ROOTS OF EHRHART POLYNOMIALS OF SMOOTH FANO POLYTOPES

GÁBOR HEGEDÜS AND ALEXANDER M. KASPRZYK

Abstract. V. Golyshev conjectured that for any smooth polytope P with $\dim(P) \leq 5$ the roots $z \in \mathbb{C}$ of the Ehrhart polynomial for P have real part equal to $-1/2$. An elementary proof is given, and in each dimension the roots are described explicitly. We also present examples which demonstrate that this result cannot be extended to dimension six.

1. Introduction

Let P be a d-dimensional convex lattice polytope in $\mathbb{R}^d$. Let $L_P(m) := \left|mP \cap \mathbb{Z}^d\right|$ denote the number of lattice points in P dilated by a factor of $m \in \mathbb{Z}_{\geq 0}$. In general the function L_P is a polynomial of degree d, called the Ehrhart polynomial [Ehr67].

The roots of Ehrhart polynomials have recently been the subject of much study (for example [BHW07, BD08, HHO10, Pfe07]), with a significant portion of this work being based on exhaustive computer calculations using the known classifications of polytopes. It has been conjectured in [BDLD$^+$05] that if $z \in \mathbb{C}$ is a root of L_P, then the real part $\mathrm{Re}(z)$ is bounded by $-d \leq \mathrm{Re}(z) \leq d-1$; Braun has shown [Bra08] that z lies inside the disc centred at $-1/2$ of radius $d(d-1/2)$.

Definition 1.1. A convex lattice polytope P containing the origin in its strict interior is called *reflexive* if the dual polytope

$$P^\vee := \{u \in \mathbb{R}^d \mid \langle u, v\rangle \leq 1 \text{ for all } v \in P\}$$

is also a lattice polytope.

There are many interesting and well-known characterisations of reflexive polytopes (for example [HK10, Theorem 3.5]). They are of particular relevance to toric geometry: reflexive polytopes correspond to Gorenstein toric Fano varieties (see [Bat94]) and have been classified up to dimension four.

Any reflexive polytope P satisfies

$$L_P(m) = L_{\partial P}(m) + L_P(m-1) \text{ for all } m \in \mathbb{Z}_{>0}, \tag{1.1}$$

where ∂P denotes the boundary of P. As a consequence, Macdonald's Reciprocity Theorem [Mac71] tells us that $L_P(-m-1) = (-1)^d L_P(m)$. In particular we observe that the roots of L_P are symmetrically distributed with respect to the line $\mathrm{Re}(z) = -1/2$.

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Theorem 1.2 ([BHW07, Proposition 1.8]). *Let P be a d-dimensional convex lattice polytope such that for all roots z of L_P, $\mathrm{Re}(z) = -1/2$. Then, up to unimodular translation, P is a reflexive polytope with* $\mathrm{vol}(P) \leq 2^d$.

Theorem 1.3 ([HHO10, Theorem 0.1]). *In each dimension d and for each integer $0 \leq 2k \leq d$ there exists a reflexive polytope P such that L_P possesses exactly $2k$ distinct roots in $\mathbb{C} \setminus \mathbb{R}$, all with real part equal to $-1/2$. The remaining $d - 2k$ roots of L_P are also distinct and contained in the interval $(-1, 0)$.*

Definition 1.4. A d-dimensional convex lattice polytope P is called *smooth* if the vertices of any facet of P form a $\mathbb{Z}$-basis of the ambient lattice $\mathbb{Z}^d$.

Clearly any smooth polytope is simplicial and reflexive. Smooth polytopes are in bijective correspondence with non-singular toric Fano varieties, and have been classified up to dimension eight [Øbr07].

V. Golyshev conjectured in [Gol09, §5] that, for any smooth polytope P of dimension $d \leq 5$, the roots $z \in \mathbb{C}$ of L_P satisfy $\mathrm{Re}(z) = -1/2$ (the "canonical line hypothesis"). Notice that it is not required that $z \notin \mathbb{R}$. We prove Golyshev's conjecture without resorting to the known classifications – see Sections 2 and 3 below.

