated with each fake weighted projective space X is a weight $(\lambda_0, \lambda_1, \ldots, \lambda_n)$. It transpires that a great deal of information about the singularities of X can be deduced by considering the singularities of $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. We see that in order to classify all fake weighted projective spaces with terminal (or canonical) singularities, it is sufficient to restrict our attention to those weights for which the corresponding weighted projective space is terminal (or canonical). Similarly for Gorenstein fake weighted projective spaces.

Each fake weighted projective space comes equipped with a measure we call its multiplicity. This can be regarded as a measure of how far away it is from being a *bona fide* weighted projective space. The multiplicity is one if and only if the space is a weighted projective space. Bounds on the multiplicity are established in Theorem 4.4.13, which depends only upon knowledge of the weights.

This relationship between weights and multiplicity is vitally important in classification, since it demonstrates that enlarging our attention to fake weighted projective spaces involves relatively little effort. The bound on the multiplicity, combined with the work of Conrads ([Con02]), means that the problem of classifying all fake weighted projective spaces (with appropriate limits on their singularities) is reduced to an understanding of the weights involved.

1.3 Classifications in Low Dimension

In Chapter 5 we consider toric Fano surfaces. Using a new method we derive the well-known classifications in the terminal and canonical cases. This can be regarded as an introduction to the techniques used in Chapters 6–8.

The methodology is uncomplicated. Restricting ourselves to those toric Fano varieties of dimension n with terminal (or canonical) singularities, we can order the corresponding Fano polytopes by inclusion. Since the classification is known *a priori* to be finite, there must exist a finite collection of polytopes which are minimal with respect to this ordering. By understanding the permissible weights of fake weighted projective space of dimension less than or equal to n, it is possible to “grow” these minimal polytopes in order to obtain a complete classification.

The process of “growing” the polytopes is purely mechanical, and best left to a computer. The difficulty lies in finding the minimal polytopes. This issue is addressed in Chapter 8. Their description is inductive, requiring an understanding of the lower-dimensional minimal polytopes. Fake weighted projective space plays a crucial role – they form the building blocks of the minimal polytopes. Once more, we see that understanding the weights of weighted projective space is vital.

Using the technique described above we classify all toric Fano threefolds with at worst terminal singularities in Chapter 6. The main features of the classification are summarised in Theorem 6.1.1. In Sections 8.5–8.6 the minimal canonical polytopes in three dimensions are found; it is now possible for a computer to complete the classification. Knowing the minimal canonical polytopes allows us to establish an upper bound on the degree of a toric Fano threefold with at worst canonical singularities (Theorem 8.5.5).

In Section 7.1 we give a new proof of an old party trick: the connection between Fano polygons and the number twelve. By employing the new notion of minimality (and the corresponding concept of maximality), a concise proof is possible.

1.4 The Admissible Weights

As alluded to above, understanding the admissible weights of weighted projective space is crucial. Chapter 9 looks at this problem in more detail. Section 9.2 serves as an introduction to reflexive simplices. In particular we summarise the recent work of Nill ([Nil04, Nil05]) and reinterpret an important result of Conrads in terms of multiplicity.

In Section 9.3 we consider which weights give rise to terminal (or canonical) weighted projective space. First we review the bounds which are known to exist on the weights. Corollary 9.3.2 establishes a bound on the sum of the weights whilst Theorem 9.3.6 gives a new upper bound on the individual weights.

Sections 9.5–9.9 attempt to generalise the process of deriving weights. This is particularly successful in the reflexive (Gorenstein) case; for example Proposition 9.8.1 and Theorem 9.8.4. The techniques developed, combined with the bounds at the start of the chapter, allow efficient computations to be performed.

1.5 The Ehrhart Polynomial

Chapter 10 can be regarded as being separate to the rest of this thesis. It provides an introduction to the Ehrhart polynomial and Ehrhart series associated with a lattice polytope. This is a particularly powerful tool; a polytope's equivalent to the Hilbert series. Using this machinery, it is possible to derive a characterisation of reflexive three-dimensional Fano polytopes in terms of their volume and the number of lattice points on the boundary (Corollary 10.3.9).

The Ehrhart series of polytopes have recently received particular attention from Batyrev ([BN06, Bat06]), as well as forming the subject of a forthcoming book by Beck and Robins ([BR]). In many respects this chapter is simply an advertisement for the tools involved.

CHAPTER 2

What is a Toric Variety?

2.1 Cones, Fans and Toric Varieties

Definition 2.1.1. A *toric variety* of dimension n over an algebraically closed field $k = \overline{k}$ is a normal variety X that contains a torus $T \cong (k^*)^n$ as a dense open subset, together with an action $T \times X \to X$ of T on X that extends the natural action of T on itself.

Let $M \cong \mathbb{Z}^n$ be a lattice, and $N = \mathrm{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^n$ be its dual lattice. We regard M as the lattice of Laurent monomials in $X_1, \ldots, X_n$. Thus points in M correspond to monomials $X_1^{a_1} \ldots X_n^{a_n}$ for some $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. Let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, and similarly for $N_{\mathbb{R}}$.

Definition 2.1.2. A *cone* (or more precisely, a *finitely generated rational polyhedral cone*) in $\mathbb{N}_{\mathbb{R}}$ is a set of the form:

$$\left\{ \sum_{i=1}^{k} \lambda_i v_i \in N_{\mathbb{R}} \middle| \lambda_i \geq 0 \right\},$$

for some finite collection of elements $\{v_1, \ldots, v_k\}$ in N.

Associated to each cone $\sigma \subset N_{\mathbb{R}}$ is its *dual* in $M_{\mathbb{R}}$:

Definition 2.1.3. The *dual cone* $\sigma^\vee \subset M_{\mathbb{R}}$ to a cone $\sigma \subset N_{\mathbb{R}}$ is given by:

$$\sigma^\vee := \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

As might be expected, the dual of a convex polyhedral cone is a convex polyhedral cone (*Farkas' Theorem*, see [Ful93, pg.11]), and $(\sigma^\vee)^\vee = \sigma$ (e.g. [Ful93, pg.9]).

We can regard the lattice points of M contained in a dual cone $\sigma^\vee$ as monomials in the coordinate ring of some affine variety. This is achieved by first defining the semigroup $S_\sigma := \sigma^\vee \cap M$. This semigroup is finitely generated, by *Gordon's Lemma* (e.g. [Ful93, pg.12]).

We now define the corresponding affine ring $A_\sigma := \mathbb{C}[S_\sigma]$. We denote by χ^u the element in the $\mathbb{C}$-algebra corresponding to the semigroup element $u \in S_\sigma$. We require that $\chi^u \chi^{u'} := \chi^{u+u'}$. The elements of $\mathbb{C}[S_\sigma]$ are thus given by finite sums $\sum c_i \chi^{u_i}$, where $c_i \in \mathbb{C}, u_i \in S_\sigma$.

Finally the affine variety U_σ corresponding to a cone σ is given by:

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma]) .$$

It is often useful to interpret the points of U_σ as semigroup homomorphisms, i.e. we identity $U_\sigma = \mathrm{Hom}_{\mathrm{sgp}}(S_\sigma, \mathbb{C})$. This identification immediately suggests two special points in U_σ:

$$\begin{aligned} x_0 : S_\sigma &\to \mathbb{C} \\ u &\mapsto 1. \end{aligned}$$

$$\begin{aligned} x_\sigma : S_\sigma &\to \mathbb{C} \\ u &\mapsto \begin{cases} 1, & \text{if } -u \in S_\sigma; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We call x_0 the *base point* of U_σ and x_σ the *distinguished point* of U_σ.

We now wish to describe a way of patching these affine varieties together. In the lattice N we define a fan Δ:

Definition 2.1.4. A *fan* Δ is defined to be a finite collection of cones in $N_\mathbb{R}$ such that:

(i) If $\sigma \in \Delta$, then $\sigma \cap (-\sigma) = \{0\}$. Such a cone is said to be *strongly convex*[1];

(ii) If $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$;

(iii) If $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma' \in \Delta$.

For each cone $\sigma \in \Delta$ there exists an affine variety U_σ constructed as above. These affine varieties can be glued together via the following Lemma:

[1] A cone is strongly convex if it contains no one-dimensional vector space.

Lemma 2.1.5. *If τ is a face of σ then the map $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.*

Proof. See [Ful93, pg.18]. $\square$

Since any two cones $\sigma, \sigma' \in \Delta$ share a common face, there are injections $\phi : U_{\sigma\cap\sigma'} \to U_\sigma$ and $\varphi : U_{\sigma\cap\sigma'} \to U_{\sigma'}$. The identification is then given by:

$$\begin{aligned} f : \phi(U_{\sigma\cap\sigma'}) &\to \varphi(U_{\sigma\cap\sigma'}) \\ x &\mapsto \varphi(\phi^{-1}(x)), \end{aligned}$$

with inverse $y \mapsto \phi(\varphi^{-1}(y))$. Patching for all $\sigma \in \Delta$ gives the toric variety denoted by X_Δ.

This patching can also be constructed at the level of $\mathbb{C}$-algebras. For any two cones $\sigma, \sigma' \in \Delta$ there exists $u \in (-\sigma')^\vee \cap \sigma^\vee$ such that $\sigma \cap u^\perp = \sigma' \cap \sigma = \sigma' \cap u^\perp$. This result is known as the *Separation Lemma* (see [Ful93, pg.13]). Hence $\mathbb{C}[S_\sigma]_{\chi^u} \cong \mathbb{C}[S_{\sigma\cap\sigma'}] \cong \mathbb{C}[S_{\sigma'}]_{\chi^u}$.

Note that since $\{0\} \subset N_\mathbb{R}$ is a face of every cone in Δ, so $U_{\{0\}}$ can be regarded as sitting inside each affine variety U_σ. Now $S_{\{0\}} = M$, where we regard the lattice M as a semigroup with $2n$ generators. In particular if N is generated (as a lattice) by $e_1, \ldots, e_n$, then M is generated (as a semigroup) by $\pm e_1^*, \ldots, \pm e_n^*$. Setting $X_i := \chi^{e_i^*}$ and $X_i^{-1} := \chi^{-e_i^*}$ we see that:

$$U_{\{0\}} = \operatorname{Spec}\Big(\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]\Big) \cong (\mathbb{C}^*)^n.$$

Thus the (algebraic) torus is a principal open subset of all the U_σ. This is why X_Δ is called a *toric* variety.

Definition 2.1.6. Let Δ be a fan in $N_\mathbb{R}$. The *support* of Δ is given by:

$$|\Delta| := \bigcup_{\sigma\in\Delta} \sigma \subset N_\mathbb{R}.$$

2.2 Divisors on Toric Varieties

Let $T \cong (\mathbb{C}^*)^n$ be the algebraic torus of dimension n. Associated with T are two groups:

Definition 2.2.1. The *character group* of T is the group:

$$M := \mathrm{Hom}(T, \mathbb{C}^*).$$

The *one–parameter subgroup* of T is the group:

$$N := \mathrm{Hom}(\mathbb{C}^*, T).$$

Indeed, our choice of labelling is no coincidence (see [Ful93, §2.3] for an alternative derivation):

Lemma 2.2.2. *$M \cong \mathbb{Z}^n$ where, for any $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, we have that:*

$$\chi^u(t_1, \ldots, t_n) = t_1^{u_1} \ldots t_n^{u_n},$$

and $N \cong \mathbb{Z}^n$ where, for any $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$, we have that:

$$\lambda^v(t) = (t^{v_1}, \ldots, t^{v_n}).$$

Proof. Any morphism $M \ni \chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$ corresponds to a ring homomorphism:

$$\chi^* : \mathbb{C}[Y, Y^{-1}] \to \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}].$$

By definition, $1 = \chi^*(YY^{-1}) = \chi^*(Y)\chi^*(Y^{-1})$, and since $\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = \mathbb{C}[X_1, \ldots, X_n]_{X_1 \ldots X_n}$ we have that there exist $G, H \in \mathbb{C}[X_1, \ldots, X_n]$ such that:

$$G \cdot H = X_1^{a_1} \ldots X_n^{a_n} \qquad \text{for some } a_1, \ldots, a_n \in \mathbb{Z}.$$

This forces G and H to be monomials. Hence $\chi^*(Y)$ (and $\chi^*(Y^{-1})$) is a monomial in $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$.

Conversely any monomial $X_1^{u_1} \ldots X_n^{u_n}$ in $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ clearly defines a ring homomorphism, sending $Y \mapsto X_1^{u_1} \ldots X_n^{u_n}$ and $Y^{-1} \mapsto X_1^{-u_1} \ldots X_n^{-u_n}$.

Similarly for $\lambda \in N$. $\square$

Let $\chi \in M$ and $\lambda \in N$. The composition $\chi \circ \lambda : \mathbb{C}^* \to \mathbb{C}^*$ gives a map $t \mapsto t^k$ for some $k \in \mathbb{Z}$. We define $\langle \chi, \lambda \rangle := k$. In fact the map is given by:

$$\begin{aligned} \langle \cdot, \cdot \rangle : M \times N &\to \mathbb{Z} \\ (\chi^u, \lambda^v) &\mapsto u_1 v_1 + \cdots + u_n v_n, \end{aligned}$$

and is a perfect pairing (see [Ful93, pg. 37]).

Let X be a toric variety with torus $T \cong (\mathbb{C}^*)^n$. Consider some point $u \in M \cong \mathbb{Z}^n$. This corresponds to a morphism $\chi^u : T \to \mathbb{C}^*$, and since T is a dense subvariety of X, we can regard χ^u as a rational function on X.

Since X is normal, associated to this rational function is a divisor $\operatorname{div}(\chi^u)$. It is supported on the complement $X \setminus T$, which we can write as the union of a finite set of irreducible divisors, i.e.:

$$X \setminus T = D_1 \cup \ldots \cup D_r.$$

Hence we can write:

$$\operatorname{div}(\chi^u) = \sum_{i=1}^{r} a_i D_i,$$

where the $a_i := \operatorname{ord}_{D_i}(\operatorname{div}(\chi^u)) \in \mathbb{Z}$ are the order of vanishing of χ^u along D_i.

In fact, for each $i = 1, \ldots, r$, there exists a unique element $v_i \in N$ such that $\langle u, v_i \rangle = a_i$. It transpires that these lattice points $v_i \in N$ generate the one-dimensional cones (or *rays*) of the fan Δ corresponding to X. See [Ful93, §3.3] for a proof of this remarkable claim.

Via the torus embedding, a one-parameter subgroup λ^v in T can be interpreted as a mapping of $\mathbb{C}^*$ to X_Δ. To be precise, for $z \in \mathbb{C}^*$, the element of X_Δ corresponding to $\lambda^v(z)$ is the point $\lambda^v(z) \cdot x_0$ on the orbit of the base point x_0. A key question is whether $\lim_{z\to 0} \lambda^v(z)$ exists. This question has a surprisingly concise answer (for a proof, see [Ful93, pg. 38]):

Proposition 2.2.3. *If $v \in \sigma^\circ$ for some $\sigma \in \Delta$, then $\lim_{z\to 0} \lambda^v(z)$ exists and equals the distinguished point x_σ. Otherwise the limit does not exist.*

This result allows us to completely recover the fan Δ from the torus action.

Consider the embedded torus $U_{\{0\}} = T$. It is well known that the coordinate ring $\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is a unique factorisation domain. Hence the *divisor class group*:

$$\mathrm{A}_{n-1}(T) := \mathrm{WDiv}(T)/\mathrm{Div}_0(T),$$

is trivial ([Har77, Proposition II.6.2]). Thus any divisor D on X restricts to a principal divisor on T. This principal divisor may be extended to a principal divisor on X. Subtracting this from D yields a divisor linearly equivalent to D, which is T-invariant. Hence there exists an isomorphism between classes of Weil (or Cartier) divisors on X and classes of T-invariant Weil (or Cartier) divisors. We obtain (see [Ful93, pg. 63] for details):

Proposition 2.2.4. *Let $X := X_\Delta$ be a toric variety, where Δ is a maximum dimensional fan in $N_\mathbb{R}$ (i.e. Δ is not contained in any proper subspace of $N_\mathbb{R}$). Then there is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="] & \mathrm{Div}^T(X) \arrow[r] \arrow[d, hook] & \mathrm{Pic}\, X \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathrm{WDiv}^T(X) \arrow[r] & \mathrm{A}_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

In particular, $\operatorname{rk} \operatorname{Pic} X \leq \operatorname{rk} \mathrm{A}_{n-1}(X) = r - n$*, where $r = |\Delta^{(1)}|$ is the number of rays in the fan Δ.*

If the fan Δ is simplicial (i.e. each cone of Δ is simplicial) then we have equality, and hence $\operatorname{Pic} X \cong \mathbb{Z}^{r-n}$. Note that the Picard group of a toric variety is always torsion free provided that Δ contains at least one maximum dimensional cone, whilst $\mathrm{A}_{n-1}(X)$ can have torsion even if X is simplicial. If X is smooth then $\operatorname{Pic} X = \mathrm{A}_{n-1}(X)$. For more information on calculating the rank of the Picard group from the fan Δ, see [Ewa96, Theorem VII.2.16].

If all maximal cones of Δ are of maximum dimension, then $\operatorname{Pic} X \cong H^2(X, \mathbb{Z})$ (e.g. [Ful93, pg. 64]).

Definition 2.2.5. A function $h : |\Delta| \to \mathbb{R}$ is called a *linear support function* (*of* Δ) if it is linear on each cone $\sigma \in \Delta$ and takes integer values at lattice points.

Proposition 2.2.6. *There is a bijective correspondence between torus invariant Cartier divisors and linear support functions.*

Proof. Let D be a torus invariant Cartier divisor on X_Δ. Then $D\mid_\sigma = \operatorname{div}(\chi^{-u_\sigma})$ for some $u_\sigma \in M \cap \sigma^\perp$. On $U_{\sigma\cap\sigma'}$ we have that $\operatorname{div}(\chi^{-u_\sigma})$ and $\operatorname{div}(\chi^{-u_{\sigma'}})$ agree.

Recall that M is the dual lattice of N. Thus we can regard (by extension) u_σ as linear function from $N_\mathbb{R}$ to $\mathbb{R}$. For each $\sigma \in \Delta$ define $\psi_\sigma : \sigma \to \mathbb{R}$ to be equal to the function u_σ restricted to σ; $\psi_\sigma(v) = \langle u_\sigma, v\rangle$ for all $v \in \sigma$. Then ψ_σ and $h_{\sigma'}$ agree on $\sigma \cap \sigma'$. We write ψ_D for the resulting linear function $\psi_D : |\Delta| \to \mathbb{R}$.

The converse is obvious. Consult, for example, [Ewa96, §VII.4]. $\square$

A torus invariant Cartier divisor $D = \sum_{i=1}^r a_i D_i$ on X_Δ also determines a rational convex polytope in $M_\mathbb{R}$ defined by:

$$P_D := \{u \in M_\mathbb{R} \mid \langle u, v_i\rangle \geq -a_i \text{ for all } i\}, \tag{2.2.1a}$$

$$= \{u \in M_\mathbb{R} \mid u \geq \psi_D \text{ on } |\Delta|\}. \tag{2.2.1b}$$

From the top row in the commutative diagram in Proposition 2.2.4, if $\mathcal{L}$ is a line bundle on X_Δ then there exists a support function ψ such that $\mathcal{L}$ is isomorphic to $\mathcal{O}(D_\psi)$, where D_ψ represents the torus invariant Cartier divisor associated with ψ and is given by:

$$D_\psi = -\sum_{i=1}^r \psi(\rho_i) D_i. \tag{2.2.2}$$

Several key results are connected with the polytope associated with the divisor D_ψ, which we present here. Proofs can be found in any of the standard texts; for example [Ful93, pp. 66–70].

Proposition 2.2.7. *The global sections of the line bundle $\mathcal{O}(D_\psi)$ are given by:*

$$H^0(X_\Delta, \mathcal{O}(D_\psi)) = \bigoplus_{u \in P_\psi \cap M} \mathbb{C} \cdot \chi^u.$$

Sketch proof. A section of $\mathcal{O}(D)$ can be expressed as a rational function f on X_Δ such that $(f) + D \geq 0$. Thus there exists some $u \in M$ such that we can write $\mathrm{div}(\chi^u) + D \geq 0$. It must be that $\langle u, \rho_i\rangle \geq -a_i$ for each (primitive generator of the) ray ρ_i in Δ. Hence we obtain an isomorphism. $\square$

Proposition 2.2.8. *With notation as above, $\mathcal{O}(D_\psi)$ is globally generated if and only if for any $\sigma \in \Delta$ there exists $u_\sigma \in P_\psi$ such that for all $v \in \sigma$, $\langle u_\sigma, v\rangle = \psi(v)$.*

Proposition 2.2.9. *With notation as above, $\mathcal{O}(D_\psi)$ is ample if and only if for any maximal cone $\sigma \in \Delta$ there exists $u_\sigma \in M$ such that:*

(i) $u_\sigma \geq \psi$ *on* $|\Delta|$;

(ii) $\langle u_\sigma, v\rangle = \psi(v)$ *if and only if* $v \in \sigma$.

An immediate consequence of Propositions 2.2.8 and 2.2.9 is that:

Corollary 2.2.10. *If X is a toric variety then every ample line bundle is globally generated.*

Another remarkable fact, perhaps best elucidated in [Dan78, §11], is that:

Proposition 2.2.11. *Let D be a divisor on a toric variety X of dimension n. Then the self-intersection number (D^n) is given by:*

$$(D^n) = n!\mathrm{vol}\, P_D,$$

where the volume is given relative to the lattice M.

2.3 Toric Morphisms

We shall give a summary of the key facts concerning toric morphisms. For a comprehensive treatment, consult [Ewa96, VI.2 and VI.6] or [Ful93, §1].

Definition 2.3.1. Let $\Phi : \mathbb{C}^k \to \mathbb{C}^m$ be a monomial mapping, and let $U_\sigma \hookrightarrow \mathbb{C}^k$ and $U_{\sigma'} \hookrightarrow \mathbb{C}^m$ be affine toric varieties. If $\Phi(U_\sigma) \subset U_{\sigma'}$ then $\varphi := \Phi\,|_{U_\sigma}$ is a morphism called an (*affine*) *toric morphism* from U_σ to $U_{\sigma'}$.

Every toric morphism $\varphi : U_\sigma \to U_{\sigma'}$ uniquely determines a monomial homomorphism $\varphi^* : A_{\sigma'} \to A_\sigma$, and hence a homomorphism of lattices $N \to N'$ which sends the cone σ into the cone σ'. The converse also holds.

Proposition 2.3.2. *For lattice cones σ and σ', the following conditions are equivalent:*

(i) $\sigma \cong \sigma'$;

(ii) $A_\sigma \cong A_{\sigma'}$;

(iii) $U_\sigma \cong U_{\sigma'}$ *(via a toric isomorphism).*

Definition 2.3.3. Let Δ and Δ' be fans in $N_\mathbb{R}$ and $N'_\mathbb{R}$ respectively. Let $\varphi : N \to N'$ be a lattice homomorphism which sends each cone σ in Δ into some cone σ' in Δ'. Then φ is said to induce a *map of fans* from Δ to Δ'.

It is usual to confuse the distinction between the map of fans induced by a lattice homomorphism, and the lattice homomorphism itself. We have the following result:

Proposition 2.3.4. *Let $\varphi : N \to N'$ induce a map of fans from Δ in $N_{\mathbb{R}}$ to Δ' in $N'_{\mathbb{R}}$. Then φ gives rise to a map:*

$$\psi : X_\Delta \to X_{\Delta'},$$

whose restriction $\psi_\sigma := \psi\ |_{U_\sigma}$ to any affine piece U_σ of X_Δ is an affine toric morphism. In particular, this map is continuous with respect to both the complex and the Zariski topology on X_Δ and $X_{\Delta'}$.

Definition 2.3.5. With notation as in Proposition 2.3.4, we call ψ a *toric morphism*.

Toric morphism can be characterised as those morphisms which are equivariant with respect to a suitable homomorphism of the embedded tori. We have the following important result:

Proposition 2.3.6 ([Dem70])**.** *If two complete toric varieties are isomorphic as abstract varieties, then they are isomorphic as toric varieties.*

Remark 2.3.7. Proposition 2.3.6 is a consequence of fact that the automorphism group is a linear algebraic group with maximal torus; *Borel's Theorem* ([Bor91, Corollary 11.3]) tells us that in such a group any two maximal tori are conjugate. An alternative proof can be found in [Ber03, Theorem 4.1]. An analogous result holds for affine toric varieties, however the proof is much harder (see [Dem82, Dan82] or [Gub98]).

Recall the definition of what it means for a morphism to be proper (e.g. [Har77, §II.4]):

Definition 2.3.8. A morphism $\varphi : X \to Y$ is said to be *proper* if for every base change $Y' \to Y$ the resulting morphism $X \times_Y Y' \to Y'$ is closed.

We have the following result:

Proposition 2.3.9. *A toric morphism $\varphi_* : X_\Delta \to X_{\Delta'}$ is proper if and only if $\varphi^{-1}(|\Delta'|) = |\Delta|$.*

Proof. The proof in one direction is immediate from Proposition 2.2.3. For the converse direction, consult [Ful93, §2.4]. □

Example 2.3.10. *We give two important examples of proper morphisms.*

First, let Δ be any fan in $N_{\mathbb{R}}$, and let $N' \subset N$ be a sublattice of the lattice N with finite index. Then the projection of Δ in $N'_{\mathbb{R}}$ defines a fan Δ'. The resulting toric morphism $X_\Delta \to X_{\Delta'}$ is proper.

Second, consider two fans Δ and Δ' in $N_{\mathbb{R}}$ such that $|\Delta| = |\Delta'|$. We call Δ a subdivision *of Δ' if for every cone $\sigma \subset \Delta$ there exists some cone $\sigma' \in \Delta'$ such that $\sigma \subset \sigma'$. The obvious toric morphism $X_\Delta \to X_{\Delta'}$ is proper.*

Definition 2.3.11. A variety X is said to be *complete* if it is proper over $\operatorname{Spec} \mathbb{C}$.

Definition 2.3.12. A fan Δ in $N_{\mathbb{R}}$ is said to be *complete* if $|\Delta| = N_{\mathbb{R}}$.

Proposition 2.3.13. *A toric variety X_Δ is complete if and only if the corresponding fan Δ is complete.*

Proof. Once again, the implication in one direction is immediate from Proposition 2.2.3, since if the fan is complete then every one–parameter subgroup has a limit point. The converse follows readily from Definition 2.3.11 and Proposition 2.3.9, since the one–point space $\{0\}$ is a toric variety, and the map from any toric variety to $\{0\}$ is a toric morphism. For details see any of the standard references; for example [Ewa96, Theorem VI.9.1]. □

When the toric variety X is complete, we have a nice reinterpretation of Propositions 2.2.8 and 2.2.9 in terms of the polytope P_D.

Proposition 2.3.14. *Let X_Δ be a complete toric variety of dimension n. With notation as in Propositions 2.2.8 and 2.2.9:*

(i) *$\mathcal{O}(D)$ is generated by global sections if and only if P_D is the convex hull of the set $\left\{u_\sigma \mid \sigma \in \Delta^{(n)}\right\}$;*

(ii) *$\mathcal{O}(D)$ is ample if and only if $u_\sigma \neq u_{\sigma'}$ for $\sigma \neq \sigma'$ in $\Delta^{(n)}$ and P_D is an n-dimensional polytope with vertices u_σ, $\sigma \in \Delta^{(n)}$.*

Proof. Immediate from [Ful93, pp. 68 and 70]. □

There is a similarly satisfying condition for a toric variety to be projective:

Proposition 2.3.15. *Let X_Δ be a complete toric variety with associated fan Δ in $N_{\mathbb{R}}$. X_Δ is projective if and only if Δ is the set of cones spanned by the faces of a polytope $P \subset N_{\mathbb{R}}$ with vertices in N and the origin in its interior.*

Proof. Immediate from Proposition 2.3.14 and [Ful93, pg. 72]. □

CHAPTER 3

Toric Singularities

3.1 $\mathbb{Q}$-Factorial Varieties

Definition 3.1.1. A Weil divisor D is said to be $\mathbb{Q}$-*Cartier* if there exists some $m \in \mathbb{N}$ such that mD is a Cartier divisor.

Definition 3.1.2. A normal variety X is said to be $\mathbb{Q}$-*factorial* if any Weil divisor of X is $\mathbb{Q}$-Cartier.

$\mathbb{Q}$-factoriality is essentially a local property. Indeed, in [Deb01] such an X would be referred to as being *locally* $\mathbb{Q}$-*factorial*.

The notion of $\mathbb{Q}$-factoriality is important when studying birational morphisms $\pi : Y \to X$. If X is $\mathbb{Q}$-factorial, then every irreducible component of the exceptional locus $\operatorname{Exc} \pi$ of π has codimension one in Y. That is, the exceptional locus is a union of prime divisors. See [Deb01, §1.40].

3.2 Terminal Singularities

When X is a normal singular variety, care is needed over the definition of the canonical divisor K_X. Over the nonsingular locus $X_{\text{reg}} := X \setminus \operatorname{Sing} X$ the sheaf of canonical forms $\Omega^n_{X_{\text{reg}}}$ is a line bundle. Hence there exists a divisor K_X on X_{reg} such that:

$$\Omega^n_{X_{\text{reg}}} \cong \mathcal{O}_{X_{\text{reg}}}(K_X).$$

Since X is normal, $\operatorname{codim}_X \operatorname{Sing} X \geq 2$. Hence the divisor K_X can be regarded as a Weil divisor on the whole of X.

Armed with this definition of K_X, we define:

Definition 3.2.1. A normal variety X of dimension n has *terminal singularities* if the canonical divisor K_X is $\mathbb{Q}$-Cartier, and if there exists a projective birational morphism $f : V \to X$ from a smooth variety V such that in the *ramification formula*:

$$K_V = f^* K_X + \sum a_i E_i,$$

all the $a_i > 0$.

Remark 3.2.2. The E_i of the ramification formula are the *f-exceptional divisors* in V, i.e. the irreducible components of $E := \operatorname{Exc} f$ of codimension one.

The ramification formula can be regarded as a generalisation of the situation when blowing up a point p on a nonsingular projective surface T. If $\mu : S \to T$ is the blowup of T at p we obtain the following relation between the canonical divisors of S and T (see [Har77, Proposition V.3.3]):

$$K_S = \mu^* K_T + E.$$

It is not immediately obvious how to take the pullback of a $\mathbb{Q}$-Cartier divisor, in this case K_X. Instead we take the pullback $f^*(mK_X)$ (where $m > 0$ is such that mK_X is Cartier) and formally divide by m – we calculate:

$$mK_v = f^*(mK_X) + \sum m a_i E_i.$$

For further details on calculating the coefficients a_i in the ramification formula, consult [Mat02, §4] or [Deb01, §7.2].

Definition 3.2.3. Let K_X be a $\mathbb{Q}$-Cartier divisor. Let j_X be the smallest positive integer such that $j_X K_X$ is Cartier. We call j_X the *Gorenstein index* (*of* X). We call the variety X *Gorenstein* if $j_X = 1$.

Remark 3.2.4. If X is Gorenstein, then by definition K_X is Cartier. Since all varieties under consideration are *Cohen–Macaulay* (see [Har77, pp. 184–5] or [Ful93, pg. 30]), this implies that all of the local rings are *Gorenstein* (see [Eis95, §21] or [CK99, Appendix A]). Hence the terminology.

Although not immediately apparent, the choice of desingularisation in Definition 3.2.1 is irrelevant. If $a_i > 0$ holds for all i for one choice of f, then it holds for all choices. For a proof of this fact, see [Deb01, pg. 178]. Hence we can rewrite Definition 3.2.1 as:

Definition 3.2.5. A normal variety X of dimension n has *terminal singularities* if the canonical divisor K_X is $\mathbb{Q}$-Cartier, and if for any projective birational morphism $f : V \to X$ from a smooth variety V we have that the coefficients a_i of the ramification formula are all positive.

Suppose that X has only terminal singularities. Let $\{U_\alpha\}$ be a covering of X by Zariski open subsets. Taking:

$$f_\alpha = f\,|_{U_\alpha} : f^{-1}(U_\alpha) \to U_\alpha,$$

we see that each U_α has only terminal singularities.

Conversely, suppose that each U_α has only terminal singularities. Let $f_\alpha : V_\alpha \to U_\alpha$ be a projective birational morphism as in Definition 3.2.5. Let $g : W \to X$ be any projective birational morphism from a nonsingular variety W. Then each:

$$g_\alpha = g\,|_{U_\alpha} : g^{-1}(U_\alpha) \to U_\alpha,$$

satisfied the conditions of Definition 3.2.5, since the resolution $f_\alpha : V_\alpha \to U_\alpha$ does. Hence so does $g : W \to X$. Thus X has only terminal singularities.

Thus we have seen that the notion of terminal singularities is essentially a local property.

In dimension two, a consequence of *Castelnuovo's Contractibility Criterion* ([Mat02, Theorem 1.1.6]) is that a normal surface has only terminal singularities if and only if it is smooth ([Mat02, Theorem 4.6.5]). We can see an instance of this in Corollary 5.1.3.

A general hyperplane section of a projective variety X with terminal singularities has terminal singularities, as does the general fibre of a morphism $X \to Y$ ([Rei80, Theorem 1.13]).

Terminal singularities play an important role in birational geometry (Section 3.4). A great deal of classification results exist in various cases; for example the results of [Mor82, Rei83b, MS84, MMM88]. In [Mor85b] it was shown that, with two exceptions, isolated canonical cyclic quotient singularities in dimension three are all either Gorenstein or terminal. Mori [Mor85a] and Reid [Rei87, Chapter II] addressed the issue of classifying threefold terminal singularities. Part of this work is the classification in the toric case – the *Terminal Lemma* (see [Rei87, pg. 380], [MS84, Theorem 2.3], or [Bor00, Theorem 3.1]):

Theorem 3.2.6. *Every three-dimensional $\mathbb{Q}$-factorial terminal toric singularity is isomorphic to a quotient of $\mathbb{A}^3$ by a group μ_r which acts linearly with weights $\frac{1}{r}(1, a, -a)$ for some $r \in \mathbb{N}$ and $a \in \mathbb{Z}/r\mathbb{Z}$, with $\gcd\{a, r\} = 1$.*

Here μ_r is the group of r-th roots of unity. The notation $\frac{1}{r}(1, a, -a)$ means that $\xi \in \mu_r$ multiplies the first coordinate by ξ, the second coordinate by ξ^a, and the third by ξ^{r-a}. The proof of Theorem 3.2.6 relies on Proposition 6.4.3.

Less is known in dimension four. For a good summary, consult [Bor00, Bor05]. Abelian quotient singularities that are both terminal and Gorenstein were classified in [MS84]. Terminal quotient singularities of type $\mathbb{Z}/p\mathbb{Z}$, where p is prime, have been classified up to possibly finitely many exceptions [MMM88, San90, Bor99]. There exist a large, but finite, number of "sporadic" terminal singularities, one three-parameter series, two two-parameter series, and 29 exceptional stable quintuples.

A higher dimensional analogue to Theorem 3.2.6 has been conjectured in [Bor05, Conjecture 2].

3.3 Canonical Singularities

The definition of a canonical singularity is given by loosening the condition on terminal singularities.

Definition 3.3.1. A normal variety X of dimension n has *canonical singularities* if the canonical divisor K_X is $\mathbb{Q}$-Cartier, and if there exists a projective birational morphism $f : V \to X$ from a smooth variety V such that in the ramification formula:

$$K_v = f^* K_X + \sum a_i E_i,$$

all the $a_i \geq 0$.

As remarked upon in the terminal case, the choice of desingularisation does not affect whether the coefficients a_i are non-negative. It is thus possible to reformulate Definition 3.3.1 in the style of Definition 3.2.5. It is also easy to see that the notion of canonical singularities is a local one, just as in the terminal case.

Remark 3.3.2. It should be noted that if for some f we have that $a_i = 0$ for all exceptional divisors, f is called a *crepant resolution*. This is a particular property of f, and does not imply that all the $a_i = 0$ in general.

In dimension two, canonical singularities are rational double points, also called Du Val singularities [DV34, Art66].

In dimension three, [Mor85b] classified the isolated cyclic quotient singularities which are canonical. A complete classification of all three-dimensional canonical toric singularities is presented in [II87, Theorem 4.1]. This latter classification includes those which are not $\mathbb{Q}$-factorial.

3.4 Why Are These Singularities Interesting?

It is perfectly natural to ask why we have chosen to study terminal and canonical singularities. What are the properties that make these objects worth investigating?

The importance of the notion of terminal singularities lies chiefly in the fact that, together with $\mathbb{Q}$-factoriality, it characterises the singularities of the objects in the smallest category that contains the category of nonsingular projective varieties and in which the Minimal Model Program (or Mori Program) operates (see [Mat02, Theorem 4.1.3] or [Deb01, Proposition 7.44]).

Some kind of singularities must be allowed in the Minimal Model Program – for example, a divisorial contraction on a smooth projective variety may lead to terminal singularities. The singularities allowed must be stable under flips and divisorial contractions, and terminal singularities satisfy this requirement. It is inappropriate to go into the details of the Minimal Model Program here; for details consult the excellent books [Mat02, Deb01, CKM88], and the papers [Wiś02, FS04, Fuj03]. It suffices to say that terminal singularities are completely natural objects to study in this context.

Canonical singularities, as we have seen, arise from a slight relaxation of the definition of terminal singularity. Their importance lies in the fact that they characterise the singularities of the canonical models of varieties of general type whose canonical rings are finitely generated $\mathbb{C}$-algebras.

To every smooth projective variety X we associate the *canonical ring*:

$$R(X, K_X) := \bigoplus_{m \geq 0} H^0(X, mK_X).$$

This is a birational invariant ([Deb01, §7.1]). Assuming that this is a finitely generated algebra, we construct a projective variety:

$$X_{\mathrm{can}} := \operatorname{Proj} R(X, K_X).$$

This variety depends only upon the birational equivalence class of X, and is called the *canonical model* of X.

Definition 3.4.1. Let X be a projective variety of dimension n. A Cartier divisor D on X is *big* if:

$$\liminf_{m \to +\infty} \frac{h^0(X, mD)}{m^n} > 0.$$

A $\mathbb{Q}$-Cartier D is *big* if there exists some positive integer m such that mD is Cartier and big.

Definition 3.4.2. Let X be a complete variety. A $\mathbb{Q}$-Cartier divisor D on X is *nef* if:

$$(D \cdot C) \geq 0,$$

for all irreducible curves $C \subset X$.

We say that a smooth projective variety X is *of general type* if the canonical divisor K_X is big. Ample divisors are big, and the condition that a divisor is both big and nef can be seen as a generalisation of ampleness – many vanishing theorems which hold for ample divisors hold for nef and big divisors. (See [Deb01, §1.29–1.30].) Thus varieties of general type are of considerable interest.

It transpires that singularities occur on the canonical models X_{can} of varieties X of general type, and those singularities are precisely canonical singularities. See [Rei87, Rei80] for a proof of this statement. This idea allows a reformulation of what it means for a variety to possess canonical singularities: A variety X with ample canonical divisor K_X has canonical singularities if and only if it is the canonical model of a nonsingular projective variety V of general type whose canonical ring is a finitely generated algebra.

3.5 A Hierarchy of Singularities

Let X be a normal variety. If X is smooth we can regard X as possessing terminal singularities. If X has terminal singularities then we can regard X as possessing canonical singularities. This hierarchy can be extended by further relaxing the restrictions on the a_i in the ramification formula.

Definition 3.5.1. Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. The *discrepancy* of X is the minimum of the a_i in the ramification formula, for all possible desingularisations f of X.

The discrepancy can be computed from any resolution f whose exceptional locus has normal crossings (these resolutions exist when working in characteristic zero by *Hironaka's Theorem* [Deb01, §7.22]) by the formula [Deb03, pg. 94]:

$$\operatorname{discrep} X = \begin{cases} \min\{1, a_i\}, & \text{if the } a_i \text{ are all} \geq -1 \\ -\infty, & \text{otherwise.} \end{cases}$$

X has terminal (resp. canonical) singularities if $\operatorname{discrep} X > 0$ (resp. $\operatorname{discrep} X \geq 0$). If $\operatorname{discrep} X = 1$ then X is smooth. This suggests a systematic way of extending the hierarchy of singularities:

Definition 3.5.2. Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. We say X has *log terminal* (resp. *log canonical*) singularities if $\operatorname{discrep} X > -1$ (resp. $\operatorname{discrep} X \geq -1$).

Remark 3.5.3. There are many variations on the definition of log terminal (*lt*) singularities. These include *divisorially log terminal* (*dlt*) and *weakly Kawamata log terminal* (*wklt*) singularities. If X is $\mathbb{Q}$-factorial then the three notions of lt, dlt, and wklt coincide. In general we have:

$$\text{dlt} \iff \text{wklt} \implies \text{lt}.$$

See [Mat02, §4.3] for the definitions and a proof of these claims.

Definition 3.5.4. A normal projective variety X with log terminal singularities such that the anticanonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor is said to be *Fano*.

Remark 3.5.5. The definition of what it means for a variety X to be Fano varies in the literature, particularly in the toric literature – sometimes it is necessary for X to be smooth (e.g. [Bat81, Bat99, Ewa96, Sat99, Wiś02]). Care must be taken to check the definition being used.

A smooth Fano surface is usually called a *del Pezzo surface*. These have been classified (see [Har77, Remark V.4.7.1]) – they are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^2$ blown up in at most eight points (in general position).

Smooth Fano threefolds have also been classified. There are seventeen families with Picard number one, and eighty-nine other families ([Isk79b, Isk79a, MU83, Šok79, Cut89, Tak89, MM04]). There are eighteen smooth toric Fano threefolds (see [Bat81, WW82]), and 128 smooth toric Fano fourfolds ([Bat99]).

In [KMM92] it was shown that the degree $(-K_X)^n$ of any smooth Fano variety X of dimension n is bounded. If we restrict our attention to those X with Picard number one, we obtain the explicit bound [Deb03, Corollary 20]:

$$(-K_X)^n \leq (n(n+1))^n.$$

Also in the case of smooth Fano varieties, the number of deformation types of dimension n is known to be bounded ([Deb03, Theorem 21]) by:

$$(n+2)^{(n+2)^{n2^{3n}}}.$$

For a general Fano variety it is not known whether the number of deformation types remains bounded, but this is conjectured to be the case. A step towards proving this can be found in [McK02]. We shall see later that this is true in the toric case (c.f. Theorem 3.6.10).

3.6 Toric Singularities

We now wish to give toric descriptions of the singularities introduced above, beginning with combinatorial interpretations of smoothness and of Definition 3.1.2. Because of the vital importance of these results, outline proofs have been included.

Proposition 3.6.1. *A toric variety X_Δ, with associated fan Δ in $N_\mathbb{R}$, is smooth of and only if each cone $\sigma \in \Delta$ is regular.*

Outline proof. This is a very well known result. See, for example, [Ful93, §2.1].

