PROJECTING FANOS IN THE MIRROR

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ABSTRACT. A new structure connecting toric degenerations of smooth Fano threefolds by projections was introduced by I. Cheltsov et al. in "Birational geometry via moduli spaces"; using Mirror Symmetry, these connections were transferred to the side of Landau–Ginzburg models, and a nice way to connect the Picard rank one Fano threefolds was described. We apply this approach to all smooth Fano three-folds, connecting their degenerations by toric basic links. In particular, we find many Gorenstein toric degenerations of the smooth Fano threefolds we consider. We implement mutations in this framework too. It turns out that appropriately chosen toric degenerations of the Fanos are connected by toric basic links from a few roots. We interpret the relations we found in terms of Mirror Symmetry.

CONTENTS

1. INTRODUCTION

One of the central topics of research in birational geometry are Fano varieties — varieties with ample anticanonical bundle. They play a crucial role in the Minimal Model Program and present a rich geometric picture. Fano varieties are central in Mirror Symmetry — many constructions of mirror duality are either Calabi–Yau manifolds or are for Fano varieties.

The classification problem for smooth Fano varieties goes back to the XIX century. Riemann showed that the only Fano curve is a projective line $\mathbb{P}^1$. Pasquale del Pezzo classified the smooth Fano surfaces (now called *del Pezzo* surfaces in his honour). He showed that these surfaces (with very ample anti-canonical class) are non-degenerate surfaces of degree n in $\mathbb{P}^n$; now we also add two trigonal examples of degrees 2 and 1. These surfaces have degree at most 9 and form an irreducible family in each degree, with the exception of degree 8 where there are two irreducible families. The modern (that is, classical, not pre-classical) description of del Pezzo surfaces is as blow-ups of general-enough points on $\mathbb{P}^2$, together with a quadric surface. That is, they are projections of an anticanonically embedded $\mathbb{P}^2 \subset \mathbb{P}^9$ from general-enough points (again together with a quadric). If we choose any points as centres of projection we obtain singular del Pezzo surfaces because, in this case, the projection may contract lines through the point; however we arrive at the same family, so smooth del Pezzo surfaces can be obtained as smoothings of projected singular ones.

Classification of Fano threefolds is more tricky. It was initiated by Gino Fano and developed later by Iskovskikh [43, 44] (the modern definition of Fano varieties is due to Iskovskikh, who named them after Fano). Soon afterwards, Mori and Mukai, using Iskovskikh approach and the Minimal Model Program, classified all smooth Fano threefolds [58]; there are 105 families (the final family, accidentally overlooked

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in the original classification, was found in 2002 [59]). There is currently no classification known in higher dimensions, however Kollár–Miyaoka–Mori show that there is a finite number of families of Fano varieties in any given dimension. It is expected that even in dimension 4 the number of families of Fano varieties is very large.

Unlike the two-dimensional case, there is no structure in the list of Fano threefolds (see [46]) systematically relating one with each other. An approach to obtaining such a structure is described in [15]. Briefly, the idea is as follows: similarly to the two-dimensional case, one hopes to relate all Fano varieties to some "specific" varieties (not necessary smooth); a class of simple relations between these new varieties should include projections from singular points, tangent spaces to smooth points, lines, and conics.

To place the problem on the combinatorial level, we choose toric Fano varieties as the "specific" varieties. We call the simple projections between toric varieties *toric basic links*. One can describe the needed projections in terms of the spanning polytopes. An example of a nice subtree in the projections tree relating Picard rank one Fano varieties (see Figure 1) is found in [15]. Moreover, one can add mutations to this picture (see §4); that is, deformations from one toric degeneration to another. In this paper we study projections systematically for all Fano threefolds. In particular, we prove the following. Given a toric variety T let us call a projection in an anticanonical embedding from tangent space to invariant smooth point, invariant cDV point, or an invariant smooth line an *F-projection*.