Theorem 1.5 (Golyshev). *Let P be a smooth polytope of dimension $d \leq 5$. If $z \in \mathbb{C}$ is a root of $L_P(m)$ then* $\mathrm{Re}(z) = -1/2$.

Explicit descriptions of the roots are given in Corollaries 2.6 and 3.8. We summarise them in the following theorem.

Theorem 1.6. *Let P be a smooth d-dimensional polytope, and suppose that $z = -1/2 + \beta i \in \mathbb{C}$ is a root of L_P. Let $f_0 := \left|\partial P \cap \mathbb{Z}^d\right|$ and $b_2 := \left|\partial(2P) \cap \mathbb{Z}^d\right|$.*
If $d = 2$ then

$$\beta^2 = -\frac{1}{4} + \frac{2}{f_0}.$$

If $d = 3$ then $\beta = 0$ or

$$\beta^2 = -\frac{1}{4} + \frac{6}{f_0 - 2}.$$

If $d = 4$ then

$$\beta^2 = -\frac{17}{4} + \frac{3b_2}{b_2 - 2f_0} \pm \sqrt{1 - \frac{12(f_0 + 2)}{b_2 - 2f_0} + \frac{36f_0^2}{(b_2 - 2f_0)^2}}.$$

If $d = 5$ then $\beta = 0$ or

$$\beta^2 = -\frac{5}{4} + \frac{10(f_0 - 2)}{6 + b_2 - 4f_0} \pm \sqrt{1 - \frac{20(f_0 + 4)}{6 + b_2 - 4f_0} + \frac{100(f_0 - 2)^2}{(6 + b_2 - 4f_0)^2}}.$$

The following example demonstrates that we cannot extend Theorem 1.5 to dimension 6.

Example 1.7. There exist exactly four smooth polytopes in dimension six having roots z of the Ehrhart polynomial such that $\mathrm{Re}(z) \neq -1/2$; in each case $z \notin \mathbb{R}$. The polytopes have IDs 1895, 1930, 4853, and 5817 in the Graded Ring Database[1]. The corresponding Ehrhart polynomials are:

$$1+\frac{31}{10}m+\frac{257}{60}m^2+\frac{5}{2}m^3+\frac{19}{12}m^4+\frac{2}{5}m^5+\frac{2}{15}m^6,$$
$$1+\frac{7}{2}m+\frac{175}{36}m^2+\frac{35}{12}m^3+\frac{35}{18}m^4+\frac{7}{12}m^5+\frac{7}{36}m^6,$$
$$1+\frac{7}{2}m+\frac{21}{4}m^2+\frac{15}{4}m^3+\frac{5}{2}m^4+\frac{3}{4}m^5+\frac{1}{4}m^6,$$
$$1+\frac{31}{10}m+\frac{257}{60}m^2+\frac{5}{2}m^3+\frac{19}{12}m^4+\frac{2}{5}m^5+\frac{2}{15}m^6.$$

The second polytope has roots where $\mathrm{Re}(z) > 0$, and where $\mathrm{Re}(z) < -1$. This demonstrates that the more general "canonical strip hypothesis" does not hold in dimension six.

2. Dimensions Two and Three

One of the fundamental pieces of numerical data associated with a polytope is the f-vector, which enumerates the number of faces of P. We begin by deriving an expression for the Ehrhart polynomial of a smooth polytope in terms of its f-vector.

Definition 2.1. Let P be a d-dimensional convex polytope. Define $f_{-1} := 1$, $f_d := 1$, and f_i equal to the number of i-dimensional faces of P, for any $0 \le i \le d-1$. The *f-vector* of P is the sequence $(f_{-1}, f_0, \dots, f_d)$.