Since the condition is local, we need only prove the statement for any cone $\sigma \in \Delta$ and the corresponding affine variety U_σ. One direction is easy. Let $n = \dim N$ and let $\dim \sigma = k \leq n$. Since σ is regular, so is its dual cone $\sigma^\vee$ in $M_\mathbb{R}$. Thus we see that $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, which is smooth.

For the converse, let us first suppose that $k = n$. Let $\mathfrak{m}$ be the maximal ideal generated by $\sigma^\vee \cap M$ (i.e. $\mathfrak{m}$ corresponds to the distinguished point x_σ). Note that the primitive generators ρ_i of the rays of $\sigma^\vee$ represent irreducible elements in $\mathfrak{m}/\mathfrak{m}^2$. Since U_σ is assumed to be smooth and of dimension n, so $n = \dim \mathfrak{m}/\mathfrak{m}^2$. Hence $\sigma^\vee$ has at most n rays, and so must possess exactly n rays, and the corresponding ρ_i generate $\sigma^\vee \cap M$. Hence they must form a basis for M, and so $\sigma^\vee$ is regular. Hence σ is regular.

Finally we deal with the case when $\dim \sigma = k < n$. This is handled by considering the sublattice $N' \subset N$ generated by the cone σ in N. Let $N(\sigma) = N/N'$ be the quotient lattice. By considering the maps:

$$\begin{array}{c} N' \to N \to N(\sigma) \\ \sigma' \mapsto \sigma \mapsto \{0\}, \end{array}$$

where σ' is the image of σ in N', one obtains the corresponding fibre bundle:

$$U_{\sigma'} \to U_\sigma \to T_{N(\sigma)}.$$

If U_σ is nonsingular then $U_{\sigma'}$ is nonsingular, and the preceding paragraph applies to σ'. □

Proposition 3.6.2. *A toric variety X_Δ, with associated fan Δ in $N_\mathbb{R}$, is $\mathbb{Q}$-factorial if and only if each cone $\sigma \in \Delta$ is simplicial.*

Outline proof. See [Mat02, Lemma 14.1.1] for a self-contained proof. See also [Oda78, pg. 27] and [Dai02].

Once again, since the condition is local, we need only prove the statement for any cone $\sigma \in \Delta$ and the corresponding affine variety U_σ. Suppose first that σ is not simplicial. By considering the one-skeleton of σ we observe that there exist two rays ρ_1 and ρ_2 such that no two-dimensional face of σ contains both ρ_i. The ρ_i correspond to irreducible divisors D_1 and D_2 whose orbit decompositions are given by ([Ful93, pg. 54]):

$$D_i = \bigcup_{\substack{\rho_i \prec \tau \\ \tau \in \Delta}} O(\tau).$$

Thus we obtain:

$$D_1 \cap D_2 = \bigcup_{\substack{\rho_1 + \rho_2 \subset \tau \\ \tau \in \Delta}} O(\tau).$$

Now $\rho_1 + \rho_2 \subset \sigma$, hence $D_1 \cap D_2$ is not empty. Because we insisted that no two-dimensional face of σ contains both ρ_1 and ρ_2, so it must be that any τ is of at least dimension three. Hence $D_1 \cap D_2$ is of codimension at least three. If U_σ were $\mathbb{Q}$-factorial, then some multiple of the D_i is locally factorial. By *Krull's Principal Ideal Theorem* [Eis95, §8.2.2], codim $D_1 \cap D_2 = 2$. Thus U_σ cannot be $\mathbb{Q}$-factorial.

Suppose now that $\sigma \in \Delta$ is simplicial, with generators $v_1, \ldots, v_n \in N$ (we may assume without loss of generality that $\dim \sigma = \dim N$). We shall show that U_σ is $\mathbb{Q}$-factorial (see [Ful93, pg. 62]). Let $N' \subset N$ be the lattice generated by the v_i, and let σ' be the image of σ in $N'_{\mathbb{R}}$. The inclusion of lattices $N' \to N$ corresponds to a finite morphism $\phi : U_{\sigma'} \to U_\sigma$. Observe that $U_{\sigma'}$ is nonsingular by Proposition 3.6.1. Let $D \subset U_\sigma$ be a divisor on U_σ, and let f be such that $\phi^* D = \operatorname{div} f$ on $U_{\sigma'}$. Then, locally:

$$\deg \phi \cdot D = \phi_*(\phi^* D) = \operatorname{div}(\operatorname{Norm} f).$$

Hence U_σ is $\mathbb{Q}$-factorial. □

Remark 3.6.3. We reiterate the observation proceeding Proposition 2.2.4. If X_Δ is $\mathbb{Q}$-factorial then the Picard group of X_Δ is free abelian of rank ([Ful93, pg. 65]):

$$\rho_X = \left| \left\{ \rho \mid \rho \in \Delta^{(1)} \right\} \right| - \dim N.$$

We now present a combinatorial condition for the canonical divisor of a toric variety to be $\mathbb{Q}$-Cartier, and, when this is the case, how to calculate the discrepancy. A proof can be found in [Deb03, Proposition 12].

Proposition 3.6.4. *Let $X := X_\Delta$ be the toric variety associated with the fan Δ in $N_{\mathbb{R}}$. Then:*

(i) The canonical divisor K_X is a $\mathbb{Q}$-Cartier divisor if and only if for each maximal cone $\sigma \in \Delta$, spanned by primitive lattice elements $\rho_1, \ldots, \rho_k$, there exists an element $u_\sigma \in M_{\mathbb{Q}}$ such that $u_\sigma(\rho_i) = 1$ for all i.

(ii) If K_X is $\mathbb{Q}$-Cartier, then X has log terminal singularities and:

$$\operatorname{discrep} X = -1 + \min \left\{ u_\sigma(v) \middle| \sigma \in \Delta \textit{ is maximal}, v \in \sigma \cap N \setminus \{0, \rho_1, \ldots, \rho_k\} \right\}.$$

Remark 3.6.5. Let X be the projective toric variety associated with a polytope P with vertices in N (see Proposition 2.3.15). By (i) of Proposition 3.6.4 the divisor K_X is $\mathbb{Q}$-Cartier if and only if, for each facet of P, the primitive generators ρ_i (of the rays of Δ passing through the vertices of the facet) lie on an affine hyperplane.

The following result is immediate (see for example [Mat02, Proposition 14.3.1], [Rei83a, pg. 401], or [Rei80, pg. 294]).

Corollary 3.6.6. *Let X_Δ be a projective toric variety with associated fan Δ in $N_\mathbb{R}$. Suppose that K_X is $\mathbb{Q}$-Cartier. Then:*

(i) X_Δ has terminal singularities if and only if $N \cap \sigma \cap \{v \in N_\mathbb{R} \mid u_\sigma(v) \leq 1\} = \{0\} \cup \{\rho \in N \mid \rho \text{ is primitive}, \mathbb{R}_{\geq 0}\rho \prec \sigma\}$ for all $\sigma \in \Delta$.

(ii) X_Δ has canonical singularities if and only if $N \cap \sigma \cap \{v \in N_\mathbb{R} \mid u_\sigma(v) < 1\} = \{0\}$ for all $\sigma \in \Delta$.

We are in a position to give a combinatorial interpretation of Definition 3.5.4.

Proposition 3.6.7. *Let X be a projective toric variety with associated polytope $P \subset N_\mathbb{R}$. X is a Fano variety if and only if the vertices of P are primitive elements of N.*

Outline proof. Proposition 3.6.4 tells us that, provided K_X is $\mathbb{Q}$-Cartier, X has log terminal singularities. Thus we need not worry about this condition.

Let X be a projective variety whose rays are generated by the ρ_i. Associated with each ray is an irreducible torus-invariant divisor D_i, and the anticanonical divisor $-K_X$ is given by (e.g. [Ful93, pg. 85]):

$$-K_X = \sum_i D_i.$$

Suppose that X is Fano. Then $-K_X$ is $\mathbb{Q}$-Cartier, hence some multiple, $-mK_X$ say, is Cartier. Equation (2.2.2) tells us that $\phi_{-mK}(\rho) = -m$ for each generator of the rays in Δ. From Proposition 2.2.9 we see that there exists a convex lattice polytope in $N_\mathbb{R}$ whose faces generate Δ, and whose vertices are given by the ρ_i – namely $\{v \in N_\mathbb{R} \mid \phi_{-mK}(v) = -m\}$.

Conversely by Remark 3.6.5 we see that K_X is $\mathbb{Q}$-Cartier. From equation (2.2.1a) we see that $P_{-K} = P^\vee$. But $P^\vee$ is a convex polytope with vertices in $M_\mathbb{Q}$, and so some multiple of $P^\vee$ is a convex lattice polytope in M. Hence we see that $-K_X$ is $\mathbb{Q}$-Cartier and (by [Ful93, pg. 72]) ample. $\square$

Remark 3.6.8. Strictly speaking the result given in [Ful93, pg. 85] is for *smooth* toric varieties. One simply needs to take an appropriate disingularisation of X. Let Δ be the fan associated with X, and let Δ' be a regular stellar subdivision of Δ (see [Ewa96, §V.6 and Theorem VI.8.5]). Then $f : X_{\Delta'} \to X$ is a resolution of singularities, and:

$$-K_X = f_*(-K_{X_{\Delta'}}) = f_*\Big(\sum_{\rho\in\Delta'^{(1)}} V(\rho)\Big) = \sum_{\rho\in\Delta^{(1)}} V(\rho).$$

Suppose that X is a projective toric variety such that K_X is $\mathbb{Q}$-Cartier. Choose any facet of the associated polytope P, and let σ be the cone spanned by that facet. Let u_σ be the equation of the hyperplane containing the facet (notation as in Proposition 3.6.4). Then for any positive integer k we have that $\operatorname{discrep} X \geq -1 + 1/k$ only if $u_\sigma(v) \geq 1/k$ for all $v \in \sigma \cap N \setminus \{0\}$. Since for a Fano toric variety the vertices of P are primitive (Proposition 3.6.7), we obtain:

Lemma 3.6.9. *Let X be a Fano toric variety with associated polytope $P \subset N_\mathbb{R}$. Let $k > 0$ and suppose that* $\operatorname{discrep} X \geq -1 + 1/k$. *Then $P^\circ \cap kN = \{0\}$.*

Given a bound on the discrepancy, a very general finiteness result was shown in [BB92]:

Theorem 3.6.10 ([BB92])**.** *Given positive integers n and k there are only finitely many (up to isomorphism) toric Fano varieties of dimension n and discrepancy $\geq -1 + 1/k$.*

In fact in [Bor00, §2] it was observed that Theorem 3.6.10 can be deduced from Lemma 3.6.9 and a combinatorial result of D. Hensley:

Theorem 3.6.11 ([Hen83])**.** *Given positive integer n and $\varepsilon > 0$ there are only finitely many (up to the action of $GL(n,\mathbb{Z})$) convex lattice polytopes P of dimension n such that $(\varepsilon P) \cap \mathbb{Z}^n = \{0\}$.*

3.7 Fano Polytopes

In light of the results of Section 3.6, the following definitions are natural:

Definition 3.7.1. Let $P \subset N_\mathbb{R}$ be an n-dimensional convex lattice polytope containing the origin as an interior lattice point, such that the vertices of P are all primitive elements of N. Then we call P a *Fano polytope* (c.f. Proposition 3.6.7).

Definition 3.7.2. Let $P \subset N_\mathbb{R}$ be an n-dimensional Fano polytope. Then:

(i) If the vertices of any facet of P form a $\mathbb{Z}$-basis of N, we call P a *regular* (or *smooth*) Fano polytope (c.f. Proposition 3.6.1);

(ii) If every facet of P is an $(n-1)$-simplex, we call P *simplicial* (or $\mathbb{Q}$-*factorial*) Fano polytope (c.f. Proposition 3.6.2);

(iii) If the dual polytope $P^\vee \subset M_\mathbb{R}$ is also a lattice polytope, P is called a *reflexive* (or *Gorenstein*) Fano polytope (c.f. Definition 3.2.3);

(iv) If the only lattice points on or in P consist of the vertices and the origin we call P a *terminal* Fano polytope (c.f. Corollary 3.6.6, (i));

(v) If the only interior lattice point of P is the origin, we say that P is *canonical* (c.f. Corollary 3.6.6, (ii)).

Remark 3.7.3. There clearly exists a correspondence between isomorphism classes of Fano varieties and isomorphism classes of Fano polytopes. More systematic notation was introduced in [Bor00, §2], which includes log terminal and log canonical Fano varieties in the labelling. For our purposes this is unnecessary.

3.8 Calculating the Dual Polytope

Let $P \subset N_\mathbb{R}$ be an n-dimensional Fano polytope. We represent P by an $n \times k$ matrix, where k is equal to the number of vertices of P. Each column of the matrix gives the coordinate of a vertex of P. Of course this matrix is unique only up to permutation of the order of the vertices. We denote the matrix by P, and regard the matrix representation as synonymous with the polytope.

Let $P^\vee := \{u \in M_\mathbb{R} \mid u(v) \geq -1 \text{ for all } v \in P\}$ be the dual polytope to P. We shall show how to determine the vertices of $P^\vee$.

Let $\{F_i\}_{i\in I}$ be the set of $(n-1)$-dimensional faces of P, where I is some (finite) indexing set. For each face F_i, let the matrix F_i be given by any affine independent subset of n vertices of F_i such that those vertices span the hyperplane containing F_i. Write c_{ij} for the j-th column of F_i, $j = 1, \ldots, n$. Then the vertex u_i of $P^\vee$ dual to the face F_i is uniquely determined by:

$$u_i(c_{ij}^t) = -1, \qquad \text{for } j = 1, \ldots, n. \tag{3.8.1}$$

We have that F_i is non-degenerate, hence there exists an inverse matrix F_i^{-1}. Let a_{ij} denote the rows of F_i^{-1}. Then, for each $j \in \{1, \ldots, n\}$:

$$a_{ij}^t(c_{il}) = \begin{cases} 1, & \text{if } l = j; \\ 0, & \text{otherwise.} \end{cases}$$

Setting $u_i := -\sum_{j=1}^n a_{ij}^t$ we see that condition (3.8.1) is satisfied. Hence:

$$P^\vee = \operatorname{conv}\{u_i \mid i \in I\}.$$

Since P is Fano, so $-K_{X(P)}$ is ample. By Proposition 2.3.14 we see that the u_i are precisely the vertices of $P_{-K} = P^\vee$.

Example: The Degree of $\mathbb{P}^n$

By considering the dual of the polytope P associated with $\mathbb{P}^n$ we shall give a 'toric' proof that the degree of $\mathbb{P}^n$ is $(n+1)^n$. In the course of this calculation, an explicit description of $P^\vee$ and its volume will be given, and mult $P^\vee$ will be calculated.

Remark 3.8.1. In order to simplify the calculations, results from Chapter 4 will be used. This example is perhaps best delayed until after reading the material in that chapter. In particular Proposition 4.4.10 shall be used implicitly, and Proposition 4.2.5 and equation (4.4.3) shall be used explicitly.

The polytope P has vertices given by the standard basis $e_1, \ldots, e_n$, along with the point $-e_1 - \ldots - e_n$ (see [Ful93, pg. 22]). Since P is a simplex, there are precisely $n+1$ faces.

Any face of P either contains $-e_1 - \ldots - e_n$, or it doesn't. In the latter case there is only one face, whose associated matrix is given by the identity matrix. Hence the corresponding dual vertex is $-e_1 - \ldots - e_n$. In the former case the matrix is given, up to reordering of the basis, by:

$$\left(\begin{array}{ccc:c} & & & -1 \\ & I_{n-1} & & \vdots \\ & & & -1 \\ \hdashline 0 & \ldots & 0 & -1 \end{array}\right),$$

where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. This matrix is easily seen to be self-inversing, and gives the point $-e_1 - \ldots - e_{n-1} + ne_n$.

We see that the dual polytope $P^\vee$ is given by:

$$\left(\begin{array}{cccc|c} n & -1 & \dots & -1 & -1 \\ -1 & n & \ddots & \vdots & \vdots \\ \vdots & \ddots & & -1 & \\ -1 & \dots & -1 & n & -1 \\ \hline -1 & & \dots & -1 & -1 \end{array}\right). \tag{3.8.2}$$

We now wish to calculate the volume of (3.8.2). In light of equation (4.4.3) it is sufficient to calculate $\operatorname{mult} P^\vee$. This can be obtained by considering the determinant of any face of $P^\vee$. We shall consider the face given by the first n columns of (3.8.2).

By performing suitable matrix operations we see that the determinant is given by:

$$\det\left(\begin{array}{c|ccc} 1 & -1 & \dots & -1 \\ \hline 0 & & & \\ \vdots & & (n+1)I_{n-1} & \\ 0 & & & \end{array}\right).$$

This is readily seen to be $(n+1)^{n-1}$. Hence $\operatorname{mult} P^\vee = (n+1)^{n-1}$. In particular, by Proposition 4.2.5 and (4.4.3):

$$\operatorname{vol} P^\vee = \frac{(n+1)^n}{n!}.$$

By Proposition 2.2.11, $(-K_{\mathbb{P}^n})^n = (n+1)^n$.

3.9 The Gorenstein Condition

A particularly interesting class of toric Fano varieties are those which are Gorenstein (see Definition 3.2.3). These varieties have attracted the attention of numerous researchers (e.g. [Bat94, KS97, Con02, Cas03b, Cas04, Nil05]).

From Proposition 3.6.7 we see that Gorenstein toric Fano varieties are in bijective correspondence with Fano polytopes whose duals are also lattice polytopes (since the anticanonical divisor is Cartier). We call such polytopes *reflexive* (Definition 3.7.2).

Remark 3.9.1. Gorenstein toric Fano varieties necessarily include those toric Fano varieties which are smooth. In fact it transpires (see [Nil05, Proposition 1.2]) that Gorenstein toric Fano varieties possess at worst canonical singularities (i.e. $P^\circ \cap N = \{0\}$). Hence, by Theorem 3.6.10, the number of isomorphism classes of Gorenstein toric

Fano varieties (of fixed dimension) is finite.

Many equivalent descriptions exist for the conditions in Definition 3.7.2 (for example the list in [HM04]). Here we present one of the more interesting variations:

Proposition 3.9.2. *Let $P \subset N_{\mathbb{R}}$ be an n-dimensional Fano polytope. P is reflexive if and only if:*

$$\operatorname{vol} P = \frac{\operatorname{vol} \partial P}{n},$$

where $\operatorname{vol} \partial P$ denotes the surface area of P normalised with respect to the sublattice containing each facet of P.

Proof. Let F be a facet of P. Then there exists a vertex u of $P^\vee \subset M_{\mathbb{R}}$ such that $u(v) = -1$ for all $v \in P$. Let $k \in \mathbb{Z}_{>0}$ be the smallest integer such that $ku \in M$. Clearly P is reflexive if and only if $k = 1$ for all facets F of P. k is the *height* of the facet F.

Consider the simplex $S_F := \operatorname{conv}(F \cup \{0\})$. This has volume:

$$\operatorname{vol} S_F = \frac{k}{n} \operatorname{vol} F.$$

The result follows by observing that:

$$\operatorname{vol} P = \sum_{F \text{ a facet of } P} \operatorname{vol} S_F.$$

□

We shall meet this result again in Section 10.3.

Remark 3.9.3. The proof of Proposition 3.9.2 actually proves a slightly stronger result, namely that for any n-dimensional Fano polytope $P \subset N_{\mathbb{R}}$,

$$\operatorname{vol} P \leq \frac{\operatorname{vol} \partial P}{n},$$

with equality if and only if P is reflexive.

Suppose that P is a reflexive polytope. Then so is $P^\vee$. Hence we have two Gorenstein toric Fano varieties which are, in some sense, "mirror" to each other – namely $X(P)$ and $X(P^\vee)$. This pairing has important connections with *Mirror Symmetry* (the duality between Calabi–Yau threefolds discovered by physicists). The connection between toric geometry and Mirror Symmetry was first described in [Bat94]. See the superb exposition [CK99], or the survey paper [Gan00].

For low dimensions, it is not unreasonable to attempt to classify all reflexive polytopes. Clearly for dimension one there is only one possibility, the interval $[-1, 1]$ associated with $\mathbb{P}^1$. In dimension two it can easily be observed that the sixteen canonical polygons (see Chapter 5) are reflexive. Indeed, without this fact Theorem 7.1.1 would be meaningless. Corollary 5.1.3 gives a non-exhaustive proof of the fact that any canonical toric Fano surface is Gorenstein.

A classification of all terminal Gorenstein toric Fano threefolds emerges as a by-product of Chapter 6. There are exactly 100 isomorphism classes (see Theorem 6.1.1). This result was proved independently in [Nil05, Theorem 4.2].

In [KS97] a general approach to classifying reflexive polytopes is outlined. The technique is not too dissimilar to that described in Chapter 8. This approach has proved efficient enough to allow a complete classification of Gorenstein toric Fano varieties in dimensions three and four. In dimension three ([KS98]) a total of 4319 polytopes were discovered. In dimension four ([KS00]) the figure is an astonishing 473,800,776 reflexive polytopes.

Primitive Collections

One of the most successful approaches to classifying reflexive polytopes comes from studying "primitive collections". Originally introduced by Batyrev ([Bat91]) whilst investigating smooth projective toric varieties, they have become a standard tool; for example [Hoş98, CK99, Sat00, Cas03a, Nil05].

Definition 3.9.4. Let Δ be a complete fan in $N_{\mathbb{R}}$. A subset $\{\rho_1, \ldots, \rho_k\} \subset \Delta^{(1)}$ consisting of rays of Δ is called a *primitive collection* if:

(i) For any $\sigma \in \Delta$, $\mathrm{cone}\{\rho_1, \ldots, \rho_k\} \not\subseteq \sigma$;

(ii) For each $i = 1, \ldots, k$ there exists some $\sigma \in \Delta$ such that $\mathrm{cone}\{\rho_1, \ldots, \widehat{\rho_i}, \ldots, \rho_k\} \subset \sigma$.

Primitive collections, and the associated notion of *primitive relations* (see [Bat91, Definition 2.8]), capture a great deal of the combinatorial structure of a fan.

In [Bat93] Batyrev demonstrated how to define the "quantum cohomology ring" of a smooth projective toric variety in terms of primitive collections. When the variety is Fano, this ring agrees with the "small quantum cohomology ring" defined using the Gromov–Witten invariants. An approachable summary can be found in [CK99, §8.1].

In [Nil05] a generalisation of the notion of primitive collection was used to analyse lattice points on the boundary of reflexive polytopes. In particular, collections of order two (i.e. for which $k = 2$ in Definition 3.9.4) were found to give a meaningful generalisation. From this, the following important result was obtained:

Theorem 3.9.5 ([Nil05, Proposition 3.1])**.** *Let $P \subset N_{\mathbb{R}}$ be a reflexive polytope, and let $x, y \in \partial P \cap N$ be two distinct lattice points on its boundary. Then precisely one of the following holds:*

(i) *x and y lie in a common face of P;*

(ii) *$x + y = 0$;*

(iii) *$x + y \in \partial P$.*

Using this result it is possible to prove that only certain combinatorial types can be realised as reflexive polytopes ([Nil05, Corollary 3.4]).

CHAPTER 4

$\mathbb{Q}$-Factorial Toric Fano Varieties with Picard Number One

4.1 Weighted Projective Space

Let $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ be a finite set of positive integers. Let $S(\lambda_0, \lambda_1, \ldots, \lambda_n)$ be the polynomial algebra $k[X_0, \ldots, X_n]$ over a field k (for our purposes $k = \mathbb{C}$), graded by the condition:

$$\deg X_i := \lambda_i, \qquad \text{for } i = 0, \ldots, n.$$

Definition 4.1.1. The projective variety $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n) := \operatorname{Proj}(S(\lambda_0, \lambda_1, \ldots, \lambda_n))$ is called *weighted projective space* (*of type* $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$).

We write $\mathbb{P}^n$ for $\mathbb{P}(1, 1, \ldots, 1)$, which is the usual *projective space of dimension* n.

Perhaps the most comprehensive study of weighted projective space is to be found in [Dol82]. The following two lemmas are taken from this source.

Lemma 4.1.2. *Let $a \in \mathbb{Z}_{>0}$ be a positive integer. Then:*

$$\mathbb{P}(a\lambda_0, a\lambda_1, \ldots, a\lambda_n) \cong \mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n).$$

In light of this lemma, we may assume from now on that:

$$\gcd\{\lambda_0, \lambda_1, \ldots, \lambda_n\} = 1.$$

Lemma 4.1.3. *Suppose that* $\gcd\{\lambda_0, \lambda_1, \ldots, \lambda_n\} = 1$ *and that d is a common factor of λ_i for all $i \neq j$. Then:*

$$\mathbb{P}\Big(\frac{\lambda_0}{d}, \ldots, \frac{\lambda_{j-1}}{d}, \lambda_j, \frac{\lambda_{j+1}}{d}, \ldots, \frac{\lambda_n}{d}\Big) \cong \mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n).$$

An immediate consequence of Lemma 4.1.3 is the following:

Corollary 4.1.4. $\mathbb{P}(\lambda_0, \lambda_1) \cong \mathbb{P}^1$ *for any λ_0, λ_1.*

Definition 4.1.5. A weighted projective space of type $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is said to be *well formed* if no n of $\lambda_0, \lambda_1, \ldots, \lambda_n$ have a common factor.

Interpretation

Several interpretations of Definition 4.1.1 are given in [Dol82]. We reproduce here the most useful way of viewing weighted projective spaces for our purposes.

Let $\mu_i := \mu_{\lambda_i} \subset \mathbb{G}_m$ be the group consisting of the λ_i-roots of unity. Let $\mu_0 \times \mu_1 \times \ldots \times \mu_n$ act on $\mathbb{P}^r$ by:

$$\begin{gathered}(\xi_0, \xi_1, \ldots, \xi_n) \cdot (x_0, x_1, \ldots, x_n) \mapsto (\xi_0 x_0, \xi_1 x_1, \ldots, \xi_n x_n) \\ \text{where } \xi_i = \exp(2\pi i \kappa_i / \lambda_i), \quad 0 \leq \kappa_i < \lambda_i.\end{gathered}$$

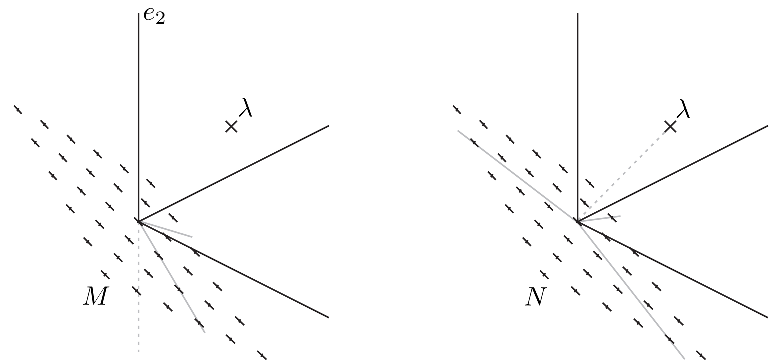
Then:

$$\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)^{\mathrm{an}} = \mathbb{P}^r / (\mu_0 \times \mu_1 \times \ldots \times \mu_n).$$

From this we conclude that the affine piece $(x_i \neq 0)$ of $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$ is the quotient of $\mathbb{A}^{n+1}$ by $\frac{1}{\lambda_i}(\lambda_0, \ldots, \widehat{\lambda}_i, \ldots, \lambda_n)$, where $\widehat{\lambda}_i$ indicates that the entry λ_i is omitted. For more details concerning this conclusion, see [Rei02, § 2].

Using this interpretation, the following proposition is obvious. We mention it here because a toric proof will be given in Section 4.2.

Proposition 4.1.6. *Let $\mathbb{P} = \mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$ be well formed. The affine variety $(x_i \neq 0) \subset \mathbb{P}$ is smooth if and only if $\lambda_i = 1$.*



(a) The dual cone $M_{\mathbb{R}} \cap \operatorname{cone}\{e_0, e_1, \pm e_2\}$.

(b) The three cones, in $N_{\mathbb{R}}$, of the fan Δ.

Figure 4-1: Constructing the dual fan, and the fan, of weighted projective space.

4.2 Toric Construction of Weighted Projective Space

Weighted projective spaces are in fact complete toric varieties. We shall now give two toric descriptions of how to construct the fan associated with weighted projective space. These two descriptions are essentially the same, however the first is perhaps more obviously identical to the definition in Section 4.1. The constructions involved are quite natural from the point of view of invariant theory; further details concerning the connection between these definitions and GIT quotients can be found in [Muk03, pp. 113–115] and [Dol03, pp. 38–39]. See also [Cox95a, Cox95b].

Constructing the Dual Fan

Let $\widetilde{M} \cong \mathbb{Z}^{n+1}$ be an $(n+1)$-dimensional lattice generated by $\{e_0, e_1, \ldots, e_n\}$, and let $\widetilde{N} := \operatorname{Hom}(\widetilde{M}, \mathbb{Z}) \cong \mathbb{Z}^{n+1}$. Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ be a primitive element of the lattice $\widetilde{N}$, i.e.:

$$\gcd\{\lambda_0, \lambda_1, \ldots, \lambda_n\} = 1.$$

We must insist, in addition, that $\lambda_i > 0$ for all i. Define the lattice $M \subset \widetilde{M}$ to be the kernel of the map λ. For each i take the cone given by the intersection of $M \otimes_{\mathbb{Z}} \mathbb{R}$ with $\operatorname{cone}\{e_0, e_1, \ldots, \pm e_i, \ldots, e_n\}$ (as illustrated in Figure 4-1(a)).

The resulting collection of cones in $M_{\mathbb{R}}$ is the dual fan associated with weighted projective space $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. See [MS05, §10] for a more comprehensive treatment.

Constructing the Fan

Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ be a primitive element of the lattice $\widetilde{N} \cong \mathbb{Z}^{n+1}$. The quotient $N := \widetilde{N}/\mathbb{Z}\lambda$ is a lattice of dimension n. If in addition all $\lambda_i > 0$, then $\mathbb{R}_{\geq 0}\lambda$ partitions the positive quadrant of $\widetilde{N} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{n+1}$ into $n+1$ cones. These cones form a fan $\widetilde{\Delta}$ whose one-dimensional faces are spanned by λ and the standard basis vectors. The fan $\widetilde{\Delta}$ projects to a fan Δ covering $N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. This is illustrated in Figure 4-1(b).

The complete toric variety X_Δ associated with the fan Δ is the weighted projective space $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$.

Remark 4.2.1. For the purposes of actual calculations the above descriptions may prove to be too abstract to be readily applicable. Fortunately [Con02, §3] gives an explicit algorithm for constructing the fan of a weighted projective space: essentially this is achieved by repeated application of *Cramer's Rule* (*c.* 1750). The calculations involved are exceptionally tedious and perhaps best left to a computer – for this reason they will not be reproduced here.

From the method of constructing the fan, we obtain the following simple, yet powerful, result:

Proposition 4.2.2. *Let Δ be the fan in $N_{\mathbb{R}}$ associated with the weighted projective space $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. Let $\rho_0, \rho_1, \ldots, \rho_n \in N$ be the primitive generators of the rays of Δ. Then:*

(i) $\lambda_0\rho_0 + \lambda_1\rho_1 + \ldots + \lambda_n\rho_n = 0$*;*

(ii) The ρ_i generate the lattice N.

Proof. Since $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \ker(\widetilde{N} \to \widetilde{N}/\mathbb{Z}\lambda)$ we have that (i) holds. (ii) is similarly immediate. □

In fact [BB92, Proposition 2] tells us that Proposition 4.2.2 characterises the fan of weighted projective space:

Proposition 4.2.3 ([BB92])**.** *For any set $\{\lambda_0, \lambda_1, \ldots, \lambda_n\} \subset \mathbb{Z}_{>0}$ such that:*

$$\gcd\{\lambda_0, \lambda_1, \ldots, \lambda_n\} = 1,$$

let $\rho_0, \rho_1, \ldots, \rho_n$ and $\rho'_0, \rho'_1, \ldots, \rho'_n$ be two sets of primitive lattice elements such that each set satisfies conditions (i) and (ii) of Proposition 4.2.2. Then there exists a transformation in $GL(n, \mathbb{Z})$ that sends ρ_i to ρ'_i for $i = 0, 1, \ldots, n$.

Let us prove one final fact before giving a 'toric' proof of Proposition 4.1.6.

Proposition 4.2.4. *Let Δ be the fan in $N_{\mathbb{R}}$ associated with the weighted projective space $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. Suppose that $\rho_1, \rho_2, \ldots, \rho_n$ generate the lattice N, where each ρ_i is the primitive generator of a ray of Δ. Then $\lambda_0 = 1$*

Proof. Without loss of generality we may assume that, under the map $\widetilde{N} \to \widetilde{N}/\mathbb{Z}\lambda$, we have $\widetilde{e}_i \mapsto \rho_i$, where $\{\widetilde{e}_0, \widetilde{e}_1, \ldots, \widetilde{e}_n\}$ forms the standard basis for $\widetilde{N}$. Since $\{\rho_1, \ldots, \rho_n\}$ forms a basis of N, there exist $c_i \in \mathbb{Z}$ such that $c_1\widetilde{e}_1 + \ldots + c_n\widetilde{e}_n - \widetilde{e}_0 \in \ker(\widetilde{N} \to \widetilde{N}/\mathbb{Z}\lambda)$. By considering projection onto the first factor (i.e. onto $\mathbb{Z}\widetilde{e}_0$) we see that $\lambda_0 = 1$. $\square$

Proof of Proposition 4.1.6. If $\lambda_i = 1$ then, by Proposition 4.2.2 (i), we have that ρ_i lies in the lattice generated by the $\rho_j, j \neq i$. By Proposition 4.2.2 (ii) we have that $\rho_0, \rho_1, \ldots, \rho_n$ generates N, and hence $\rho_0, \rho_1, \ldots, \widehat{\rho}_i, \ldots, \rho_n$ generates N. But this implies that the cone generated by the $\rho_j, j \neq i$ is smooth.

In the opposite direction, since the cone generated by the $\rho_j, j \neq i$ is smooth so the $\rho_j, j \neq i$ form a basis of N. By Proposition 4.2.4 we see that $\lambda_i = 1$. $\square$

We conclude this section with a result on the volume of the simplex associated with weighted projective space $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. Note that this is a result on the simplex in $N_{\mathbb{R}}$, not the dual in $M_{\mathbb{R}}$, and should not be confused with the degree of $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$.

Proposition 4.2.5. *Let $P \subset N_{\mathbb{R}}$ be the n-simplex associated with $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. Then:*

$$\operatorname{vol} P = \frac{1}{n!} \sum_{i=0}^{n} \lambda_i.$$

Proof. Let T be an arbitrary n-simplex. It is a well known result that:

$$\operatorname{vol} T = \frac{1}{n!} \left| \det \begin{pmatrix} & T & & \\ \hdashline 1 & 1 & \ldots & 1 \end{pmatrix} \right|. \tag{4.2.1}$$

Let F_i be the face of P which does not contain the vertex ρ_i. Subdivide P into $n+1$ simplices P_i, where each P_i is the n-simplex conv $\{0\} \cup F_i$. Regarding each F_i as an $n \times n$ matrix, from equation (4.2.1) we have that:

$$\operatorname{vol} P_i = \frac{1}{n!} \left| \det F_i \right|, \qquad \text{for } i = 0, 1, \ldots, n.$$

But $|\det F_i|$ is precisely the order of the group $\mu_i \subset \mathbb{G}_m$ acting on the affine piece $(x_i \neq 0)$. Thus we have that $|\det F_i| = \lambda_i$ for $i = 0, 1, \ldots, n$. Since:

$$\operatorname{vol} P = \sum_{i=0}^{n} \operatorname{vol} P_i,$$

the result follows. □

4.3 $\mathbb{P}(1, \lambda_1, \lambda_2, \lambda_3)$ with Terminal Quotient Singularities

Let $\mathbb{P} := \mathbb{P}(1, \lambda_1, \lambda_2, \lambda_3)$ be a three-dimensional weighted projective space, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_{>0}$, $\gcd\{\lambda_1, \lambda_2, \lambda_3\} = 1$ (i.e. $\mathbb{P}$ is well formed). Assume in addition that $\mathbb{P}$ has at worst terminal singularities.

Let Δ be the fan in $N_{\mathbb{R}} \cong \mathbb{R}^3$ associated with $\mathbb{P}$. Let $\rho_0, \rho_1, \rho_2, \rho_3 \in N$ be the primitive lattice points generating the one-skeleton of Δ. Then Proposition 4.2.2 tells us that, without loss of generality, we may assume:

$$\rho_0 + \lambda_1 \rho_1 + \lambda_2 \rho_2 + \lambda_3 \rho_3 = 0.$$

Proposition 4.2.4 tells us that there exists a lattice isomorphism sending ρ_1 to e_1, ρ_2 to e_2, and ρ_3 to e_3, where the e_i form the usual basis for $\mathbb{Z}^3$. The polytope whose faces span Δ can be represented by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & -\lambda_1 \\ 0 & 1 & 0 & -\lambda_2 \\ 0 & 0 & 1 & -\lambda_3 \end{pmatrix}.$$

Lemma 4.3.1. *For each λ_i $(i = 1, 2, 3)$ we have that either:*

(a) $\lambda_1 + \lambda_2 + \lambda_3 \equiv 0 \pmod{\lambda_i}$*; or*

(b) $\gcd\{\lambda_i, \lambda_j\} = 1$ *and* $\lambda_k + 1 \equiv 0 \pmod{\lambda_i}$, *for some* $j \neq k$ *distinct from* i.

Proof. Since $\mathbb{P}$ is has at worst terminal singularities, by Theorem 3.2.6 we know that they are of the form:

$$\frac{1}{r}(1, a, -a). \tag{4.3.1}$$

Without loss of generality let us fix $i = 1$. Then we know that $\frac{1}{\lambda_1}(1, \lambda_2, \lambda_3)$ can be written in the form (4.3.1), up to permutation of the elements and the finite group action.

If $\frac{1}{\lambda_1}(1, \lambda_2, \lambda_3)$ is already in the form (4.3.1) then we are done, since this gives us (a) from the statement. Let us assume otherwise. Without loss of generality, there exists an element $\lambda_2^{-1} \in \mathbb{Z}/(\lambda_1)$ such that:

$$\lambda_2 \lambda_2^{-1} \equiv 1 \pmod{\lambda_1} \tag{4.3.2a}$$

$$\lambda_2^{-1} \equiv -\lambda_3 \lambda_2^{-1} \pmod{\lambda_1}. \tag{4.3.2b}$$

Now (4.3.2a) tells us that $\gcd\{\lambda_1, \lambda_2\} = 1$, so λ_2^{-1} is a unit of $\mathbb{Z}/(\lambda_1)$. Rearranging (4.3.2b) gives:

$$\lambda_2^{-1}(\lambda_3 + 1) \equiv 0 \pmod{\lambda_1}.$$

Since λ_2^{-1} is a unit, it cannot be a zero divisor. Hence $\lambda_3 + 1 \equiv 0 \pmod{\lambda_1}$. □

Using Lemma 4.3.1 it is possible to classify all possible values of λ_1, λ_2 and λ_3. This result appears as Theorem 4.3.5, and is summarised in Table 4.1.

Proposition 4.3.2. *Suppose that condition (a) of Lemma 4.3.1 is satisfied when* $i = 1, 2, 3$. *Then* $(\lambda_1, \lambda_2, \lambda_3)$ *is equal to* $(1, 1, 1)$, $(1, 1, 2)$, *or* $(1, 2, 3)$.

Proof. By the hypothesis there exists $k, k', k'' \in \mathbb{Z}_{>0}$ such that:

$$\lambda_2 + \lambda_3 = k\lambda_1, \ \lambda_1 + \lambda_3 = k'\lambda_2, \text{ and } \lambda_1 + \lambda_2 = k''\lambda_3.$$

Hence we obtain:

$$2(\lambda_1 + \lambda_2 + \lambda_3) = k\lambda_1 + k'\lambda_2 + k''\lambda_3.$$

Thus $\min\{k, k', k''\} \in \{1, 2\}$. We may assume without loss of generality that $k = \min\{k, k', k''\}$.

Suppose that $k = 1$. Then $2\lambda_3 = (k' - 1)\lambda_2$, and $2\lambda_2 = (k'' - 1)\lambda_3$. Hence we see that $4\lambda_2 = (k' - 1)(k'' - 1)\lambda_2$. Thus either $k' = k'' = 3$ or $k' = 5, k'' = 2$. The first possibility gives us that $\lambda_3 = \lambda_2$ and $\lambda_1 = 2\lambda_2$. But $\gcd\{\lambda_1, \lambda_2, \lambda_3\} = 1$, hence $\lambda_2 = 1, \lambda_1 = 2, \lambda_3 = 1$. The second possibility gives us that $\lambda_3 = 2\lambda_2$ and that $\lambda_1 = 3\lambda_2$. By coprimality we see that $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 2$.

Now suppose that $k = 2$. Then $k' = k'' = 2$, and we see that $\lambda_1 = \lambda_2 = \lambda_3$. Since $\gcd\{\lambda_1, \lambda_2, \lambda_3\} = 1$ it must be that $\lambda_1 = \lambda_2 = \lambda_3 = 1$. □

Proposition 4.3.3. *Suppose that condition (a) of Lemma 4.3.1 is satisfied when $i = 2$ and 3, and that condition (b) is satisfied when $i = 1$. Then, up to permutation, $(\lambda_1, \lambda_2, \lambda_3)$ is equal to one of $(1, 1, 2)$, $(1, 2, 3)$, $(2, 3, 5)$, or $(3, 4, 5)$.*

Proof. We have that:

$$\lambda_3 + 1 = k\lambda_1, \tag{4.3.3a}$$

$$\lambda_1 + \lambda_3 = k'\lambda_2, \tag{4.3.3b}$$

$$\lambda_1 + \lambda_2 = k''\lambda_3. \tag{4.3.3c}$$

Suppose that $k' = 1$. Equations (4.3.3a) and (4.3.3b) give us that $\lambda_2 + 1 = (k + 1)\lambda_1$. From (4.3.3a) we see that there are three possibilities.

(i) $\lambda_1 = \lambda_3 = 1$, and hence $\lambda_2 = 2$.

(ii) $k = 1$, in which case equation (4.3.3a) gives that $2 = (k'' - 3)\lambda_3$. Thus we have that either $\lambda_3 = 2$, which implies that $\lambda_2 = 5$ and $\lambda_1 = 3$; or $\lambda_3 = 1$, which implies that $\lambda_2 = 3$ and $\lambda_1 = 2$.

(iii) $\lambda_1 < \lambda_3$, and hence $k'' = 2$ or 1. If $k'' = 1$ then equations (4.3.3b) and (4.3.3c) give $2\lambda_1 = 0$, an impossibility. If $k'' = 2$ then these same equations give $2\lambda_1 = \lambda_3$ and $\lambda_2 = 3\lambda_1$. Hence, since $\gcd\{\lambda_1, \lambda_2, \lambda_3\} = 1$, we see that $(\lambda_1, \lambda_2, \lambda_3) = (1, 3, 2)$.