Theorem 1.1 (Theorem 8.1)**.** *Given any smooth Fano threefold X, there exists a Gorenstein toric degeneration of X that can be obtained by a sequence of mutations and F-projections from a toric degeneration of one of 15 smooth Fano threefolds (see Table 2). The directed graph connecting all Fano varieties with very ample anticanonical class via the projections and mutation is presented in Table 3. Each of the toric degenerations we use can be equipped with a toric Landau–Ginzburg model.*

Theorem 1.2 (Theorem 8.3)**.**   (i) *For any smooth Fano threefold with very ample anticanonical class there exists a choice of Gorenstein toric degeneration such that all these degenerations are connected by sequences of F-projections. The directed graph of such projections can be chosen as a union of 15 trees with roots shown in Table 2. The directed graph connecting all Fano varieties with very ample anticanonical class via projection and mutation is presented in Figure 5. Each of the toric degenerations can be equipped with a toric Landau–Ginzburg model.*

  (ii) *For any smooth Fano threefold with very ample anticanonical class there exists a choice of toric degeneration such that all these degenerations are connected by sequences of projections in the anticanonical embedding with toric centres which are either tangent spaces to smooth points, or cDV points, or smooth lines, or smooth conics. The directed (sub)graph of such projections connecting degenerations of all smooth Fano threefolds can be chosen to have five roots which are: $\mathbb{P}^3$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$, the quadric threefold, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^3$ blown-up in a line. The directed graph connecting all Fano varieties via the projections and mutations is presented in Figure 6. Each of the toric degenerations can be equipped with a toric Landau–Ginzburg model.*

Note that $\mathbb{P}^3$ blown-up in a line is not a projection of $\mathbb{P}^3$ since a line in anticanonical embedding has degree 4.

As the theorems suggest, the toric degenerations providing basic links correspond to toric Landau–Ginzburg models (that is, Laurent polynomials related to toric degenerations and representing mirrors; see below), and the theorems show that two Fano varieties are related by a toric basic link if their toric Landau–Ginzburg models are closely related too.

The idea of presenting Landau–Ginzburg models dual to Fano varieties, or dual to varieties close to Fano, as Laurent polynomials goes back to Givental. In [33] he suggested a Landau–Ginzburg model for a smooth toric variety as a complex torus with a complex-valued function (a *superpotential*) represented by a Laurent polynomial with support equal to the spanning polytope of the toric variety. His construction was generalized to varieties admitting nice toric degenerations; in this case one associates the Laurent polynomial with the fan of the toric degeneration of the Fano variety. This idea goes back to Batyrev–Borisov's approach [9] to mirror duality for toric varieties as duality of the corresponding polytopes and good deformational behavior of Gromov–Witten invariants (ses [8] and references therein). In this spirit

Laurent presentations of Landau–Ginzburg models for Grassmannians were constructed by Eguchi–Hori–Xiong [30], and for Grassmannians and partial flag varieties by Batyrev–Ciocan-Fontanine–Kim–van Straten [10, 11].

The crucial part of the duality between Fano varieties and Landau–Ginzburg models in this approach is an identification of (a part of) Gromov–Witten theory for the Fano varieties and periods of the dual one-dimensional family. If the total space of the family is a complex torus, then the Landau–Ginzburg model, as we mentioned above, is (in some basis) represented by a Laurent polynomial. In this case the main period of the family is a generating series for constant terms of powers of Laurent polynomials, see §3. In [33] Givental proves the coincidence of the two series for smooth toric varieties; the same was done in [10, 11] for Grassmannians and for partial flag varieties.

In [33] Givental suggested an approach to constructing Landau–Ginzburg models for (almost) Fano complete intersections in toric varieties. This approach was generalized to complete intersections in Grassmannians and (partial) flag varieties in [10, 11], see also [12]. The output of Givental's construction is not a complex torus with a function but a complete intersection in a torus with a function. These Landau–Ginzburg models satisfy the Gromov–Witten period condition via Quantum Lefshetz Theorem, which enables one to pass from Gromov–Witten theory of a variety to Gromov–Witten theory of an ample (Fano) hypersurface.