Lemma 2.2. *Let P be a d-dimensional smooth polytope. Then*

$$L_P(m) = \sum_{i=-1}^{d-1} f_i \binom{m}{i+1} \quad \textit{and} \quad L_{\partial P}(m) = \sum_{i=0}^{d-1} f_i \binom{m-1}{i}.$$

Proof. Clearly

$$L_{\partial P}(m) = f_0 + \sum_F \left|(mF)^\circ \cap \mathbb{Z}^d\right|,$$

where the sum is taken over all i-dimensional faces F of P, $i > 0$, and Q° denotes the (relative) interior of Q. Since P is smooth, $F \cap \mathbb{Z}^d$ forms part of a basis for the underlying lattice $\mathbb{Z}^d$ for any face F. Hence

$$L_{\partial P}(m) = \sum_{i=0}^{d-1} f_i \binom{m-1}{i}.$$

[1] http://www.grdb.co.uk/search/toricsmooth?id_cmp=in&id=1895,1930,4853,5817

To calculate $L_P(m)$ we make use of (1.1):

$$\begin{aligned} L_P(m) &= 1 + \sum_{k=1}^{m} L_{\partial P}(k) = 1 + \sum_{k=1}^{m}\sum_{i=0}^{d-1} f_i \binom{k-1}{i} \\ &= 1 + \sum_{i=0}^{d-1} f_i \sum_{k=1}^{m} \binom{k-1}{i} \\ &= 1 + \sum_{i=0}^{d-1} f_i \binom{m}{i+1} \\ &= \sum_{i=-1}^{d-1} f_i \binom{m}{i+1}. \end{aligned}$$

□

The f-vectors of low-dimensional smooth polytopes were calculated in [HK10, Theorem 4.2]. As a consequence we obtain the following formulae for the Ehrhart polynomial:

Corollary 2.3. *Let P be a d-dimensional smooth polytope. Define $b_2 := \left|\partial(2P) \cap \mathbb{Z}^d\right|$. If $d = 2$ then*

$$L_P(m) = 1 + \frac{1}{2} f_0 m + \frac{1}{2} f_0 m^2.$$

If $d = 3$ then

$$L_P(m) = 1 + \frac{1}{6}(f_0 + 10)m + \frac{1}{2}(f_0 - 2)m^2 + \frac{1}{3}(f_0 - 2)m^3$$

If $d = 4$ then

$$L_P(m) = 1 + \frac{1}{12}(8f_0 - b_2)m + \frac{1}{24}(14f_0 - b_2)m^2 - \frac{1}{12}(2f_0 - b_2)m^3 - \frac{1}{24}(2f_0 - b_2)m^4.$$

If $d = 5$ then

$$\begin{aligned} L_P(m) = 1 + \frac{1}{60}(14f_0 - b_2 + 94)m + \frac{1}{24}(16f_0 - b_2 - 30)m^2 + \frac{1}{3}(f_0 - 2)m^3 \\ - \frac{1}{24}(4f_0 - b_2 - 6)m^4 - \frac{1}{60}(4f_0 - b_2 - 6)m^5. \end{aligned}$$

Casagrande provides sharp bounds on the number of vertices f_0 of a smooth polytope in terms of the dimension:

Theorem 2.4 ([Cas06]). *Let P be a d-dimensional smooth polytope. Then*

$$f_0 \leq \begin{cases} 3d, & \text{if } d \text{ is even;} \\ 3d - 1, & \text{if } d \text{ is odd.} \end{cases}$$

We now prove Theorem 1.5 in dimensions 2 and 3.

Proposition 2.5. *Let P be a smooth polytope of dimension two or three. If $z \in \mathbb{C}$ is a root of $L_P(m)$ then* $\mathrm{Re}(z) = -1/2$.

Proof. $d = 2$: By Corollary 2.3 we know that

$$L_P(m) = 1 + \frac{1}{2} f_0 m + \frac{1}{2} f_0 m^2.$$

Let $\alpha + \beta i \in \mathbb{C}$ be a root of L_P, where $\alpha, \beta \in \mathbb{R}$. Assume that $\beta \neq 0$. By considering the imaginary part we obtain

$$\beta(1 + 2\alpha) = 0,$$

hence $\alpha = -1/2$ as required. The real part simplifies to

$$\beta^2 = \frac{2}{f_0} - \frac{1}{4}.$$

Theorem 2.4 tells us that this is always positive, thus we obtain both roots of L_P.