Now suppose that $k' \geq 2$. We have that:

$$2\lambda_1 + \lambda_2 + 2\lambda_3 + 1 = k\lambda_1 + k'\lambda_2 + k''\lambda_3. \tag{4.3.4}$$

Hence $\min\{k, k', k''\} \in \{1, 2\}$. If $\min\{k, k', k''\} = 2$ then equation (4.3.4) tells us that $k = k' = k'' = 2$ and $\lambda_2 = 1$. Thus, rearranging equations (4.3.3a)–(4.3.3c), we see that $\lambda_1 = \lambda_3 = 1$.

If $\min\{k,k',k''\}=1$ then first let us suppose $k=1$. First we shall demonstrate that $k=k''=1$ is impossible. From equations (4.3.3a) and (4.3.3c) we obtain that:

$$\lambda_2+1=(k''-1)\lambda_3, \tag{4.3.5}$$

and hence that $k''\geq 2$. Now, equations (4.3.4) and (4.3.3a) gives:

$$(k'-1)\lambda_2+(k''-2)\lambda_3=2+\lambda_3.$$

From equation (4.3.5) we deduce that:

$$(k'-2)\lambda_2+2(k''-2)\lambda_3=3.$$

Since $k',k''\geq 2$ and since $2(k''-2)\lambda_3$ is necessarily even, we see that the only possibilities are either that $(k'-2)\lambda_2=3$ and $k''=2$, or that $k'=k''=3, \lambda_2=\lambda_3=1$. Hence $(\lambda_1,\lambda_2,\lambda_3)$ equals $(5,3,4)$, $(3,1,2)$ or $(2,1,1)$.

Finally, let us suppose that $k''=1$. We have already seen that this forces $k\geq 2$. $\square$

Proposition 4.3.4. *Suppose that condition (b) of Lemma 4.3.1 is satisfied by at least two values of i. Then* $(\lambda_1,\lambda_2,\lambda_3)$ *is equal to* $(1,2,3),(2,3,5)$, *or* $(3,4,5)$.

Proof. First let us suppose that $\lambda_1+1\equiv 0 \pmod{\lambda_2}$ and that $\lambda_2+1\equiv 0 \pmod{\lambda_1}$. By assumption we have that $\lambda_1,\lambda_2\neq 1$. Thus $\lambda_1+1=k\lambda_2$ and $\lambda_2+1=k'\lambda_1$ for some $k,k'\in\mathbb{Z}_{\geq 1}$. Hence $\lambda_2+1=k'(k\lambda_2-1)$. Since $\lambda_2\geq 2$ we see that there are only two possibilities. Either $k'=3,\lambda_2=2,k=1$, which forces $\lambda_1=1$ and hence is impossible; or $k'=1,\lambda_2=2,k=2$ and so $\lambda_1=3$. Thus let us consider the singularity $\frac{1}{\lambda_3}(1,2,3)$. For this to be of the form (4.3.1) we observe that λ_3 must be one of $1,2,3,4$ or 6. But $\gcd\{\lambda_1,\lambda_3\}=1=\gcd\{\lambda_2,\lambda_3\}$. Hence the only possibility is that $\lambda_3=1$.

Now let us suppose that $\lambda_1+1\equiv 0 \pmod{\lambda_2}$ and that $\lambda_1+1\equiv 0 \pmod{\lambda_3}$. We may assume that $\lambda_2+\lambda_3\equiv 0 \pmod{\lambda_1}$, since otherwise we are in the previous case. We have that:

$$k\lambda_2=\lambda_1+1=k'\lambda_3, \qquad \text{for some } k,k'\in\mathbb{Z}_{\geq 1}.$$

If $\lambda_1=1$ then $\lambda_2=\lambda_3=2$, contradicting the fact that $\gcd\{\lambda_2,\lambda_3\}=1$. If $\lambda_1=2$ then $\lambda_2=\lambda_3=3$, again a contradiction. Thus we may assume that $\lambda_1\geq 3$.

If $k,k'\geq 3$ then $\lambda_2+\lambda_3\leq(1/3)(2\lambda_1+2)<\lambda_1$. Hence it must be that one of k,k' is equal to either 1 or 2. We shall show that the first case is impossible. Without loss of generality suppose that $k=1$. Then $\lambda_2 = \lambda_1 \equiv 1 \pmod{\lambda_1}$. Hence $\lambda_3 + 1 \equiv 0 \pmod{\lambda_1}$, which we have already ruled against.

Suppose without loss of generality that $k=2$. Then $2\lambda_2 = \lambda_1 + 1$. Our previous paragraph forces $k' \geq 2$, and if $k'=2$ then $\gcd\{\lambda_2,\lambda_3\} > 1$. Hence $k' \geq 3$. But:

$$\frac{\lambda_1+1}{2} + \frac{\lambda_1+1}{k'} \leq \frac{\lambda_1+1}{2} + \frac{\lambda_1+1}{3} = \frac{5\lambda_1+5}{6}.$$

Hence $\lambda_1 \leq 5$. Since $2 \mid \lambda_1 + 1$ the only possibilities are that $\lambda_1 = 3$ or 5. The former case gives $\lambda_2 = 2, \lambda_3 = 2$ or 4. This is a contradiction. The latter case gives $\lambda_2 = 3, \lambda_3 = 2, 3$, or 6. All but the first choice of λ_3 are impossible.

The final possibility is to consider the case when $\lambda_1 + 1 \equiv 0 \pmod{\lambda_2}$ and $\lambda_2 + 1 \equiv 0 \pmod{\lambda_3}$. If we are not to reduce to one of the two cases considered in the preceding paragraphs, we need entertain only two possibilities:

(i) $\lambda_2 + \lambda_3 \equiv 0 \pmod{\lambda_1}$;

(ii) $\lambda_3 + 1 \equiv 0 \pmod{\lambda_1}$.

Let us consider case (i). We have that $\lambda_1 + 1 = k\lambda_2$ and $\lambda_2 + 1 = k'\lambda_3$ for some $k, k' \in \mathbb{Z}_{>0}$. Suppose that $k=1$. Then $1 + \lambda_3 \equiv 0 \pmod{\lambda_1}$ and we are in case (ii) to be addressed below. Hence we may assume that $k \geq 2$.

Suppose that $k' = 1$ and that $k = 2$. Then $\lambda_1 + 1 = 2\lambda_2$ and $\lambda_1 + 1 = \lambda_3$. Thus, by assumption, we have that $(1-\lambda_2) + (\lambda_2 + 1) \equiv 0 \pmod{\lambda_1}$ and so $\lambda_1 = 2$. But this is impossible, since it forces $\lambda_2 = 3/2$. Consider $k' = 1$ and $k \geq 3$. We have that $2\lambda_2 + 1 \equiv 0 \pmod{\lambda_1}$, and that $\lambda_2 \leq (1/3)(\lambda_1 + 1)$. Hence:

$$2\lambda_2 + 1 \leq \frac{2\lambda_1+5}{3}.$$

Hence $\lambda_1 \leq 5$. Recalling that $\lambda_2, \lambda_3 \neq 1$ we see that the only possibility is that $(\lambda_1,\lambda_2,\lambda_3) = (5,2,3)$.

Now assume that $k, k' \geq 2$. Then:

$$\lambda_2 + \lambda_3 \leq \frac{\lambda_1+1}{2} + \frac{\lambda_2+1}{2} \leq \frac{3\lambda_1+5}{4}.$$

Once again we see that $\lambda_1 \leq 5$, and the only possibility it that $(\lambda_1,\lambda_2,\lambda_3) = (5,2,3)$.

Finally, we consider case (ii). Suppose that $k, k' = 1$. Then $\lambda_1 + 1 = \lambda_2$ and $\lambda_2 + 1 = \lambda_3$. Hence $\lambda_3 + 1 = \lambda_1 + 3$ and so $\lambda_1 = 3$. This gives $\lambda_2 = 4, \lambda_3 = 5$. Now assume that

$\mathbb{P}^3$	$\mathbb{P}(1,1,1,2)$	$\mathbb{P}(1,1,2,3)$
$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$

$\mathbb{P}(1,2,3,5)$	$\mathbb{P}(1,3,4,5)$
$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -5 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{pmatrix}$

Table 4.1: The weighted projective spaces of the form $\mathbb{P}(1,\lambda_1,\lambda_2,\lambda_3)$ with at worst terminal singularities, and the associated polytopes in $N_{\mathbb{R}}$.

at least one of $k, k' \geq 2$. We see that:

$$\lambda_3 + 1 \leq \frac{\lambda_1 + 5}{2}.$$

Thus $\lambda_1 \leq 5$. Recalling that, in this case, $\gcd\{\lambda_i, \lambda_j\} = 1$ for $i \neq j$, we see that the only possibilities are that $(\lambda_1, \lambda_2, \lambda_3) = (4,5,3)$ or $(5,3,4)$. □

Theorem 4.3.5. *The only weighted projective spaces in dimension three, one of whose weights is* 1*, with at worst terminal singularities, are* $\mathbb{P}^3$*,* $\mathbb{P}(1,1,1,2)$*,* $\mathbb{P}(1,1,2,3)$*,* $\mathbb{P}(1,2,3,5)$*, and* $\mathbb{P}(1,3,4,5)$*.*

Proof. Lemma 4.3.1 gives the possible relations between the weights. Propositions 4.3.2–4.3.4 analyse all the possibilities. □

4.4 Fake Weighted Projective Space

Let $N \cong \mathbb{Z}^n$ be an n-dimensional lattice. Let $\{\rho_0, \rho_1, \ldots, \rho_n\} \subset N$ be a set of primitive lattice points such that $N_{\mathbb{R}} = \sum_{i=0}^{n} \mathbb{R}_{\geq 0}\rho_i$. There exist $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{Z}_{>0}$, with $\gcd\{\lambda_0, \lambda_1, \ldots, \lambda_n\} = 1$, such that:

$$\lambda_0\rho_0 + \lambda_1\rho_1 + \ldots + \lambda_n\rho_n = 0.$$

Define the n-dimensional cones:

$$\sigma_i := \operatorname{cone}\{\rho_0, \rho_1, \ldots, \widehat{\rho}_i, \ldots, \rho_n\}, \qquad \text{for } i = 0, 1, \ldots, n.$$

Let Δ be the fan in $N_{\mathbb{R}}$ generated by the σ_i, $i = 0, 1, \ldots, n$. Then Δ is a complete n-dimensional fan.

Definition 4.4.1. The projective toric variety associated with the fan Δ is called a *fake weighted projective space* (*of type* $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$)[1].

An immediate consequence of this definition is that fake weighted projective spaces are $\mathbb{Q}$-factorial toric Fano varieties with Picard number one. Of course, the collection of weighted projective spaces is a sub-collection of the collection of fake weighted projective spaces. Naturally, there exist fake weighted projective spaces which are not weighted projective spaces:

Proposition 4.4.2. *There exist at least two fake weighted projective spaces with weights* $(1,1,1,1)$. *They are* $\mathbb{P}^3$ *and the toric variety associated with the tetrahedron:*

$$P := \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}.$$

Proof. First we shall show that the two fake weighted projective spaces are not isomorphic as toric varieties. If they where, there would exist some element A of $GL(3, \mathbb{Z})$ such that $AP \sim T$, where $\sim$ denotes equality up to possible permutation of the columns, and T is the matrix:

$$T := \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Any 3×3 sub-matrix of T has determinant with absolute value 1 (this is clear from the description of T, but is equally obvious from the fact that $\mathbb{P}^3$ is smooth, and hence any cone of the fan is regular). However, any 3×3 sub-matrix of P has

[1] I would like to thank Weronika Krych for introducing me to the name "fake weighted projective space" in her talk [Kry03].

determinant with absolute value five (in fact the singularities of the fake weighted projective space associate with P are of the form $\frac{1}{5}(1,2,3)$ – one in each of the four affine patches $(x_i \neq 1)$). Hence $|\det A| = 1/5$; an absurdity.

Finally we use Proposition 2.3.6 to conclude that these two fake weighted projective spaces cannot be isomorphic even as abstract algebraic varieties. □

In fact the example above has appeared in various forms on several occasions in the literature. For example [BB92, pg. 178], [BCF+05, pg. 189], and [Mat02, Remark 14.2.3]. The most interesting formulation, however, is to be found in [Rei87, §4.15]:

Example 4.4.3 ([Rei87])**.** *Let $M \subset \mathbb{Z}^4$ be the three-dimensional lattice defined by:*

$$M := \left\{ (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \,\middle|\, \sum_{i=1}^{4} m_i = 5 \text{ and } \sum_{i=1}^{4} i m_i \equiv 0 \pmod 5 \right\}.$$

Let $\Sigma \subset M_{\mathbb{R}}$ be the simplex whose four vertices are given by the points $(0, \ldots, 5, \ldots, 0)$ (i.e. the points corresponding to the monomials x_i^5, $i = 1, \ldots, 4$). The toric variety constructed from Σ is $\mathbb{P}^3/\mu_5$ (c.f. Corollary 4.4.7), where the μ_5-action is given by:

$$\epsilon : x_i \mapsto \epsilon^i x_i \qquad \text{for } (x_1, x_2, x_3, x_4) \in \mathbb{P}^3.$$

The polytope in $N_{\mathbb{R}}$ whose faces generate the fan of $\mathbb{P}^3/\mu_5$ is:

$$\begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}.$$

Mori Theory and Fake Weighted Projective Space

Fake weighted projective spaces play a natural role in toric Mori theory. The following result is adapted from [Rei83a, (2.6)].

Proposition 4.4.4 (After [Rei83a])**.** *Let X be a projective toric variety whose associated fan Δ is simplicial (i.e. X is $\mathbb{Q}$-factorial). If R is an extremal ray of $NE(X)$ (the cone of effective one-cycles) then there exists a toric morphism $\varphi_R : X \to Y$ with connected fibres, which is an elementary contraction in the sense of Mori theory: $\varphi_{R*}\mathcal{O}_X = \mathcal{O}_Y$, and for a curve $C \subset X$, $\varphi_R C = \text{pt.} \in Y$ if and only if $C \in R$.*

Let:

$$\begin{array}{rcl} A & \longrightarrow & B \\ \cap & & \cap \\ \varphi_R : X & \longrightarrow & Y \end{array}$$

be the loci on which φ_R is an isomorphism. Then $\varphi_R\,|_A\colon A \to B$ is a flat morphism, all of whose fibres are fake weighted projective spaces of dimension $\dim A - \dim B$*.*

The original statement in [Rei83a, (2.6)] claimed that all the fibres of $\varphi_R\,|_A$ were weighted projective spaces. This oversight has been noted (and corrected) in, amongst other places, [Mat02, Remark 14.2.4], [Fuj03, §1], and [Kry02].

We now consider what can be said concerning the singularities of a fake weighted projective space. Let Δ in $N_{\mathbb{R}}$ be the fan of X, a fake weighted projective space with weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$. Let $\rho_0, \rho_1, \ldots, \rho_n$ be primitive elements of N which generate the one-skeleton of Δ. We have that:

$$\sum_{i=0}^{n} \lambda_i \rho_i = 0. \tag{4.4.1}$$

Let $N' \subset N$ be the lattice generated by the ρ_i. Let Δ' be the projection of Δ onto $N'_{\mathbb{R}}$. By construction the corresponding ρ'_i of Δ' generate the lattice N' and satisfy equation (4.4.1). Hence, by Proposition 4.2.2 and Proposition 4.2.3, Δ' is the fan of $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. This observation allows us to prove the following key result.

Proposition 4.4.5. *Let X be any fake weighted projective space with weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$. There exists a toric morphism $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n) \to X$.*

Proof. The map of lattices $N' \to N$ generated by $\rho'_i \mapsto \rho_i$ induces a injective map of fans $\Delta' \to \Delta$. Hence there exists a toric morphism $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n) \to X$. □

Remark 4.4.6. Let X be as in Proposition 4.4.5. Then there exists a sequence of maps of fans:

$$\Delta_{\mathbb{P}^n} \to \Delta_{\mathbb{P}(\lambda_0,\lambda_1,\ldots,\lambda_n)} \to \Delta_X. \tag{4.4.2}$$

The first map corresponds to the subdivision of the lattice by inserting the extra generators $(1/\lambda_i)\rho_i$ (see [Ful93, pg. 35]), and the second map corresponds to that described above. Thus we obtain a sequence of proper toric morphisms:

$$\mathbb{P}^n \to \mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n) \to X.$$

The construction of the morphism in the proof of Proposition 4.4.5 gives us the following two corollaries, the second of which will be of particular importance to our studies:

Corollary 4.4.7 (c.f. [Con02, Proposition 4.7])**.** *Let $X(P)$ be any fake weighted projective space with weights $(\lambda_0, \lambda_1, \dots, \lambda_n)$. Then $X(P)$ is the quotient of $\mathbb{P}(\lambda_0, \lambda_1, \dots, \lambda_n)$ by the action of the finite group N/N'.*

Proof. This is immediate from the proof of Proposition 4.4.5 and the preceding remarks. □

Corollary 4.4.8. *Let X be any fake weighted projective space with weights $(\lambda_0, \lambda_1, \dots, \lambda_n)$. X has at worst terminal (resp. canonical) singularities only if $\mathbb{P}(\lambda_0, \lambda_1, \dots, \lambda_n)$ has at worst terminal (resp. canonical) singularities.*

Proof. Using the notation of the proof of Proposition 4.4.5, we have that any lattice point in the fan of $\mathbb{P}(\lambda_0, \lambda_1, \dots, \lambda_n)$ can be expressed as a integer sum of the ρ_i', and hence corresponds to a lattice point in N with the same integer relationship between the ρ_i. □

Corollary 4.4.8 tells us that if we wish to classify all fake weighted projective spaces with at worst terminal (resp. canonical) singularities, it is sufficient to find only those weights $(\lambda_0, \lambda_1, \dots, \lambda_n)$ for which the corresponding weighted projective space possesses at worst terminal (resp. canonical) singularities. In essence, there do not exist any "extra" weights.

A similar result holds for Gorenstein fake weighted projective space:

Corollary 4.4.9. *With notation as above, X is Gorenstein only if $\mathbb{P}(\lambda_0, \lambda_1, \dots, \lambda_n)$ is Gorenstein.*

Proof. If X is Gorenstein then the associated n-simplex P_X is reflexive. Hence $P_X^\vee$ is a lattice n-simplex in $M_\mathbb{R}$. Dualising (4.4.2) tells us that the n-simplex $P^\vee_{\mathbb{P}(\lambda_0, \lambda_1, \dots, \lambda_n)}$ must also be a lattice polytope. Hence $\mathbb{P}(\lambda_0, \lambda_1, \dots, \lambda_n)$ is Gorenstein. □

There is a fascinating result concerning the weights of dual simplices, due to Conrads:

Proposition 4.4.10 ([Con02, Lemma 5.3])**.** *Let $X(P)$ be any fake weighted projective space with weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ and associated n-simplex P. Then the fake weighted projective space $X(P^\vee)$ also has weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$.*

Corollary 4.4.7 provides the motivation for the following definition:

Definition 4.4.11. Let $P \subset N_{\mathbb{R}}$ be a n-simplex whose vertices $\rho_0, \rho_1, \ldots, \rho_n$ are contained in the lattice N. We define the *multiplicity* of P to be the index of the lattice generated by the ρ_i in the lattice N. We write:

$$\operatorname{mult} P := [N : \mathbb{Z}\rho_0 + \mathbb{Z}\rho_1 + \ldots + \mathbb{Z}\rho_n].$$

Lemma 4.4.12. *Let P be the n-simplex associated with a fake weighted projective space X. X is a weighted projective space if and only if* $\operatorname{mult} P = 1$.

Proof. Immediate from Proposition 4.2.2 and Proposition 4.2.3. □

In fact there exists a bound on how large $\operatorname{mult} P$ can be, which depends only on the λ_i and the number of interior lattice points.

Theorem 4.4.13. *Let P be the n-simplex associated with a fake weighted projective space X with weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$. Then:*

$$\operatorname{mult} P \leq \frac{|N \cap P^\circ|\, h^{n-1}}{\lambda_1 \lambda_2 \ldots \lambda_n}, \qquad \textit{where } h := \sum_{i=0}^{n} \lambda_i.$$

Remark 4.4.14. Note that the omission of λ_0 in the denominator is deliberate. Of course it makes sense to choose the λ_i such that $\lambda_0 \leq \lambda_j$ for all $j > 0$, however this is not a necessity.

Corollary 4.4.15. *With notation as above, assume that X has at worst canonical singularities. Then:*

$$\operatorname{mult} P \leq \frac{h^{n-1}}{\lambda_1 \lambda_2 \ldots \lambda_n}.$$

Proof. Immediate from Theorem 4.4.13, given the fact that $|N \cap P^\circ| = 1$. □

Before this theorem can be proved, we need the following result, which can be found in [Hen83, Theorem 3.4], [LZ91, Lemma 2.3], or [Pik01, Lemma 5]. Throughout, the volume is – as always – given relative to the underlying lattice.

Proposition 4.4.16. *Let* $P := \operatorname{conv}\{\rho_0, \rho_1, \ldots, \rho_n\} \subset N_{\mathbb{R}}$ *be any simplex such that:*

$$\sum_{i=0}^{n} \lambda_i \rho_i = 0, \qquad \text{for some } \lambda_i \in \mathbb{Z}_{>0}.$$

Let $h := \sum_{i=0}^{n} \lambda_i$ *and* $k := |N \cap P^\circ|$. *Then:*

$$\operatorname{vol} P \leq \frac{kh^n}{n!\lambda_1\lambda_2 \ldots \lambda_n}.$$

To prove Proposition 4.4.16 we require a generalisation of Minkowski's Theorem (observe that if $k = 1$ then Minkowski's Theorem will suffice):

Theorem 4.4.17 ([vdC35]). *Let* k *be any positive integer and let* $K \subset N_{\mathbb{R}}$ *be any centrally symmetric convex body such that* $\operatorname{vol} K > 2^n k$. *Then* K *contains at least* k *pairs of points in the lattice* N.

Proof of Proposition 4.4.16. Consider the convex body:

$$K := \left\{ \sum_{i=1}^{n} \mu_i(\rho_i - \rho_0) \,\middle|\, |\mu_i| \leq \frac{\lambda_i}{h} \right\}.$$

This is centrally symmetric around the origin, with volume:

$$\operatorname{vol} K = \left(n! \prod_{i=1}^{n} \frac{2\lambda_i}{h} \right) \operatorname{vol} S.$$

If $\operatorname{vol} K > 2^n k$ then, by Theorem 4.4.17, at least k pairs of lattice points lie in the interior of P. But this contradicts the definition of k. Hence $\operatorname{vol} K \leq 2^n k$ and the result follows. □

Proof of Theorem 4.4.13. Proposition 4.4.5 and Proposition 4.2.5 tells us that:

$$\operatorname{vol} P = \frac{h}{n!} \operatorname{mult} P. \tag{4.4.3}$$

Applying Proposition 4.4.16 gives the result. □

We conclude with a rather neat result of Conrads, for which we need the following definition.

Definition 4.4.18. For $n, k \in \mathbb{Z}_{>0}$ we denote by $\mathrm{Herm}(n,k)$ the set of all lower triangular matrices $H = (h_{ij}) \in GL(n, \mathbb{Q}) \cap M(n \times n; \mathbb{Z}_{\geq 0})$ with $\det H = k$, where $h_{ij} \in \{0, \ldots, h_{jj} - 1\}$ for all $j = 1, \ldots, n-1$ and all $i > j$. We call $\mathrm{Herm}(n,k)$ then set of *Hermite normal forms* of dimension n and determinant k.

Theorem 4.4.19 ([Con02, Theorem 4.4])**.** *Let $X(P')$ be any fake weighted projective space with weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ and associated n-simplex P'. Let P the n-simplex associated with $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. Then there exists $H \in \mathrm{Herm}(n, \mathrm{mult}\, P')$ such that $P' = HP$ (up to the action of $GL(n, \mathbb{Z})$).*

For a nice corollary to Theorem 4.4.19, also due to Conrads, see Proposition 9.2.1.

Finiteness of Fake Weighted Projective Space

It is worth observing that Theorem 4.4.13 provides a proof that the number of fake weighted projective spaces whose polytope contains a fixed number of lattice points is finite. This is achieved by combining [Pik01, Theorem 6] – which establishes a bound on the volume of an n-simplex once the number of interior lattice points is fixed, and is reproduced as Theorem 9.3.1 below – with Proposition 4.2.5. This establishes a bound on h, and hence a bound on the index of the sublattice. We perform this argument in the case when our fake weighted projective space possesses at worst canonical singularities.

Proposition 4.4.20. *The number of fake weighted projective spaces of fixed dimension n with at worst canonical singularities is finite.*

Proof. Combining [Pik01, Theorem 6] with Proposition 4.2.5 immediately gives us an upper bound on h – see Corollary 9.3.2 for the details. Thus the possible choices of the λ_i is finite, and Corollary 4.4.15 bounds $\mathrm{mult}\, X$; let k_n be this upper bound. Proposition 4.2.3 tells us that each X generates a unique sublattice N'. But the number of sublattices of a fixed finite index is finite, and so the number of possible X such that $\mathrm{mult}\, X = k \leq k_n$ is finite. □

Once finiteness of the number of possible fake weighted projective spaces has been established, it is a simple matter to demonstrate that the number of toric Fano varieties (with some limit on the class of singularities permitted) are also finite.

Finiteness results such as this are not new. Indeed, there seem to be as many proofs as there are statements of this fact. A particularly elegant proof of the finiteness of fake weighted projective spaces with at worst terminal singularities can be found in [Mat02, Proposition 14.5.2]. Debarre ([Deb03, Corollary 13]) uses the results of [Hen83] and [LZ91] – which are similar to those of [Pik01] – to prove finiteness. Borisov and Borisov avoid the use of explicit bounds in their proof [BB92, §4].

4.5 Classifying Certain Three-Dimensional Fake Weighted Projective Spaces

Let us restrict our attention to attempting to classify all fake weighted projective spaces with some fixed weight $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ which possess, at worst, terminal singularities. By virtue of Corollary 4.4.8 it is sufficient to consider only those weights for which $\mathbb{P}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ has at worst terminal singularities.

Suppose that $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ is such a weight. Without loss of generality we insist that $\lambda_0 \leq \ldots \leq \lambda_3$. Fix the lattice $N = \mathbb{Z}^3$ and let $P := \operatorname{conv}\{x_0, x_1, x_2, x_3\}$ be the tetrahedron associated with one of our fake weighted projective spaces, where $x_i \in \mathbb{Z}^3$ are such that:

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0. \tag{4.5.1}$$

Let $k = \operatorname{mult} P$. Proposition 4.4.5 and Proposition 4.2.5 tells us that $\operatorname{vol} P = kh/6$, where $h = \lambda_0 + \ldots + \lambda_3$. Thus we have that:

$$\left| \det \begin{pmatrix} x_0^t & x_1^t & x_2^t & x_3^t \\ \hdashline 1 & 1 & 1 & 1 \end{pmatrix} \right| = kh. \tag{4.5.2}$$

By virtue of Pick's Theorem and the fact that P is a terminal simplex, we may insist that $x_0 = e_1$ and $x_1 = e_2$ (see Lemma 6.2.1 for a proof). It is then a simple matter to calculate determinant (4.5.2) and, using equation (4.5.1), see that $x_2^{(3)} = k\lambda_3$ and that $x_3^{(3)} = -k\lambda_2$, where $x_i^{(j)}$ denotes the j^{th} coefficient of x_i.

By the action of a suitable element in $GL(3, \mathbb{Z})$ we can insist that $0 \leq x_2^{(1)}, x_2^{(2)} < x_2^{(3)} = k\lambda_3$. We can also insist without loss of generality that $x_2^{(1)} \leq x_2^{(2)}$. When a value of x_2 is chosen, so x_3 is fixed via equation (4.5.1). Thus we can choose a value for k and, for each of the $(1/2)k\lambda_3(k\lambda_3 + 1)$ possible values of x_2, test whether the resulting tetrahedron is terminal. Corollary 4.4.15 establishes a reasonably low bound on k.

We do not employ this method here. We simply observe that it is a realistic approach to achieving a classification. In fact we shall develop a more subtle method in Section 6.3; this technique will offer significantly less possibilities to check, at the expense of requiring quite a bit more theory.

CHAPTER 5

Toric Fano Surfaces

5.1 Toric Fano Surfaces with Terminal Singularities

We shall prove the well-known result that there are exactly five terminal toric Fano surface and, in addition, that these are all smooth. Smooth Fano surfaces are known as *del Pezzo surfaces*. There are ten deformation types of del Pezzo surfaces, given by $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown-up in $0, 1, \ldots, 8$ points in general position (see [Har77, Remark V.4.7.1]). The first five of these are toric surfaces.

By Proposition 3.6.7 it is enough to demonstrate that, up to the action of $GL(2, \mathbb{Z})$, there are only five terminal Fano polygons (see Definition 3.7.2). Observing that these are all regular will prove smoothness. Trivially all polygons in $\mathbb{Z}^2$ are simplicial (i.e. all the faces are one-simplices – line segments), hence all toric Fano surfaces are $\mathbb{Q}$-factorial.

We shall make use of the following staple result:

Theorem 5.1.1 (Pick's Theorem)**.** *Let A be the area of a simply closed lattice polygon. Let B denote the number of lattice points on the polygon's edges. Let I denote the number of points in the interior of the polygon. Then:*

$$A = I + \frac{1}{2}B - 1.$$

We take e_1 and e_2 to be the standard basis elements of $\mathbb{Z}^2$. The first step is to prove the following rather trivial result:

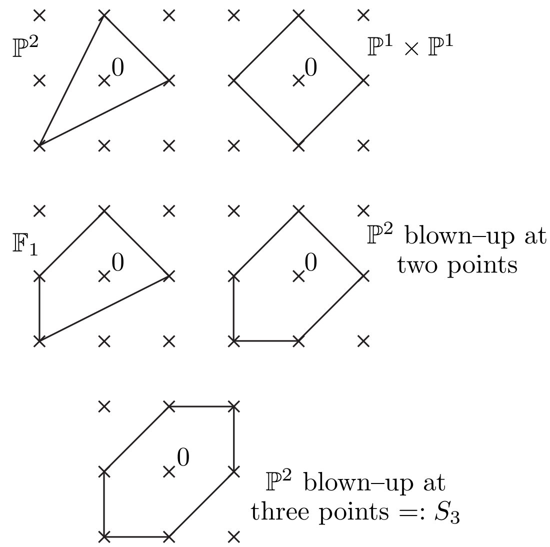


Figure 5-1: The five two-dimensional smooth Fano polygons.

Lemma 5.1.2. *If* $\{0, x_1, x_2\} \subset \mathbb{Z}^2$ *are the vertices of a triangle which contains no non-vertex lattice points then the triangle is equivalent, up to the action of* $GL(2, \mathbb{Z})$, *to* $\operatorname{conv}\{0, e_1, e_2\}$.

Proof. Applying Theorem 5.1.1 with $B = 3$ and $I = 0$ we see that the triangle has area $1/2$. Hence:

$$\det\begin{pmatrix} x_1^t & x_2^t \end{pmatrix} = \pm 1.$$

Thus there exists an element of $GL(2, \mathbb{Z})$ sending x_1 to e_1 and x_2 to e_2, and we have our result. $\square$

The results of the following corollary are well known, and can be proved in numerous ways (see for example [Nil05, Lemma 1.17]).

Corollary 5.1.3. *Every terminal toric Fano surface is smooth. Every canonical toric Fano surface is Gorenstein.*

Proof. An immediate consequence of Lemma 5.1.2 is that every face of a terminal Fano polygon is regular (that is, the vertices of any face form a $\mathbb{Z}$-basis for the lattice). Thus every terminal toric Fano variety is smooth.

Let P be a canonical Fano polygon, and let $x, y \in \partial P$ be two adjacent lattice points on any face of P. Then by Lemma 5.1.2 we see that x and y form a $\mathbb{Z}$-basis of the lattice N. Hence the supporting hyperplane associated with that face corresponds to a *lattice* *point* in the dual lattice M. Thus the dual polygon $P^\vee$ is a lattice polygon, and so P is reflexive. □

We now present a classification of the terminal toric Fano surfaces (which, in light of the above comments, are necessarily smooth and $\mathbb{Q}$-factorial). This result appears extensively in the literature (for example [Ewa96, KS97, Sat00, Nil05]), with each occurrence seeming to bring with it a new method of proof. The proof of Proposition 5.1.4 given here closely follows the method in [Ewa96, pp. 192-3].

Proposition 5.1.4. *There are five terminal Fano polygons, up to the action of $GL(2,\mathbb{Z})$. Up to equivalence, these are the polygons depicted in Figure 5-1. All the polygons are regular.*

Proof. By Lemma 5.1.2 we may take e_1 and e_2 to be two adjacent vertices of the polytope P. Let $a = (\alpha_1, \alpha_2) \neq e_2$ be adjacent to e_1. Then:

$$\alpha_2 = \det\begin{pmatrix} e_1^t & a^t \end{pmatrix}.$$

The triangle $\operatorname{conv}\{0, e_1, a\}$ contains no non-vertex lattice points, hence by Theorem 5.1.1 we have that $\alpha_2 = \pm 1$.

Suppose that $\alpha_2 = 1$. Convexity requires that:

$$a \in \left\{(x,y) \in \mathbb{Z}^2 \mid x + y < 1\right\}. \tag{5.1.1}$$

Hence $\alpha_1 \leq -1$. Again by convexity, for any lattice point $p \in P$:

$$p \in \left\{(x,y) \in \mathbb{Z}^2 \mid x + (1-\alpha_1)y > 1\right\}.$$

In particular 0 is not contained in the interior of P. This is a contradiction, and so $\alpha_2 = -1$. By (5.1.1) we see that $\alpha_1 \leq 1$.

Similarly we obtain that the vertex $b \neq e_1$ adjacent to e_2 has coordinates $(-1, \beta_2) \in \mathbb{Z}^2$, where $\beta_2 \leq 1$.

We denote the vertex adjacent to a (resp. b) and distinct from e_1 (resp. e_2) by $a' = (\alpha_1', \alpha_2')$ (resp. $b' = (\beta_1', \beta_2')$). We divide the possible values for α_1 and β_2 into four cases:

(i) Take $\alpha_1 = \beta_2 = 1$. Then:

$$\alpha_1' + \alpha_2' = \det\begin{pmatrix} a^t & a'^t \end{pmatrix}.$$

Since the triangle with vertices $\{0, a, a'\}$ contains no non-vertex lattice points, by Theorem 5.1.1 we have that $\alpha'_1 + \alpha'_2 = \pm 1$. Convexity gives us that:

$$a' \in \left\{(x, y) \in \mathbb{Z}^2 \mid x + y < 1\right\},$$

and so $\alpha'_1 + \alpha'_2 = -1$. Similarly we obtain:

$$\beta'_1 + \beta'_2 = \det\begin{pmatrix} b^t & b'^t \end{pmatrix} = -1.$$

The convexity of P requires that $a', b' \in \{-e_1, -e_2\}$. We have that either $a' \neq b'$ and $a' = -e_2$, $b' = -e_1$, or that $a' = b'$ are equal to either $-e_1$ or $-e_2$. In either case we finish with one of the following two polygons:

$$\mathrm{conv}\{e_1, \pm e_2, \pm(e_1 - e_2)\} \text{ or } \mathrm{conv}\{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}.$$

(ii) Take $\beta_2 = 1$ and $\alpha_1 \leq 0$. Then convexity gives us that $\beta'_1 + \beta'_2 = -1$. By convexity of P we must have $\beta'_1 \geq -1$. Hence $\alpha'_1 \geq 0$ and so $\alpha'_1 = 0$. Thus $a' = -e_2$ and $b' = -e_1$ or $-e_2$. Hence we obtain:

$$\mathrm{conv}\{e_1, \pm e_2, -(e_1 + e_2)\} \text{ or } \mathrm{conv}\{\pm e_1, \pm e_2, -(e_1 + e_2)\}.$$

Note that the second polygon is equivalent to the first possibility of case (i).

(iii) By symmetry, taking $\alpha_1 = 1$ and $\beta_2 \leq 0$ is equivalent to case (ii) and yields no new polygons.

(iv) Finally take $\alpha_1 \leq 0$ and $\beta_2 \leq 0$. Then convexity gives us that:

$$a, b \in \{-e_1, -e_2, -e_1 - e_2\}.$$

Along with the polygons already found, we obtain two additional possibilities:

$$\mathrm{conv}\{e_1, e_2, -e_1 - e_2\} \text{ and } \mathrm{conv}\{\pm e_1, \pm e_2\}.$$

□

Observe how the polygons in Figure 5-1 can be "grown", by successive addition of vertices, from the two polygons at the top of the figure. Using this method it should be possible to achieve the same classification result by first finding this pair of "minimal" polygons, and then adding additional vertices.

Alternative proof of Proposition 5.1.4. Let P be any terminal Fano polygon, and let x be a vertex of P. Consider the line passing through x and the origin. This intersects ∂P at some (not necessarily lattice) point y. There are two possibilities:

(i) The point y is a lattice point. In which case $y = -x$ is a vertex of P.

(ii) The point y lies in a face of P generated by the vertices x', x''. Then $\mathrm{conv}\{x, x', x''\}$ is a terminal Fano triangle.

In order for P to be minimal with respect to addition or subtraction of vertices, P contains a terminal Fano triangle if and only if P is a terminal Fano triangle. Otherwise P is centrally symmetric (by case (i)). Let us first consider the second possibility.

Let P be a minimal centrally symmetric terminal Fano polytope. By Lemma 5.1.2 we may assume $\pm e_1$ and $\pm e_2$ are vertices of P, where e_1 and e_2 form a face of P. By symmetry, $-e_1$ and $-e_2$ generate a face of P, hence any additional pairs of vertices of P must lie in the interior of $\mathrm{cone}\{e_1, -e_2\}$ and $\mathrm{cone}\{-e_1, e_2\}$. Suppose x and $-x$ are such a pair, with $x \in \mathrm{cone}\{e_1, -e_2\}$. But then $\mathrm{conv}\{x, -e_1, e_2\}$ is a terminal Fano triangle. This is a contradiction. Hence the only possibility is the polygon $\mathrm{conv}\{\pm e_1, \pm e_2\}$.

Now suppose that P is a minimal terminal Fano triangle. Again by Lemma 5.1.2 we may assume that e_1 and e_2 are two vertices of P. Let x be the remaining vertex. By applying Lemma 5.1.2 to the two lattice-free triangles containing x (or otherwise) it is a simple matter to see that the only possibility is $x = -e_1 - e_2$.

Finally the classification is completed by "growing" the two minimal polygons by adding vertices according to the possibilities (i) and (ii) above. □

5.2 Toric Fano Surfaces with Canonical Singularities

Let P be a canonical Fano polygon. A little care is needed here with our concept of "minimality", since the number of vertices need not be a strictly increasing function – i.e. the introduction of a new vertex might lead to a reduction in the number of vertices as points are subsumed into faces.

Definition 5.2.1. Let $x \in \operatorname{vert} P$. The polygon obtained by *subtracting the vertex x* from P is given by:

$$\operatorname{conv}(P \cap \mathbb{Z}^2 \setminus \{x\}).$$

Definition 5.2.2. We say that P is *minimal* if, for all vertices x of P, the polygon obtained by subtracting x is not a canonical Fano polygon. We say that P is *maximal* if, for all lattice points $x \in \mathbb{Z}^2$, the polygon $\operatorname{conv}(P \cup \{x\})$ is either equal to P or is not a canonical Fano polygon.

Lemma 5.2.3. *The process of dualisation establishes a self-inverting map between the set of minimal canonical Fano polygons and the set of maximal canonical Fano polygons:*

$$\begin{aligned} \cdot^{\vee} : \{\textit{minimal polygons}\} &\longleftrightarrow \{\textit{maximal polygons}\} \\ P &\mapsto P^{\vee}. \end{aligned}$$

Proof. It is essential that all canonical Fano polygons are reflexive (Corollary 5.1.3). Observe that if P and Q are two polygons such that $Q \subset P$, then $P^{\vee} \subset Q^{\vee}$. Note also that $(P^{\vee})^{\vee} = P$ (up to equivalence). The result follows. □

Using these tools, we shall produce a classification of the canonical Fano polygons. As with the terminal classification, this result is well documented in the literature and more often than not appears alongside an original method of proof (for example [KS97, Sat00, PRV00, Nil05]).

Proposition 5.2.4. *There are sixteen canonical Fano polygons, up to the action of $GL(2, \mathbb{Z})$. Up to equivalence, these are the polygons listed in Table 5.1. All the polygons are reflexive.*

Proof. As in the alternative proof of Proposition 5.1.4, we see that for any lattice point x on the boundary of P either $-x \in \partial P$ or P contains a canonical Fano triangle which contains x. In this second case we may insist that x is a vertex of the triangle, otherwise we reduce to the first case.

Thus we see that a minimal Fano polygon P comes in one of two forms. Either P is a triangle, or P is centrally symmetric and free of canonical Fano triangles. In the latter case we quickly see (by applying Lemma 5.1.2 to two adjacent lattice points on a face of P) that the only possibility is $\operatorname{conv}\{\pm e_1, \pm e_2\}$.

Suppose that P is a minimal Fano triangle. Choose two adjacent lattice points both contained on a face of P. By Lemma 5.1.2 we may assume these points are e_1

Comments	Vertices	Comments	Vertices
3 Points Minimal	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$	6 Points	$\begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$
4 Points Minimal	$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$	6 Points	$\begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$
4 Points Minimal	$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix}$	7 Points	$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 & -1 \end{pmatrix}$
4 Points	$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$	7 Points	$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & -1 & 2 & 1 \end{pmatrix}$
5 Points	$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$	8 Points	$\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 0 & 2 & -1 \end{pmatrix}$
5 Points	$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix}$	8 Points Maximal	$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & -1 \end{pmatrix}$
6 Points	$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}$	8 Points Maximal	$\begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$
6 Points	$\begin{pmatrix} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix}$	9 Points Maximal	$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$

Table 5.1: The sixteen canonical Fano polygons, up to the action of $GL(2, \mathbb{Z})$.

and e_2. Without loss of generality we may also insist that e_1 is a vertex of P. Let $a = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ be the lattice point adjacent to e_1 distinct from e_2. Since a and e_1 form a $\mathbb{Z}$-basis for the lattice, so $\alpha_2 = -1$. By considering $\operatorname{conv}\{e_1, e_2, a\} \subset P$ we see that $\alpha_1 \in \{-1, -2\}$. In either case minimality forces e_2 to be a vertex of P, and we obtain the two possibilities:

$$\operatorname{conv}\{e_1, e_2, -e_1 - e_2\} \text{ or } \operatorname{conv}\{e_1, e_2, -2e_1 - e_2\}.$$

By Lemma 5.2.3 we know that the maximal polygons are given by the duals of the three minimal polygons. To complete the classification it simply remains to observe the polygons obtained by successive subtraction of vertices, starting from these maximal polygons. □

Remark 5.2.5. Note that the final step in the proof of Proposition 5.2.4 could be achieved in the same fashion as in the alternative proof of Proposition 5.1.4 – by “growing” the minimal polygons. Arguably the method used here is neater, however it relies on the fact that all canonical Fano polygons are reflexive. This is not the case in higher dimensions.

CHAPTER 6

Toric Fano Threefolds with Terminal Singularities

6.1 Introduction

In this chapter a complete classification of all toric Fano threefolds at worst with terminal singularities will be given. Moving to the corresponding combinatorial problem, by Corollary 3.6.6, (i), and Proposition 3.6.7 we wish to find, up to the action of $GL(3,\mathbb{Z})$, all convex lattice polytopes in $\mathbb{Z}^3$ which contain only the origin as a non-vertex lattice point (by which we mean that no lattice points lie on the surface of the polytope other than the vertices, and no lattice points are contained in the interior of the polytope besides the origin).

An equivalent classification for surfaces was given in Section 5.1; precisely five polygons were found, of which two are minimal (the Fano triangle and the Fano square, which make an appearance in Section 6.5) and one is maximal, in the sense of Definitions 8.2.5 and 6.6.1. The approach used for this classification relies on the basic result that, up to the action of $GL(2,\mathbb{Z})$, there is a unique lattice point free triangle (see Lemma 5.1.2), namely $\operatorname{conv}\{0,e_1,e_2\}$. This fails to hold in dimension three (see [Sca85]). It is also worth observing that in dimension two all polytopes are simplicial (and hence the corresponding toric variety is $\mathbb{Q}$-factorial by Proposition 3.6.2), something which is clearly not the case in three dimensions.

The classification presented in this chapter is inspired by the work of A. Borisov and L. Borisov [BB, BB92]. Results given in [BB92, Bor00] assure us that a finite classification is possible (see Theorem 3.6.10). The combinatorial approach we adopt is based on that formulated in [BB]. In this unpublished work, the essential steps described can be outlined thus:

(i) Observe that every polytope can be "grown" from a "minimal" polytope;

(ii) These minimal polytopes divide into tetrahedra and non-tetrahedra;

(iii) The minimal tetrahedra can be classified in terms of their barycentric coordinates;

(iv) The minimal non-tetrahedra can be determined directly;

(v) A recursive algorithm can be written, allowing a computer to "grow" these minimal polytopes and hence classify all polytopes of interest.

Although inspired by [BB], the methods employed here are original; in particular the techniques used to derive the tetrahedra (both their barycentric coordinates and their vertices), and the final classification, are new.

The result of Proposition 6.2.5 is a specific case of [BB92, Proposition 3]. However the proof presented here is of an elementary combinatorial nature, in keeping with the style of the remainder of this chapter. In addition the results of Table 6.4 are obtained more explicitly than in [BB92]; again the justification for repeating these results lies in the methods used to obtain them. With a nice restatement of Proposition 6.2.5 (concerning tetrahedra containing one non-vertex lattice point) we obtain a result which closely mirrors one of [Sca85] (concerning tetrahedra containing no non-vertex lattice points), although once more the methods of proof are very different.

Lattice point tetrahedra which contain only one internal lattice point have received considerable attention (e.g. [Rez86, BCF$^+$05]). In 1986 the possible barycentric coordinates (see Proposition 6.2.12) were classified in [Rez86]. Despite continued attention (e.g. [LZ91, Pik01, Maz04, BCF$^+$05]), the resulting tetrahedra remained unknown to the combinatorialists (who were unaware of [BB92]) until the results in this chapter were published ([Kas03]). Reznick recently completed the classification ([Rez06]) using techniques markedly different from those presented here and in [BB92]. Some of these results are discussed in Section 6.4.

For practical reasons the final classification is not presented here, but has been made available on the Internet (see the end of Section 6.6 for the address). We conclude this introduction by presenting a summary of the main features of this list (see also Table 6.7):

Theorem 6.1.1. *Up to isomorphism there exist exactly* 233 *toric* $\mathbb{Q}$*-factorial Fano threefolds, of which* 18 *are smooth. There exist an additional* 401 *having terminal singularities that are not* $\mathbb{Q}$*-factorial. There are exactly* 100 *Gorenstein toric Fano threefolds.*

There exist twelve $\mathbb{Q}$*-factorial minimal cases: eight with Picard number one, two with Picard number two, and two with Picard number three. There exists one minimal case which is not* $\mathbb{Q}$*-factorial, corresponding to a polytope with five vertices.*

There exist nine maximal cases, corresponding to polytopes with 8 (*three occurrences*)*,* 9*,* 10 (*two occurrences*)*,* 11*,* 12 *and* 14 *vertices. Only those with* 8 *vertices are* $\mathbb{Q}$*-factorial.*

6.2 Classifying the Tetrahedra: The Barycentric Coordinates

When we refer to e_1, e_2 or e_3 we mean the standard basis elements of $\mathbb{Z}^3$. If x is a point in $\mathbb{Z}^3$, by $x^{(1)}, x^{(2)}$ and $x^{(3)}$ we mean the integers such that $x = \left(x^{(1)}, x^{(2)}, x^{(3)}\right)$. For any $q \in \mathbb{Q}$ we define $\lfloor q \rfloor := \max\{a \in \mathbb{Z} \mid a \leq q\}$ and $\lceil q \rceil := \min\{a \in \mathbb{Z} \mid a \geq q\}$. The *fractional part* of q, which we shall denote $\{q\}$, is given by $q - \lfloor q \rfloor$.

We will make frequent appeals to the following well-known consequence of Theorem 5.1.1:

Lemma 6.2.1. *Any lattice point free triangle with vertices* $\{0, x_1, x_2\} \subset \mathbb{Z}^3$ *is equivalent under the action of* $GL(3, \mathbb{Z})$ *to the triangle with vertices* $\{0, e_1, e_2\}$*.*

Let $\{x_1, \ldots, x_4\} \subset \mathbb{Z}^3$ be the lattice point vertices of a tetrahedron containing the origin. Let $\mu_1, \ldots, \mu_4 \in \mathbb{Q}$ give the (unique) *barycentric coordinate* of the origin with respect to the x_i;

$$\begin{aligned} \text{ie. } \mu_1 x_1 + \ldots + \mu_4 x_4 &= 0, \\ \mu_1 + \ldots + \mu_4 &= 1, \\ \mu_1 \geq 0, \ldots, \mu_4 &\geq 0. \end{aligned}$$

Choose $\lambda_1, \ldots, \lambda_4 \in \mathbb{Z}_{>0}$ coprime such that $\mu_i = \lambda_i / h$, where $h = \lambda_1 + \ldots + \lambda_4$.

Lemma 6.2.2. *For any* $\kappa \in \{2, \ldots, h-2\}$ *we have that* $\sum_{i=1}^{4} \{\lambda_i \kappa / h\} \in \{1, 2, 3\}$*.*

Proof. Since $\sum_{i=1}^{4} \lambda_i \kappa / h = \kappa \in \mathbb{Z}_{>0}$ it follows that $\sum_{i=1}^{4} \{\lambda_i \kappa / h\} \in \{0, 1, 2, 3\}$. Suppose for some $\kappa \in \{2, \ldots, h-2\}$, $\{\lambda_i \kappa / h\} = 0$ for $i = 1, 2, 3, 4$. We have that $h \mid \kappa \lambda_i$ for each i, so let p be a prime such that $p \mid h$, so that $h = p^r h'$ where $p \nmid h'$. Then $p^r \mid \kappa \lambda_i$. Suppose that $p^r \nmid \kappa$. Then $p \mid \lambda_i$ for each i. Hence $p \mid \gcd\{\lambda_1, \ldots, \lambda_4\} = 1$, a contradiction. Thus $p^r \mid \kappa$. By induction on the prime divisors of h we see that $h \mid \kappa$, so in particular $h \leq \kappa$, which is a contradiction. □

For convenience we make the following definition:

Definition 6.2.3. We say a tetrahedron is *Fano* if the vertices lie at lattice points and the only non-vertex lattice point it contains is the origin, which lies strictly in the interior of the tetrahedron.

Remark 6.2.4. For brevity, and because this chapter concerns itself only with terminal Fano varieties, we have dropped the word "terminal" from Definition 6.2.3 – similarly for Definition 6.5.1. It is hoped that no confusion with Definition 3.7.1 will arise as a consequence.

Proposition 6.2.5. *If the tetrahedron associated with the λ_i is Fano then*

(i) $\sum_{i=1}^{4} \{\lambda_i \kappa / h\} = 2$ *for all* $\kappa \in \{2, \ldots, h-2\}$*, and*

(ii) $\gcd\{\lambda_i, \lambda_j\} = 1$ *for* $i \neq j$*.*

Proof. Let the λ_i be associated with a Fano tetrahedron. Since the origin is strictly in the interior the λ_i are all non-zero. By Lemma 6.2.2 we only need to consider the cases where $\sum_{i=1}^{4} \{\lambda_i \kappa / h\} = 1$ or 3.

Suppose that $\sum_{i=1}^{4} \{\lambda_i \kappa / h\} = 3$ for some κ. Since $\{\lambda_i \kappa / h\} < 1$, it must be that $\{\lambda_i \kappa / h\} \neq 0$ for $i = 1, \ldots, 4$. Hence $\{\lambda_i (h-\kappa)/h\} = 1 - \{\lambda_i \kappa / h\}$ for all i, and so $\sum_{i=1}^{4} \{\lambda_i (h-\kappa)/h\} = 1$.

Suppose for some $\kappa \in \{2, \ldots, h-2\}$ the sum is 1. Let $\chi_i = \{\lambda_i \kappa / h\}$. Then $(\chi_1, \ldots, \chi_4)$ is the (unique) barycentric coordinate for some point in the tetrahedron. We shall show that it is a non-vertex lattice point not equal to the origin.

We have that $\sum_{i=1}^{4} \lfloor \lambda_i \kappa / h \rfloor x_i$ is a lattice point, call it $a \in \mathbb{Z}^3$. We also have that $\sum_{i=1}^{4} \lambda_i \kappa / h x_i = 0$. Thus

$$\sum_{i=1}^{4} \chi_i x_i = \sum_{i=1}^{4} \frac{\lambda_i \kappa}{h} x_i - \sum_{i=1}^{4} \left\lfloor \frac{\lambda_i \kappa}{h} \right\rfloor x_i = -a \in \mathbb{Z}^3.$$

By the uniqueness of barycentric coordinates we have that $-a$ is a non-vertex point, since each $\chi_i < 1$. Furthermore suppose $-a = 0$, so that $\chi_i = \lambda_i/h$ for $i = 1, 2, 3, 4$. For each i, $\lambda_i\kappa/h - \lfloor \lambda_i\kappa/h \rfloor = \lambda_i/h$, so we obtain that $\lfloor \lambda_i\kappa/h \rfloor = \lambda_i(\kappa-1)/h$ and hence that $h \mid \lambda_i(\kappa-1)$. As in the proof of Lemma 6.2.2 we find that $h \mid \kappa - 1$, and so in particular $h + 1 \leq \kappa$. This contradicts our range for κ. Hence $-a$ must be a non-vertex, non-zero lattice point in the tetrahedron, contradicting our hypothesis.

Now suppose for a contradiction that $\gcd\{\lambda_1, \lambda_2\} \neq 1$. We have

$$\frac{\lambda_3}{\gcd\{\lambda_1,\lambda_2\}}x_3 + \frac{\lambda_4}{\gcd\{\lambda_1,\lambda_2\}}x_4 = -\frac{\lambda_1}{\gcd\{\lambda_1,\lambda_2\}}x_1 - \frac{\lambda_2}{\gcd\{\lambda_1,\lambda_2\}}x_2 \in \mathbb{Z}^3.$$

Since the triangle with vertices $\{0, x_3, x_4\}$ is lattice point free, by Lemma 6.2.1 there exists an element of $GL(3,\mathbb{Z})$ mapping $x_3 \mapsto e_1$ and $x_4 \mapsto e_2$. Hence it must be that $\gcd\{\lambda_1,\lambda_2\} \mid \lambda_3$ and $\gcd\{\lambda_1,\lambda_2\} \mid \lambda_4$, thus $\gcd\{\lambda_1,\ldots,\lambda_4\} \neq 1$. □

Corollary 6.2.6. *Let $(\lambda_1, \ldots, \lambda_4)$ be associated with a Fano tetrahedron. Then*

(i) $\sum_{i=1}^4 \lceil \lambda_i\kappa/h \rceil = \kappa + 2$ *for all* $\kappa \in \{2, \ldots, h-2\}$, *and*

(ii) $\gcd\{\lambda_i, h\} = 1$ *for* $i = 1, \ldots, 4$.

Proof. Proposition 6.2.5 tells is that $\sum_{i=1}^4 \{\lambda_i\kappa/h\} = 2$ for all $\kappa \in \{2, \ldots, h-2\}$. Since $\{\lambda_i\kappa/h\} < 1$ it must be that $\{\lambda_i\kappa/h\} = 0$ for at most one value of i. We shall show that, in fact, $\{\lambda_i\kappa/h\} \neq 0$ for all $i = 1, \ldots, 4$.

Suppose (with possible relabelling of the indices) that $\{\lambda_4\kappa/h\} = 0$. This implies that $\sum_{i=1}^3 \{\lambda_i\kappa/h\} = 2$ and hence $\sum_{i=1}^3 \{\lambda_i(h-\kappa)/h\} = 1$, whilst $\{\lambda_4(h-\kappa)/h\} = 0$. But then $\sum_{i=1}^4 \{\lambda_i(h-\kappa)/h\} = 1$, contradicting Proposition 6.2.5. Hence $\{\lambda_i\kappa/h\} \neq 0$ for $i = 1, \ldots, 4$.

By hypothesis we have $\{\lambda_1\kappa/h\} + \ldots + \{\lambda_4\kappa/h\} = 2$, and by definition $\lambda_1\kappa/h + \ldots + \lambda_4\kappa/h = \kappa$. Hence we obtain $\lfloor \lambda_1\kappa/h \rfloor + \ldots + \lfloor \lambda_4\kappa/h \rfloor = \kappa - 2$. We have just demonstrated that $\{\lambda_i\kappa/h\} \neq 0$, and so $\lfloor \lambda_i\kappa/h \rfloor = \lceil \lambda_i\kappa/h \rceil - 1$ for $i = 1, \ldots, 4$. This proves the first part of the proposition.

Finally suppose that, for some i, $\gcd\{\lambda_i, h\} \neq 1$. Then taking $\kappa = h/\gcd\{\lambda_i, h\} \in \{2, \ldots, h-2\}$ we have $\{\lambda_i\kappa/h\} = 0$. Hence $\gcd\{\lambda_i, h\} = 1$. □

Proposition 6.2.7 ([CK99, Lemma 3.5.6])**.** *Let $X = \mathbb{P}(\lambda_0, \ldots, \lambda_n)$ be a weighted projective space, and let $h = \sum_{i=0}^n \lambda_i$. Then X is Gorenstein Fano if and only if $\lambda_i \mid h$ for all i.*

Corollary 6.2.8. *With the exception of those fake weighted projective spaces with weights* $(1,1,1,1)$ *(e.g.* $\mathbb{P}^3$*), no three dimensional fake weighted projective space with at worst terminal singularities is Gorenstein.*

Proof. This is an immediate consequence of Corollory 6.2.6 (ii), Proposition 6.2.7, and Corollary 4.4.9. □

Remark 6.2.9. Proposition 3.9.2 also provides a justification for why there exists only one reflexive terminal Fano tetrahedron: Theorem 5.1.1 tells us that any face must have area $1/2$, hence the tetrahedron has volume $2/3$ and by Proposition 4.2.5 this is the tetrahedron associated with $\mathbb{P}^3$.

This limitation does not exist in higher dimensions – see, for example, Proposition 6.4.1.

Although not required, it is worth observing the similarity between Corollary 6.2.6 and the following:

Proposition 6.2.10 ([Sca85])**.** *Let a lattice point tetrahedron containing no non-vertex lattice points have the vertices of Lemma* 6.3.3 *with* $x, y, z \geq 1$*. Let* $d := x + y + z - 1$*. Then*

(i) $\lceil \kappa x/d \rceil + \lceil \kappa y/d \rceil + \lceil \kappa z/d \rceil = \kappa + 2$ *for all* $\kappa \in \{1, \ldots, d-1\}$*, and*

(ii) $\gcd\{x, d\} = \gcd\{y, d\} = \gcd\{z, d\} = 1$.

Let $h \geq 4$. By making use of Corollary 6.2.6 we can construct bounds on the λ_i. We may assume without loss of generality that $\lambda_1 \leq \ldots \leq \lambda_4$. For each $\kappa \in \{2, \ldots, h-2\}$ and each i let $a_{i,\kappa} := \lceil \lambda_i \kappa / h \rceil$. The following conditions are immediate:

$$\begin{aligned} &a_{1,\kappa} \leq \ldots \leq a_{4,\kappa}, \\ &a_{1,\kappa} + \ldots + a_{4,\kappa} = \kappa + 2, \\ &(a_{1,2}, a_{2,2}, a_{3,2}, a_{4,2}) = (1,1,1,1)\,. \end{aligned} \tag{6.2.1}$$

We have also that $(h/\kappa)(a_{i,\kappa} - 1) < \lambda_i < (h/\kappa) a_{i,\kappa}$, and so:

$$h \max_{2 \leq n \leq \kappa} \frac{1}{n}(a_{i,n} - 1) < \lambda_i < h \min_{2 \leq n \leq \kappa} \frac{1}{n} a_{i,n}.$$

Recalling that $\lambda_i / h = \mu_i$ gives us:

$$\frac{1}{\kappa}(a_{i,\kappa} - 1) < \mu_i < \frac{1}{\kappa} a_{i,\kappa}, \tag{6.2.2}$$

$$\max_{2 \leq n \leq \kappa} \frac{1}{n}(a_{i,n} - 1) < \mu_i < \min_{2 \leq n \leq \kappa} \frac{1}{n} a_{i,n}. \tag{6.2.3}$$

This suggests a recursive method of determining an upper bound for h. Assume $h \geq 4$ is associated with a Fano tetrahedron. Then it is possible to construct a sequence $\{(a_{1,\kappa}, \ldots, a_{4,\kappa})\}_{2 \leq \kappa \leq h-2}$ satisfying the conditions (6.2.1) and (6.2.3) for all $\kappa \in \{2, \ldots, h-2\}$. Moreover we have that for each $\kappa \in \{2, \ldots, h-3\}$ there exists some $i \in \{1, \ldots, 4\}$ such that:

$$a_{j,\kappa+1} = \begin{cases} a_{j,\kappa}, & \text{for } j \neq i; \\ a_{j,\kappa} + 1, & \text{for } j = i. \end{cases}$$

Lemma 6.2.11. *Let $a, k \in \mathbb{N}$ be such that $a < k$. Then $a/k > a/(k+1)$ and $a/k < (a+1)/(k+1)$.*

An immediate consequence of Lemma 6.2.11 is that:

$$\frac{1}{\kappa+1} a_{i,\kappa+1} = \frac{1}{\kappa+1}(a_{i,\kappa} + 1) \geq \min_{2 \leq n \leq \kappa} \frac{1}{n} a_{i,n},$$

and hence, using (6.2.2) and (6.2.3), we obtain:

$$\frac{1}{\kappa+1}(a_{i,\kappa+1} - 1) = \frac{1}{\kappa+1} a_{i,\kappa} < \mu_i < \min_{2 \leq n \leq \kappa+1} \frac{1}{n} a_{i,n} = \min_{2 \leq n \leq \kappa} \frac{1}{n} a_{i,n}.$$

Thus we have the requirement that:

$$\frac{1}{\kappa+1} a_{i,\kappa} < \min_{2 \leq n \leq \kappa} \frac{1}{n} a_{i,n}. \tag{6.2.4}$$

Conditions (6.2.1) and (6.2.4) are independent of h, so by writing a simple recursive function on a computer it is possible to test these conditions for large values of κ, using all the sequences obtained for κ to check whether a sequence exists for $\kappa + 1$. If no such sequence exists we have found an upper bound for h, namely $h \leq \kappa + 2$.

It is worth observing that this method for finding a bound for h really does do that; when all possible sequences have terminated it is impossible to proceed any further. No *a priori* guarantee that this search along all possible sequences will terminate has been given here.

It is also worth noting that the bound this method gives is not the tightest, but this deficiency is balanced by the fact that it providing a technique which is independent of h.

This yields a bound for $h \leq 30$. Proposition 6.2.12 now follows from Proposition 6.2.5 by the easy task of checking all possible λ_i up to this bound. An alternative proof of Proposition 6.2.12 can be found in [BB92].

Proposition 6.2.12. *Let $\lambda_1 \leq \ldots \leq \lambda_4$ be associated with a Fano tetrahedron. Then $(\lambda_1, \ldots, \lambda_4)$ is equal to one of the following:*

$$\begin{array}{llll} (1,1,1,1), & (1,1,1,2), & (1,1,2,3), & (1,2,3,5), \\ (1,3,4,5), & (2,3,5,7), & (3,4,5,7). & \end{array}$$

6.3 Classifying the Tetrahedra: The Coordinates of the Vertices

Let $\{x_1, \ldots, x_4\} \subset \mathbb{Z}^3$ be the lattice point vertices of a Fano tetrahedron. Assume that the indices have been chosen such that $\lambda_1 \leq \ldots \leq \lambda_4$. We represent this tetrahedron by the 3×4 matrix $\begin{pmatrix} x_1^t & \ldots & x_4^t \end{pmatrix}$, where x_i^t denotes the vertex x_i regarded as a column vector.

Proposition 6.3.1. *Let $\lambda_1 \leq \ldots \leq \lambda_4$ be associated with a Fano tetrahedron. Then, by means of the action of $GL(3, \mathbb{Z})$, we can transform the tetrahedron to the form:*

$$\begin{pmatrix} 1 & 0 & k''\lambda_4 - a\lambda_1 & -k''\lambda_3 - b\lambda_1 \\ 0 & 1 & k'\lambda_4 - a\lambda_2 & -k'\lambda_3 - b\lambda_2 \\ 0 & 0 & k\lambda_4 & -k\lambda_3 \end{pmatrix},$$

where $a, b \in \mathbb{Z}, a > 0$ are such that $a\lambda_3 + b\lambda_4 = 1$, and $k, k', k'' \in \mathbb{Z}_{\geq 0}$ are such that:

$$0 \leq k''\lambda_4 - a\lambda_1 < k\lambda_4, \tag{6.3.1a}$$

$$\text{and } 0 \leq k'\lambda_4 - a\lambda_2 < k\lambda_4, \tag{6.3.1b}$$

with one of these inequalities equal to zero only if $\lambda_4 = 1$.

Proof. By virtue of Lemma 6.2.1 we may assume without loss of generality that our tetrahedron has vertices $\{e_1, e_2, x, y\}$ with $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 x + \lambda_4 y = 0$. Thus we see that $\lambda_3 x^{(3)} = -\lambda_4 y^{(3)}$, and so $y^{(3)} = -(\lambda_3/\lambda_4)x^{(3)} \in \mathbb{Z}$. Hence $\lambda_4 \mid \lambda_3 x^{(3)}$, but $\gcd\{\lambda_3, \lambda_4\} = 1$ and so it must be that $\lambda_4 \mid x^{(3)}$. Thus there exists some $k \in \mathbb{Z}$ such that:

$$x^{(3)} = k\lambda_4,$$
$$y^{(3)} = -k\lambda_3.$$

We may take $x^{(3)}$ positive, and so $k \in \mathbb{Z}_{\geq 0}$.

We also have that $\lambda_2 + \lambda_3 x^{(2)} + \lambda_4 y^{(2)} = 0$, so that $\lambda_3 x^{(2)} + \lambda_4 y^{(2)} = -\lambda_2$. Now since $\gcd\{\lambda_3, \lambda_4\} = 1$ there exist $a, b \in \mathbb{Z}, a > 0$ such that $\lambda_3 a + \lambda_4 b = 1$. This gives us that $\lambda_3(-\lambda_2 a) + \lambda_4(-\lambda_2 b) = -\lambda_2$, so that $\lambda_3(x^{(2)} + a\lambda_2) + \lambda_4(y^{(2)} + b\lambda_2) = 0$. Thus there exists some $k' \in \mathbb{Z}$ such that:

$$x^{(2)} = k'\lambda_4 - a\lambda_2,$$
$$y^{(2)} = -k'\lambda_3 - b\lambda_2.$$

Similarly we obtain that there exists some $k'' \in \mathbb{Z}$ such that:

$$x^{(1)} = k''\lambda_4 - a\lambda_1,$$
$$y^{(1)} = -k''\lambda_3 - b\lambda_1.$$

By applying:

$$\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{Z}),$$

for suitably chosen $c, d \in \mathbb{Z}$ we can arrange matters so that (with possible relabelling of k' and k''):

$$0 \leq k'\lambda_4 - a\lambda_2 < k\lambda_4,$$
$$0 \leq k''\lambda_4 - a\lambda_1 < k\lambda_4.$$

Now suppose that $k'\lambda_4 - a\lambda_2 = 0$. Since $\gcd\{\lambda_2, \lambda_4\} = 1$ there must exist some constant $m \in \mathbb{Z}$ such that $k' = m\lambda_2$ and $a = m\lambda_4$. In particular this gives us that $\lambda_4(m\lambda_3 + b) = 1$, so that $\lambda_4 = 1$. Similarly if $k''\lambda_4 - a\lambda_1 = 0$. □

The exceptional case in Proposition 6.3.1 occurring when $\lambda_1 = \ldots = \lambda_4 = 1$ will be dealt with now.

Proposition 6.3.2. *Using the notation introduced above, the only exceptional case is given, up to equivalence, by the tetrahedron with vertices* $\{e_1, e_2, e_3, -e_1 - e_2 - e_3\}$.

Proof. Using the notation introduced in the proof of Proposition 6.3.1, we may take $a = 1, b = 0$ and so taking $k'\lambda_4 - a\lambda_2 = 0$ implies that $k' = 1$. Thus we see that our tetrahedron has the form:

$$\begin{pmatrix} 1 & 0 & k'' - 1 & -k'' \\ 0 & 1 & 0 & -1 \\ 0 & 0 & k & -k \end{pmatrix},$$

where k'' and k are to be determined.

The triangle defined by the origin, the first and the third vertices in the above matrix is lattice point free. Thus,

$$\det \begin{pmatrix} 1 & k'' - 1 \\ 0 & k \end{pmatrix} = \pm 1.$$

This forces $k = 1$ and the resulting tetrahedron is equivalent to that given in the statement. □

The following two results are taken from [Sca85]. A proof is given for the first result because we need to know explicitly the steps required for the transformation.

Lemma 6.3.3 (cf. [Sca85])**.** *A lattice point tetrahedron containing no non-vertex lattice points can, by means of a translation and the action of* $GL(3, \mathbb{Z})$*, be transformed to the form:*

$$\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{pmatrix},$$

where $x, y, z \in \mathbb{Z}, x, y \geq 0, z \geq 1$.

Proof. By applying a translation if necessary and considering Lemma 6.2.1, we may assume without loss of generality that the tetrahedron is in the form:

$$\begin{pmatrix} 0 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & z \end{pmatrix},$$

where $z \geq 1$, but the conditions on x and y remain to be determined.

Let $x \mapsto x \pmod z$ and $y \mapsto y \pmod z$. Observe that this is equivalent to the (left) action of:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{Z}),$$

for suitably chosen $a, b \in \mathbb{Z}$. Thus we can assume that $0 \leq x < z$ and $0 \leq y < z$. Suppose that $z < x + y$. Then set:

$$\mu_1 := 1 - \mu_2 - \mu_3 - \mu_4, \qquad \mu_2 := 1 - \frac{x}{z}, \qquad \mu_3 := 1 - \frac{y}{z}, \qquad \mu_4 := \frac{1}{z}.$$

Clearly $\sum \mu_i = 1$, and $\mu_2, \mu_3, \mu_4 \geq 0$. We have also that $\mu_1 = (x + y - z - 1)/z \geq 0$. But then:

$$\mu_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

gives us a non-vertex lattice point in the interior of the tetrahedron, a contradiction. Thus it must be that $z \geq x + y$.

Finally we apply the unimodular transformation $z \mapsto -x - y + z + 1$ which yields the result. $\square$

Proposition 6.3.4 (cf. [Sca85], Simplification of Howe's Theorem)**.** *Let a lattice point tetrahedron containing no non-vertex lattice points have the vertices of Lemma* 6.3.3 *with* $x, y, z \geq 1$. *Then* $\{x, y, z\} \cap \{1\} \neq \emptyset$.

Let us now consider a Fano tetrahedron presented in the form given in Proposition 6.3.1. In addition we shall assume that we are not looking at the case handled in Proposition 6.3.2. The tetrahedron with vertices given by $\{0, e_1, e_2, (x, y, z)\}$, where:

$$x := k''\lambda_4 - a\lambda_1 \geq 1, \quad y := k'\lambda_4 - a\lambda_2 \geq 1, \quad z := k\lambda_4 \geq 1,$$

is lattice point free. By following the proof of Lemma 6.3.3 we see that it is equivalent to:

$$\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z - x - y + 1 \end{pmatrix},$$

and that $z \geq x + y$. Proposition 6.3.4 tells us that $\{x, y, z - x - y + 1\} \cap \{1\} \neq \emptyset$.

<table>
<tr><th>$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$</th><th>$a$</th><th>$b$</th><th>$(1 + a\lambda_1)/\lambda_4$</th><th>$(1 + a\lambda_2)/\lambda_4$</th><th>$a(\lambda_1 + \lambda_2)/\lambda_4$</th></tr>
<tr><td>$(1,1,1,1)$</td><td>1</td><td>0</td><td colspan="2">2</td><td>2</td></tr>
<tr><td>$(1,1,1,2)$</td><td>1</td><td>0</td><td colspan="2">1</td><td>1</td></tr>
<tr><td>$(1,1,2,3)$</td><td>2</td><td>-1</td><td colspan="2">1</td><td>-</td></tr>
<tr><td>$(1,2,3,5)$</td><td>2</td><td>-1</td><td>-</td><td>1</td><td>-</td></tr>
<tr><td>$(1,3,4,5)$</td><td>4</td><td>-3</td><td>1</td><td>-</td><td>-</td></tr>
<tr><td>$(2,3,5,7)$</td><td>3</td><td>-2</td><td>1</td><td>-</td><td>-</td></tr>
<tr><td>$(3,4,5,7)$</td><td>3</td><td>-2</td><td>-</td><td>-</td><td>3</td></tr>
</table>

Table 6.1: The values depending on a.

Thus:

$$\begin{aligned} \text{either } k'' = (1 + a\lambda_1)/\lambda_4 \in \mathbb{Z} \quad &\text{if and only if} \quad x = 1; \\ \text{or } k' = (1 + a\lambda_2)/\lambda_4 \in \mathbb{Z} \quad &\text{if and only if} \quad y = 1; \\ \text{or } k - k' - k'' = -a(\lambda_1 + \lambda_2)/\lambda_4 \in \mathbb{Z} \quad &\text{if and only if} \quad z - x - y + 1 = 1. \end{aligned}$$

The result of applying this to the barycentric coordinates found in Proposition 6.2.12 is given in Table 6.1. Observe that the only cases of ambiguity are for $(1,1,1,1)$ and $(1,1,1,2)$.

Proposition 6.3.5. *Let $\lambda_1 \leq \ldots \leq \lambda_4$ be associated with a Fano tetrahedron presented in the form given in Proposition* 6.3.1*. Then:*

$$\begin{aligned} 0 \leq k\lambda_3 - k''\lambda_3 - b\lambda_1 < k\lambda_3, \\ \textit{and } 0 \leq k\lambda_3 - k'\lambda_3 - b\lambda_2 < k\lambda_3, \end{aligned}$$

with one of these inequalities equal to zero only if $\lambda_3 = 1$, in which case the tetrahedron is equivalent either to that given in Proposition 6.3.2 *or to:*

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

Proof. Since $a\lambda_3 + b\lambda_4 = 1$ we have that $a = (1 - b\lambda_4)/\lambda_3$. By substituting this into equation (6.3.1a) we obtain $\lambda_1/\lambda_4 \leq k''\lambda_3 + b\lambda_1 < k\lambda_3 + \lambda_1/\lambda_4$. Splitting this into two inequalities yields:

$$k\lambda_3 - k''\lambda_3 - b\lambda_1 > -\lambda_1/\lambda_4,$$
$$\text{and } k\lambda_3 - k''\lambda_3 - b\lambda_1 \leq k\lambda_3 - \lambda_1/\lambda_4.$$

Recall that $\lambda_1/\lambda_4 \in (0,1]$. Hence we have that $0 \leq k\lambda_3 - k''\lambda_3 - b\lambda_1 < k\lambda_3$. If instead we start with equation (6.3.1b) we derive that $0 \leq k\lambda_3 - k'\lambda_3 - b\lambda_2 < k\lambda_3$.

Now suppose that $k\lambda_3 - k''\lambda_3 - b\lambda_1 = 0$. Then we have that $(k - k'')\lambda_3 = b\lambda_1$, and since $\gcd\{\lambda_1, \lambda_3\} = 1$ there must exist some $c \in \mathbb{Z}$ such that $k - k'' = c\lambda_1$ and $b = c\lambda_3$. But then $a\lambda_3 + c\lambda_3\lambda_4 = 1$, which forces $\lambda_3 = 1$ (as required). The only cases where $\lambda_3 = 1$ are when $a = 1, b = 0$. Hence $k = k''$.

There are two possible choices for λ_4. First consider the case where $\lambda_4 = 1$. We have that $k \geq k'' + k' - 2$, and $k' \geq 2$. Thus $k' = 2$. Hence we see that our Fano tetrahedron is equivalent to the form:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & k & -k \end{pmatrix}.$$

The triangle with vertices given by the origin and the second and fourth column of the above matrix is lattice point free. By Lemma 6.2.1 it must be that $k = 1$, which gives a tetrahedron equivalent to that derived in Proposition 6.3.2.

Finally, consider the case where $\lambda_4 = 2$. We have that $k \geq k'' + k' - 1$, and $k' \geq 1$. Thus $k' = 1$. Hence we see that our Fano tetrahedron is equivalent to the form:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2k & -k \end{pmatrix}.$$

As before we see that $k = 1$ and the result follows. □

We consider a Fano tetrahedron presented in the form given in Proposition 6.3.1 and assume we are not looking at the case handled in Proposition 6.3.5. By Proposi-

<table>
<tr><th>$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$</th><th>$a$</th><th>$b$</th><th>$(1 + b\lambda_1)/\lambda_3$</th><th>$(1 + b\lambda_2)/\lambda_3$</th><th>$b(\lambda_1 + \lambda_2)/\lambda_3$</th></tr>
<tr><td>$(1,1,1,1)$</td><td>1</td><td>0</td><td colspan="2">1</td><td>0</td></tr>
<tr><td>$(1,1,1,2)$</td><td>1</td><td>0</td><td colspan="2">1</td><td>0</td></tr>
<tr><td>$(1,1,2,3)$</td><td>2</td><td>-1</td><td colspan="2">0</td><td>-1</td></tr>
<tr><td>$(1,2,3,5)$</td><td>2</td><td>-1</td><td>0</td><td>-</td><td>-1</td></tr>
<tr><td>$(1,3,4,5)$</td><td>4</td><td>-3</td><td>-</td><td>-2</td><td>-3</td></tr>
<tr><td>$(2,3,5,7)$</td><td>3</td><td>-2</td><td>-</td><td>-1</td><td>-2</td></tr>
<tr><td>$(3,4,5,7)$</td><td>3</td><td>-2</td><td>-1</td><td>-</td><td>-</td></tr>
</table>

Table 6.2: The values depending on b.

<table>
<tr><th>$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$</th><th>$k$</th><th>$k'$</th><th>$k''$</th></tr>
<tr><td rowspan="2">$(1,1,1,1)$</td><td>k</td><td>$k-2$</td><td>2</td></tr>
<tr><td>k</td><td>3</td><td>$k-1$</td></tr>
<tr><td rowspan="2">$(1,1,1,2)$</td><td>k</td><td>$k-1$</td><td>1</td></tr>
<tr><td>k</td><td>$k-1$</td><td>2</td></tr>
<tr><td rowspan="2">$(1,1,2,3)$</td><td>k</td><td>k</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>$(1,2,3,5)$</td><td>k</td><td>1</td><td>k</td></tr>
<tr><td rowspan="2">$(1,3,4,5)$</td><td>k</td><td>$k-2$</td><td>1</td></tr>
<tr><td>k</td><td>$k+2$</td><td>1</td></tr>
<tr><td>$(2,3,5,7)$</td><td>k</td><td>$k+1$</td><td>1</td></tr>
<tr><td>$(3,4,5,7)$</td><td>k</td><td>2</td><td>$k+1$</td></tr>
</table>

Table 6.3: The relationships among k, k' and k''.

tion 6.3.4 we have that:

$$\begin{aligned} \text{either } k - k'' &= \frac{1 + b\lambda_1}{\lambda_3} \in \mathbb{Z}; \\ \text{or } k - k' &= \frac{1 + b\lambda_2}{\lambda_3} \in \mathbb{Z}; \\ \text{or } k - k' - k'' &= b\frac{\lambda_1 + \lambda_2}{\lambda_3} \in \mathbb{Z}. \end{aligned}$$

The result of applying this to the barycentric coordinates found in Proposition 6.2.12 is presented in Table 6.2. The results of Table 6.1 and Table 6.2 complement each other beautifully, allowing the relationships amongst k, k' and k'' shown in Table 6.3 to be established.

We are now in a position to calculate the vertices of the Fano tetrahedra (up to the action of $GL(3, \mathbb{Z})$). We will proceed by taking each barycentric coordinate in turn and combining the results of Table 6.3 and Proposition 6.3.1. The final results are collected together in Table 6.4. It is worth comparing this with the results of [Suz04].

Recall that the matrix $\begin{pmatrix} x_1^t & \dots & x_4^t \end{pmatrix}$ is used to represent the tetrahedron with vertices $\{x_1, \dots, x_4\}$. In what follows, references to the vertex x_i should be regarded as references to the i^{th} column of this matrix.

(i) First we consider the case with barycentric coordinate $(1,1,1,1)$. From the results of Table 6.3 and Proposition 6.3.1 we have that our Fano tetrahedron has two possible forms, both of which are equivalent to:

$$\begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & k & -k \end{pmatrix}.$$

We observe that x_3 tells us that $\gcd\{3,k\} = 1$ and x_4 tells us that $\gcd\{2,k\} = 1$. Furthermore, taking $k = 1$ gives us a tetrahedron equivalent to that found in Proposition 6.3.2. Suppose that $k \geq 7$. Then $(4/k)x_2 + (2/k)x_3 + (1/k)x_4 = e_3$, which contradicts our tetrahedron being Fano. Thus the only remaining possibility is that $k = 5$, which by inspection we see does indeed give us a Fano tetrahedron.

(ii-a) Now we consider the case with barycentric coordinate $(1,1,1,2)$. By Table 6.3 and Proposition 6.3.1 we see once more that our Fano tetrahedron can take two possible forms. First we consider the form equivalent to:

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 2k & -k \end{pmatrix}.$$

If we take $k = 1$ we obtain a Fano tetrahedron equivalent to that found in Proposition 6.3.5. Suppose that $k = 2$. Then $(1/2)(1,-3,4) + (1/2)(-1,1,-2) = (0,-1,1)$ is a non-vertex, non-zero lattice point in the interior of the tetrahedron, and hence it is not Fano. The third column tells us that $\gcd\{3,k\} = 1$. Finally, the tetrahedron is not Fano if $k \geq 4$ since then $(2/k)x_2 + (1/k)x_3 + (1/k)x_4 = e_3$.

(ii-b) Now we consider the second possibility, which is equivalent to:

$$\begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 2k & -k \end{pmatrix}.$$

When $k = 1$ we obtain a Fano tetrahedron equivalent to the one previously found. x_3 and x_4 tell us that $\gcd\{3, k\} = 1$ and $\gcd\{2, k\} = 1$, respectively, and if $k \geq 7$ we have the non-vertex, non-zero internal lattice point given by $(3/k)x_1 + (1/k)x_3 + (3/k)x_4 = -e_3$. Thus the only remaining possibility is $k = 5$, which contains the lattice point $(1/5)(1, 0, 0) + (2/5)(3, -3, 10) + (1/5)(-2, 1, -5) = (1, -1, 3)$.

(iii) For barycentric coordinate $(1, 1, 2, 3)$ the two possibilities are (up to equivalence):

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3k & -2k \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -2 \end{pmatrix}.$$

The third column tells us that k must be odd, but if $k \geq 3$ we have the interior lattice point $(1/k)x_2 + (1/k)x_3 + (1/k)x_4 = e_3$. Thus the only possibility is that $k = 1$, but the resulting tetrahedron is equivalent to that already found.

(iv) When we have barycentric coordinate $(1, 2, 3, 5)$ our tetrahedron equivalent to:

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 5k & -3k \end{pmatrix}.$$

The third column tells us that k is odd, and if $k \geq 3$ we have the internal lattice point $(1/k)x_1 + (1/k)x_3 + (1/k)x_4 = 2e_3$. By inspection we see that the case where $k = 1$ is Fano.

(v-a) For barycentric coordinate $(1, 3, 4, 5)$ we have two possibilities. First we consider the case where our tetrahedron is equivalent to:

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -22 & 17 \\ 0 & 0 & 5k & -4k \end{pmatrix}.$$

x_3 tells us that k is odd. If $k \geq 7$ then it is not Fano, since $(5/k)x_2 + (1/k)x_3 + (1/k)x_4 = e_3$. If $k = 5$ then $(1/5)\,(1,-22,25) + (1/5)\,(-1,17,-20) = (0,-1,1)$, and if $k = 3$ then $k = 3$ then $(1/3)e_1 + (1/3)e_2 + (1/3)\,(-1,17,-12) = (0,6,-4)$. By inspection we see that the case where $k = 1$ is Fano.

(v-b) The second possibility is the tetrahedron equivalent to:

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 5k & -4k \end{pmatrix}.$$

We require that k is odd, but if $k \geq 3$ we obtain the point $(1/k)x_2 + (1/k)x_3 + (1/k)x_4 = e_1$, and when $k = 1$ we obtain the tetrahedron found above.

(vi) Continuing in the same vein, for barycentric coordinate $(2,3,5,7)$ we have:

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 7k & -5k \end{pmatrix}.$$

This tells us that k is odd, and if $k \geq 3$ we obtain the internal lattice point $(1/k)x_2 + (1/k)x_3 + (1/k)x_4 = 2e_3$. Thus $k = 1$ is the only possibility, and we see by inspection that it is indeed Fano.