$d = 3$: In this case Corollary 2.3 tells us that

$$L_P(m) = 1 + \frac{1}{6}(f_0 + 10)m + \frac{1}{2}(f_0 - 2)m^2 + \frac{1}{3}(f_0 - 2)m^3,$$

giving real and imaginary parts:

$$1 + \frac{1}{6}(f_0 + 10)\alpha + \frac{1}{2}(f_0 - 2)(\alpha^2 - \beta^2) + \frac{1}{3}(f_0 - 2)(\alpha^2 - 3\beta^2)\alpha = 0, \tag{2.1}$$

$$\frac{1}{6}(f_0 + 10)\beta + (f_0 - 2)\alpha\beta + \frac{1}{3}(f_0 - 2)(3\alpha^2 - \beta^2)\beta = 0. \tag{2.2}$$

Assume that $\beta \neq 0$. Equation (2.2) gives us

$$(f_0 - 2)\beta^2 = \frac{1}{2} f_0 + 5 + 3(f_0 - 2)\alpha + 3(f_0 - 2)\alpha^2. \tag{2.3}$$

Substituting (2.3) into (2.1) gives

$$\frac{1}{12}(2\alpha + 1)\left(4(f_0 - 2)(2\alpha + 1)^2 + 26 - f_0\right) = 0.$$

Clearly $\alpha = -1/2$ is one possible solution. The discriminant of $4(f_0 - 2)(2\alpha + 1)^2 + 26 - f_0$, regarded as a quadratic in $2\alpha + 1$, is $16(f_0 - 2)(f_0 - 26)$. This is negative when $2 \leq f_0 \leq 26$, and by Theorem 2.4 this covers all possible values of f_0. Hence $\alpha = -1/2$ is the only solution. The values for β are determined by (2.3):

$$\beta^2 = \frac{26 - f_0}{4f_0 - 8}.$$

If we allow $\beta = 0$ then (2.1) becomes

$$\frac{1}{24}(2\alpha + 1)\left((f_0 - 2)(2\alpha + 1)^2 + 26 - f_0\right) = 0.$$

Once more the discriminant of the quadratic component tells us that the only solution is when $\alpha = -1/2$. $\square$

The proof of Proposition 2.5 gives us explicit equations for the roots of L_P.

Corollary 2.6. *Let P be a smooth d-dimensional polytope, and suppose that $z = -1/2 + \beta i \in \mathbb{C}$ is a root of L_P. If $d = 2$ then*

$$\beta^2 = -\frac{1}{4} + \frac{2}{f_0}.$$

If $d = 3$ then $\beta = 0$ or

$$\beta^2 = -\frac{1}{4} + \frac{6}{f_0 - 2}.$$

3. Dimensions Four and Five

In order to prove Theorem 1.5 in dimension 4 we require a some additional results. Throughout we write $b_2 := \left|\partial(2P) \cap \mathbb{Z}^d\right|$, where d is the dimension of P.

Lemma 3.1 ([HK10, Corollary 4.4])**.** *Let P be a four-dimensional smooth polytope. Then*

$$5f_0 - 10 \leq b_2 \leq 5f_0.$$

Lemma 3.2. *Let P be a four-dimensional smooth polytope. Then*

$$(b_2 - 8f_0)^2 > 24(b_2 - 2f_0).$$

Proof. From Lemma 3.1 we have that

$$\begin{aligned}(b_2 - 8f_0)^2 &= (b_2 - 16f_0)b_2 + 64f_0^2 \\ &\geq (10 - 5f_0)(10 + 11f_0) + 64f_0^2 \\ &= 9f_0^2 + 60f_0 + 100 \\ &= (3f_0 + 10)^2\end{aligned}$$

Clearly $72f_0 < (3f_0 + 10)^2$, and since $24(b_2 - 2f_0) \leq 72f_0$ (by Lemma 3.1) we obtain the result. □

We shall also make use of the following trivial observation:

Lemma 3.3. *Let $g(x) := ax^4 + bx^2 + c \in \mathbb{R}[x]$ be a polynomial such that $a > 0$, $b < 0$, $c > 0$ and $b^2 - 4ac > 0$. Then g has four distinct real roots.*

Proposition 3.4. *Let P be a four-dimensional smooth polytope. If $z \in \mathbb{C}$ is a root of $L_P(m)$ then* $\mathrm{Re}(z) = -1/2$.