(vii) Finally consider barycentric coordinate $(3,4,5,7)$. This gives us:

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 7k & -5k \end{pmatrix}.$$

Once more we see that k must be odd, and that if $k \geq 3$ then it is not Fano since we have $(1/k)x_1 + (1/k)x_3 + (1/k)x_4 = 2e_3$. When $k = 1$ we do indeed get a Fano tetrahedron.

6.4 Some Remarks on Lattice Points in Tetrahedra

As mentioned in Section 6.1, (terminal) Fano tetrahedra (or *one-point lattice tetrahedra*) have been independently studied by many combinatorialists. The barycentric coordi-

$(1/4)\,(1,1,1,1)$	$(1/4)\,(1,1,1,1)$	$(1/5)\,(1,1,1,2)$	$(1/7)\,(1,1,2,3)$
$\begin{pmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&-1\end{pmatrix}$	$\begin{pmatrix}1&0&1&-2\\0&1&-3&2\\0&0&5&-5\end{pmatrix}$	$\begin{pmatrix}1&0&1&-1\\0&1&1&-1\\0&0&2&-1\end{pmatrix}$	$\begin{pmatrix}1&0&1&-1\\0&1&-2&1\\0&0&3&-2\end{pmatrix}$
$(1/11)\,(1,2,3,5)$	$(1/13)\,(1,3,4,5)$	$(1/17)\,(2,3,5,7)$	$(1/19)\,(3,4,5,7)$
$\begin{pmatrix}1&0&-2&1\\0&1&1&-1\\0&0&5&-3\end{pmatrix}$	$\begin{pmatrix}1&0&1&-1\\0&1&-2&1\\0&0&5&-4\end{pmatrix}$	$\begin{pmatrix}1&0&1&-1\\0&1&-2&1\\0&0&7&-5\end{pmatrix}$	$\begin{pmatrix}1&0&-2&1\\0&1&2&-2\\0&0&7&-5\end{pmatrix}$

Table 6.4: The vertices of the Fano tetrahedra, up to the action of $GL(3,\mathbb{Z})$

nates of Section 6.2 were discovered in [Rez86]. An initial attempt in [LZ91, pg. 1023] to bound the volume of the possible tetrahedra gave an upper volume of 14^{48}. This bound was refined considerably in [Pik01, Lemma 5] (see Proposition 4.4.16), giving a maximum volume of $85/6$. The two tetrahedra with barycentric coordinates $(1/4)(1,1,1,1)$ were found in [Maz04], and the problem received renewed attention in [BCF$^+$05]. The classification of Section 6.3 was eventually reproduced in [Rez06][1].

The details of the techniques employed differ somewhat from those of Sections 6.2–6.3. It can be shown that any tetrahedron can be presented in the form:

$$T_{a,b,c} := \operatorname{conv}\{0, e_1, e_2, (a,b,c)\}\,.$$

[Ree57] observed that $T_{1,1,n}$ is lattice-free for any $n \in \mathbb{Z}_{\geq 0}$. In fact:

Proposition 6.4.1 ([Ree57, pp. 389–390])**.** *A lattice tetrahedron is lattice-free if and only if it can be presented in the form $T_{0,0,1}$ or $T_{a,b,c}$ for some $c \geq 2$, $0 \leq a, b \leq c-1$ and $\gcd\{a,c\} = \gcd\{b,c\} = \gcd\{1-a-b, c\} = 1$.*

This result is equivalent to Lemma 6.3.3, and to the following important result of White:

Definition 6.4.2. A polytope $P \subset N_{\mathbb{R}}$ has *lattice width* given by the minimum of the lengths of its projections to $\mathbb{R}$ using linear functions on N with integer coefficients.

Proposition 6.4.3 ([Whi64])**.** *Every lattice-free tetrahedron has lattice width one.*

A one-point tetrahedron $T_{a,b,c}$ can be decomposed as four lattice-free tetrahedra, which can each be written in the form of Proposition 6.4.1. Careful analysis of the possibilities yields a set of relations amongst the parameters a, b, c. By having previously

[1] I would like to express my gratitude to Bruce Reznick for providing me with an early draft of [Rez06].

established a bound of 85/6 on the volume of the tetrahredon, it is possible to limit these relations and concluded the classification ([Rez06, Theorem 7]). In the above notation, the tetrahedra of Table 6.4 are equivalent to:

$$T_{3,3,4},\ T_{2,2,5},\ T_{2,4,7},\ T_{2,6,11},\ T_{2,7,13},\ T_{2,9,17},\ T_{2,13,19},\ \text{and } T_{3,7,20}.$$

Reznick makes the following interesting observations:

Corollary 6.4.4 ([Rez06, Corollary 15])**.** *If T is a one-point lattice tetrahedron, then T has lattice width two.*

Proposition 6.4.5 ([Rez06, Theorem 16])**.** *The lattice width of $T_{a,b,c}$ is at most $2\left\lceil c^{1/3}\right\rceil$.*

These results lead to the following conjecture:

Conjecture 6.4.6 ([Rez06, Conjecture 17])**.** *If T is a k-point lattice tetrahedron (i.e. T contains k interior lattice points) then its lattice width is at most $k+1$, and there exists at least one interior lattice point on each of the consecutive lattice planes in any minimal direction.*

6.5 Classifying the Minimal Polytopes

We extend Definition 6.2.3 to any polytope P (see Remark 6.2.4):

Definition 6.5.1. We say a lattice point polytope P in $\mathbb{Z}^3$ is *Fano* if P is convex and the only non-vertex lattice point it contains is the origin, which lies strictly in the interior of the polytope.

Given any Fano polytope P with vertices $\{x_1,\ldots,x_k\}$ we make the following definition:

Definition 6.5.2. We say P is *minimal* if, for all $j\in\{1,\ldots,k\}$, the polytope P' given by the vertices $\{x_1,\ldots,x_k\}\setminus\{x_j\}$ is not Fano.

Definition 6.5.3. Let $M=\{p_1,\ldots,p_k\}$ be a finite set of points in $N_\mathbb{Q}$. The *convex hull* of M is given by $\operatorname{conv} M := \left\{\sum_{j=1}^k \nu_j p_j \mid \nu_j\geq 0 \text{ for all } j, \sum_{j=1}^k \nu_j = 1\right\}\subset N_\mathbb{Q}$.

Let us consider a minimal Fano polytope P. Since $0\in P$ there exist non-coplanar vertices $x_1,\ldots,x_4$ of P such that $0\in\operatorname{conv}\{x_1,\ldots,x_4\} =: P'$.

Either P is equivalent to one of the tetrahedra in Table 6.4, or it is not. If it is not, then minimality gives us that it does not contain a Fano tetrahedron; in particular P' is not Fano. We assume that this is the case.

Since P' is not a Fano tetrahedron it must be that either the origin lies on a face of P' or on an edge of P'. If the origin lies on a face of P' then P contains a Fano triangle. Thus there exist three vertices of P which lie in a plane containing the origin, and the origin lies strictly in the interior of the triangle defined by these three points. This possibility will be discussed in further detail below.

Assume now that P does not contain a Fano triangle. Then it must be that the origin lies on one of the edges of P', say on the edge defined by x_1 and x_2. Since the origin lies in the strict interior of P there must exist distinct vertices $y_1, \ldots, y_4$ of P not equal to x_1 or x_2 such that $\text{conv}\{x_1, x_2, y_1, y_2\}$ is a Fano square and $\text{conv}\{x_1, x_2, y_1, y_2, y_3, y_4\}$ is a Fano octahedron. Minimality gives that P is a Fano octahedron, and these will be classified in Lemma 6.5.4.

We return now to considering in more detail the case where P contains a Fano triangle, say that defined by $\{x_1, x_2, x_3\}$. Since the origin lies in the strict interior of P there must exist vertices y_1 and y_2 lying on either side of the plane containing our Fano triangle. Minimality then gives us that P is precisely the polygon with vertices $\{x_1, x_2, x_3, y_1, y_2\}$.

Now consider the line passing through the origin and y_1. This line crosses the polytope P at points $y_1 \in \mathbb{Z}^3$ and x not necessarily in $\mathbb{Z}^3$. There are three possible locations for x:

(i) x is equal to y_2. Then $y_2 = -y_1$. These will be classified in Lemma 6.5.5.

(ii) x lies on the edge with endpoints $\{x_1, y_2\}$. Then $\text{conv}\{x_1, y_1, y_2\}$ is a Fano triangle. We use the fact that the origin has barycentric coordinate $(1/3, 1/3, 1/3)$ with respect to $\{x_1, y_1, y_2\}$. Thus the line passing through x_1 and the origin bisects the line with endpoints $\{y_1, y_2\}$ at a point x', say. Now the length of the line joining $\{x_1, 0\}$ is twice the length of the line joining $\{x', 0\}$. Similarly by considering the Fano triangle $\text{conv}\{x_1, x_2, x_3\}$, the line passing through x_1 and the origin bisects the line with endpoints $\{x_2, x_3\}$ at a point x'', say, and we have that the distance from x_1 to the origin is twice the length of the line joining the origin to x''. Hence we see that $\{x_2, x_3, y_1, y_2\}$ are coplanar and form a parallelogram. These will be classified in Lemma 6.5.6.

(iii) x' lies strictly in the interior of the triangle $\text{conv}\{x_1, x_2, y_2\}$. This then forces

$\mathrm{conv}\{x_1, x_2, y_1, y_2\}$ to be a Fano tetrahedron, contradicting our assumption.

Lemma 6.5.4. *The vertices of the minimal Fano octahedra (up to the action of $GL(3, \mathbb{Z})$) are given by:*

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix}.$$

Proof. By making use of Lemma 6.2.1 and recalling that P does not contain a Fano triangle, we may take the vertices of P to be $\{e_1, -e_1, e_2, -e_2, x_1, x_2\}$. We observe that $x_1 = -x_2$, for otherwise we would have that P contains a Fano tetrahedron. So take $x = -x_2 = x_1 = (a, b, c)$. First we shall show that, without loss of generality, we may take a, b, c such that:

$$0 \leq a \leq b \leq \frac{c}{2}. \tag{6.5.1}$$

Trivially we may assume that $0 \leq a \leq b$. Suppose that $b > c/2$. Then $b - c > -c/2$ and so $c - b < c/2$. This process corresponds to the action of $GL(3, \mathbb{Z})$ transforming

$$\begin{pmatrix} 1 & 0 & -1 & 0 & a & -a \\ 0 & 1 & 0 & -1 & b & -b \\ 0 & 0 & 0 & 0 & c & -c \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & -1 & 0 & a & -a \\ 0 & -1 & 0 & 1 & c-b & -(c-b) \\ 0 & 0 & 0 & 0 & c & -c \end{pmatrix}.$$

Hence we may assume that the inequality (6.5.1) holds.

Now consider the point e_3. Either $x = e_3$ or e_3 lies outside of P. The first possibility gives us the first Fano octahedron. The second possibility tells us that e_3 must lie on the opposite side to the origin of the plane defined by $\{-e_1, -e_2, x\}$. This plane intersects the z-axis at the point $(0, 0, c/(a + b + 1))$. This gives us that $c \leq a + b$. Combining this with (6.5.1) gives us that $b \leq a$ and so $b = a$. This in turn gives us that $c \leq 2b$ and $2b \leq c$, and so we obtain $2a = 2b = c$. Thus (up to the action of $GL(3, \mathbb{Z})$) we have that $a = 1, b = 1, c = 2$, which gives us the second Fano octahedron. □

Lemma 6.5.5. *If P is a minimal Fano polytope with vertices $\{x_1, x_2, x_3, y_1, -y_1\}$ such that $\{x_1, x_2, x_3\}$ are the vertices of a Fano triangle, then P is equal (up to the action of $GL(3, \mathbb{Z})$) to one of:*

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 3 & -3 \end{pmatrix}.$$

Proof. By making use of Lemma 6.2.1 we may assume that the vertices of P are given by $\{e_1, e_2, -e_1 - e_2, x, y\}$. If $y \neq -x$ then P would contain a Fano tetrahedron, which contradicts minimality. Let $x = (a, b, c)$. We claim that, without loss of generality, we may take a, b, c such that $0 < a \leq b \leq c$ and:

$$a + b \leq c. \tag{6.5.2}$$

Clearly we can take $0 < a \leq b$ and $c > 0$. Suppose that $a + b > c$. Then we have that $(c - a) + (c - b) < c$. By using the fact that $y = -x$ and applying the transformation:

$$\begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & -c \\ 0 & 0 & -1 \end{pmatrix} \in GL(3, \mathbb{Z}),$$

we see that we may assume that the inequality (6.5.2) holds.

Now consider the point e_3. Either $x = e_3$ or e_3 lies outside of P. The first case gives us the first Fano polytope in the statement. The second case tells us that we have e_3 lies on the opposite side to the origin of the plane defined by $\{e_1, -e_1 - e_2, x\}$. This plane intersects the z-axis at the point $(0, 0, c/(2b - a + 1))$, and so:

$$2b - a \geq c. \tag{6.5.3}$$

Now consider the point $x' = e_2 + e_3$. Either $x' = x$, which gives a Fano polytope equivalent to the one previously found, or x' lies outside of P. If this is the case we have that x' lies on the opposite side to the origin of the plane defined by $\{e_2, -e_1 - e_2, x\}$. This plane intersects the line passing through the origin and $e_2 + e_3$ at the point $(0, k, k)$ where $k := c/(2a - b + c + 1)$. Hence:

$$b \leq 2a. \tag{6.5.4}$$

Now suppose both e_3 and x' lie outside P. Combining the inequalities (6.5.2) and (6.5.3) gives us that $2a \leq b$, and so by (6.5.4) we obtain that $2a = b$. Thus (up to the action of $GL(3, \mathbb{Z})$) we have that $a = 1, b = 2, c = 3$. A quick check confirms that this is indeed Fano. $\square$

Lemma 6.5.6. *If P is a minimal Fano polytope with vertices $\{x_1, x_2, x_3, x_4, x_5\}$ such that $\{x_2, x_3, x_4, x_5\}$ are coplanar and give the vertices of a parallelogram, then P is equal* (*up to the* *action of* $GL(3,\mathbb{Z})$) *to:*

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Proof. Since P does not contain a Fano tetrahedron it must be that opposite corners of the parallelogram, along with x_1, give us a Fano triangle. Thus we can (by virtue of Lemma 6.2.1) write P in the form:

$$\begin{pmatrix} 1 & 0 & -1 & a+1 & -a \\ 0 & 1 & -1 & b+1 & -b \\ 0 & 0 & 0 & c & -c \end{pmatrix},$$

where $0 < a+1 \leq b+1 \leq c$.

Consider the point $-e_3$. Either $a = 0, b = 0, c = 1$, which gives the Fano polytope in the statement, or $-e_3$ lies outside P. Thus $-e_3$ lies on the opposite side to the origin of the plane defined by $\{e_1, e_2, (-a, -b, -c)\}$. This plane intersects the z-axis at the point $(0, 0, c/(a+b+1))$. Thus we have that $-c > -a-b-1$ and so:

$$c \leq a+b. \tag{6.5.5}$$

Now let $x' = e_1 + e_2 + e_3$. Either $a = 0, b = 0, c = 1$, which gives the Fano polytope in the statement, or x' lies outside P. Thus x' lies on the opposite side to the origin of the plane defined by $\{e_1, e_2, (a+1, b+1, c)\}$. Thus the plane intersects the line through the origin and x' at the point (k, k, k), where $k := c/(2c-a-b-1)$. Thus we see that $c < 2c-a-b-1$ and so:

$$c > a+b+1. \tag{6.5.6}$$

Now suppose both $-e_3$ and x' lie outside P. But then both inequalities (6.5.5) and (6.5.6) must be satisfied, which is impossible. □

Combining the results of Table 6.4 and Lemmas 6.5.4–6.5.6 we obtain Table 6.5.

Comments	Vertices
4 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}$
4 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 7 & -5 \end{pmatrix}$
4 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 7 & -5 \end{pmatrix}$
4 Vertices Regular	$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$
4 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 5 & -4 \end{pmatrix}$
4 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 5 & -3 \end{pmatrix}$
4 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix}$
4 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \end{pmatrix}$
5 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & -1 & -2 \\ 0 & 0 & 3 & 0 & -3 \end{pmatrix}$
5 Vertices Regular	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$
5 Vertices	$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$
6 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & 0 & -2 \end{pmatrix}$
6 Vertices Regular	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$

Table 6.5: The vertices of the minimal Fano polytopes, up to the action of $GL(3, \mathbb{Z})$

6.6 Classifying All Fano Polytopes

Given any Fano polytope P with vertices $\{x_1, \ldots, x_k\}$ we make the following definition (cf. Definition 8.2.5):

Definition 6.6.1. We say P is *maximal* if, for all $x_{k+1} \in \mathbb{Z}^3 \setminus \{x_1, \ldots, x_k\}$, the polytope P'' given by the vertices $\{x_1, \ldots, x_k, x_{k+1}\}$ is not Fano.

We will also make the following non-standard definition:

Definition 6.6.2. Let $P = \mathrm{conv}\{x_1, \ldots, x_k\}$ and P'' be Fano polytopes and let $x_{k+1} \in \mathbb{Z}^3$ be a point such that, up to the action of $GL(3, \mathbb{Z})$, $P'' = \mathrm{conv}\{x_1, \ldots, x_k, x_{k+1}\}$. Then we say that P is the *parent* of P'', and that P'' is the *child* of P.

Clearly a polytope P is minimal if and only if it has no parents, and is maximal if and only if it has no children.

Let P be any Fano polytope. Then the following results are immediate:

(i) Any Fano polytope can be obtained from a (not necessarily unique) minimal Fano polytope by consecutive addition of vertices.

(ii) The number of possible vertices that can be added to P to create a Fano polytope P'' is finite. For suppose P has vertices $\{x_1, \ldots, x_n\}$ and the vertex x_{n+1} is to be added. Then the line through x_{n+1} and the origin, extended in the direction away from x_{n+1}, crosses ∂P at some point x', not necessarily in $\mathbb{Z}^3$. x' corresponds to either a vertex point of P, lies on an edge of P, or lies on a face.

The first possibility gives us that $x_{n+1} = -x_i$ for some $i \in \{1, \ldots, n\}$. The second possibility tells us that $\mathrm{conv}\{x_i, x_j, x_{n+1}\}$ is an Fano triangle for some distinct $i, j \in \{1, \ldots, n\}$, and hence that $x_{n+1} = -x_i - x_j$. The final possibility splits naturally into two cases.

The first case corresponds to being able to choose three vertices x_i, x_j, x_k defining the face such that $\mathrm{conv}\{x_i, x_j, x_k, x_{n+1}\}$ is a Fano tetrahedron (where i, j, k are necessarily distinct, $i, j, k \in \{1, \ldots, n\}$). Hence $\lambda_{\sigma 1} x_i + \lambda_{\sigma 2} x_j + \lambda_{\sigma 3} x_k + \lambda_{\sigma 4} x_{n+1} = 0$ for some $(\lambda_1, \ldots, \lambda_4)$ in Proposition 6.2.12 and some $\sigma \in S_4$.

The second case corresponds to such a selection being impossible. In this case the face containing x' has four vertices which, up to possible renumbering, correspond to the vertices x_1, x_2, x_3 and x_4, and x' equals the intersection of the lines joining x_1 to x_3 and x_2 to x_4. Thus $\mathrm{conv}\{x_1, x_3, x_{n+1}\}$ is a Fano triangle, and so $x_{n+1} = -x_1 - x_3$ (or, equivalently, equals $-x_2 - x_4$).

Comments	Vertices
8 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 & 2 & -1 \end{pmatrix}$
8 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 1 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 \end{pmatrix}$
8 Vertices Simplicial	$\begin{pmatrix} 1 & 0 & 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & 1 & -1 & -1 & 1 \\ 0 & 0 & 5 & -5 & -2 & 2 & 1 & -1 \end{pmatrix}$
9 Vertices	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & -1 & 0 & -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$
10 Vertices	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$
10 Vertices	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$
11 Vertices	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \end{pmatrix}$
12 Vertices	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}$
14 Vertices	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & -1 \end{pmatrix}$

Table 6.6: The vertices of the maximal Fano polytopes, up to the action of $GL(3, \mathbb{Z})$

(iii) If $\{x_1, \ldots, x_n\}$ are the vertices of P, and the Fano polytope P'' is created by adding the vertex x_{n+1}, then:

$$P'' \setminus P \subset \bigcup_{i,j} \operatorname{conv}\{0, x_i, x_j, x_{n+1}\}.$$

Vertices	4	5	6	7	8	9	10	11	12	13	14
Polytopes	8	38	95	144	151	107	59	21	8	2	1
Simplicial	8	35	75	74	35	5	1	0	0	0	0
Minimal	8	3	2	0	0	0	0	0	0	0	0
Maximal	0	0	0	0	3	1	2	1	1	0	1

Table 6.7: The number of Fano polytopes in $\mathbb{Z}^3$.

Using these results and our list of minimal Fano polytopes, it is a relatively straightforward task to write a recursive function to allow a computer to calculate all the Fano polytopes up to the action of $GL(3, \mathbb{Z})$. In particular, (ii) asserts that the calculation will terminate, since the list is finite; a stronger finiteness result to include ε-log-canonical toric Fano varieties ($0 < \varepsilon \leq 1$) is given by Theorem 3.6.10.

The source code for such a function is available on the Internet at:

`http://www.maths.bath.ac.uk/~mapamk/code/Polytope_Classify.c`.

Using this code a complete classification was obtained in under 20 minutes on an average personal computer. The classification is available at:

`http://www.maths.bath.ac.uk/~mapamk/pdf/Fano_List.pdf` (or `.ps`).

A searchable version of the classification, listing various geometric data associated with each variety, can be accessed via [Bro06]. The geometric data available for each variety are: the degree; the Gorenstein index; the Fano index; the Hilbert series; the basket of singularities.

The maximal polytopes are reproduced in Table 6.6, and a summaries of the results are given in Theorem 6.1.1 and in Table 6.7.

CHAPTER 7

Some Bounds on Fano Polytopes

7.1 Fano Polygons and the Number Twelve

Let $P \subset N_{\mathbb{R}}$ be a Fano polygon. In Section 5.2 it was shown that there exist precisely sixteen such polygons (up to equivalence), which are listed in Table 5.1. Corollary 5.1.3 tells us that each of these sixteen polygons is reflexive – i.e. the dual polygon $P^{\vee} \subset M_{\mathbb{R}}$ is also a Fano polygon. These facts are well-documented in the literature, along with the following intriguing result[1]:

Theorem 7.1.1. *Let $P \subset N_{\mathbb{R}}$ be a Fano polygon. Then:*

$$|\partial P \cap N| + \left|\partial P^{\vee} \cap M\right| = 12.$$

In [PRV00] four proofs are given to Theorem 7.1.1. The first method is exhaustion: by studying Table 5.1 one can explicitly calculate the dual in each case and hence verify the theorem. Two methods of proof are essentially given in [Ful93, pp. 42–4]: one of these methods is 'toric' in nature and will be reproduced below. The fourth proof involves the study of modular forms. A fifth proof is given in [HS02].

In addition to the 'toric' proof, we give a new proof which relies on Lemma 5.2.3.

A 'Toric' Proof

We require the following trivial application of Theorem 5.1.1:

[1] Victor Batyrev informs me that this result appeared in one of his early unpublished works.

Lemma 7.1.2. *Let P be a Fano polygon. Then:*

$$\operatorname{vol} P = \frac{|\partial P \cap N|}{2}.$$

Remark 7.1.3. Observe that $|\partial P \cap N| = \operatorname{vol} \partial P$. Here $\operatorname{vol} \partial P$ is calculated by summing the relative lattice volume of each face of P. This is an instance of a more general result: see Proposition 3.9.2.

Proof of Theorem 7.1.1. Let $P \subset N_{\mathbb{R}}$ be a Fano polygon. The associated toric surface $X := X(P)$ need not be smooth, since P need not be regular. If this is the case one simply takes an appropriate disingularisation of X. This is achieved by taking a regular stellar subdivision of the fan Δ_X associated with X (see [Ewa96, §V.6 and Theorem VI.8.5]). In this case such a subdivision corresponds to inserting a ray in the fan Δ_X through each point $\partial P \cap N$ (when no ray already exists). Let us assume that X is just such a desingularisation.

Let $\{\rho_1, \ldots, \rho_d\}$ be the set of primitive lattice generators corresponding to each ray of Δ_X, where $d = |\partial P \cap N|$. Associated with each ρ_i is an irreducible torus-invariant divisor D_i. By [Ful93, pg. 85] we have that the anticanonical divisor $-K$ of X is given by:

$$-K = \sum_{i=1}^{d} D_i.$$

Associated to this divisor is the polytope (see (2.2.1a)):

$$P_{-K} = \{u \in M_{\mathbb{R}} \mid u(\rho_i) \geq -1 \text{ for } i = 1, \ldots, d\}.$$

Since $P = \operatorname{conv}\{\rho_1, \ldots, \rho_d\}$ we have that:

$$P_{-K} = \{u \in M_{\mathbb{R}} \mid u(v) \geq -1 \text{ for all } v \in P\} = P^{\vee}.$$

Proposition 2.2.11 and Lemma 7.1.2 give that:

$$(-K \cdot -K) = 2! \operatorname{vol} P^{\vee} = \left|\partial P^{\vee} \cap M\right|. \tag{7.1.1}$$

We can also compute $(-K \cdot -K)$ directly from *Noether's formula* [Ful93, pg. 86] (or alternatively by observing that the orbit closure $D_i \cong \mathbb{P}^1$ for $i = 1, \ldots, d$). This gives:

$$(-K \cdot -K) = 12 - d. \tag{7.1.2}$$

Finally by equating equations (7.1.1) and (7.1.2) we obtain:

$$12 - |\partial P \cap N| = \left|\partial P^\vee \cap M\right|.$$

□

A New Proof

The proof we give here relies on Definition 5.2.2. Before we proceed, we need to introduce some additional notation.

Definition 7.1.4. Let $\{P_i\}_{i=0}^d$ be a set of distinct n-dimensional Fano polytopes such that P_0 is minimal, P_d is maximal, and:

$$P_0 \subset P_1 \subset \ldots \subset P_{d-1} \subset P_d. \tag{7.1.3}$$

We call (7.1.3) a *filtration*.

Clearly every Fano polytope is a member of some filtration.

Definition 7.1.5. Consider two filtrations whose elements are given by $\{P_i\}_{i=0}^d$ and $\left\{Q_j\right\}_{j=0}^h$. We say that the first filtration is a *refinement* of the second if $P_0 = Q_0$, $P_d = Q_h$, and there exist $0 < i_1 < \ldots < i_{h-1} < d$ such that $P_{i_j} = Q_j$ for $j = 1, \ldots, h-1$.

We call a filtration *fine* if it admits no non-trivial refinement.

Lemma 7.1.6. *A filtration of Fano polytopes $\{P_i\}_{i=0}^d$ is fine if and only if, for each $i = 0, \ldots, d-1$:*

$$|\partial P_{i+1} \cap N| = |\partial P_i \cap N| + 1.$$

Proof. It is sufficient to prove the following. Let P and Q be any two Fano polytopes such that $P \subset Q$. Then there exists a Fano polytope P' such that $P \subset P' \subseteq Q$ and $|\partial P' \cap N| = |\partial P \cap N| + 1$.

Let $I = (Q \setminus P) \cap N$ be the set of all lattice points of Q which are not contained in P. Let $P_x = \operatorname{conv}(P \cup \{x\})$ for any $x \in N$. By considering $\operatorname{vol} P_x$ for all $x \in I$ we see that there exists some $x \in I$ for which this quantity is minimised. Suppose that $x \neq y \in I$ is such that $y \in P_x$. Then $P_y \subset P_x$, and minimality of the volume gives $\operatorname{vol} P_x = \operatorname{vol} P_y$. Hence $P_y = P_x$ and so $x = y$. To conclude, set $P' = P_x$. □

Lemma 7.1.7. *A filtration of Fano polygons $\{P_i\}_{i=0}^{d}$ is fine if and only if the dual filtration,*

$$P_d^\vee \subset P_{d-1}^\vee \subset \ldots \subset P_1^\vee \subset P_0^\vee, \tag{7.1.4}$$

is fine.

Proof. We shall begin by observing that (7.1.4) really does give us a filtration (albeit in $M_\mathbb{R}$ rather than in $N_\mathbb{R}$). Since $P_i \subset P_{i+1}$ so $P_{i+1}^\vee \subset P_i^\vee$. By Corollary 5.1.3 we know that each $P_i^\vee$ is a Fano polygon. Lemma 5.2.3 tells us that $P_d^\vee$ is minimal and $P_0^\vee$ is maximal.

Suppose that (7.1.4) is not fine. Then there exists a refinement, and dualising this refinement tells us that the original filtration could not have been fine. Similarly if the original filtration is not fine then neither is (7.1.4). □

We are now in a position to give a very quick proof of Theorem 7.1.1.

Proof of Theorem 7.1.1. Let $P \subset N_\mathbb{R}$ be any Fano polygon. Then P lies in some fine filtration $\{P_i\}_{i=0}^{d}$. Lemmas 7.1.6 and 7.1.7 tell us that:

$$\begin{aligned} |\partial P_i \cap N| + \left|\partial P_i^\vee \cap M\right| &= |\partial P_0 \cap N| + i + \left|\partial P_0^\vee \cap M\right| - i \\ &= |\partial P_0 \cap N| + \left|\partial P_0^\vee \cap M\right|. \end{aligned}$$

We saw in the proof of Proposition 5.2.4 that there are exactly three minimal Fano polygons, and it is a simply matter to calculate their duals. We see that Theorem 7.1.1 holds for all minimal polygons, and hence holds for all Fano polygons. □

7.2 A Three-Dimensional Analogue

In [BCF+05, pg. 185] Haase reported the following result, which he attributed to Dais and is a consequence of [BD96]:

Theorem 7.2.1 ([BCF+05, Theorem 4.3])**.** *Let $P \subset N_\mathbb{R}$ be a three-dimensional reflexive Fano polytope. Then:*

$$\sum_{E \text{ an edge of } P} \operatorname{vol} E \cdot \operatorname{vol} E^\vee = 24.$$

Proof. Either by exhaustion using the classification of all 4,319 reflexive Fano polytopes given in [KS98], or by using the results of [BD96] (see Theorem 7.2.4 below). □

	(Vertex) 0	(Edge) 1	$\cdots$	(Facet) $n-1$	n
Num. d-faces of $P_{\mathbb{P}^n}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\cdots$	$\binom{n+1}{n}$	$\binom{n+1}{n+1}$
Vol. d-face of $P_{\mathbb{P}^n}$	$\frac{1}{0!}$	$\frac{1}{1!}$	$\cdots$	$\frac{1}{(n-1)!}$	$\frac{n+1}{n!}$
Vol. d-face of $P^\vee_{\mathbb{P}^n}$	$\frac{(n+1)^0}{0!}$	$\frac{(n+1)^1}{1!}$	$\cdots$	$\frac{(n+1)^{n-1}}{(n-1)!}$	$\frac{(n+1)^n}{n!}$
Num. d-faces of $P_{(\mathbb{P}^1)^n}$	$2^1\binom{n}{1}$	$2^2\binom{n}{2}$	$\cdots$	$2^n\binom{n}{n}$	1
Vol. d-face of $P_{(\mathbb{P}^1)^n}$	$\frac{1}{0!}$	$\frac{1}{1!}$	$\cdots$	$\frac{1}{(n-1)!}$	$\frac{2^n}{n!}$
Vol. d-face of $P^\vee_{(\mathbb{P}^1)^n}$	2^0	2^1	$\cdots$	2^{n-1}	2^n

Table 7.1: The volume of the faces of the polytopes associated with $\mathbb{P}^n$ and $(\mathbb{P}^1)^n$, and volume of the faces of the dual polytopes.

	(Vertex) 0	(Edge) 1	(Face) 2	(Facet) 3	4
Num. d-faces of $P_{\mathbb{P}^4}$	5	10	10	5	1
Vol. d-face of $P_{\mathbb{P}^4}$	1	1	$1/2$	$1/6$	$5/24$
Vol. d-face of $P^\vee_{\mathbb{P}^4}$	1	5	$5^2/2$	$5^3/6$	$5^4/24$
Num. d-faces of $P_{(\mathbb{P}^1)^4}$	8	24	32	16	1
Vol. d-face of $P_{(\mathbb{P}^1)^4}$	1	1	$1/2$	$1/6$	$2/3$
Vol. d-face of $P^\vee_{(\mathbb{P}^1)^4}$	1	2	4	8	16

Table 7.2: The volumes of Table 7.1 when $n = 4$.

Remark 7.2.2. Haase proposes that it might be possible to interpret vol $E^\vee$ as a measure of curvature; the resulting expression would resemble a Gauß–Bonnet formula in the style of [PRV00].

By rephrasing Theorems 7.1.1 and 7.2.1 one might be inclined to suggest the following conjecture (perhaps also considering more terms in the summation):

Conjecture 7.2.3. *Fix $n \geq 2$. The following sum is constant for all n-dimensional reflexive Fano polytopes P:*

$$\sum_{E \text{ an edge of } P} \operatorname{vol} E \cdot \operatorname{vol} E^\vee + \sum_{E \text{ an edge of } P^\vee} \operatorname{vol} E \cdot \operatorname{vol} E^\vee.$$

By considering the polytopes associated with $\mathbb{P}^n$ and $(\mathbb{P}^1)^n$, we see that Conjecture 7.2.3 and its obvious variations are unlikely to hold. Table 7.1 contains the relevant information for general dimensions; the values for $(\mathbb{P}^1)^n$ and its dual are easily calculated, as are those for $\mathbb{P}^n$. The volumes of the faces of $P^\vee_{\mathbb{P}^n}$ are calculated using the multiplicity obtained in the concluding example of Section 3.8. In particular, the volumes of the faces when $n = 4$ (Table 7.2) show that even a four-dimensional analogue must be non-trivial.

An appropriate generalisation, requiring concepts from Mirror Symmetry, was given by Batyrev and Dais:

Theorem 7.2.4 ([BD96, Corollary 7.10])**.** *Fix $n \geq 3$. Let P be an n-dimensional reflexive Fano polytope. Then:*

$$e_{st}(\overline{Z}_f) = \sum_{i=1}^{n-2} \sum_{\dim F = i} (-1)^i \mathrm{vol}\, F \cdot \mathrm{vol}\, F^\vee,$$

where F is a face of P, and $e_{st}(\overline{Z}_f)$ is the string-theoretic Euler number of the generic Calabi–Yau embedded in X_P.

7.3 The Number of Lattice Points on a Fano Polytope

The proof of Proposition 7.3.1 below was adapted from [CLR02, Theorem 2] in 2004. The objects under study in [CLR02] are "distinct pair-sum" polytopes, and the idea behind the proof is attributed to a 1971 *Putnam Problem*. Proposition 7.3.1 appeared independently in [Nil05, Corollary 6.3], in which the idea was attributed to [Bat99, Proposition 2.1.11] (where it was proved for regular Fano polytopes); the method of proof is identical.

Proposition 7.3.1. *Let $P \subset N_\mathbb{R}$ be a terminal Fano polytope of dimension n. Then:*

$$|\mathrm{vert}\, P| = |\partial P \cap N| \leq 2^{n+1} - 2.$$

This bound is obtained only if P is centrally symmetric.

Proof. Consider the vertices of P reduced modulo 2; that is, reduce the components of each vertex modulo 2. Clearly no vertex is equivalent to $0 \pmod 2$. By the *Pigeon Hole Principle*, if $|\mathrm{vert}\, P| \geq 2^n$ there must exist at least two vertices $v_1 \neq v_2$ which are equivalent modulo 2. Hence $v_1 + v_2$ is equivalent to 0, and so $w := (1/2)(v_1 + v_2)$ is an interior lattice point of P. The only possibility is that $w = 0$, which forces $v_2 = -v_1$. The result follows. $\square$

From the following example we see that the bound given in Proposition 7.3.1 is sharp.

Example 7.3.2. *Let $S = \{0,1\}^n \setminus \{0\}$, and set $P = \mathrm{conv}(\pm S) \subset \mathbb{N}_{\mathbb{R}}$. Then P is a terminal Fano polytope of dimension n with $2(2^n - 1)$ vertices.*

With a slight modification, a similar statement can be made concerning centrally symmetric canonical Fano polytopes. Here it is important to remember that $\mathrm{vert}\, P$ does not necessarily equal $\partial P \cap N$. This result also appeared independently in [Nil05, Corollary 6.4].

Proposition 7.3.3. *Let $P \subset N_{\mathbb{R}}$ be a centrally symmetric canonical Fano polytope of dimension n. Then:*

$$|\partial P \cap N| \leq 3^n - 1.$$

Proof. Consider the points of $\partial P \cap N$ reduced modulo 3. No vertex is equivalent to $0 \pmod 3$. By the Pigeon Hole Principle, if $|\partial P \cap N| \geq 3^n$ then there must exist at least two vertices v_1 and v_2 such that $v_1 \equiv v_2 \pmod 3$. Hence $w := (1/3)(v_1 - v_2)$ is an interior lattice point of P, and so we conclude that $v_1 = v_2$. $\square$

Once again we see that this bound is sharp:

Example 7.3.4. *Let P be the n-cube $\mathrm{conv}\{\pm e_1 \pm \ldots \pm e_n\}$. Then P is a centrally symmetric canonical Fano polytope with $3^n - 1$ vertices.*

Remark 7.3.5. It is not clear how remove the requirement that P be centrally symmetric from Proposition 7.3.3. One can certainly consider reduction modulo 3, and ask when two points v_1 and v_2 exist such that $v_1 \equiv -v_2 \pmod 3$. One sees that this forces $v_1 = -v_2$.

Suppose that v_1, v_2 and v_3 are all equivalent modulo 3. Clearly there cannot now exist a point v such that $v \equiv -v_1 \pmod 3$. If v_1, v_2 and v_3 are not contained in a common face of P then $(1/3)(v_1 + v_2 + v_3) = 0$. Hence for any v such that $v \equiv v_1 \pmod 3$ we see that $v = v_i$ for $i = 1, 2$ or 3.

The problem which remains is the possibility that v_1, v_2 and v_3 lie in a common face F of P. It is then not clear what prevents an arbitrary number of points $v \equiv v_1 \pmod 3$ accumulating on F.

In the case of simplicial reflexive Fano polytopes, the following bound on the number of vertices has recently been established:

Theorem 7.3.6 (c.f. [Cas04, Theorem 1])**.** *Let $P \subset N_{\mathbb{R}}$ be an n-dimensional simplicial reflexive Fano polytope. Then:*

$$|\mathrm{vert}\, P| = \begin{cases} 3n, & \text{if } n \text{ is even;} \\ 3n-1, & \text{if } n \text{ is odd.} \end{cases}$$

For more on the history of this remarkable result, consult [Deb03].

CHAPTER 8

Minimal Toric Varieties

8.1 Introduction

In Chapter 6 a classification of toric Fano threefolds with at worst terminal singularities was given. The method employed relied on an approach first outlined in [BB]. It depends on the polytopal description of a toric Fano variety (see Proposition 3.6.7), and can be summarised in two steps:

(i) Classify all the "minimal" polytopes;

(ii) Inductively "grow" these minimal polytopes by successive addition of vertices.

The second step (see Section 6.6) is immediately generalisable to any dimension, and will not be discussed here. All the work is concentrated in the first step. It is natural to ask what can be said about the minimal polytopes of arbitrary dimension.

Here we shall prove, in Proposition 8.2.11, an inductive description of these minimal polytopes. It shall be seen that an understanding of these minimal polytopes reduces to an understanding of the $\rho = 1$ cases – the fake weighted projective spaces (see Section 4.4) associated with n-simplices. Various results which can be immediately obtained from this decomposition are given.

We shall then proceed, in Section 8.6, to use a modification of Proposition 8.2.11 in order to classify all minimal canonical Fano three–dimensional polytopes. The minimal polytopes found can then be fed to a computer in order to establish a complete classification of toric Fano threefolds with canonical singularities; the approach is analogous to that of Chapter 6.

Knowing the minimal canonical polytopes allows us to establish an upper bound on the degree of a toric Fano threefold with at worst canonical singularities. Theorem 8.5.5 tells us that the degree is at most 72. This is obtained in the case $\mathbb{P}(1,1,1,3)$.

8.2 Decomposition of Minimal Toric Varieties

Let $x_0, \ldots, x_n \in \mathbb{R}^n$ be such that $P := \operatorname{conv}\{x_0, \ldots, x_n\} \subset \mathbb{R}^n$ is an n-simplex with $0 \in P^\circ$. To this simplex we associate the complete fan $\Delta := \Delta(P)$ generated by:

$$\{\sigma_{x_k} := \operatorname{cone}\{x_0, \ldots, \hat{x}_k, \ldots, x_n\} \subset \mathbb{R}^n\}_{k=0}^{n},$$

where $\hat{x}_k$ indicates that the vertex x_k is omitted. Note that each σ_{x_k} is strictly convex. By a traditional abuse of notation we write σ_k in place of σ_{x_k}.

Definition 8.2.1. $-\Delta := \{-\sigma \subset \mathbb{R}^n \mid \sigma \in \Delta\}$.

The following results are immediate from the definition:

(i) $-\Delta$ is a fan;

(ii) Δ is complete if and only if $-\Delta$ is complete.

Lemma 8.2.2. *$x_k \in (-\sigma_i)^\circ$ if and only if $i = k$.*

Proof. Since $-\Delta$ is complete it must be that $x_k \in -\sigma_i$ for some i. To prove our claim it thus suffices to show that $x_k \notin -\sigma_i$ if $i \neq k$.

Suppose for a contradiction that $x_k \in -\sigma_i$ for some $i \neq k$. There thus exist $c_j \in \mathbb{R}, c_j \geq 0$ such that:

$$-x_k = \sum_{\substack{j=0 \\ j \neq i}}^{n} c_j x_j.$$

Hence $\{x_0, \ldots, \hat{x}_i, \ldots, x_n\}$ lie in an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. In particular we have that either P is $(n-1)$-dimensional, or that 0 lies in a facet of P. Both cases yield a contradiction. □

Lemma 8.2.3. *If $x \in (-\sigma_k)^\circ$ then:*

$$P' := \operatorname{conv}\{x_0, \ldots, \hat{x}_k, \ldots, x_n, x\},$$

is an n-dimensional convex polytope with $0 \in P'^\circ$, and $\Delta(P')$ is a complete fan.

Proof. Let $x \in (-\sigma_k)^\circ$. This tells us that $x \neq 0$ and that there exist $c_j \in \mathbb{R}, c_j > 0$ such that:

$$-x = \sum_{\substack{j=0 \\ j \neq k}}^{n} c_j x_j.$$

By dividing through by the sum of the c_j's, and observing that $x \in P'$, we see immediately that $0 \in P'$.

Assume that $0 \notin P'^\circ$. The origin must lie on a facet of P' which contains x. Thus there exists some non-empty subset $F \subsetneq \left\{0, \ldots, \hat{k}, \ldots, n\right\}$ with $0 \in \operatorname{conv}(F \cup \{x\})^\circ$. Hence there exist, for all $j \in F$, $c_j \in \mathbb{R}, c_j > 0$ such that:

$$-x = \sum_{j \in F} c_j x_j.$$

Setting $\rho := \operatorname{cone} F \in \Delta$ we see that $x \in -\rho$. Observing that $\dim \rho < n$ we have a contradiction, and so $0 \in P'^\circ$. That $\Delta(P')$ is complete follows immediately. □

By combining Lemma 8.2.2 and Lemma 8.2.3 we obtain the following:

Proposition 8.2.4. *Let $x \in \mathbb{R}^n$. $x \in (-\sigma_k)^\circ$ if and only if:*

$$P' := \operatorname{conv}\{x_0, \ldots, \hat{x}_k, \ldots, x_n, x\},$$

is an n-simplex with $0 \in P'^\circ$, and $\Delta(P')$ is a complete fan.

Definition 8.2.5. Let $P = \operatorname{conv}\{x_0, \ldots, x_k\} \subset N_\mathbb{R}$ be a terminal Fano polytope of dimension n. We say that P is *minimal* if, for all $j \in \{0, \ldots, k\}$, $\operatorname{conv}\{x_0, \ldots, \hat{x}_j, \ldots, x_k\}$ is not a terminal Fano polytope of dimension n.

Remark 8.2.6. As with conventional use of (Euclidean) interior (e.g. P°), our use of Fano and minimal will often be relative to some obvious subspace. Such occurrences, as in the case with interior, should not cause any confusion.

Example 8.2.7. *Let $P := \operatorname{conv}\{\pm e_1, \pm e_2\} \subset N_\mathbb{R}$. We have that $0 \in P^\circ$ and P the terminal Fano polygon associated with $\mathbb{P}^1 \times \mathbb{P}^1$. Let $P' := \operatorname{conv}\{\pm e_1\} \subset P$. We have that $0 \in P'^\circ$ and P' is a one-dimensional terminal Fano polytope (associated with $\mathbb{P}^1$). Both P and P' are examples of minimal polytopes.*

Using the results of Section 2.3 we can translate the notion of minimality into the language of toric varieties.

Definition 8.2.8. Let X be a toric Fano variety with at worst terminal singularities. We say that X is *minimal* if for every toric Fano variety $Y \neq X$ there exists a surjective toric morphism $X \twoheadrightarrow Y$ only if $\dim X > \dim Y$.

It will be useful to introduce some notation with which we can refer to particular classes of toric Fano varieties.

Definition 8.2.9. We define $\mathcal{P}_n$ to be the set of all $\mathbb{Q}$-factorial toric Fano n-folds with at worst terminal singularities whose Picard number $\rho = 1$. We define $\mathcal{T}_n$ to be the set of all minimal toric Fano n-folds with at worst terminal singularities.

Remark 8.2.10. In the language of Section 4.4, $\mathcal{P}_n$ is the set of all fake weighted projective spaces with at worst terminal singularities.

Because of the correspondence between toric Fano varieties and Fano polytopes (see Section 3.7), we shall blur the distinction between the two. When stating that two polytopes P and P' are equal, we mean that they are equal only up to the action of $GL(n, \mathbb{Z})$ (i.e. the corresponding varieties are isomorphic). When a polytope is represented as a matrix, this matrix is unique only up to permutation of the columns (vertices) and the action of $GL(n, \mathbb{Z})$.

We are now in a position to prove the main result of this section.

Proposition 8.2.11. *Any $T \in \mathcal{T}_n$ is either an element of $\mathcal{P}_n$, or can be written as $T = \operatorname{conv}(P \cup T')$ for some $P \in \mathcal{P}_k$ and $T' \in \mathcal{T}_{n-k+r}$, where $0 \leq r < k < n$.*

Proof. Let us assume that $T \notin \mathcal{P}_n$. Then we can write T as $T := \operatorname{conv}\{x_0, \ldots, x_l\} \in \mathcal{T}_n$, where $l > n$. Let:

$$S = \left\{ (\sigma_0, \ldots, \sigma_n) \mid 0 \leq \sigma_0 < \ldots < \sigma_n \leq l \text{ and } x_{\sigma_0}, \ldots, x_{\sigma_n} \text{ do not all lie in some } \mathbb{A}^{n-1} \right\}.$$

We have that $T = \bigcup_{\sigma \in S} \operatorname{conv}\{x_{\sigma_0}, \ldots, x_{\sigma_n}\}$, and since $0 \in T$ it must be that $0 \in \operatorname{conv}\{x_{\sigma_0}, \ldots, x_{\sigma_n}\}$ for some permutation $\sigma \in S$. We may take this σ to be the identity.

Minimality of T ensures that $0 \notin \operatorname{conv}\{x_0, \ldots, x_n\}^\circ$. Hence the origin must lie on some facet, and we may assume (with a possible reordering) that $0 \in \operatorname{conv}\{x_0, \ldots, x_k\}^\circ$ for some $k < n$. Thus we see that $P := \operatorname{conv}\{x_0, \ldots, x_k\} \in \mathcal{P}_k$.

Let $T'' := \operatorname{conv}\{x_{k+1}, \ldots, x_l\}$, so $T = \operatorname{conv}(P \cup T'')$. Let Γ be the k-dimensional subspace of $N_\mathbb{R}$ containing P. P being minimal and T being minimal ensures that

$\{x_0, \dots, x_k\} \cap T'' = \emptyset$ and that $\{x_{k+1}, \dots, x_l\} \cap \Gamma = \emptyset$. It must also be that $T''^{\circ} \cap \Gamma \neq \emptyset$ (otherwise $0 \in P$ would lie in a facet of T - a contradiction). Let $m := \dim(T''^{\circ} \cap \Gamma)$. A simple dimension count reveals that $\dim T'' = n - k + m$.

By minimality of T and the result of Proposition 8.2.4 it must be that $T''^{\circ} \cap \Gamma \subset -\sigma$ for some $-\sigma \in (-\Delta(P))^{(k-1)}$. Since $\sigma \cap -\sigma = \{0\}$ we have that either $T''^{\circ} \cap \Gamma = \{0\}$ or $0 \notin T''^{\circ} \cap \Gamma$. The first case gives us that $T'' \in \mathcal{T}_{n-k}$, so by setting $T' = T''$ we are done. For the second possibility we proceed as follows: Note that $k > r := \dim \sigma \geq m$, $r \neq 0$. We may assume that $\sigma = \operatorname{cone}\{x_0, \dots, x_{r-1}\}$, and construct the polytope $T' := \operatorname{conv}\{x_0, \dots, x_{r-1}, x_{k+1}, \dots, x_l\}$. By construction $T'' \subset T'$ and $T' \in \mathcal{T}_{n-k+r}$. $\square$

Remark 8.2.12. Proposition 8.2.11 tells us that, when $T \notin \mathcal{P}_n$, we have:

(i) $P \cap T' = \{0\}$ if and only if $r = 0$. In this case T can be given by:

$$\left(\begin{array}{c|c} \overbrace{\begin{array}{c} P \\ \hline 0 \end{array}}^{k+1} & \overbrace{\;T'\;}^{l-k} \end{array}\right).$$

(ii) $0 \in (P \cap T')^{\circ}$ if and only if $r > 0$. In this case we have that P and T' have precisely r vertices in common:

$$\left(\begin{array}{cc} \overbrace{\begin{array}{c} P \\ \hline 0 \end{array}}^{k-r+1} & \begin{array}{c|c} & \overbrace{\quad T' \quad}^{l-r} \\ \hline \underbrace{0}_{r} & \end{array} \end{array}\right).$$

Furthermore, the proof of Proposition 8.2.11 tells us that:

$$\operatorname{span}_{\mathbb{R}}(P) \cap \operatorname{conv}\{x_{k+1}, \dots, x_l\} \subset -\operatorname{cone}\{x_{k-r+1}, \dots, x_k\}.$$

In both cases T' is equivalent, under the action of $GL(n, \mathbb{Z})$, to some $(n - k + r) \times (l - k + r)$ matrix in $\mathcal{T}_{n-k+r}$.

8.3 Some Immediate Applications

Corollary 8.3.1. *Let $T \in \mathcal{T}_n$. Then $|\operatorname{vert} T| \leq 2n$.*

Proof. We proceed by induction on the dimension n. The result is trivially true for $\mathcal{T}_1 = \{\mathbb{P}^1\}$. Suppose that $n > 1$.

T decomposes into P and T' as in Proposition 8.2.11. By the inductive hypothesis, T' has at most $2(n-k+r)$ vertices, and so $l - k \leq 2(n-k) + r$. Thus we see that $|\operatorname{vert} T| \leq k + 1 + 2(n-k) + r$. By setting $r = k - 1$ the right hand side achieves its maximum value of $2n$. □

Remark 8.3.2. The bound obtained in Corollary 8.3.1 is sharp; for any n we have $\left(\mathbb{P}^1\right)^n \in \mathcal{T}_n$.

Corollary 8.3.1 could also be deduced from *Steinitz's Theorem*: any point in the interior of a convex polytope $\operatorname{conv} S$ lies in the interior of the convex hull of at most $2n$ points of S (where $n = \dim \operatorname{conv} S$).

Corollary 8.3.3. *Let $T \in \mathcal{T}_n$ be such that $|\operatorname{vert} T| = 2n$. Then, in the notation of Proposition 8.2.11, $k = 1$ for all possible choices of P.*

Proof. We proceed by induction on the dimension of n. The result is trivially true in the case $n = 1$. Suppose that $n > 1$.

First we shall show that $l - k + r = |\operatorname{vert} T'| = 2(n-k+r)$. If this is not the case then Corollary 8.3.1 tells us that $|\operatorname{vert} T'| < 2(n-k+r)$. We readily obtain the desired contradiction:

$$2n = l + 1 < (k - r + 1) + 2(n - k + r) = 2n - k + r + 1 \leq 2n.$$

The inductive hypothesis applied to T' tells us that the vertices of T' come in pairs: $x \in \operatorname{vert} T'$ if and only if $-x \in \operatorname{vert} T'$. Hence $r = 0$ (as $P \in \mathcal{P}_k$). Finally we observe that $2n = k + 1 + 2(n-k)$, which forces $k = 1$, and we have our result. □

As a consequence of Corollary 8.3.3 we have that any $T \in \mathcal{T}_n$ with $2n$ vertices is centrally symmetric, and is built up by the successive "introduction of $\mathbb{P}^1$'s". Such a construction is simplicial, and so the corresponding variety is $\mathbb{Q}$-factorial. A trivial example of this was given in Remark 8.3.2, where we simply took the cross product; this example is the only regular, and hence smooth, case.

With the exception of when $k = 1$, no element of $\mathcal{P}_k$ is centrally symmetric. Thus $T \in \mathcal{T}_n$ is centrally symmetric if and only if $|\operatorname{vert} T| = 2n$.

Corollary 8.3.4. *Let $T \in \mathcal{T}_n$. T is centrally symmetric if and only if $|\operatorname{vert} T| = 2n$. If $|\operatorname{vert} T| = 2n$ then T is $\mathbb{Q}$-factorial, and smooth only when $T = \left(\mathbb{P}^1\right)^n$.*

Remark 8.3.5. In [Nil05, §6] it is reported that Wirth found a characterisation of centrally symmetric *reflexive* Fano polytopes with $2n$ vertices. In particular, these polytopes can always be embedded in the n-cube $\operatorname{conv}\{\pm e_1 \pm \ldots \pm e_n\}$.

We shall now restrict our attention to those $T \in \mathcal{T}_n$ with $|\operatorname{vert} T| = 2n$ obtainable from $\left(\mathbb{P}^1\right)^{n-1} \in \mathcal{T}_{n-1}$. We can represent such a T by the matrix:

$$\left(\begin{array}{c:ccc} \pm v_1 & \pm 1 & & \\ \vdots & & \ddots & \\ \pm v_{n-1} & & & \pm 1 \\ \hdashline \pm v_n & & 0 & \end{array}\right),$$

where $v_1, \ldots, v_n \in \mathbb{Z}_{\geq 0}$, $\gcd\{v_1, \ldots, v_n\} = 1$, and $0 \leq v_i \leq v_n/2$ for $1 \leq i < n$. Without loss of generality we may also insist that $v_1 \leq \ldots \leq v_n$.

Let $P_k = \operatorname{conv}\{\pm e_1, \ldots, \pm e_k\}$ be a polytope representation of $\left(\mathbb{P}^1\right)^n \in \mathcal{T}_n$. We have the recurrence relation:

$$\operatorname{vol} P_k = \frac{1}{k}(\operatorname{vol} P_{k-1})(\operatorname{vol} P_1).$$

Since $\operatorname{vol} P_1 = 2$ we see that $\operatorname{vol} P_k = (1/k!)2^k$. Hence we obtain:

$$\operatorname{vol} T = \frac{1}{n!} 2^n v_n.$$

Combining this with *Minkowski's Convex Body Theorem* gives us:

Lemma 8.3.6. *With notation as above, $v_n \leq n!$.*

A crude bound on the number of possibilities for T is given by:

$$2 \sum_{k=1}^{\frac{1}{2}n!} (k+1)^{n-1}.$$

This is a gross overestimate; the condition that the v_i are coprime and the requirement that $T \in \mathcal{T}_n$ are ignored.

Suppose that the following conditions are satisfied:

$$\begin{aligned}
&v_1, \ldots, v_n \in \mathbb{Z}_{\geq 0}, \\
&\gcd\{v_1, \ldots, v_n\} = 1, \\
&v_1 \leq \ldots \leq v_n, \\
&0 \leq v_i \leq \tfrac{v_n}{2} \text{ for all } 1 \leq i < n, \\
&v_n \leq n!.
\end{aligned} \tag{8.3.1}$$

How can we decide whether the resulting T lies in $\mathcal{T}_n$? For this to be the case $T \cap N = \operatorname{vert} T \cup \{0\}$. If $v_n = 1$ then $v_0 = \ldots = v_{n-1} = 0$ by (8.3.1), and we obtain $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$. Suppose that $v_n > 1$. Given any $\kappa, \lambda \in \mathbb{Z}_{\geq 0}$ such that $1 \neq \kappa \mid v_n$, $\lambda \in \{1, \ldots, \kappa - 1\}$, we have that:

$$\sum_{i=1}^{n-1} \min\left(\left\{\frac{\lambda v_i}{\kappa}\right\}, 1 - \left\{\frac{\lambda v_i}{\kappa}\right\}\right) > 1 - \frac{\lambda}{\kappa}, \tag{8.3.2}$$

where $\{\mu\} := \mu - \lfloor \mu \rfloor$ denotes the fractional part of μ.

Proposition 8.3.7. *With notation as above, suppose that $v_n > 1$. Then $T \in \mathcal{T}_n$ if and only if conditions* (8.3.1) *and* (8.3.2) *are satisfied for all $1 \neq \kappa \mid v_n, \lambda \in \{1, \ldots, \kappa - 1\}$.*

Example 8.3.8. *If we take $n = 2$,* (8.3.1) *tells us that $v_2 \leq 2$. If we take $v_2 = 1$ then, as noted above, there is only one possible choice for v_1 and we obtain $\mathbb{P}^1 \times \mathbb{P}^1$. If we take $v_2 = 2$ then v_1 can be either* 0 *or* 1*, but in both cases the summation in* (8.3.2) *totals* 0*.*

Note that $\mathcal{T}_1 = \{\mathbb{P}^1\}$. We thus see that we have proven that the only element of $\mathcal{T}_2$ with four vertices is $\mathbb{P}^1 \times \mathbb{P}^1$.

Now let us take $n = 3$. We have that $v_3 \leq 6$, and when $v_3 = 1$ we obtain the usual cross product. If $v_3 = 6$ then $v_1, v_2 \leq 3$. By setting $\kappa = 6$, $\lambda = 1$ we contradict inequality (8.3.2) *for all choices with the exception of $v_1 = v_2 = 3$, when $\kappa = 6$, $\lambda = 2$ will suffice. If $v_3 = 5$, $v_1, v_2 \leq 2$ then $\kappa = 5$, $\lambda = 1$ contradicts the inequality in every case. When $v_3 = 4$ the situation is similar to when $v_3 = 6$. When $v_3 = 3$ then take $\kappa = 3$, $\lambda = 1$. Now suppose $v_3 = 2$. There is only one possible choice for κ and λ, namely $\kappa = 2$, $\lambda = 1$, and we see that* (8.3.2) *is satisfied if and only if $v_1 = v_2 = 1$.*

Combining this with our result for $n = 2$ we see that we have proven that there are only two elements of $\mathcal{T}_3$ with six vertices, namely $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $v_1 = v_2 = 1$, $v_3 = 2$.

The minimal polytopes of Section 5.1 and Section 6.5 can be easily found using the techniques of Proposition 8.2.11:

Proposition 8.3.9. $\mathcal{T}_2 \setminus \mathcal{P}_2 = \{\mathbb{P}^1 \times \mathbb{P}^1\}$.

n	2	3	4	5	6
Extensions	1	2	6	211	446,948

Table 8.1: The number of extensions in $\mathcal{T}_n$ of $\left(\mathbb{P}^1\right)^{n-1}$.

Proof. This follows immediately from Corollary 8.3.1 and the first part of Example 8.3.8. $\square$

Proposition 8.3.10. *Elements of $\mathcal{T}_3 \setminus \mathcal{P}_3$ come in three forms:*

(i) The convex hull of two triangles which share a common vertex and intersect in a one-dimensional subspace;

(ii) The convex hull of a triangle and a $\mathbb{P}^1$ which intersect only at the origin;

(iii) The convex hull of $\mathbb{P}^1 \times \mathbb{P}^1$ and a $\mathbb{P}^1$ which intersect only at the origin.

In particular, those of form (iii) have already been explicitly calculated; they are the two polytopes given in the second part of Example 8.3.8.

Proof. Combine Proposition 8.2.11 with Proposition 8.3.9. $\square$

Remark 8.3.11. In fact we can readily obtain explicit descriptions of the polytopes of forms (i) and (ii) if we are told that $\mathcal{P}_2 = \left\{\mathbb{P}^2\right\}$. The necessary calculations (with matrices) are essentially identical to those of Lemmas 6.5.5–6.5.6. It is worth noticing that the task is made substantially easier by observing that the triangle representing $\mathbb{P}^2$ has a full set of symmetries. The result for form (i) becomes almost immediate when the final observation of Remark 8.2.12 is considered. We find that there are two polytopes (up to equivalence) of form (i), which we shall call F_1 and F_2.

Attempts at calculating those elements of $\mathcal{T}_n$ with $2n$ vertices for $n \geq 4$ is hindered somewhat by the fact that we need to consider more than just extensions of $\left(\mathbb{P}^1\right)^{n-1} \in \mathcal{T}_{n-1}$. For the case $n = 4$ this is not too serious an encumbrance, as we need only consider the extensions of one other polytope (namely $v_1 = v_2 = 1$, $v_3 = 2$). Such calculations are possible in theory by an easy modification of the argument used above in obtaining Proposition 8.3.7. Problems arise due to the rapid increase in the number of extensions we will need to consider, as born witness by Table 8.1. The complete list of all elements of $\mathcal{T}_4$ with 8 vertices has been calculated using the method outlined above, and is presented in Table 8.2.

Table 8.2: The 15 elements of $\mathcal{T}_4$ with 8 vertices.

$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$			
v'_1	v'_2	v'_3	v'_4
0	0	0	1
0	1	1	2
1	1	1	2
1	1	1	3
1	1	2	4
1	2	2	5

$v_1 = v_2 = 1, v_3 = 2$			
v'_1	v'_2	v'_3	v'_4
0	0	0	1
0	0	1	2
0	1	1	2
1	1	1	2
1	1	1	3
1	1	1	4
1	2	1	4
2	2	1	4
1	2	1	5

Proposition 8.3.12. *Elements of $\mathcal{T}_4 \setminus \mathcal{P}_4$ come in four forms:*

(i) The convex hull of two $\mathbb{P}^2$'s which intersect only at the origin;

(ii) The convex hull of $\mathbb{P}^1$ and an element of $\mathcal{T}_3$ which intersect only at the origin;

(iii) The convex hull of $\mathbb{P}^2$ and an element of $\mathcal{P}_3 \cup \{F_1, F_2\}$ which share one vertex and intersect in a one-dimensional subspace;

(iv) The convex hull of two elements of $\mathcal{P}_3$ which share two vertices and intersect in a two-dimensional subspace.

Proof. Combine Proposition 8.2.11 with Propositions 8.3.9 and 8.3.10, insisting that the P in each case be of the smallest possible dimension. $\square$

8.4 A Special Case: The Join of Two Fans

Definition 8.4.1. Let $\Delta = \Delta' \cdot \Delta''$ be the join of two fans Δ' and Δ'' (see [Ewa96, III.1.12–1.14; VI.6.5]) such that:

(i) Δ' is contained in a k-dimensional subspace U of $\mathbb{R}^n, 0 < k < n$;

(ii) Δ'' can be projected bijectively onto a fan $\Delta_0 \subset U^\perp$.

We call Δ_0 a *projection fan of* Δ *perpendicular to* Δ' and say that Δ has a projection fan (with respect to Δ', Δ'').

Lemma 8.4.2 ([Ewa96, Lemma VI.6.6]). *If* $\Delta = \Delta' \cdot \Delta''$ *and* Δ' *is complete (relative to* $U = \operatorname{span}_{\mathbb{R}} |\Delta'|$*) then* Δ *has a projection fan* Δ_0 *(in* $U^{\perp}$*).*

We would like to know when a decomposition given by Proposition 8.2.11 gives a join $\Delta_T = \Delta_P \cdot \Delta_{T''}$, where $T'' = \operatorname{conv}(\operatorname{vert} T' \setminus \operatorname{vert} P)$. Since all our fans Δ_P are complete, by the above lemma any join would yield a projection fan. If in addition we have that $P = \mathbb{P}^k$ and T'' is regular then the following may be applied:

Theorem 8.4.3 ([Ewa96, Theorem VI.6.7]). *Let* $\Delta, \Delta', \Delta''$ *be regular fans in* $\mathbb{R}^n$ *such that* $\Delta = \Delta' \cdot \Delta''$*, and let* Δ_0 *be the projection fan of* Δ *perpendicular to* $\operatorname{span}_{\mathbb{R}} |\Delta'|$. *Then the projection* $\pi : \Delta \to \Delta_0$ *induces a fibration of* X_Δ*, an* $X_{\Delta'}$*-fiber bundle over* X_{Δ_0}*.*

Using the notation of Proposition 8.2.11, when $l - k = 2$ we have that the resulting decomposition gives a join. It is also worth observing the trivial fact that $T'' = T'$ when $r = 0$.

We have the following result:

Corollary 8.4.4. *With notation as in Proposition 8.3.10, those elements of* $\mathcal{T}_3$ *of forms (i) and (ii) correspond to* $\mathbb{P}^2$*-fiber bundles over* $\mathbb{P}^1$*, and those of form (iii) correspond to* $\mathbb{P}^1 \times \mathbb{P}^1$*-fiber bundles over* $\mathbb{P}^1$*.*

8.5 Minimal Canonical Fano Polytopes

The decomposition given by Proposition 8.2.11 is very much a result about *vertices*. In fact, it should be observed that the proof avoids mentioning any other lattice point. This means that the result can be readily applied to canonical Fano polytopes, provided we use the following notion of minimality:

Definition 8.5.1. Let P be a canonical Fano polytope of dimension n. We say that P is *minimal* if for every $x \in \operatorname{vert} P$ the polytope $\operatorname{conv}(\operatorname{vert} P \setminus \{x\})$ is not a canonical Fano polytope of dimension n.

Unfortunately, Definition 8.5.1 does not agree with our intuition. Unlike in the terminal case, the number of vertices of a canonical polytope is not a strictly increasing function – i.e. by adding a new vertex to a canonical polytope it is possible to reduce the total number of vertices by subsuming vertices into faces. Instead we desire a generalisation of Definition 5.2.2.

n	$\mathcal{CP}_n$	$\mathcal{C}_n \setminus \mathcal{CP}_n$
1	$\mathbb{P}^1$	$\emptyset$
2	$\mathbb{P}^2, \mathbb{P}(1,1,2)$	$\mathbb{P}^1 \times \mathbb{P}^1$

Table 8.3: The elements of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Definition 8.5.2. Let $x \in \operatorname{vert} P$. The polytope obtained by *subtracting the vertex x* from P is given by:

$$\operatorname{conv}(P \cap N \setminus \{x\}).$$

Definition 8.5.3. We say that an n-dimensional canonical Fano polytope P is *minimal* if, for all vertices x of P, the polytope obtained by subtracting x is not an n-dimensional canonical Fano polytope.

Definition 8.5.3 agrees with Definition 8.2.5 when the polytope is terminal. Fortunately any minimal polytope according to Definition 8.5.3 will also be minimal with respect to Definition 8.5.1 (although the converse is not true). This means that we can apply Proposition 8.2.11 provided we remember to work with the vertices of our polytope, and not the boundary points.

Definition 8.5.4. We define $\mathcal{CP}_n$ to be the set of all minimal $\mathbb{Q}$-factorial toric Fano n-folds with at worst canonical singularities whose Picard number $\rho = 1$. We define $\mathcal{C}_n$ to be the set of all minimal toric Fano n-folds with at worst canonical singularities.

The minimal canonical Fano polygons were calculated in the proof of Proposition 5.2.4. The results are summarised in Table 8.3.

The minimal canonical Fano tetrahedra are classified in Proposition 8.6.2. It should be emphasised that not every canonical Fano tetrahedron is minimal. Using the techniques of Chapters 4 and 9, all canonical Fano tetrahedra can be classified. This classification is available on the Internet at:

`http://www.maths.bath.ac.uk/~mapamk/pdf/Canonical_Tet.pdf` (or `.ps`).

Alternatively, a complete list can be found in the appendix to [BB]. As mentioned in [BB92, pg. 278], there are 225 tetrahedra. Proposition 8.6.2 tells us that, of these, only 15 are minimal.

Proposition 8.2.11 combined with Table 8.3 allows us to calculate $\mathcal{C}_3 \setminus \mathcal{CP}_3$. Assume we have chosen P and T' such that k is as small as possible. If $k = 1$ then $r = 0$

(1/4) (1,1,1,1)	(1/4) (1,1,1,1)	(1/4) (1,1,1,1)	(1/6) (1,1,2,2)
$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 3 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -2 & 2 & 0 & 0 \\ -3 & 1 & 2 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}$

(1/8) (1,1,2,4)	(1/8) (1,1,2,4)	(1/6) (1,1,1,3)
$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -2 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ -5 & 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$

Table 8.4: The vertices of the strictly canonical minimal Fano tetrahedra, up to the action of $GL(3,\mathbb{Z})$

and we have that $P = \mathbb{P}^1$, $T \in \mathcal{C}_2$. These possibilities are classified in Lemmas 8.6.3–8.6.5. The alternative is that $k = 2$. Since $\mathcal{C}_1 = \mathcal{CP}_1$, and since $\mathcal{C}_2 \setminus \mathcal{CP}_2 = \{\mathbb{P}^1 \times \mathbb{P}^1\}$, we need only consider the cases when $r = 1$ and $T' \in \mathcal{CP}_2$. These cases will be classified in Lemmas 8.6.6–8.6.8. We find that there are exactly ten non-simplicial minimal canonical Fano polytopes in dimension three. The results are collated in Table 8.5.

Once the minimal polytopes are known, the following result is immediate:

Theorem 8.5.5. *Let X be any toric Fano threefold with at worst canonical singularities. Then $(-K_X)^3 \leq 72$.*

Proof. Let P_X be the polytope associated with X. There exists a minimal polytope Q such that $Q \subset P_X$, hence $P_X^\vee \subset Q^\vee$. Inspection gives $\operatorname{vol} Q^\vee \leq 12$. The result follows from Proposition 2.2.11. □

Remark 8.5.6. The maximum degree is obtained by $\mathbb{P}(1,1,1,3)$.

8.6 Minimal Toric Fano Threefolds with Canonical Singularities

Lemma 8.6.1. *Let $P \in \mathcal{CP}_3$ be a minimal canonical Fano tetrahedron. Then P is either one of the eight terminal Fano tetrahedra in Table 6.4, or P has weights $(1,1,1,1)$, $(1,1,2,2)$, $(1,1,2,4)$, or $(1,1,1,3)$.*

Proof. Suppose that $P = \operatorname{conv}\{x_0, x_1, x_2, x_3\}$ is not terminal. Let Δ be the fan spanned by the faces of P. Let $x \in \partial P \cap N \setminus \operatorname{vert} P$. Since $-\Delta$ is complete, so $x \in -\sigma$ for some cone $\sigma \in \Delta$ of smallest possible dimension. In particular $\dim \sigma \leq 2$, otherwise P is not minimal.

Suppose that $\sigma = \mathrm{cone}\{x_0, x_1\}$. Then $\mathrm{conv}\{x, x_0, x_1\}$ is a minimal Fano triangle and hence corresponds to the Fano triangle associated with either $\mathbb{P}^2$ or $\mathbb{P}(1,1,2)$. Observe that x does not lie strictly in the interior of a face of P, since this would force $\dim \sigma = 1$. Thus x lies on the edge joining x_2 and x_3. Minimality forces $x = (1/2)(x_2 + x_3)$.

First list us consider the case when the Fano triangle has weights $(1,1,1)$. In other words $x_0 + x_1 + x = 0$, so:

$$x_0 + x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 0.$$

Hence P has weights $(1,1,2,2)$.

Now consider the case when the Fano triangle has weights $(1,1,2)$. This Fano triangle contains one non-vertex boundary lattice point x'. Without loss of generality there are two possibilities: $x' = -x$, or $x' = -x_0$. The first possibility forces $x' = (1/2)(x_0 + x_1)$, and so P has weights $(1,1,1,1)$. The second possibility yields weights $(1,1,2,4)$.

Finally, suppose that $\dim \sigma = 1$. Then, without loss of generality, $x = -x_0$. We may assume that:

$$\mathrm{conv}\{x_1, x_2, x_3\} \cap N = \{x, x_1, x_2, x_3\},$$

otherwise there exists a lattice point x' which lies in a two-dimensional cone, and we can reduce the problem to the previous case. Hence $\mathrm{conv}\{x_1, x_2, x_3\}$ can be regarded as a Fano triangle with origin x, and so has weights $(1,1,1)$. We see that:

$$3x_0 + x_1 + x_2 + x_3 = 0,$$

and so P has weights $(1,1,1,3)$. □

Proposition 8.6.2. *Let $P \in \mathcal{CP}_3$ be a minimal canonical Fano tetrahedron. Then P is either one of the eight terminal Fano tetrahedra in Table 6.4, or P is one of the seven canonical Fano tetrahedra in Table 8.4.*

Proof. Lemma 8.6.1 lists the possible weights, and Corollary 4.4.15 establishes an upper bound on the multiplicity in each case. We then apply Theorem 4.4.19 and inspection. □

Lemma 8.6.3. *The minimal Fano polytopes obtained from adding the points* $\pm x$ *to a Fano square are equivalent to:*

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix}.$$

Proof. We observe that the Fano polytope under consideration is centrally symmetric. Hence by minimality it must be at worst terminal; these were classified in Lemma 6.5.4. □

Lemma 8.6.4. *The minimal Fano polytopes containing a Fano triangle of type* $\mathbb{P}^2$*, along with a pair of points* $\pm x$ *not lying in the plane containing the Fano triangle, are equivalent to:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 3 & -3 \end{pmatrix}.$$

Proof. We can arrange matters such that $P := \operatorname{conv}\{e_1, e_2, -e_1 - e_2, \pm x\}$, where $x = (a, b, c)$. Without loss of generality we may insist that $0 \le a \le b < c$ and:

$$a + b \le c. \tag{8.6.1}$$

Clearly $x = (0, 0, 1)$ is a solution. We now assume that $c > 1$.

Suppose that the point e_3 lies on the surface of P. By minimality, $\operatorname{conv}\{e_1, e_2, e_3, -x\}$ is not a Fano tetrahedron, and so one of a, b or c must be zero. By assumption $c \neq 0$, so assume that $a = 0$. Since $c \neq 1$ so $b \neq 0$. We have that:

$$\begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & -c \end{pmatrix},$$

must be Fano. Hence the only possibility is that $a = 0, b = 1, c = 2$, but then $-e_3 \in P$, contradicting minimality.

Suppose now that e_3 lies outside P. Thus e_3 lies on the opposite side to the origin of the plane defined by $\{e_1, -e_1 - e_2, x\}$. Thus the plane intersects the line through the origin and e_3 at the point ke_3, where $k = c/(2b - a + 1)$. We see that:

$$2b - a \geq c, \tag{8.6.2}$$

and so:

$$b \geq 2a. \tag{8.6.3}$$

Now consider the line joining the origin to the point $e_2 + e_3$. This line intersects $\operatorname{conv}\{e_2, -e_1 - e_2, x\}$ at the point $k(e_2 + e_3)$, where $k = c/(2a - b + c + 1)$. If $e_2 + e_3$ lies on the surface of P we have that:

$$b = 2a + 1. \tag{8.6.4}$$

Otherwise it must be that $2a \geq b$, and combining this with equation (8.6.3):

$$b = 2a. \tag{8.6.5}$$

Consider the line joining 0 and $-e_2 - e_3$. This line intersects $\operatorname{conv}\{e_1, -e_1 - e_2, -x\}$ at the point $-k(e_2 + e_3)$, where $k = c/(a - 2b + 2c + 1)$. The point lies on the surface of P only if:

$$2b - a = c + 1. \tag{8.6.6}$$

Alternatively $-e_2 - e_3 \notin P$ and so $2b - a \leq c$. Combining this with equation (8.6.2) gives:

$$2b - a = c. \tag{8.6.7}$$

Finally, consider the point $-e_3$. The line connecting the origin and this point intersects $\operatorname{conv}\{e_1, e_2, -x\}$ at $-ke_3$. This occurs when $k = c/(a + b + 1)$. If $-e_3$ lies on the surface of P we obtain:

$$a + b = c - 1. \tag{8.6.8}$$

Alternatively we have that $a + b \geq c$ which, combined with equation (8.6.1), gives:

$$a + b = c. \tag{8.6.9}$$

Suppose that $e_2 + e_3$ lies on the surface of P. By minimality of P, $-e_2 - e_3 \notin P$. By combining equations (8.6.4) and (8.6.7) we obtain that $3a + 2 = c$. If $-e_3 \notin P$ then combining this result with equation (8.6.9) yields a contradiction. Hence it must be that $-e_3$ lies on the surface of P. We are left with the conclusion that:

$$x = (a, 2a+1, 3a+2).$$

Now unless $a = b$ either $\operatorname{conv}\{e_1, -e_1 - e_2, -e_3, x\}$ or $\operatorname{conv}\{e_2, -e_1 - e_2, -e_3, x\}$ is a Fano tetrahedron. This forces $a = -1$, which contradicts our assumptions.

Suppose instead that $e_2 + e_3 \notin P$. In addition assume that $-e_2 - e_3$ lies on the surface of P. By combining equations (8.6.5) and (8.6.6) we see that $3a = c + 1$. If $-e_3 \notin P$ then equations (8.6.5) and (8.6.9) gives the contradictory conclusion that $3a = c$. Hence $-e_3$ lies on the surface of P, but by using equation (8.6.8) we see that $3a = c - 1$.

The only remaining possibility is that $e_2 + e_3 \notin P$ and $-e_2 - e_3 \notin P$. Equations (8.6.5) and (8.6.7) give that $x = (a, 2a, 3a)$, and coprimality forces $a = 1$. $\square$

Lemma 8.6.5. *Any minimal Fano polytope containing the minimal Fano triangle of type* $\mathbb{P}(1,1,2)$*, along with a pair of points* $\pm x$ *not lying in the same subspace as the triangle, is equivalent to one of:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & -2 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix}.$$

Proof. Arrange matters such that $P := \operatorname{conv}\{e_1, e_2, -2e_1 - e_2, x, -x\}$; $x := (a, b, c)$ is such that $0 \leq a, b < c$. Clearly $a = 0, b = 0, c = 1$ is a solution. Let us assume that $c > 1$.

Suppose that $e_3 \in P$. Then $\operatorname{conv}\{e_1, e_2, e_3, -x\}$ is not a Fano tetrahedron, by minimality of P. Hence it must be that either $a = 0$ or $b = 0$, but not both, since then $c = 1$. We see that the only possible solutions are $a = 0, b = 1, c = 2$ and $a = 1, b = 0, c = 2$; in both cases we obtain that $-e_3 \in P$ and so by minimality $x = e_3$.

Assume that $e_3 \notin P$ and consider the line connecting e_3 to the origin. If $a \geq 2b$ this line intersects $\operatorname{conv}\{-e_1, -2e_1 - e_2, x\}$ at the point ke_3, where $k = c/(c - b + 1)$. This tells us that $a - b \geq c$, which contradicts our assumptions.

It must be that $a < 2b$. The line joining e_3 and 0 intersects $\operatorname{conv}\{e_1, -2e_1 - e_2, x\}$ at the point ke_3, where $k = c/(3b - a + 1)$. Hence:

$$3b - a \geq c. \tag{8.6.10}$$

Suppose that $-e_3 \in P$. Since $a < 2b$ we see that $\operatorname{conv}\{e_1, -e_3, -2e_1 - e_2, x\}$ is a Fano tetrahedron unless $c = 1$. This contradicts out assumptions. The line joining the origin and $-e_3$ intersects $\operatorname{conv}\{e_1, e_2, -x\}$ at the point $k(-e_3)$, where $k = c/(a+b+1)$. Thus we obtain:

$$a + b \geq c. \tag{8.6.11}$$

Suppose that $-e_1 - e_3 \in P$. Since $a < 2b$ we see that $\operatorname{conv}\{e_1, -2e_1 - e_2, -e_1 - e_3, x\}$ is a Fano tetrahedron. By minimality this is not the case, and so $-e_1 - e_3 \notin P$. The line connecting the origin with $-e_1 - e_3$ intersects $\operatorname{conv}\{-e_1, e_2, -x\}$ at the point $k(-e_1 - e_3)$, where $k = c/(c+b-a+1)$. Hence:

$$b \geq a. \tag{8.6.12}$$

Finally, let us consider the point $-e_1 - e_2 - e_3$. This point must lie outside P, for otherwise $\operatorname{conv}\{e_1, e_2, -e_1 - e_2 - e_3, x\}$ would be a Fano tetrahedron. We consider the line connecting 0 and this point. If $2b - a > c$ then the line intersects $\operatorname{conv}\{-e_1, -2e_1 - e_2, -x\}$ at the point $k(-e_1 - e_2 - e_3)$, where $k = c/(b-a+1)$. But this yields $b - a \geq c$, a contradiction. Hence it must be that $2b - a \leq c$, and the line intersects $\operatorname{conv}\{e_1, -2e_1 - e_2, -x\}$. This occurs when $k = c/(a - 3b + 2c + 1)$, and gives us:

$$c \geq 3b - a. \tag{8.6.13}$$

Combining equations (8.6.10) and (8.6.13) tells us that $c = 3b - a$, and by applying equation (8.6.11) we see that $a \geq b$. Of course equation (8.6.12) now tells us that $a = b$, and so $x = (a, a, 2a)$. This forces $a = 1$. □

Lemma 8.6.6. *The minimal Fano polytopes containing two copies of the Fano triangle of type $\mathbb{P}^2$ are equivalent to:*

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Proof. Let us fix the lattice such that $P := \operatorname{conv}\{e_1, e_2, -e_1 - e_2, x, y\}$, where $x := \operatorname{conv}\{a+1, b+1, c\}$, $y := \operatorname{conv}\{-a, -b, -c\}$, and $0 < a+1 \leq b+1 \leq c$. Clearly $a = 0, b = 0, c = 1$ is a solution. Assume that $c > 1$.

Suppose that the point $-e_3$ lies on the surface of P. Then $\operatorname{conv}\{e_1, -e_1 - e_2, -e_3, x\}$ is not a Fano tetrahedron. Hence either $a = -1$ or $c = 0$. Both options are impossible. Thus $-e_3$ lies outside P. The line connecting $-e_3$ with the origin intersects $\mathrm{conv}\{e_1, e_2, y\}$ at the point $-ke_3$, where $k = c/(a+b+1)$. We see that:

$$c \leq a + b. \tag{8.6.14}$$

Consider the point $e_1 + e_2 + e_3$. The line joining this point and the origin intersects $\mathrm{conv}\{e_1, e_2, x\}$ at $k(e_1 + e_2 + e_3)$, where $k = c/(2c - (a+1) - (b+1) + 1)$. If $e_1 + e_2 + e_3 \notin P$ then $(a+1) + (b+1) \leq c$, contradicting equation (8.6.14). Hence $e_1 + e_2 + e_3$ lies on the surface of P, and $(a+1) + (b+1) - 1 = c$. But again we find that this contradicts equation (8.6.14). $\square$

Lemma 8.6.7. *Any minimal Fano polytope containing one copy of each of the two minimal Fano triangles is equivalent to:*

$$\begin{pmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Proof. Arrange matters so that $P := \mathrm{conv}\{e_1, e_2, -2e_1 - e_2, x, y\}$. There are two cases to consider:

(i) $x + y + e_2 = 0$;

(ii) $x + y + e_1 = 0$.

Observe that in case (i), the line joining e_1 and $-2e_1 - e_2$ intersects span $\{e_2\}$ at the point $-(1/3)e_2$, whereas the line joining x and y intersects span $\{e_2\}$ at $-(1/2)e_2$. Hence $P \setminus \{-2e_1 - e_2\}$ is still Fano, which contradicts minimality of P. Indeed, this case reduces to those polytopes discussed in Lemma 8.6.4.

We now address case (ii).

We have that $x = (a, b, c)\,, y = (-a-1, -b, -c)\,$, and can insist that $0 \leq a, b < c$. Clearly $a = 0, b = 0, c = 1$ is a solution, so suppose that $c > 1$. If e_3 lies on the surface of P then minimality of P requires that $\mathrm{conv}\{e_1, e_2, e_3, y\}$ is not a Fano tetrahedron. Hence either $a = -1, b = 0$, or $c = 0$. All but $b = 0$ are impossible. If we set $b = 0$ then the following must be Fano:

$$\begin{pmatrix} 1 & 0 & -a-1 \\ 0 & 1 & -c \end{pmatrix}.$$

The two possibilities reduce to $a = 0, b = 0, c = 1$, which has already been noted. The assumption that $c > 1$ forces $e_3 \notin P$.

Note that the point $-e_1$ lies on the line joining e_2 and $-2e_1 - e_2$, whilst the line joining x to y intersects the plane span $\{e_1, e_2\}$ at $-(1/2)e_1$. Hence this line (minus the end points) is contained strictly in the interior of P.