Proof. In four dimensions the Ehrhart polynomial simplifies to

$$L_P(m) = 1 + \frac{1}{12}(8f_0 - b_2)m(m+1) - \frac{1}{24}(2f_0 - b_2)m^2(m+1)^2.$$

If $z = \alpha + i\beta$ is a root of L_P then, by considering the real and imaginary parts, we obtain

$$24 + 12f_0((\alpha+1)\alpha - \beta^2) - (2f_0 - b_2)\alpha(\alpha+1)(\alpha(\alpha+1) - 2 - 6\beta^2) - (2f_0 - b_2)\beta^2(\beta^2+1) = 0, \tag{3.1}$$

$$\left(6f_0 - (2f_0 - b_2)\left((\alpha+1)\alpha - \beta^2 - 1\right)\right)(2\alpha+1)\beta = 0. \tag{3.2}$$

Clearly $\alpha = -1/2$ is a possible solution to equation (3.2), in which case β satisfies (by (3.1))

$$16(b_2 - 2f_0)\beta^4 + 8(5b_2 - 34f_0)\beta^2 + 3(128 + 3b_2 - 22f_0) = 0. \tag{3.3}$$

This quadratic in β^2 has distinct real solutions if and only if

$$(b_2 - 8f_0)^2 - 24(b_2 - 2f_0) > 0.$$

By Lemma 3.2 we know that this is always true.

Now we consider the signs of the coefficients of (3.3). The leading coefficient is equal to $8f_2$, and so is positive. The coefficient of β^2 is always negative by Lemma 3.1, and the constant term is positive by Lemma 3.2. Hence, by Lemma 3.3, there are four distinct real solutions to equation (3.1).

We have found four distinct roots when $\mathrm{Re}(z) = -1/2$. Since L_P is of degree four, we are done. □

Finally we consider dimension five.

Lemma 3.5 ([HK10, Corollary 4.4])**.** *Let P be a five-dimensional smooth polytope. Then*

$$42f_0 - 105 \le 7b_2 \le 52f_0 - 90.$$

Lemma 3.6. *Let P be a five-dimensional smooth polytope. Then*

$$100(f_0-2)^2 + (6 + b_2 - 4f_0)^2 > 20(6 + b_2 - 4f_0)(f_0 + 4).$$

Proof. We begin by observing that the statement is equivalent to

$$\big(10(f_0 - 2) - (6 + b_2 - 4f_0)\big)^2 > 120(6 + b_2 - 4f_0),$$

which in turn is equivalent to

$$\big(13(f_0-2) - (b_2 - f_0)\big)\big(13(f_0-2) - (b_2 - f_0) + 120\big) > 1200(f_0 - 2).$$

From Lemma 3.5 we have that

$$13(f_0 - 2) - (b_2 - f_0) \ge \frac{46}{7}f_0 - \frac{92}{7},$$

which is always positive since $f_0 \ge 6$. Hence

$$\begin{aligned}&\big(13(f_0-2) - (b_2 - f_0)\big)\big(13(f_0-2) - (b_2 - f_0) + 120\big) - 1200(f_0 - 2)\\ &\qquad \ge \Big(\frac{46}{7}f_0 - \frac{92}{7}\Big)\Big(\frac{46}{7}f_0 - \frac{92}{7} + 120\Big) - 1200(f_0-2)\\ &\qquad = \frac{4}{49}(f_0 - 2)(529f_0 - 6098).\end{aligned}$$

This is positive for all $f_0 \ge 12$.