The point $e_1 + e_2 + e_3$ lies outside P, otherwise $\operatorname{conv}\{e_1, -2e_1 - e_2, e_1 + e_2 + e_3, y\}$ is a Fano tetrahedron contained in P. The line connecting this point to 0 must intersect $\operatorname{conv}\{e_1, e_2, x\}$. This occurs at $k(e_1 + e_2 + e_3)$, where $k = c/(2c - a - b + 1)$. We thus have:

$$a + b \leq c. \tag{8.6.15}$$

The point $-e_1 - e_3$ must lie outside P, otherwise P contains the Fano tetrahedron $\operatorname{conv}\{e_1, -2e_1 - e_2, -e_1 - e_3, x\}$, contradicting minimality of P. The line originating at 0 and passing through $-e_1 - e_3$ intersects ∂P in either $\operatorname{conv}\{e_1, e_2, y\}$ or $\operatorname{conv}\{-e_1, e_2, y\}$. The first possibility gives the point of intersection to be $k(-e_1 - e_3)$, where $k = c/(a + b - c + 2)$, and we have that $a + b + 1 \geq 2c$. Combining this with equation (8.6.15) yields a contradiction.

Consider the second possibility; the line connecting $-e_1 - e_3$ and the origin intersects $\operatorname{conv}\{-e_1, e_2, y\}$ at the point $k(-e_1 - e_3)$ where $k = c/(c + b - a)$. We have that:

$$b \geq a + 1. \tag{8.6.16}$$

Finally, consider the point $e_2 + e_3$. This point must lie outside P; if $e_2 + e_3$ were contained in P, then $\operatorname{conv}\{e_1, e_2 + e_3, -2e_1 - e_2, y\}$ would be a Fano tetrahedron. The line joining the point with the origin intersects $\operatorname{conv}\{e_1, -2e_1 - e_1, x\}$ or $\operatorname{conv}\{-e_1, e_2, x\}$. In the first case the point of intersection is given by $k(e_2 + e_3)$, where $k = c/(c - a - b + 1)$. Hence $a + b \leq 0$, which is an impossibility (since $c \neq 1$).

The alternative is that the line intersects $\operatorname{conv}\{-e_1, e_2, x\}$. This occurs at the point $k(e_2 + e_3)$, where $k = c/(a - b + c + 1)$, and we see that $a \geq b$. By considering equation (8.6.16) we obtain our final contradiction. □

Lemma 8.6.8. *Any minimal Fano polytope containing two copies of the minimal Fano triangle of type* $\mathbb{P}(1,1,2)$ *is equivalent to:*

$$\begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & -2 & 1 & -3 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix}.$$

Comments	Vertices	Comments	Vertices
5 Vertices Simplicial Terminal	$\begin{pmatrix}1&0&0&0&-1\\0&1&0&0&-1\\0&0&1&-1&0\end{pmatrix}$	5 Vertices Simplicial	$\begin{pmatrix}1&0&0&-2&-1\\0&1&0&-1&0\\0&0&1&0&-1\end{pmatrix}$
5 Vertices Simplicial Terminal	$\begin{pmatrix}1&0&-1&1&-1\\0&1&-1&2&-2\\0&0&0&3&-3\end{pmatrix}$	5 Vertices	$\begin{pmatrix}1&0&0&-2&-2\\0&1&0&-1&0\\0&0&1&0&-1\end{pmatrix}$
5 Vertices Simplicial	$\begin{pmatrix}1&0&0&0&-2\\0&1&0&0&-1\\0&0&1&-1&0\end{pmatrix}$	5 Vertices	$\begin{pmatrix}1&0&-2&1&-3\\0&1&-1&1&-1\\0&0&0&2&-2\end{pmatrix}$
5 Vertices Simplicial	$\begin{pmatrix}1&0&-2&1&-1\\0&1&-1&1&-1\\0&0&0&2&-2\end{pmatrix}$	6 Vertices Simplicial Terminal	$\begin{pmatrix}1&0&0&-1&0&0\\0&1&0&0&-1&0\\0&0&1&0&0&-1\end{pmatrix}$
5 Vertices Terminal	$\begin{pmatrix}1&0&0&-1&1\\0&1&0&-1&1\\0&0&1&0&-1\end{pmatrix}$	6 Vertices Simplicial Terminal	$\begin{pmatrix}1&0&-1&0&1&-1\\0&1&0&-1&1&-1\\0&0&0&0&2&-2\end{pmatrix}$

Table 8.5: The minimal canonical Fano polytopes of dimension three which are not tetrahedra.

Proof. Fix the lattice such that $P := \operatorname{conv}\{e_1, e_2, -2e_1 - e_2, x, y\}$. If $x + y + 2e_2 = 0$ then $-e_2$ is contained on the surface of P. We already know that $-e_1$ lies on the surface of P, and hence minimality reduced us to the case considered in Lemma 8.6.5. Thus $x + y + 2e_1 = 0$ and $x = (a, b, c)\,, y = (-a-2, -b, -c)$, where $0 \leq a, b < 0$. Clearly $a = 0, b = 0, c = 1$ is a solution. Let us assume that $c > 1$.

Suppose that $-e_3 \in P$. Then unless $a = 2b$ either $\operatorname{conv}\{e_1, e_2, -2e_1 - e_2, x\}$ or $\operatorname{conv}\{e_2, e_3, -2e_1 - e_2, x\}$ is a Fano tetrahedron. Thus it must be that $a = 2b$, and in particular $b + 1 \leq c$. But then $(b/c)(-2e_1 - e_2) + (1/c)x = e_3$, contradicting minimality of P. Hence $-e_3 \notin P$. The line joining $-e_3$ to the origin intersects $\operatorname{conv}\{e_1, e_2, y\}$ at the point $k(-e_3)$, where $k = c/(a + b + 3)$. Hence we conclude that:

$$a + b + 2 \geq c. \tag{8.6.17}$$

The point $e_1 + e_2 + e_3$ does not lie in P, otherwise either:

$$\operatorname{conv}\{e_1, -2e_1 - e_2, e_1 + e_1 + e_3, y\}\,,$$

or:

$$\operatorname{conv}\{-e_1, -2e_1 - e_2, e_1 + e_2 + e_3, y\}\,,$$

would be a Fano tetrahedron. Consider the line connecting 0 and $e_1 + e_2 + e_3$. This line intersects $\text{conv}\{e_1, e_2, x\}$ at the point $k(e_1 + e_2 + e_3)$, where $k = c/(2c - a - b + 1)$. In particular,

$$a + b \leq c. \tag{8.6.18}$$

If $e_2 + e_3 \in P$ then $\text{conv}\{e_1, -2e_1 - e_2, e_2 + e_3, y\}$ would be a Fano tetrahedron. This is not permissible. The line connecting $e_2 + e_3$ and the origin intersects $\text{conv}\{-e_1, e_2, x\}$ at the point $k(e_2 + e_3)$, where $k = c/(a - b + c + 1)$. We conclude that:

$$a \geq b. \tag{8.6.19}$$

In particular $a \neq 0$, since the alternative would force $c = 1$.

Finally we consider the point $-e_1 - e_3$. The line connecting this point with the origin intersects $\text{conv}\{-e_1, e_2, y\}$ if $a + 2 \leq c$, or $\text{conv}\{e_1, e_2, y\}$ if $a + 2 > c$. The first possibility gives the point of intersection as $k(-e_1 - e_3)$, where $k = c/(b + c - a - 1)$. If $-e_1 - e_3$ lies on the surface of P, we see that $b = a + 1$. This contradicts equation (8.6.19). Hence it must be that $-e_1 - e_3$ lies outside P. In this case, $b \geq a + 2$, and once again this contradicts equation (8.6.19). It must be that $a + 2 > c$, which implies that $a = c - 1$. Equation (8.6.18) forces $b \leq 1$, and by applying equation (8.6.17) we see that the only possibility is $a = 1, b = 1, c = 2$. □

CHAPTER 9

On the Barycentric Coordinates of Fano Simplices

9.1 Introduction

In Chapter 8 we saw that in order to understand the minimal polytopes an understanding of fake weighted projective space is crucial (Proposition 8.2.11). The simplices associated with fake weighted projective space form the building blocks from which the minimal polytopes can be derived. Furthermore, knowledge of the weights of weighted projective space is required in order to “grow” these minimal polytopes.

Theorem 4.4.13 established a bound on the multiplicity of fake weighted projective space. Combined with the work of Conrads ([Con02]), the problem of classifying the fake weighted projective spaces is reduced to an understanding of the weights involved. Thus in this chapter we consider the possible weights in more detail.

Corollary 9.3.2 establishes a crude bound on the sum of the weights, whilst Theorem 9.3.6 gives a new upper bound on the individual weights. Section 9.4 is dedicated to proving Theorem 9.3.6. Sections 9.5–9.9 attempt to generalise the process of deriving weights. In the Gorenstein case, Proposition 9.8.1 and Theorem 9.8.4 establish bounds which allow us to perform some new classifications.

The techniques developed in this chapter culminate in several classifications: Proposition 9.7.10, Proposition 9.7.11, Proposition 9.8.5 and, via use of a computer algorithm, Proposition 9.9.1.

9.2 Reflexive Fano Simplices

Reflexive simplices have been studied in some detail in [Con02, Nil04] and in [Bat94, §5.4]. Nearly all the results stem from Proposition 6.2.7, which is reproduced here:

Proposition 6.2.7 ([CK99, Lemma 3.5.6])**.** *Let $X = \mathbb{P}(\lambda_0, \ldots, \lambda_n)$ be a weighted projective space, and let $h = \sum_{i=0}^n \lambda_i$. Then X is Gorenstein Fano if and only if $\lambda_i \mid h$ for all i.*

From Corollary 4.4.9 we see that in order to classify the weights of all Gorenstein fake weighted projective spaces, it is sufficient to understand those weights satisfying Proposition 6.2.7. The three articles cited above provide a very thorough analysis.

Let P' be an n-simplex with weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$, and let P be the n-simplex associated with $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. By using Theorem 4.4.19 in the case where P' is reflexive, Conrads proved the following:

Proposition 9.2.1 ([Con02, Proposition 5.5])**.** *With notation as above;*

$$\operatorname{mult} P' \mid \operatorname{mult} P^\vee.$$

Proof. Since P' is reflexive, so P must be reflexive by Corollary 4.4.9. By Theorem 4.4.19 there exists some $H \in \operatorname{Herm}(n, \operatorname{mult} P')$ such that $P' = HP$. Hence $P'^\vee = H^\vee P^\vee$. Now $H^\vee = (H^t)^{-1}$ (this is true for linear transformations in general), and so $\det H^\vee = 1/\operatorname{mult} P'$.

Thus $\det P'^\vee = \det P^\vee / \operatorname{mult} P'$. By Proposition 4.4.10 and equation (4.4.3) we obtain:

$$\operatorname{mult} P'^\vee = \frac{\operatorname{mult} P^\vee}{\operatorname{mult} P'}.$$

Observing that $\operatorname{mult} P'^\vee \in \mathbb{Z}_{>0}$ gives the result. □

As mentioned in Remark 4.2.1, given weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ it is a straight forward task (best achieved by a computer) to calculate the simplex P associated with weighted projective space $\mathbb{P}(\lambda_0, \lambda_1, \ldots, \lambda_n)$. It is equally trivial to calculate its dual $P^\vee$ and the volume $\operatorname{vol} P^\vee$. The multiplicity of $P^\vee$ can then be calculated by equation (4.4.3).

Proposition 9.2.1 thus provides the means to calculate all possible P' efficiently by prescribing permitted values of $\operatorname{mult} P'$. Theorem 4.4.19 can then be employed to calculate the reflexive simplices P'.

Remark 9.2.2. The algorithm outlined above relies on the simplices under consideration being reflexive – without this Proposition 9.2.1 does not hold (indeed, is meaningless), and so Conrads establishes no bounds on the multiplicity of P'. In turn, this makes the application of Theorem 4.4.19 impossible.

It should be noted that, for general canonical simplices, all is not lost. Corollary 4.4.15 provides a bound, albeit much cruder, on $\operatorname{mult} P'$, and we can proceed from there.

The remaining ingredient is a description of the possible weights. Of course, Proposition 6.2.7 will suffice provided some bound can be established on h. In Corollary 9.3.2 a (very inefficient) bound is given. In fact the bound of Corollary 9.3.2 applies to general canonical simplices, not merely those which are reflexive.

Inspired by the work of [HM04], a very elegant approach to bounding the weights of reflexive simplices was demonstrated in [Nil04][1]. We need the following definitions:

Definition 9.2.3. A family of positive natural numbers $(k_0, k_1, \ldots, k_n)$ is called a *unit partition* if $\sum_{i=0}^{n} 1/k_i = 1$.

Given any Gorenstein weighted projective space, Proposition 6.2.7 allows you to construct the family $k_i := \lambda_i / h$, $i = 0, 1, \ldots, n$. Clearly $(k_0, k_1, \ldots, k_n)$ is a unit partition.

Definition 9.2.4. The recursive sequence (9.2.1) is called the *Sylvester sequence*.

$$y_0 := 2, \quad y_n := 1 + \prod_{k=0}^{n-1} y_k. \tag{9.2.1}$$

From the Sylvester sequence (see [Slo06, sequence number A000058] for more information), we obtain a second sequence defined by:

$$t_n := y_n - 1 = \prod_{k=0}^{n-1} y_k. \tag{9.2.2}$$

[1] The preprint [Nil04] was published after the majority of the work in Sections 9.7–9.9 had been produced. The techniques behind the two approaches are very different. The methods in Sections 9.7–9.9 apply to general canonical simplices, although the most satisfying results only apply to the reflexive case.

n	0	1	2	3	4	5	6
No. unit partitions	1	1	3	14	147	3462	294314
$t_n := y_n - 1$	1	2	6	42	1806	3263442	10650056950806

Table 9.1: The number of possible unit partitions of length $n \leq 6$, and the value t_n (which bounds the entries in the partitions) – c.f. Table 9.8.

Proposition 9.2.5 ([Nil04, Proposition 3.4]). *Let $(k_0, k_1, \ldots, k_n)$ be a unit partition. Then:*

$$n + 1 \leq \max \{k_0, k_1, \ldots, k_n\} \leq t_n,$$

with equality in the second case only for the Sylvester partition $(y_0, \ldots, y_{n-1}, t_n)$. *Furthermore, if $k_0 \leq k_1 \leq \ldots \leq k_n$ then:*

$$k_i \leq (n - i + 1)t_i, \qquad \text{for } i \in \{0, 1, \ldots, n\}.$$

The bound in Proposition 9.2.5 is the best possible, since the Sylvester partition corresponds to the weights of a reflexive simplex.

Remark 9.2.6. Calculating number of unit partitions of fixed length is a well-known problem in number theory (see [Slo06, sequence number A002966]). Compare the values tabulated in Table 9.1 with those in Table 9.8. Proposition 6.2.7 guarantees that unit partitions and Gorenstein Fano weighted projective spaces are in bijective correspondence.

9.3 Upper Bounds on the Barycentric Coordinates

Let $P := \mathrm{conv}\{x_0, \ldots, x_n\}$ be a Fano n-simplex in $N_\mathbb{R} \cong \mathbb{R}^n$. Let $\{\lambda_0, \ldots, \lambda_n\} \subset \mathbb{Z}_{>0}$ be such that $(\lambda_0/h, \ldots, \lambda_n/h)$ is the barycentric coordinate of the origin 0 with respect to the vertices $x_0, \ldots, x_n$ of P, where $h = \sum_{i=0}^n \lambda_i$. Hence:

$$\sum_{i=0}^{n} \frac{\lambda_i}{h} x_i = 0.$$

Without loss of generality we insist that $\lambda_0 \leq \ldots \leq \lambda_n$ and that $(\lambda_0, \ldots, \lambda_n) = 1$.

In [Pik01, Theorem 6] an upper bound is given for the volume of P:

Theorem 9.3.1 ([Pik01]). *For any Fano n-simplex P we have:*

$$\operatorname{vol} P \leq \frac{1}{n!} 2^{3n-2} 15^{(n-1)2^{n+1}}.$$

Combining this result with Proposition 4.2.5 immediately gives us an upper bound on h. This bound is far from tight. Indeed it is essentially unusable, and is mentioned here only for completeness.

Corollary 9.3.2. *With notation as above;*

$$h \leq 2^{3n-2} 15^{(n-1)2^{n+1}}.$$

In the case where P is a reflexive n-simplex, combining Proposition 4.2.5 with [Nil04, Theorem C] provides much better bounds:

Proposition 9.3.3. *Suppose that P is a reflexive n-simplex. With notation as above;*

$$h \leq t_n,$$

where t_n is defined in (9.2.2).

A lower bound on λ_0/h was also presented in [Pik01, Theorem 2]:

Theorem 9.3.4 ([Pik01]). *With notation as above;*

$$\frac{\lambda_0}{h} \geq \frac{1}{8 \cdot 15^{2^{n+1}}}.$$

When P is a reflexive n-simplex, [Nil04, Proposition 3.4] establishes the following lower bounds:

Proposition 9.3.5 ([Nil04, Proposition 3.4]). *Suppose that P is a reflexive n-simplex. With notation as above, for any $k \in \{0, \ldots, n\}$ we have that:*

$$\frac{\lambda_{n-k}}{h} \geq \frac{1}{(k+1)t_{n-k}},$$

where t_j is defined in (9.2.2).

We shall prove the following upper bounds holds for general canonical n-simplices:

Theorem 9.3.6. *With notation as above, for any $k \in \{0, \ldots, n-2\}$ we have that:*

$$\frac{\lambda_{n-k}}{h} \leq \frac{1}{k+2},$$

with strict inequality if P is terminal.

9.4 Proof of Theorem 9.3.6

We shall require the following results:

Lemma 9.4.1. *With notation as above, let $\{x_{i_0}, \ldots, x_{i_m}\} \subset \{x_0, \ldots, x_n\}$ be any subset of the vertices of P. Then for no non-trivial collection of integers $\mu_0, \ldots, \mu_m \in \mathbb{Z}$ does $\mu_0 x_{i_0} + \ldots + \mu_m x_{i_m} = 0$.*

Proof. Since P is Fano at least one $\mu_i < 0$; let us assume that $\mu_0 < 0$, and that $m' \geq 1$ of the μ_i are negative. We shall prove that there exists a linear combination of $x_0, \ldots, \widehat{x_{i_0}}, \ldots, x_n$ with at most $m' - 1$ negative coefficients. Inductively, this is impossible. Our claim is obvious, since the following will suffice:

$$-\mu_0 \sum_{j=0}^{n} \lambda_j x_j + \lambda_{i_0} \sum_{j=0}^{m} \mu_j x_{i_j}.$$

□

Lemma 9.4.2. *Let $\sigma = \mathrm{cone}\{x_1, \ldots, x_m\}$ be a strictly convex cone. If $x \in -\sigma$ then $0 \in \mathrm{conv}\{x, x_1, \ldots, x_m\}$.*

Proof. If $x = 0$ then we are done. Assume not. Since $x \in -\sigma$, so $x = \sum_{i=1}^{m} \mu_i(-x_i)$ for some $\mu_i \geq 0$, with at least one $\mu_i \neq 0$. Assume without loss of generality that $\mu_1 \neq 0$. We have that:

$$x = -b - \mu_1 x_1, \qquad \text{for some } b \in \mathrm{cone}\{x_2, \ldots, x_m\}.$$

Hence we see that $x + b + \mu_1 x_1 = 0$.

Since $b \in \mathrm{cone}\{x_2, \ldots, x_m\}$, so $\nu b \in \mathrm{conv}\{0, x_2, \ldots, x_m\}$ for all sufficiently small $\nu > 0$. In particular take any such ν with $\nu < 1$ and $\nu\mu_1 < 1$. This gives us:

$$0 = \nu x + \nu b + \nu\mu_1 x_1 \in \mathrm{conv}\{x, x_1, \ldots, x_m\}.$$

□

We are now in a position to prove Theorem 9.3.6. We shall present our proof assuming that P is terminal; by the end, the result when P is canonical should be apparent.

Proof of Theorem 9.3.6. Since $\sum_{i=0}^{n} \lambda_i x_i = 0$, so:

$$\sum_{i=0}^{n-k-1} \lambda_i x_i = \sum_{j=n-k}^{n} -\lambda_j x_j.$$

Now $\sum_{i=0}^{n-k-1} \lambda_i = h - \sum_{j=n-k}^{n} \lambda_j$, giving:

$$x := \sum_{j=n-k}^{n} \frac{-\lambda_j}{h-l} x_j \in \operatorname{conv}\{x_0, \ldots, x_{n-k-1}\}, \qquad \text{where } l := \sum_{j=n-k}^{n} \lambda_j.$$

Since P is simplicial, $\operatorname{conv}\{x_0, \ldots, x_{n-k-1}\}$ is a face of P. Since the λ_i are all strictly positive, x lies strictly in the interior of this face.

Let us suppose for a contradiction that:

$$\lambda_{n-k+i} \geq \frac{h}{k+2}, \qquad \text{for all } i \in \{0, \ldots, k\}. \tag{9.4.1}$$

Consider the $(k+1)$-dimensional lattice Γ generated by $e_0, \ldots, e_k$. There exists a map of lattices $\gamma : \Gamma \to N$ given by sending $e_i \mapsto x_{n-k+i}$. Note that this map is injective by Lemma 9.4.1. Let $x' := \sum_{i=0}^{k} -\lambda_{n-k+i}/(h-l)e_i$. We shall show that the non-zero lattice point $p := -\sum_{i=0}^{k} e_i$ lies in $\operatorname{conv}\{x', e_0, \ldots, e_k\}$. Hence $\gamma(p) \neq 0$ is a lattice point in $\operatorname{conv}\{x, x_{n-k}, \ldots, x_n\} \subset P$.

Since $p \notin \operatorname{conv}\{e_0, \ldots, e_k\}$, so $\gamma(p)$ is not contained in $\operatorname{conv}\{x_{n-k}, \ldots, x_n\}$. The only remaining possibility which does not contradict P being Fano is that $\gamma(p) = x$. But if P is terminal we have a contradiction.

Consider λ_n. By (9.4.1) we have that:

$$\lambda_n - h \geq \frac{-h(k+1)}{k+2}. \tag{9.4.2}$$

Summing (9.4.1) over $0 \leq i < n$ gives:

$$l - \lambda_n \geq \frac{hk}{k+2}. \tag{9.4.3}$$

Combining equations (9.4.2) and (9.4.3) gives us that $l - h \geq -h/(k+2)$. Observing that $l - h < 0$, we obtain $k + 2/h \leq 1/(h-l)$. Thus, for any $j \in \{n-k, \ldots, n\}$, we have that:

$$-1 \geq \frac{-\lambda_j}{h-l}.$$

Thus the coefficients of x' are all ≤ -1.

Let τ be the lattice translation of Γ which sends 0 to $\sum_{i=0}^{k} e_i$. Applying Lemma 9.4.2 to $\operatorname{cone}\{\tau e_0, \ldots, \tau e_k\}$, if $\tau(x') \in -\operatorname{cone}\{\tau e_0, \ldots, \tau e_k\}$ then $p \in \operatorname{conv}\{x', e_0, \ldots, e_k\}$ and we are done. Hence assume that this is not the case.

Let $H_i \subset \Gamma_{\mathbb{R}}$ be the hyperplane containing the $k+1$ points $e_0, \ldots, \widehat{e_i}, \ldots, e_k$, and p; let H_i^+ be the half–space in $\Gamma_{\mathbb{R}}$ whose boundary is H_i and which contains the point $2p$. Then:

$$-\operatorname{cone}\{\tau e_0, \ldots, \tau e_k\} = \tau\Big(\bigcap_{i=0}^{k} H_i^+\Big).$$

Since $\tau(x') \notin -\operatorname{cone}\{\tau e_0, \ldots, \tau e_k\}$, we have that $x' \notin H_i^+$ for some i. Assume, with possible reordering of the indices, that $x' \notin H_0^+$.

H_0 is given by:

$$\left\{ \sum_{j=1}^{k} \mu_j e_j - \Big(1 - \sum_{j=1}^{k} \mu_j\Big) \sum_{i=0}^{k} e_i \;\middle|\; \mu_i \in \mathbb{R} \right\}.$$

Let $q := \sum_{i=0}^{k} \nu_i e_i$ be any point in $\Gamma_{\mathbb{R}}$. By projecting q onto H_0 along e_0 we can always choose our μ_i such that:

$$\mu_j + \sum_{i=1}^{k} \mu_i - 1 = \nu_j, \qquad \text{for } 1 \leq j \leq k. \tag{9.4.4}$$

Comparing the sign of $\sum_{i=1}^{k} \mu_i - 1$ with ν_0 tells us on which side of the hyperplane H_0 the point q lies.

We have that $2p$ lies on the opposite side of H_0 to x'. Setting $\nu_j = -2$ for all j in equation (9.4.4) tells us that:

$$\sum_{i=1}^{k} \mu_i = \frac{-k}{k+1}.$$

Hence we see that:

$$\sum_{i=1}^{k} \mu_i - 1 = \frac{-k}{k+1} - 1 > -2.$$

We thus require that:

$$\sum_{i=1}^{k} \mu_i - 1 < \frac{-\lambda_{n-k}}{h-l}. \tag{9.4.5}$$

(I.e. x' lies on the opposite side of H_0 to $2p$.)

Comparing coefficients with x', we see that:

$$\mu_j + \sum_{i=1}^{k} \mu_i - 1 = \frac{-\lambda_{n-k+j}}{h-l}, \qquad \text{for } 1 \leq j \leq k. \tag{9.4.6}$$

Summing equation (9.4.6) for all $1 \leq j \leq k$ and combining this with (9.4.5) gives:

$$\sum_{j=1}^{k} \frac{-\lambda_{n-k+j}}{h-l} + k < \frac{-(k+1)\lambda_{n-k}}{h-l} + k + 1.$$

Simplifying, and recalling that $\sum_{j=n-k}^{n} \lambda_j = l$, gives us that:

$$\lambda_{n-k} < \frac{h}{k+2}. \tag{9.4.7}$$

Equation (9.4.7) contradicts (9.4.1), concluding the proof. □

9.5 Determining Possible Barycentric Coordinates

Let $P := \operatorname{conv}\{x_0, \ldots, x_n\}$ be a Fano n-simplex in $N_{\mathbb{R}} \cong \mathbb{R}^n$. Let $\{\lambda_0, \ldots, \lambda_n\} \subset \mathbb{Z}_{>0}$ be such that $(\lambda_0/h, \ldots, \lambda_n/h)$ is the barycentric coordinate of the origin 0 with respect to the vertices $x_0, \ldots, x_n$ of P, where $h = \sum_{i=0}^{n} \lambda_i$. Hence:

$$\sum_{i=0}^{n} \frac{\lambda_i}{h} x_i = 0.$$

Without loss of generality we insist that $\lambda_0 \leq \ldots \leq \lambda_n$ and that $(\lambda_0, \ldots, \lambda_n) = 1$.

Recall the following definitions:

Definition 9.5.1. Let $q \in \mathbb{Q}$. We define $\lfloor q \rfloor := \max\{a \in \mathbb{Z} \mid a \leq q\}$ and $\lceil q \rceil := \min\{a \in \mathbb{Z} \mid a \geq q\}$. The *fractional part* of q, denoted $\{q\}$, is given by $q - \lfloor q \rfloor$.

We begin by generalising the results of Section 6.2.

Lemma 9.5.2. *For any $\kappa \in \{2, \ldots, h-2\}$ we have that $\sum_{i=0}^{n} \{\lambda_i \kappa / h\} \in \{1, \ldots, n\}$.*

Proof. Since $\sum_{i=0}^{n} \lambda_i\kappa/h = \kappa \in \mathbb{N}$ it follows that $\sum_{i=0}^{n} \{\lambda_i\kappa/h\} \in \{0, 1, \ldots, n\}$. Suppose for some $\kappa \in \{2, \ldots, h-2\}$, $\{\lambda_i\kappa/h\} = 0$ for $i = 0, \ldots, n$. We have that $h \mid \kappa\lambda_i$ for each i, so let p be a prime such that $p \mid h$, so that $h = p^r h'$ where $p \nmid h'$. Then $p^r \mid \kappa\lambda_i$. Suppose that $p^r \nmid \kappa$. Then $p \mid \lambda_i$ for each i. Hence $p \mid (\lambda_0, \ldots, \lambda_n) = 1$, a contradiction. Thus $p^r \mid \kappa$. By induction on the prime divisors of h we see that $h \mid \kappa$, so in particular $h \leq \kappa$, which is a contradiction. □

Proposition 9.5.3. *Suppose that P has at worst terminal singularities, then* $\sum_{i=0}^{n} \{\lambda_i\kappa/h\} \in \{2, \ldots, n-1\}$ *for all* $\kappa \in \{2, \ldots, h-2\}$.

Proof. By Lemma 9.5.2 we only need to consider the cases when $\sum_{i=0}^{n} \{\lambda_i\kappa/h\} = 1$ or n. Suppose that $\sum_{i=0}^{n} \{\lambda_i\kappa/h\} = n$ for some κ. Since $\{\lambda_i\kappa/h\} < 1$ it must be that $\{\lambda_i\kappa/h\} \neq 0$ for all i. Hence $\{\lambda_i(h-\kappa)/h\} = 1 - \{\lambda_i\kappa/h\}$, giving us that $\sum_{i=0}^{n} \{\lambda_i(h-\kappa)/h\} = 1$.

Suppose for some $\kappa \in \{2, \ldots, h-2\}$ the sum is 1. Let $\chi_i = \{\lambda_i\kappa/h\}$. Then $(\chi_1, \ldots, \chi_4)$ is the (unique) barycentric coordinate for some point in the tetrahedron. We shall show that it is a non-vertex lattice point not equal to the origin.

We have that $\sum_{i=0}^{n} \lfloor \lambda_i\kappa/h \rfloor x_i$ is a lattice point, call it $a \in \mathbb{Z}^3$. We also have that $\sum_{i=0}^{n} \lambda_i\kappa/hx_i = 0$. Thus:

$$\sum_{i=1}^{4} \chi_i x_i = \sum_{i=1}^{4} \frac{\lambda_i\kappa}{h} x_i - \sum_{i=1}^{4} \left\lfloor \frac{\lambda_i\kappa}{h} \right\rfloor x_i = -a \in \mathbb{Z}^3.$$

By the uniqueness of barycentric coordinates we have that $-a$ is a non-vertex point, since each $\chi_i < 1$. Furthermore suppose $-a = 0$, so that $\chi_i = \lambda_i$ for $i = 0, \ldots, n$. For each i, $\lambda_i\kappa/h - \lfloor \lambda_i\kappa/h \rfloor = \lambda_i/h$, so we obtain that $\lfloor \lambda_i\kappa/h \rfloor = \lambda_i(\kappa-1)/h$ and hence that $h \mid \lambda_i(\kappa-1)$. As in the proof of Lemma 9.5.2 we find that $h \mid \kappa - 1$, and so in particular $h + 1 \leq \kappa$. This contradicts our range for κ. Hence $-a$ must be a non-vertex, non-zero lattice point in the simplex, contradicting our hypothesis. □

9.6 Understanding the Step–Function

Definition 9.6.1. Define the step functions $\sigma_i : \{0, \ldots, h\} \to \mathbb{Z}$ by:

$$\sigma_i(\kappa) := \left\{ \frac{\lambda_i\kappa}{h} \right\}.$$

Define the step function $\Sigma^n : \{0, \dots, h\} \to \mathbb{Z}$ by:

$$\Sigma^n(\kappa) := \sum_{i=0}^{n} \left\{ \frac{\lambda_i \kappa}{h} \right\}.$$

Definition 9.6.2. We say that $\kappa + 1$ is a *jump point for* σ_i if $\sigma_i(\kappa+1) \le \sigma_i(\kappa)$. We say that $\kappa + 1$ is a *jump point for* Σ^n if $\Sigma^n(\kappa+1) \le \Sigma^n(\kappa)$.

Lemma 9.6.3. *Let* $\kappa \in \{0, \dots, h-1\}$. *Then:*

(i) If $\kappa + 1$ *is a jump point for* σ_i*, then* $\sigma_i(\kappa+1) < \sigma_i(\kappa)$*;*

(ii) If $\kappa + 1$ *is not a jump point for* σ_i*, then* $\sigma_i(\kappa+1) = \sigma_i(\kappa) + \lambda_i/h$*;*

(iii) If $\kappa + 1$ *is not a jump point for* Σ^n*, then* $\Sigma^n(\kappa+1) = \Sigma^n(\kappa) + 1$.

Proof. In order to prove this result, we make use of the obvious fact that, for any $a, b \in \mathbb{R}$, $\{a+b\} = \{\{a\} + \{b\}\} \le \{a\} + \{b\}$.

First, suppose for a contradiction that we have equality between $\sigma_i(\kappa+1)$ and $\sigma_i(\kappa)$. Then:

$$\left\{ \frac{\lambda_i(\kappa+1)}{h} \right\} = \left\{ \frac{\lambda_i \kappa}{h} \right\}. \tag{9.6.1}$$

Since $0 < \lambda_i/h < 1$ we have that:

$$\left\{ \frac{\lambda_i(\kappa+1)}{h} \right\} = \left\{ \left\{ \frac{\lambda_i \kappa}{h} \right\} + \frac{\lambda_i}{h} \right\}. \tag{9.6.2}$$

Combining equations (9.6.1) and (9.6.2) gives:

$$\left\{ \frac{\lambda_i \kappa}{h} \right\} + \frac{\lambda_i}{h} - 1 = \left\{ \frac{\lambda_i \kappa}{h} \right\},$$

but this is a contradiction.

To prove the third part, simply observe that if:

$$\left\{ \frac{\lambda_i(\kappa+1)}{h} \right\} > \left\{ \frac{\lambda_i \kappa}{h} \right\} + \frac{\lambda_i}{h},$$

then we obtain:

$$\left\{ \left\{ \frac{\lambda_i \kappa}{h} \right\} + \frac{\lambda_i}{h} \right\} > \left\{ \frac{\lambda_i \kappa}{h} \right\} + \frac{\lambda_i}{h},$$

which is patently absurd. Hence it must be that:

$$\left\{\frac{\lambda_i(\kappa+1)}{h}\right\} \leq \left\{\frac{\lambda_i\kappa}{h}\right\} + \frac{\lambda_i}{h}.$$

As a consequence, $\Sigma^n(\kappa+1) \leq \Sigma^n(\kappa)+1$. Since Σ^n is integer valued only, and since $\kappa+1$ is not a jump point by our hypothesis, we are done.

Finally, to prove (ii) suppose that $\sigma_i(\kappa+1) < \sigma_i(\kappa) + \lambda_i/h$. (Note that by our proof of (iii) we already know that $\sigma_i(\kappa+1) \leq \sigma_i(\kappa)+\lambda_i/h$.) Then:

$$\left\{\left\{\frac{\lambda_i\kappa}{h}\right\} + \frac{\lambda_i}{h}\right\} < \left\{\frac{\lambda_i\kappa}{h}\right\} + \frac{\lambda_i}{h}.$$

Hence we obtain the equality:

$$\left\{\left\{\frac{\lambda_i\kappa}{h}\right\} + \frac{\lambda_i}{h}\right\} = \left\{\frac{\lambda_i\kappa}{h}\right\} + \frac{\lambda_i}{h} - 1.$$

Recalling that $\kappa+1$ is not a jump point for σ_i tells us that $\sigma_i(\kappa) < \sigma_i(\kappa) + \lambda_i/h - 1$, and so $\lambda_i > h$ – a contradiction. □

Proposition 9.6.4. *Let $\kappa \in \{0,\ldots,h-1\}$. $\kappa+1$ is a jump point for Σ^n if and only if $\kappa+1$ is a jump point for σ_i, for some i.*

Proof. Suppose that $\kappa+1$ is not a jump point for Σ^n. In addition, suppose for a contradiction that there exists an i such that $\kappa+1$ is a jump point for σ_i. By Lemma 9.6.3 (i) we have that $\sigma_i(\kappa+1) < \sigma_i(\kappa)$. Combined with Lemma 9.6.3 (ii) we see that $\Sigma^n(\kappa+1) < \Sigma^n(\kappa)+1-\lambda_i/h$, which contradicts Lemma 9.6.3 (iii).

Suppose now that $\kappa+1$ is a jump point for Σ^n. Suppose for a contradiction that $\kappa+1$ is not a jump point for σ_i for all i. Then by definition $\sigma_i(\kappa+1) > \sigma_i(\kappa)$ for all i, and so $\Sigma^n(\kappa+1) > \Sigma^n(\kappa)$; a contradiction. □

9.7 The Step–Function for Reflexive Simplices

The results in Sections 9.7–9.9 stand independent of those discussed in Section 9.2, which should be regarded as providing motivation for this work. In particular, the sharp bound of Proposition 9.2.5 is not use here.

Lemma 9.7.1. *Suppose that $\lambda_i \mid h$. $\kappa \in \{1, \ldots, h\}$ is a jump point for σ_i if and only if $\sigma_i(\kappa) = 0$.*

Proof. Clearly if $\sigma_i(\kappa) = 0$ then κ is a jump point for σ_i.

Since $\lambda_i \mid h$, there exists a $k \in \{1, \ldots, h\}$ such that $k\lambda_i = h$. Let κ be the smallest jump point for σ_i such that $\sigma_i(\kappa) \neq 0$. Clearly $\kappa > k$.

Consider $\kappa' = \kappa - k \in \{1, \ldots, h\}$. Since $\sigma_i(\kappa') = \sigma_i(\kappa)$, and since $\sigma_i(\kappa' - 1) = \sigma_i(\kappa - 1)$, so it must be that κ' is a jump point for σ_i. But $\sigma_i(\kappa') = \sigma_i(\kappa) \neq 0$, contradicting the minimality of κ.

Hence $\sigma_i(\kappa) = 0$ and we are done. □

Lemma 9.7.2. $\ker \sigma_i$ *forms a cyclic subgroup of $\mathbb{Z}/(h)$.*

Proof. Define the group endomorphism $\overline{\sigma}_i : \mathbb{Z}/(h) \to \mathbb{Z}/(h)$ by $\overline{\sigma}_i(\kappa) := \lambda_i\kappa \pmod h$. Then $h\sigma_i(\kappa) = \overline{\sigma}_i(\kappa)$ and the result follows (since any subgroup of a cyclic group is cyclic). □

In light of Lemmas 9.7.1 and 9.7.2, the following definition makes sense:

Definition 9.7.3. We say that the jump points for σ_i have *order k* if $k\lambda_i = h$ for some $k \in \mathbb{Z}$.

Definition 9.7.4. Let $s(\kappa) \in \{0, \ldots, n+1\}$ be the number of $i \in \{0, \ldots, n\}$ such that κ is a jump point for σ_i.

Proposition 9.7.5. *Let P be Gorenstein. Then $\Sigma^n(\kappa + 1) = \Sigma^n(\kappa) + 1 - s(\kappa + 1)$.*

Proof. Let $\kappa \in \{0, \ldots, h-1\}$. Without loss of generality, suppose that $\kappa + 1$ is a jump point for σ_i for $i = 0, \ldots, k-1$ (with possible relabelling of the indices), and that it is not a jump point for σ_j, $j = k, \ldots, n$. Hence $k = s(\kappa + 1)$. Let $\Sigma^n(\kappa) = a$.

We have that $\Sigma^n(\kappa + 1) = \sum_{i=0}^{k-1} \sigma_i(\kappa + 1) + \sum_{j=k}^{n} \sigma_j(\kappa + 1)$. By Lemma 9.7.1 and Proposition 6.2.7 we know that $\sigma_i(\kappa + 1) = 0$ for $i = 0, \ldots, k-1$. Hence $\Sigma^n(\kappa + 1) = \sum_{j=k}^{n} \sigma_j(\kappa + 1) = \sum_{j=k}^{n} \sigma_j(\kappa) + \sum_{j=k}^{n}(\lambda_j/h)$, by Lemma 9.6.3 (ii). This gives us:

$$\Sigma^n(\kappa + 1) = a + 1 - \sum_{i=0}^{k-1} \frac{\lambda_i}{h} - \sum_{i=0}^{k-1} \sigma_i(\kappa).$$

Consider $\sigma_i(\kappa)$, where $i \in \{0, \ldots, k-1\}$. Since $\sigma_i(\kappa+1) = 0$, we deduce that $\sigma_i(\kappa) = 1 - \lambda_i/h$. From this we see that:

$$\Sigma^n(\kappa+1) = a + 1 - k.$$

□

Proposition 9.7.6. *Let P be Gorenstein. Then $\Sigma^n(h-\kappa) = n + 1 - s(\kappa) - \Sigma^n(\kappa)$.*

Proof. Let $\kappa \in \{1, \ldots, h\}$. Without loss of generality, suppose that κ is a jump point for σ_i for $i = 0, \ldots, k-1$ (with possible relabelling of the indices), and that it is not a jump point for σ_j, $j = k, \ldots, n$. i.e. $k = s(\kappa)$. For any $i \in \{0, \ldots, k-1\}$, $\sigma_i(n-\kappa) = 0$, hence we have that $\Sigma^n(h-\kappa) = \sum_{j=k}^{n} \left(1 - \left\{\lambda_j \kappa/h\right\}\right) = n - k + 1 - \Sigma^n(\kappa)$, as desired. □

Corollary 9.7.7. *Suppose that P is Gorenstein. Then, for all $\kappa \in \{1, \ldots, h-1\}$, we have that:*

$$s(\kappa) \le \begin{cases} n-3, & \text{if } P \text{ is terminal;} \\ n-1, & \text{if } P \text{ is canonical.} \end{cases}$$

Proof. Observe that $s(1) = s(h-1) = 0$. Lemma 9.5.2 tells us that $\Sigma^n(\kappa) \ge 1$, for $\kappa \in \{2, \ldots, h-2\}$. Hence $s(\kappa) \le n-1$. This completes the general canonical case.

Suppose now that P has at worst terminal singularities. By Proposition 9.5.3 we know that $\Sigma^n(\kappa) \ge 2$ for all κ. Hence it must be that $s(\kappa) \le n-2$. From Proposition 9.7.6 we have that $2 \le \Sigma^n(h-\kappa) = n + 1 - s(\kappa) - \Sigma^n(\kappa)$. Rearranging, we see that:

$$s(\kappa) + \Sigma^n(\kappa) \le n - 1. \tag{9.7.1}$$

Suppose that $s(\kappa) = n-2$ for some κ. We obtain $\Sigma^n(\kappa) \le (n-1) - (n-2) = 1$, a contradiction. Thus we see that $s(\kappa) \le n-3$ for all κ. □

Corollary 9.7.8. *If P is Gorenstein terminal then $\Sigma^n(\kappa) \le n-2$ for all $\kappa \in \{2, \ldots, h-3\}$.*

Proof. By Proposition 9.5.3 we know that $\Sigma^n(\kappa) \le n-1$ for all $\kappa \in \{2, \ldots, h-2\}$. Suppose that $\Sigma^n(\kappa) = n-1$ for some $\kappa \in \{2, \ldots, h-3\}$. By Proposition 9.7.5 we have that $s(\kappa+1) + \Sigma^n = n$. But this contradicts equation (9.7.1). □

Using Corollary 9.7.8 we can give a new proof of the following fact (see Corollary 6.2.8):

Proposition 9.7.9. *The only Gorenstein Fano weighted projective space of dimension three with at worst terminal singularities is* $\mathbb{P}^3$.

Proof. Suppose that $h \geq 5$. By Corollary 9.7.8 we see that, in particular, $\Sigma^3(2) \leq 1$. But this contradicts Proposition 9.5.3. $\square$

A similar result can be seen to hold for dimension four:

Proposition 9.7.10. *If X is a Gorenstein Fano weighted projective space of dimension four with at worst terminal singularities, then $X = \mathbb{P}^4$ or $X = \mathbb{P}(1,1,1,1,2)$.*

Proof. Combining Proposition 9.5.3 and Corollary 9.7.8 we obtain that $\Sigma^4(\kappa) = 2$ for $\kappa \in \{2,\ldots,h-3\}$. Applying this result to Proposition 9.7.6 yields $s(\kappa) = 1$ for $\kappa \in \{3,\ldots,h-3\}$, and that $s(2) = 0$.

Let us assume that $h \geq 6$. Then, since $s(3) = 1$, it must be that the jump points for σ_4 have order 3. In particular, $3 \mid h$. Suppose now that $h \geq 9$. Since $s(2) = 0$ and $s(4) = 1$, and since the jump points for σ_4 have order $3 \nmid 4$, we have that the jump points for σ_3 must have order 4. Similarly, since $s(5) = 1$, we see that the jump points for σ_2 have order 5. Since $3,5 \mid h$ we have, in particular, that $h \geq 15$. But $3 \mid 12$ and $4 \mid 12$, which tells us that $s(12) \geq 2$, contradicting the requirement that $s(12) = 1$.

Hence it must be that $h \leq 6$. There are only two possibilities; either $h = 5$ or $h = 6$. In the former case we have that $\lambda_0 = \ldots = \lambda_4 = 1$ and we obtain $\mathbb{P}^4$ as required. In the latter case we have that $\lambda_0 = \ldots = \lambda_3 = 1$ and $\lambda_4 = 2$, and we obtain $\mathbb{P}(1,1,1,1,2)$. $\square$

The previous two cases, dimensions three and four, were reasonably straightforward; the algorithmic nature of the calculations was not as evident as it might have been. The first interesting case is dimension five, addressed below. It is apparent that the method of proof could easily be handled by a computer – something which would be desirable in higher dimensions.

Proposition 9.7.11. *If X is a Gorenstein Fano weighted projective space of dimension five with at worst terminal singularities, then $X = \mathbb{P}^5, X = \mathbb{P}(1^4,2^2), X = \mathbb{P}(1^2,2^2,3^2)$ or $X = \mathbb{P}(1^3,2,3,4)$.*

k_5	k_4	k_3	h
3	4	5	60
3	4	6	12
3	4	7	84
3	5	5	15
3	5	6	30
3	5	7	105
4	4	5	20
4	4	6	12
4	5	5	20
4	5	6	60

Table 9.2: The values of k_5, k_4 and k_3, and the resulting value of h.

Proof. With notation as above, let $k_i \in \mathbb{Z}$ be such that $k_i\lambda_i = h$ for each $i \in \{0, \ldots, 5\}$. Theorem 9.3.6 tells us that $k_{5-i} > i + 2$. In particular $k_5 > 2, k_4 > 3$ and $k_3 > 4$. Combining Proposition 9.5.3 and Corollary 9.7.8 we obtain that $\Sigma^5(\kappa) \in \{2, 3\}$ for $\kappa \in \{2, \ldots, h-3\}$. Corollary 9.7.7 tells us that $s(\kappa) \leq 2$ for all $\kappa \in \{2, \ldots, h-2\}$.

Since $s(\kappa) \leq 2$ for $\kappa \in \{2, \ldots, h-2\}$, it must be that $h - 2 \leq \text{l.c.m.}\{k_5, k_4, k_3\}$. But $\text{l.c.m.}\{k_5, \ldots, k_0\} = h$. Hence $h = \text{l.c.m.}\{k_5, k_4, k_3\}$.

We shall assume that $\lambda_5 \neq 1$. Thus we have that $h \geq 7$. If $k_5 > 4$ then, by Proposition 9.7.5, $s(4) = 4 > 3$, a contradiction. Hence $k_5 = 3$ or 4.

$k_5 = 3$. We have that $3 \mid h$, and so $h \geq 9$. Now $\Sigma^5(\kappa) \leq 3$ for $\kappa \in \{2, \ldots, 6\}$. If $k_4 \geq 6$ then $\Sigma^5(5) = 4$. Hence $k_4 = 4$ or 5.

$k_4 = 4$. Since $3 \mid h$ and $4 \mid h$ we have that $h \geq 12$. If $k_3 \geq 8$ then $\Sigma^5(7) = 4$. Hence $k_3 = 5, 6$, or 7.

$k_4 = 5$. In this case we see that $h \geq 15$. If $k_3 \geq 8$ then $\Sigma^5(7) = 4$. Hence $k_3 = 5, 6$, or 7.

$k_5 = 4$. We have the $4 \mid h$ and so $h \geq 8$. Now if $k_4 \geq 6$ then $\Sigma^5(5) = 4$. Hence $k_4 = 4$ or 5.

$k_4 = 4$. Suppose that $\lambda_3 \neq 1$. Then $h \geq 12$. If $h_3 \geq 7$ then $\Sigma^5(6) = 4$. Hence $k_3 = 5$ or 6.

$k_4 = 5$. We have that $h \geq 20$. In $k_3 \geq 7$ then $\Sigma(6) = 4$. Hence $k_3 = 5$ or 6.

These results are summarised in Table 9.2.

Knowing h, the values of λ_5, λ_4 and λ_3 are trivial to calculate. Hence we also know the value of the sum $\lambda_0 + \lambda_1 + \lambda_2$. Since $\lambda_{5-i} \leq h/(i+2)$ by Theorem 9.3.6, and since $\lambda_i \mid h$, it is an easy task to calculate all possible values of λ_0, λ_1 and λ_2 satisfying these conditions.

For the sake of completeness we shall reproduce this calculation here for two illuminating cases, leaving the remainder to the reader.

$h = 84$. We have that $k_5 = 3, k_4 = 4$ and $k_3 = 7$. Hence $\lambda_5 = 28, \lambda_4 = 21$ and $\lambda_3 = 12$. This means that $\lambda_0 + \lambda_1 + \lambda_2 = 23$. We also know that $\lambda_2 \leq 16$, and $\lambda_1, \lambda_0 < 14$. There are three possibilities: $\lambda_0 = 2, \lambda_1 = 7, \lambda_2 = 14$; $\lambda_0 = 3, \lambda_1 = 6, \lambda_2 = 14$; $\lambda_0 = 4, \lambda_1 = 7, \lambda_2 = 12$. However, we also require that $\lambda_2 \leq \lambda_3 = 12$, leaving only the final possibility.

$h = 105$. We have that $k_5 = 3, k_4 = 5$ and $k_3 = 7$, giving $\lambda_5 = 35, \lambda_4 = 21$ and $\lambda_3 = 15$. We have that $\lambda_0 + \lambda_1 + \lambda_2 = 34$, with $\lambda_2 < 21$ and $\lambda_1, \lambda_0 \leq 17$. Since the only positive integers dividing 105 which are less than 21 are $15, 7, 5, 3$, and 1, we see that a total 34 cannot be made. Hence we must rule out this possibility.

The cases when $h = 15, 20$ and 60 are also seen to be impossible. The results are collected in Table 9.3.

It is now necessary to check that these values of λ_i satisfy Corollary 9.7.7. In fact, for no value of $h \geq 20$ is this the case. Once again, this is established by exhaustion; we check each possibility listed in Table 9.3 in turn. Only the final two cases will be done here:

$h = 60$. We consider here only the case when $\lambda_5 = 20, \lambda_4 = 15, \lambda_3 = 12, \lambda_2 = 10, \lambda_1 = 2$ and $\lambda_0 = 1$. Notice that this gives $k_5 = 3, k_4 = 4$ and $k_2 = 6$, and so $s(12) \geq 3$. This contradicts Corollary 9.7.7.

$h = 84$. We have that $\lambda_5 = 28, \lambda_4 = 21, \lambda_3 = \lambda_2 = 12, \lambda_1 = 7$ and $\lambda_0 = 4$. In particular, we observe that $k_5 = 3, k_4 = 4$ and $k_1 = 12$. Hence $s(12) \geq 3$, which contradicts Corollary 9.7.7.

$\square$

λ_5	λ_4	λ_3	λ_2	λ_1	λ_0	h
1	1	1	1	1	1	6
2	2	1	1	1	1	8
3	3	2	2	1	1	12
4	3	2	1	1	1	12
5	5	4	2	2	2	20
10	6	5	3	3	3	30
10	6	5	5	2	2	30
10	6	5	5	3	1	30
20	15	12	5	4	4	60
20	15	12	5	5	2	60
20	15	12	6	4	3	60
20	15	12	6	5	2	60
20	15	12	6	6	1	60
20	15	12	10	2	1	60
28	21	12	12	7	4	84

Table 9.3: The candidate weights of the Gorenstein weighted projective spaces in dimension 5 with at worst terminal singularities. Not all weights will satisfy Corollary 9.7.7.

9.8 On the Order of λ_i.

Proposition 9.8.1. *Suppose that P is Gorenstein, and that $\lambda_n \neq 1$. Then:*

$$k_n \leq \begin{cases} n-1, & \text{if } P \text{ is terminal;} \\ n+1, & \text{if } P \text{ is canonical.} \end{cases}$$

Proof. We know that $\Sigma^n(1) = 1$. First we shall suppose that P is terminal.

Since $\lambda_n \neq 1$ we know that $h \geq n+2$. Hence $h-3 \geq n-1$. Corollary 9.7.8 tells us that, in particular, $\Sigma^n(\kappa) \leq n-2$ for $\kappa \in \{2,\ldots,n-1\}$. Hence by Proposition 9.7.5 there must exist a $\kappa \in \{2,\ldots,n-1\}$ such that $s(\kappa) > 0$. In particular, $k_n \leq n-1$.

Now we shall presume that P is canonical (but not terminal). First let us assume that $h = n+2$. Since $\Sigma^n(1) = 1$ and $s(1) = 0$ we see by Proposition 9.7.6 that $\Sigma^n(n+1) = n$ and, by symmetry, $s(n+1) = 0$. Now if $s(\kappa) = 0$ for $\kappa \in \{2,\ldots,n\}$ then, by Proposition 9.7.5, $\Sigma^n(n) = n$ and hence $n = \Sigma^n(n+1) = n+1-s(n+1)$. But this forces $s(n+1) = 1$, a contradiction.

Thus we may assume that $h \geq n+3$. We know by Lemma 9.5.2 that $\Sigma^n(\kappa) \leq n$ for $\kappa \in \{2, \dots, n+1\}$. But if $s(\kappa) = 0$ for $\kappa \in \{2, \dots, n+1\}$ then, by Proposition 9.7.5, we have that $\Sigma^n(n+1) = n+1$. Thus it must be that $k_n \leq n+1$. □

Intuitively, Theorem 9.3.6 tells us that the λ_i cannot become "too spread out". If we allow λ_n to grow arbitrarily large, then λ_{n-1} must also increase in order to prevent $\lambda_n > h/2$. Similarly, since λ_{n-1} is becoming larger, so λ_{n-2} must increase, etc. If we are in the case where $\lambda_i \mid h$, we can use this notion to provide a relative bound on how large the order of λ_i can grow.

Proposition 9.8.2. *Suppose that $k_n\lambda_n = k_{n-1}\lambda_{n-1} = h$. Then:*

$$k_{n-1} \leq nk_n,$$

with strict inequality if P is terminal.

Proof. From Theorem 9.3.6 we have that:

$$2\lambda_n \leq \sum_{i=0}^{n} \lambda_i. \tag{9.8.1}$$

Since $\lambda_i \leq \lambda_{n-1}$ for all $i \leq n-1$, we have that $\lambda_n \leq n\lambda_{n-1}$. Hence $h \leq nk_n\lambda_{n-1}$, and so $k_{n-1} \leq nk_n$.

The claim concerning equality follows directly from Theorem 9.3.6. □

Proposition 9.8.3. *Suppose that $k_n\lambda_n = k_{n-1}\lambda_{n-1} = k_{n-2}\lambda_{n-2} = h$. Then:*

$$k_{n-2} \leq 2(n-1)k_{n-1}. \tag{9.8.2}$$

If, in addition, we have that $3k_n > 2k_{n-1}$, then we have the tighter bound:

$$k_{n-2} \leq \frac{(n-1)k_nk_{n-1}}{2k_n - k_{n-1}}. \tag{9.8.3}$$

In either case the inequality is strict if P is terminal.

Proof. From Theorem 9.3.6 we have that:

$$3\lambda_{n-1} \leq \sum_{i=0}^{n} \lambda_i. \tag{9.8.4}$$

Combining inequalities (9.8.1) and (9.8.4) gives $\lambda_{n-1} \leq 2\sum_{i=0}^{n-2} \lambda_i$. Recalling that $\lambda_i \leq \lambda_{n-2}$ for all $i \leq n-2$ gives us that $\lambda_{n-1} \leq 2(n-1)\lambda_{n-2}$. Hence $h \leq 2(n-1)k_{n-1}\lambda_{n-2}$, and so we see that $k_{n-2} \leq 2(n-1)k_{n-1}$.

Let us assume that $3k_n > 2k_{n-1}$. From inequality (9.8.4) and the fact that $\lambda_i \leq \lambda_{n-2}$ for $i \leq n-2$ we obtain:

$$2\lambda_{n-1} - \lambda_n \leq (n-1)\lambda_{n-2}.$$

Multiplying by $k_n k_{n-2}$, and observing that our assumption concerning k_n and k_{n-1} implies that $2k_n - k_{n-1} > 0$, we see that:

$$h \leq \frac{(n-1)k_n k_{n-1}\lambda_{n-2}}{2k_n - k_{n-1}}.$$

Hence we obtain the second inequality.

It remains to be shown that the second inequality is stricter than the first. Since $3k_n > 2k_{n-1}$, we have that $4k_n - 2k_{n-1} > k_n$. Hence $k_n/(2k_n - k_n) < 2$ and we are done.

Finally, in either case, our claim concerning equality follows directly from Theorem 9.3.6. □

Propositions 9.8.1–9.8.3 are particularly useful when wishing to distinguish between the terminal and canonical cases. The following result, combined with Theorem 9.3.6, provides a bound on the k_i whenever P is Gorenstein. The statement makes no distinction between terminal and canonical singularities.

Theorem 9.8.4. *Suppose that P is Gorenstein. Then:*

$$k_i \leq \frac{i+1}{1 - \sum\limits_{j=i+1}^{n} \frac{1}{k_j}}, \qquad \text{for } i \in \{0, \ldots, n\}.$$

Proof. First, we shall prove by induction that:

$$(i+1)\prod_{j=i+1}^{n} k_j \geq \left(\prod_{j=i+1}^{n} k_j - \sum_{j=i+1}^{n} \prod_{\substack{l=i+1 \\ l\neq j}}^{n} k_l\right) k_i, \qquad \text{for all } i \in \{0, \ldots, n\}. \tag{9.8.5}$$

Recall that $h = \sum_{i=0}^{n} \lambda_i$, and hence $h = \sum_{i=0}^{n}(h/k_i)$. Thus we see that:

$$\sum_{i=0}^{n} \frac{1}{k_i} = 1. \tag{9.8.6}$$

Rearranging gives us:

$$\prod_{i=0}^{n} k_i = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} k_j.$$

Finally, collecting together the k_0 terms gives what will form the base case of our induction:

$$\prod_{i=1}^{n} k_i = \left(\prod_{i=1}^{n} k_i - \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} k_j \right) k_0.$$

Now suppose that, for some $m \in \{0, \dots, n-1\}$, we have:

$$(m+1) \prod_{i=m+1}^{n} k_i \geq \left(\prod_{i=m+1}^{n} k_i - \sum_{i=m+1}^{n} \prod_{\substack{j=m+1 \\ j \neq i}}^{n} k_j \right) k_m.$$

Since $k_m \geq k_{m+1} > 0$ we obtain:

$$(m+1) \prod_{i=m+2}^{n} k_i \geq \prod_{i=m+1}^{n} k_i - \sum_{i=m+1}^{n} \prod_{\substack{j=m+1 \\ j \neq i}}^{n} k_j.$$

Collecting together the k_{m+1} terms on the right of the inequality gives:

$$(m+2) \prod_{i=m+2}^{n} k_i \geq \left(\prod_{i=m+2}^{n} k_i - \sum_{i=m+2}^{n} \prod_{\substack{j=m+2 \\ j \neq i}}^{n} k_j \right) k_{m+1}.$$

Thus we have proved (9.8.5).

Finally, observe that $\prod_{j=i+1}^{n} k_j > 0$, and so (9.8.5) gives us that:

$$i + 1 \geq \left(1 - \sum_{j=i+1}^{n} \frac{1}{k_j} \right) k_i.$$

Now $k_0 > 0$, and so equation (9.8.6) tells us that the term in brackets is positive.

λ_3	λ_2	λ_1	λ_0	h
1	1	1	1	4
2	2	1	1	6
3	1	1	1	6
4	2	1	1	8
4	3	3	2	12
4	4	3	1	12
5	2	2	1	10
6	3	2	1	12
6	4	1	1	12
9	6	2	1	18
10	5	4	1	20
12	8	3	1	24
15	10	3	2	30
21	14	6	1	42

Table 9.4: The possible weights of Gorenstein weighted projective spaces in dimension 3 with at worst canonical singularities.

Dividing through yields the result. □

We are now in a position to begin classifying Gorenstein weighted projective spaces with canonical singularities. The algorithm we use is essentially identical to that used in the proof of Proposition 9.7.11, and would be best executed by a computer. We shall restrict ourselves to dimension three, which can be calculated by hand without too much trouble. The inequalities we have developed in this section will prove invaluable.

Proposition 9.8.5. *If X is a Gorenstein Fano weighted projective space of dimension three with at worst canonical singularities, then the possible weights of X are listed in Table 9.4.*

Proof. Corollary 9.7.7 tells us that $s(\kappa) \leq 2$ for all $\kappa \in \{1, \ldots, h-1\}$. Theorem 9.3.6 tells us that $k_3 \geq 2, k_2 \geq 3$ and $k_1 \geq 4$. Proposition 9.8.1 tells us that $k_3 \leq 4$. Let us assume that $\lambda_3 \neq 1$, and so $h \geq 5$.

$k_3 = 2$. Since $2 \mid h$ we have that $h \geq 6$. If $h = 6$ then $\lambda_3 = 3, \lambda_2 = \lambda_1 = \lambda_0 = 1$ and we are done. Thus let us suppose that $h > 6$. Then $h \geq 8$. Proposition 9.8.2 tells us that $k_2 \leq 6$.

$k_2 = 3$. By equation (9.8.2) of Proposition 9.8.3 we have that $k_1 \leq 12$.

$k_2 = 4$. Suppose that $h = 8$. Then $\lambda_3 = 4, \lambda_2 = 2, \lambda_1 = \lambda_0 = 1$. Thus let us suppose that $h > 8$. Then it must be that $h \geq 12$. Now if $k_1 > 11$ then $\Sigma^3(11) = 4$. Hence it must be that $k_1 \leq 11$.

$k_2 = 5$. Since $2, 5 \mid h$, so $h \geq 10$. If $k_1 > 9$ then $\Sigma^3(9) = 4$. Thus we see that $k_1 \leq 9$.

$k_2 = 6$. In this case we find that $h \geq 12$. If $k_1 > 9$ then $\Sigma^3(9) = 4$ and so it must be that $k_1 \leq 9$.

$k_3 = 3$. Since $3 \mid h$ we see that $h \geq 6$. Theorem 9.8.4 tells us that $k_2 \leq 4$.

$k_2 = 3$. By equation (9.8.3) of Proposition 9.8.3 we have that $k_1 \leq 6$.

$k_2 = 4$. Since $4 \mid h$, so $h \geq 12$. Now if $k_1 > 7$ then $\Sigma^3(7) = 4$, which would be a contradiction. Thus $k_1 \leq 7$.

$k_3 = 4$. Since we have already considered the case when $h = 4$, so it must be that $h \geq 8$. Theorem 9.8.4 tells us that $k_2 \leq 4$. There is only one possibility:

$k_2 = 4$. In this case, if $k_1 > 6$ then $\Sigma^4(6) = 4$. Hence $k_1 \leq 6$.

Knowing k_3, k_2, k_1 and h, it is a trivial matter to calculate the corresponding $\lambda_i = h/k_i$. Computing λ_0 is simply a matter of evaluating $h - \lambda_3 - \lambda_2 - \lambda_1$. The results of these calculations are tabulated in Table 9.5. Observe that in a great many cases, we derive a contradiction. The remaining cases all satisfy Corollary 9.7.7, and prove our result. □

9.9 Classifications in Higher Dimensions

As alluded to in the previous sections, a computer can execute the search for weights of Gorenstein Fano weighted projective space much more efficiently than a person armed with pen and paper. C source code which could be used to generate these classifications can be found on the Internet at:

`http://www.maths.bath.ac.uk/~mapamk/code/Weights.c`

Although computing the weights in arbitrary dimension is possible, the number of possibilities grows rapidly; only one additional cases will be presented here. The proof is by the computer algorithm however, in theory, a person could repeat the process given sufficient time and dedication.

k_3	k_2	k_1	h	λ_3	λ_2	λ_1	λ_0	k_3	k_2	k_1	h	λ_3	λ_2	λ_1	λ_0
2	3	4	12	6	4	3	-1[a]	2	5	6	30	15	6	5	4[b]
2	3	5	30	15	10	6	-1[a]	2	5	7	70	35	14	10	11[c]
2	3	6	6	3	2	1	0[a]	2	5	8	40	20	8	5	7[c]
2	3	7	42	21	14	6	1	2	5	9	90	45	18	10	17[c]
2	3	8	24	12	8	3	1	2	6	6	6	3	1	1	1
2	3	9	18	9	6	2	1	2	6	7	42	21	7	6	8[c]
2	3	10	30	15	10	3	2	2	6	8	24	12	4	3	5[c]
2	3	11	66	33	22	6	5[b]	2	6	9	18	9	3	2	4[c]
2	3	12	12	6	4	1	1	3	3	4	12	4	4	3	1
2	4	4	4	2	1	1	0[a]	3	3	5	15	5	5	3	2[b]
2	4	5	20	10	5	4	1	3	3	6	6	2	2	1	1
2	4	6	12	6	3	2	1	3	4	4	12	4	3	3	2
2	4	7	28	14	7	4	3[b]	3	4	5	60	20	15	12	13[c]
2	4	8	8	4	2	1	1	3	4	6	12	4	3	2	3[c]
2	4	9	36	18	9	4	5[c]	3	4	7	84	28	21	12	23[c]
2	4	10	20	10	5	2	3[c]	4	4	4	4	1	1	1	1
2	4	11	44	22	11	4	7[c]	4	4	5	20	5	5	4	6[c]
2	5	5	10	5	2	2	1	4	4	6	12	3	3	2	4[c]

a. This yields a contradiction since $\lambda_0 \geq 1$.
b. This yields a contradiction since $\lambda_0 \mid h$.
c. This yields a contradiction since $\lambda_0 \leq \lambda_1$.

Table 9.5: The possible values of k_3, k_2, k_1 and h for a three dimensional Gorenstein Fano weighted projective space with at worst canonical singularities.

Proposition 9.9.1. *If X is a Gorenstein Fano weighted projective space of dimension six with at worst terminal singularities, then X has one of the weights given in Table 9.6.*

Tables 9.7 and 9.8 illustrate the exponential growth in the number of possible weights.

λ_6	λ_5	λ_4	λ_3	λ_2	λ_1	λ_0	h
1	1	1	1	1	1	1	7
2	1	1	1	1	1	1	8
2	2	2	1	1	1	1	10
3	1	1	1	1	1	1	9
3	2	2	2	1	1	1	12
3	3	2	1	1	1	1	12
4	2	2	1	1	1	1	12
4	3	1	1	1	1	1	12
5	3	3	1	1	1	1	15
5	5	4	2	2	1	1	20
6	3	3	2	2	1	1	18
8	4	3	3	3	2	1	24
8	4	4	3	3	1	1	24
8	6	3	3	2	1	1	24
8	6	4	3	1	1	1	24
10	5	5	3	3	2	2	30
10	6	5	3	3	2	1	30
20	15	12	5	4	3	1	60

Table 9.6: The possible weights of Gorenstein weighted projective spaces in dimension 6 with at worst terminal singularities.

Dimension	3	4	5	6	7	8	9
Number of possible weights	1	2	4	18	135	1342	21703
Maximum value of h	4	6	12	60	140	1260	12012

Table 9.7: The number of possible weights of Gorenstein Fano weighted projective space with at worst terminal singularities.

Dimension	2	3	4	5
Number of possible weights	3	14	147	3462
Maximum value of h	6	42	1806	3263442

Table 9.8: The number of possible weights of Gorenstein Fano weighted projective space – c.f. Remark 9.2.6 and Table 9.1.

CHAPTER 10

The Ehrhart Polynomial of Fano Polytopes

10.1 The Ehrhart Polynomial

Let $P \subset N_{\mathbb{R}}$ be an n-dimensional convex lattice polytope in the lattice $N \cong \mathbb{Z}^n$. We make the following definitions:

Definition 10.1.1. Let $L_P(k) := |kP \cap N|$ denote the number of lattice points in P dilated by a factor of $k \in \mathbb{Z}_{\geq 0}$. Similarly $L_{\partial P}(k) := |\partial(kP) \cap N|$ denotes the number of lattice points on the boundary of kP and $L_{P^\circ}(k)$ denotes the number of lattice points in the (strict) interior of kP.

In three dimensions Reeve found an interesting relation between $L_P(k)$ and $L_{\partial P}(k)$:

Theorem 10.1.2 ([Ree57, Theorem 1])**.** *Suppose that P is a three-dimensional lattice polytope. With notation as above, we have that:*

$$2(k-1)k(k+1)\operatorname{vol} P = 2(L_P(k) - k\,|P \cap N|) - (L_{\partial P}(k) - k\,|\partial P \cap N|),$$

and, in addition, that:

$$L_{\partial P}(k) - k^2\,|\partial P \cap N| = 2(1-k^2).$$

Reeve provides an interesting note on how Theorem 10.1.2 was conceived. Originally, with $k = 2$, he considered the over-lattice $N' := (1/2)N$. Reeve reasoned that the volume of P could be expressed as a linear combination of seventeen terms, determined as follows: The points of N' are subdivided into four classes depending on how many of the coordinates are integers, and whether they lie inside the interior of P, on a face of P, on an edge of P, or correspond to a vertex of P. The seventeenth term is a constant. Since the vertices of P lie in N, so three of the preceding terms are discounted. The coefficients of the remaining terms were found by explicit calculations for various examples.

Ehrhart provided a general result for P of arbitrary dimension. He demonstrated that the function $L_P(k)$ is a polynomial, called the *Ehrhart polynomial*:

Theorem 10.1.3 ([Ehr67]). *If P is an n-dimensional convex lattice polytope then $L_P(k)$ is a polynomial in k of degree n; i.e.:*

$$L_P(k) = c_n k^n + \ldots + c_1 k + c_0.$$

In addition, Ehrhart determined the two leading coefficients and the constant term.

Theorem 10.1.4 ([Ehr67]). *If P is an n-dimensional convex lattice polytope then, with notation as above:*

(i) $c_n = \operatorname{vol} P$;

(ii) $c_{n-1} = (1/2)\operatorname{vol} \partial P$;

(iii) $c_0 = 1$.

In (ii), $\operatorname{vol}\partial P$ *denotes the surface area of P normalised with respect to the sublattice containing each facet of P.*

In dimension two all the coefficients of the Ehrhart polynomial are determined. From this we can deduce Theorem 5.1.1.

The values of the remaining coefficients of $L_P(m)$ have recently been addressed in [PK92, Pom93, CS94, DR97, Bec00]. In particular, attention has been paid to the connection between polytopes and toric geometry. In this case one considers the variety $\mathbb{P}_P$ (see [Ful93, §1.5]) – essentially the variety $X(P^\vee)$ associated with the dual polytope. The original polytope P corresponds to P_{-K} (see equation (2.2.1a)), and the function $L_P(m)$ evaluates $h^0(-mK)$ (see Proposition 2.2.7). One then applies the *Riemann–Roch theorem*, generalising the process in [Ful93, §5.3].

Although very productive, the 'toric' approach is not without complications. The *total Chern class* is very difficult to calculate. In [Dan78] a formula is given for calculating the total Chern class when the toric variety $\mathbb{P}_P$ is smooth. When the resulting variety is $\mathbb{Q}$-factorial, [Pom93] shows how to modify the calculations of [Dan78].

Pommersheim was able to perform this calculation in the case when P is a tetrahedron, thus obtaining an explicit formula for the number of lattice points in P; this reproduced an earlier result of Mordell [Mor51].

The following theorem summarises the general properties which the c_i are are known to possess.

Theorem 10.1.5 ([BDLD+05, Theorem 3.5])**.** *Let P be an n-dimensional convex lattice polytope. With notation as above:*

(i) $c_k \leq (-1)^{n-k}s(n,k)c_n + (-1)^{n-k-1}s(n,k+1)/(n-1)!$, *where* $0 \leq k < n$;

(ii) $n!c_i \in \mathbb{Z}$, *where* $0 \leq i \leq n$;

(iii) $n(n+1)c_n \geq 2c_{n-1}$;

(iv) $\sum_{i=0}^{n}(-1)^{n-i}c_i \geq 0$.

In (i), $s(n,k)$ denote the Stirling numbers of the first kind.

Ehrhart conjectured, and Macdonald proved, a remarkable reciprocity formula connecting $L_P(k)$ and $L_{P^\circ}(k)$ (see [Dan78] for a proof exploiting Serre–Grothendieck duality):

Theorem 10.1.6 ([Mac71])**.** *Let P be an n-dimensional convex lattice polytope. Then:*

$$L_P(-k) = (-1)^n L_{P^\circ}(k).$$

10.2 The Ehrhart Polynomial for Three–Dimensional Fano Polytopes

We shall restrict our attention to the case when P is a three-dimensional Fano polytope. Thus:

$$|\partial P \cap N| = |P \cap N| - 1. \tag{10.2.1}$$

Remark 10.2.1. Care should be taken. The formula $L_P(m)$ calculates $h^0(-mK)$ for $\mathbb{P}_P$ and *not* for $X(P)$. Perhaps it is disingenuous to refer to P as a *Fano* polytope; it might be more honest to describe P as a convex lattice polytope whose only interior point is the origin. For geometric applications, P should be regarded as residing in $M_\mathbb{R}$ and *not* in $N_\mathbb{R}$. This does not affect our results.

Combining equation (10.2.1) and Theorem 10.1.2 we obtain the following formula for $L_P(k)$:

$$L_P(k) = \mathrm{vol}(P)\, k^3 + \Big(\frac{1}{2}\,|\partial P \cap N| - 1\Big) k^2 + \Big(\frac{1}{2}\,|\partial P \cap N| - \mathrm{vol}\, P + 1\Big) k + 1. \quad (10.2.2)$$

Our aim for the remainder of this section is to derive equation (10.2.2) directly from what is known concerning the Ehrhart polynomial. In particular, we rely on Theorem 10.1.4.

Definition 10.2.2. Let P be an n-dimensional polytope. We define the *f-vector* of P to be $(f_0, f_1, \ldots, f_{n-1})$, where f_i equals the number of i-faces of P.

Proposition 10.2.3. *With notation as above:*

$$\mathrm{vol}\, \partial P = |\partial P \cap N| - 2.$$

Proof. Let (f_0, f_1, f_2) denote the f-vector of P. By *Euler's formula* we have that:

$$f_0 - f_1 + f_2 = 2. \quad (10.2.3)$$

Let $\{F_i\}_{i=1}^{f_2}$ be the set of all faces of P (for some fixed enumeration). Let $f_{0,i}$ denote the number of vertices of the face F_i. A simple counting argument gives us that:

$$|\partial P \cap N| = \sum_{i=1}^{f_2} |F_i \cap N| - \frac{1}{2} \sum_{i=1}^{f_2} |\partial F_i \cap N| - \frac{1}{2} \sum_{i=1}^{f_2} f_{0,i} + f_0. \quad (10.2.4)$$

Now:

$$\begin{aligned} \sum_{i=1}^{f_2} f_{0,i} &= \sum_{\text{faces of } P} \text{number of vertices of the face,} \\ &= \sum_{\text{vertices of } P} \text{number of edges from the vertex,} \\ &= 2 f_1, \end{aligned}$$

the final equality coming from the fact that each edge joins exactly two vertices. Hence

(10.2.4) is equivalent to:

$$|\partial P \cap N| = \sum_{i=1}^{f_2} |F_i \cap N| - \frac{1}{2} \sum_{i=1}^{f_2} |\partial F_i \cap N| - f_1 + f_0. \tag{10.2.5}$$

Finally we see that:

$$\begin{aligned}
\operatorname{vol} \partial P &= \sum_{i=1}^{f_2} \operatorname{vol} F_i \\
&= \sum_{i=1}^{f_2} |F_i \cap N| - \frac{1}{2} \sum_{i=1}^{f_2} |\partial F_i \cap N| - f_2 && \text{by Theorem 5.1.1,} \\
&= |\partial P \cap N| - f_0 + f_1 - f_2 && \text{by (10.2.5),} \\
&= |\partial P \cap N| - 2 && \text{by (10.2.3).}
\end{aligned}$$

□

Proposition 10.2.4. *Let P be a three-dimensional Fano polytope. Then P has Ehrhart polynomial given by equation* (10.2.2).

Proof. Immediate from Theorem 10.1.4, Proposition 10.2.3, and equation (10.2.1). □

Remark 10.2.5. It is interesting to observe that, by equation (10.2.2), the real root of $L_P(k)$ for any three-dimensional Fano polytope lies in the open interval $(-1, 0)$. This is certainly not the case for three-dimensional lattice polytopes in general – see [BDLD+05, Figure 5].

10.3 The Ehrhart Series

Let us introduce the generating function:

$$\operatorname{Ehr}_P(z) := \sum_{k \geq 0} L_P(k) z^k.$$

We call this the *Ehrhart series* of P. The situation is analogous to that of the Hilbert series. Indeed, the following proposition can be proved using the results of [Har77, pp. 49–52].

Proposition 10.3.1. *Let P be an n-dimensional convex lattice polytope. Then there exist $a_i \in \mathbb{Z}$ such that:*

$$\mathrm{Ehr}_P(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0}{(1-z)^{n+1}}.$$

In particular,

$$L_P(k) = a_0 \binom{k+n}{n} + a_1 \binom{k+n-1}{n} + \ldots + a_{n-1} \binom{k+1}{n} + a_n \binom{k}{n}.$$

Corollary 10.3.2. *Let P be an n-dimensional convex lattice polytope. With notation as above;*

$$\mathrm{vol}\, P = \frac{1}{n!} \sum_{i=0}^{n} a_i.$$

Proof. From the expansion in Proposition 10.3.1, we see that the leading coefficient of $L_P(k)$ is:

$$\frac{1}{n!} \sum_{i=0}^{n} a_i.$$

The result follows from Theorem 10.1.4. □

Stanley demonstrated in [Sta80, Theorem 2.1] that, in the above situation, $a_i \geq 0$ for $i = 0, \ldots, n$. Furthermore the constant term $a_0 = 1$.

Corollary 10.3.3. *Let P be an n-dimensional convex lattice polytope. With notation as above;*

$$a_1 = |P \cap N| - n - 1.$$

Proof. Immediate from the expansion in Proposition 10.3.1, setting $k = 1$. □

A consequence of Theorem 10.1.6 is the following relation between $\mathrm{Ehr}_P(z)$ and $\mathrm{Ehr}_{P^\circ}(z)$ (c.f. reciprocity of the Hilbert series of a graded Cohen–Macaulay ring [Eis95, Exercise 21.17]):

Theorem 10.3.4 ([Mac71])**.** *Let P be an n-dimensional convex lattice polytope. Then:*

$$\mathrm{Ehr}_P\left(\frac{1}{z}\right) = (-1)^{n+1} \mathrm{Ehr}_{P^\circ}(z).$$

Corollary 10.3.5. *Let P be an n-dimensional convex lattice polytope. With notation as above;*

$$a_n = |P^\circ \cap N|\,.$$

Proof. By Theorem 10.3.4 we see that the first couple of terms of $\mathrm{Ehr}_{P^\circ}(z)$ are:

$$\mathrm{Ehr}_{P^\circ} z = a_n z + ((n+1)a_n + a_{n-1})z^2 + \dots.$$

Hence $a_n = |P^\circ \cap N|$, as required. □

Proposition 10.3.6. *Let P be a three-dimensional Fano polytope. Then P has Ehrhart series:*

$$\mathrm{Ehr}_P(z) = \frac{z^3 + (6\operatorname{vol} P - |\partial P \cap N| + 1)z^2 + (|\partial P \cap N| - 3)z + 1}{(1-z)^4}.$$

Proof. Immediate from Corollaries 10.3.2, 10.3.3, and 10.3.5. □

Remark 10.3.7. Using the expansion in Proposition 10.3.1, equation (10.2.2) can be recovered from Proposition 10.3.6. Using Stanley's observation that each a_i is non-negative we obtain:

$$\operatorname{vol} P \geq \frac{|\partial P \cap N| - 1}{6}.$$

When the lattice polytope P is reflexive, Hibi proved[1] the following result (c.f. Stanley's Gorenstein criterion for the Hilbert series of a graded Cohen-Macaulay ring [Eis95, Exercise 21.19]):

Theorem 10.3.8 ([Hib92]). *Let P be an n-dimensional convex lattice polytope that contains the origin in its interior. With notation as above, P is reflexive if and only if $a_i = a_{n-i}$ for all $0 \leq i \leq n$.*

Corollary 10.3.9. *Let P be a three-dimensional Fano polytope. Then P is reflexive if and only if:*

$$\operatorname{vol} P = \frac{|\partial P \cap N| - 2}{3}.$$

Proof. Apply Theorem 10.3.8 to Proposition 10.3.6. □

[1] I am grateful to Matthias Beck for alerting me to this result.

Remark 10.3.10. In [Nil04, Lemma 4.3] the "only if" direction of Corollary 10.3.9 is quoted. Unfortunately a proof is not given.

Corollary 10.3.11. *Let P be a four-dimensional Fano polytope. Then P is reflexive if and only if:*

$$\operatorname{vol} P = \frac{\operatorname{vol} \partial P}{4}.$$

Proof. Consider the Ehrhart series of a general Fano four-dimensional polytope P. Since P is a lattice polytope we have that $a_0 = 1$, and Corollary 10.3.5 gives $a_4 = 1$. By Corollary 10.3.3 we know that $a_1 = |\partial P \cap N| - 4$.

Corollary 10.3.2 gives:

$$a_2 + a_3 = 24 \operatorname{vol} P + 2 - |\partial P \cap N| . \tag{10.3.1}$$

Using the expansion in Proposition 10.3.1 we see that the coefficient of k^3 in $L_P(k)$ is given by:

$$\frac{5}{12} + \frac{1}{4}|\partial P \cap N| - 1 + \frac{1}{12}a_2 - \frac{1}{12}a_3 - \frac{1}{4}.$$

By applying Theorem 10.1.4 and simplifying we obtain:

$$a_3 - a_2 = 3|\partial P \cap N| - 6 \operatorname{vol} \partial P - 10. \tag{10.3.2}$$

Combining equations (10.3.1) and (10.3.2) yields:

$$a_3 = 12 \operatorname{vol} P - 3 \operatorname{vol} \partial P + |\partial P \cap N| - 4. \tag{10.3.3}$$

Theorem 10.3.8 tells us that P is reflexive if and only if $a_3 = |\partial P \cap N| - 4$. By equation (10.3.3) we are done. □

By considering Theorem 5.1.1 we observed in Lemma 7.1.2 that for any Fano polygon P:

$$\operatorname{vol} P = \frac{\operatorname{vol} \partial P}{2}.$$

Recall that any Fano polygon is reflexive (Corollary 5.1.3).

Of course, the previous results are instances of Proposition 3.9.2:

Proposition 3.9.2. *Let $P \subset N_{\mathbb{R}}$ be an n-dimensional Fano polytope. P is reflexive if and only* *if:*

$$\operatorname{vol} P = \frac{\operatorname{vol} \partial P}{n},$$

where $\operatorname{vol} \partial P$ *denotes the surface area of* P *normalised with respect to the sublattice containing each facet of* P.

We can prove the "only if" direction of Proposition 3.9.2 using the theory of Ehrhart series:

Theorem 10.3.12. *Let* P *be an* n*-dimensional reflexive Fano polytope. Then:*

$$\operatorname{vol} P = \frac{\operatorname{vol} \partial P}{n}.$$

Proof. Using the expansion in Proposition 10.3.1 we see that the coefficient of k^{n-1} in $L_P(k)$ is given by:

$$\frac{n}{n!}\left(\frac{n+1}{2}a_0 + \left(\frac{n+1}{2} - 1\right)a_1 + \ldots + \left(\frac{n+1}{2} - n\right)a_n\right) = \frac{n}{n!}\sum_{i=0}^{n}\left(\frac{n+1}{2} - i\right)a_i. \quad (10.3.4)$$

Since P is reflexive, so $a_i = a_{n-i}$ for $0 \leq i \leq n$ (Theorem 10.3.8). Suppose n is odd. Then equation (10.3.4) becomes:

$$\frac{n}{n!}\left(\sum_{i=0}^{(n-1)/2}\left(\frac{n+1}{2} - i\right)a_i + \sum_{i=(n+1)/2}^{n}\left(\frac{n+1}{2} - i\right)a_{n-i}\right) = \frac{n}{n!}\sum_{i=0}^{(n-1)/2} a_i.$$

Applying Theorem 10.1.4 we see that:

$$\frac{\operatorname{vol} \partial P}{n} = \frac{2}{n!}\sum_{i=0}^{(n-1)/2} a_i.$$

Theorem 10.3.8 gives:

$$\frac{\operatorname{vol} \partial P}{n} = \frac{1}{n!}\sum_{i=0}^{n} a_i.$$

Finally, by Corollary 10.3.2, we obtain:

$$\frac{\operatorname{vol} \partial P}{n} = \operatorname{vol} P.$$

Now suppose that n is even. Equation (10.3.4) yields:

$$\begin{aligned}\frac{\operatorname{vol}\partial P}{n} &= \frac{2}{n!}\Big(\sum_{i=0}^{n/2-1}\Big(\frac{n+1}{2}-i\Big)a_i + \sum_{i=n/2+1}^{n}\Big(\frac{n+1}{2}-i\Big)a_{n-i} + \frac{1}{2}a_{n/2}\Big)\\ &= \frac{2}{n!}\sum_{i=0}^{n/2-1} a_i + \frac{1}{n!}a_{n/2}\\ &= \frac{1}{n!}\sum_{i=0}^{n} a_i.\end{aligned}$$

Once again the result follows by Corollary 10.3.2. □

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