To prove the inequality when $f_0 \le 11$ we consider

$$\begin{aligned}&\big(13(f_0-2) - (b_2 - f_0)\big)\big(13(f_0-2) - (b_2 - f_0) + 120\big) - 1200(f_0 - 2)\\ &\qquad \ge \big(13(f_0-2) - (b_2 - f_0)\big)\Big(\frac{46}{7}f_0 - \frac{92}{7} + 120\Big) - 1200(f_0-2)\\ &\qquad = -\frac{2}{7}(23f_0 + 374)b_2 + \frac{4}{7}(161f_0^2 + 219f_0 - 662).\end{aligned}$$

We wish to show that

$$-\frac{2}{7}(23f_0 + 374)b_2 + \frac{4}{7}(161f_0^2 + 219f_0 - 662) > 0$$

whenever $6 \le f_0 \le 11$. It is enough to prove that, in the given range,

$$b_2 < \frac{2(161f_0^2 + 219f_0 - 662)}{23f_0 + 374}. \tag{3.4}$$

Now

$$b_2 - f_0 = f_1 \le \binom{f_0}{2},$$

and so

$$b_2 \le \frac{f_0(f_0+1)}{2}.$$

We shall show that

$$\frac{f_0(f_0+1)}{2} < \frac{2(161f_0^2 + 219f_0 - 662)}{23f_0 + 374}.$$

But this is trivial; the cubic

$$\begin{aligned} f_0(f_0+1)(23f_0+374) - 4(161f_0^2 + 219f_0 - 662) \\ = 23f_0^3 - 247f_0^2 - 502f_0 + 2648 \end{aligned}$$

is negative when $6 \le f_0 \le 11$, hence equation (3.4) holds. □

Proposition 3.7. *Let P be a five-dimensional smooth polytope. If $z \in \mathbb{C}$ is a root of $L_P(m)$ then* $\mathrm{Re}(z) = -1/2$.

Proof. Let $z = \alpha + i\beta \in \mathbb{C}$ be a root of L_P, where P is a five-dimensional smooth polytope. By Corollary 2.3 we see that α and β must satisfy

$$\begin{aligned} (2\alpha+1)\Big((6+b_2-4f_0)\big((\alpha-1)\alpha(\alpha+1)(\alpha+2) - 10(\alpha+1)\alpha\beta^2 + 5(\beta^2+1)\beta^2\big) + \\ 20(f_0-2)\left((\alpha+1)\alpha - 3\beta^2\right) + 120\Big) = 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} (14f_0 - b_2 + 94)\beta + 5(16f_0 - b_2 - 30)\alpha\beta + 20(f_0-2)(3\alpha^2 - \beta^2)\beta - \\ 10(4f_0 - b_2 - 6)(\alpha^2 - \beta^2)\alpha\beta - (4f_0 - b_2 - 6)(5\alpha^4 - 10\alpha^2\beta^2 + \beta^4)\beta = 0. \end{aligned} \tag{3.6}$$

Clearly $\alpha = -1/2, \beta = 0$ is always a solution. Suppose that $\alpha = -1/2$ and $\beta \neq 0$. Equation (3.5) holds, and from (3.6) we obtain

$$16(6+b_2-4f_0)\beta^4 + 40(22+b_2-12f_0)\beta^2 + 2134 + 9b_2 - 116f_0 = 0. \tag{3.7}$$

This quadratic in β^2 has distinct real solutions if and only if

$$100(f_0-2)^2 + (6+b_2-4f_0)^2 > 20(6+b_2-4f_0)(f_0+4),$$

which holds by Lemma 3.6.

As in the four-dimensional case we consider the signs of the coefficients of (3.7). The leading coefficient equals $8f_4$ and so is positive. The coefficient of β^2 is negative by Lemma 3.5 and the fact that $f_0 \ge 6$, and the constant term is positive (again by Lemma 3.5). Thus, by Lemma 3.3, equation (3.7) has four distinct real solutions.

Hence we have found all five roots of L_P, and in each case $\mathrm{Re}(z) = -1/2$ as required. □

From equations (3.3) and (3.7) we have

f_0	5	6	6	7	7	7	8	8	8	8	9	9	9	9	9	10	10	10	11	12
b_2	15	20	21	25	26	27	31	32	33	34	36	38	39	41	42	44	45	50	52	60

TABLE 1. The possible pairs (f_0, b_2) for the 124 four-dimensional smooth polytopes.

Corollary 3.8. *Let P be a smooth d-dimensional polytope, and suppose that $z = -1/2 + \beta i \in \mathbb{C}$ is a root of L_P. If $d = 4$ then*

$$\beta^2 = -\frac{17}{4} + \frac{3b_2}{b_2 - 2f_0} \pm \sqrt{1 - \frac{12(f_0+2)}{b_2 - 2f_0} + \frac{36f_0^2}{(b_2-2f_0)^2}}.$$

If $d = 5$ then $\beta = 0$ or

$$\beta^2 = -\frac{5}{4} + \frac{10(f_0-2)}{6+b_2-4f_0} \pm \sqrt{1 - \frac{20(f_0+4)}{6+b_2-4f_0} + \frac{100(f_0-2)^2}{(6+b_2-4f_0)^2}}.$$

4. Concluding Remarks

In four dimensions one can prove Theorem 1.5 without knowing the explicit equation for the Ehrhart polynomial. We require the following result.

Proposition 4.1 ([BHW07, Proposition 1.9])**.** *Let P be a four-dimensional reflexive polytope. Every root $z \in \mathbb{C}$ of $L_P(m)$ has $\mathrm{Re}(z) = -1/2$ if and only if*

(i) $2\left|\partial P \cap \mathbb{Z}^4\right| \le 9\,\mathrm{vol}(P) + 16$, *and*
(ii) $\left(\left|\partial P \cap \mathbb{Z}^4\right| - 4\,\mathrm{vol}(P)\right)^2 \ge 16\,\mathrm{vol}(P)$.

Alternative proof in dimension four. First we show that condition (i) of Proposition 4.1 is satisfied. Since P is smooth, $f_0 = \left|\partial P \cap \mathbb{Z}^4\right|$. It follows from Lemma 3.1 that $15f_0 \le 3b_2 + 30$. Hence $9f_0 \le 3(b_2 - 2f_0) + 30$. By Theorem 2.4 we have that $f_0 \le 12$, giving us the (very crude) inequality

$$16f_0 < 3(b_2 - 2f_0) + 128. \tag{4.1}$$

In four dimensions we have that $f_3 = b_2 - 2f_0$ ([HK10, Theorem 4.2]) and, since P is smooth, $f_3 = 24\,\mathrm{vol}(P)$. Substituting into equation (4.1) gives condition (i).

That Proposition 4.1 (ii) holds is immediate from Lemma 3.2 and the fact that $b_2 - 2f_0 = 24\,\mathrm{vol}(P)$. □

Theorem 1.6 tells us that in order to compute the roots of the Ehrhart polynomial we need only know f_0 and, in dimensions four and five, $b_2 := \left|\partial(2P) \cap \mathbb{Z}^d\right|$. Clearly $f_0 \ge d + 1$, and Theorem 2.4 provides a sharp upper bound. The values of b_2 can be calculated from Øbro's classification [Øbr07]. The possible pairs (f_0, b_2) are reproduced in Tables 1 and 2.

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f_0	6	7	7	8	8	8	8	9	9	9	9	9	10	10	10
b_2	21	27	28	33	34	35	36	40	41	42	43	44	46	49	50

f_0	10	10	10	11	11	11	11	11	11	12	12	12	13	14
b_2	51	52	53	56	58	59	60	61	62	66	67	72	76	86

Table 2. The possible pairs (f_0, b_2) for the 866 five-dimensional smooth polytopes.

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Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040 Linz, Austria

E-mail address: `gabor.hegedues@oeaw.ac.at`

School of Mathematics and Statistics, University of Sydney, Sydney NSW 2006, Australia

E-mail address: `a.m.kasprzyk@usyd.edu.au`