The natural idea is to realise birationally Landau–Ginzburg models as Laurent polynomials. This was done in [20, 42, 65–67] for smooth Fano threefolds and complete intersections in projective spaces. In the case of Fano complete intersections (Picard rank one) it was shown that the resulting Laurent polynomials are related to toric degenerations. This led to the idea that Laudau–Ginzburg models presented as Laurent polynomials should be assigned with toric degenerations used in the generalisation of Givental's approach. Another concept is the Compactification Principle which says that a correctly chosen Laurent Landau–Ginzburg model admits a fiberwise compactification to a family of compact Calabi–Yau varieties such that this family satisfy (an algebraic part of) Homological Mirror Symmetry. These two concepts, together with the initial concept of Gromov–Witten period coincidence, form the central ideas on which the present paper is based. The above ideas were initiated by Golyshev, who has suggested a program of finding (toric) Landau–Ginzburg model by guessing Laurent polynomials having prescribed constant terms and, hence, periods. The first results were obtained for smooth rank one Fano threefolds in [65, 66]; later in [42] the proof of their toricity was completed. These results clarified the connection with toric degenerations to (possibly very singular) toric varieties. The specific simple form of found Laurent polynomials leads to binomial principle suggested in [66]. This principle states that coefficients on facets of the Newton polytope of the Laurent polynomial correspond to binomial coefficients of a power of a sum of independent variables. Surprisingly this principle covers most of (but not all) smooth Fano threefolds: most of them have toric degenerations with cDV singularities, which means that integral points of the Newton polytope for Laurent polynomial are the origin and points lying on edges. This gives an algorithm for finding Landau–Ginzburg models.

Binomial principle was generalised to the Minkowski principle in [19]. It relates coefficients of the Laurent polynomial with Minkowski decompositions of facets of its Newton polytope into particular elementary summands. Moreover, all canonical polytopes that are Newton polytopes of Minkowski Laurent polynomials were found and all J-series of smooth Fano threefolds were computed. This gives, for any smooth Fano threefold, a Laurent polynomial (which is not unique) satisfying the period condition. In this paper we establish toric degeneration condition as well, see §5.

Theorem 1.3. *Let T be a Gorenstein toric variety appeared in Theorems 1.1 and 1.2. Then T is a toric degeneration of the corresponding Fano threefold.*

Remark 1.4. The assertion of Theorem 1.3 holds for much larger class of Gorenstein toric varieties, see Appendix B.

As we have mentioned, a Laurent polynomial, as a mirror dual to Fano, is not unique. However its Calabi–Yau compactification is unique. Indeed, under mild natural conditions, this holds for rank one Fano threefolds [29, 42]. This means that Landau–Ginzburg models are birational over the base field $\mathbb{C}(x)$. In other words, corresponding Laurent polynomials differ by mutations. It is proven in [2] that all Laurent polynomials with support on reflexive polytopes that produce the same period differ by (a

sequence of) mutations. So they have a common log Calabi–Yau compactification. This suggests the current and strongest concept of assigning a Laurent Landau–Ginzburg model to a reflexive polytope, the maximal mutational principle. It is described in details in §4.

The above findings can be interpreted categorically. We propose the following:

Conjecture 1.5. *The moduli spaces of Landau–Ginzburg models (defined in [25]) for directed graphs of Fano varieties from Theorem 1.2 are contained in each other with the top Landau–Ginzburg models contained in the ones obtained by projections.*

In such a way the behaviour of Landau–Ginzburg models for three-dimensional Fano varieties is very similar to the behaviour of Landau–Ginzburg models of del Pezzo surfaces. We lift this conjecture to further categorical levels. As a consequence of the connection of curve complexes and stability conditions it was noticed in [26] that stability conditions should behave well in families. Later on, the following theorem was proven by Haiden–Katzarkov–Kontsevich–Pandit [38]. Below we use the definition of stability conditions given by Bridgeland. We give the most general version of the statement. After that we will explain the connection with our situation. In what follows we give a categorical description of a family of hyperplane sections. We use the language of comonads. Here the category $\mathcal{C}_{\mathrm{special}}$ is the analogue of a singular hyperplane section and the category $\mathcal{C}_{\mathrm{general}}$ is the analogue of a general section. The category $\mathcal{C}_0$ is the global family.

Theorem 1.6. *Consider the following data:*

   (i) *a category $\mathcal{C}_{\mathrm{special}}$ which is an $(\infty, 1)$-category;*
   (ii) *a stability condition on $\mathcal{C}_{\mathrm{special}}$;*
   (iii) *a comonad T on $\mathcal{C}_{\mathrm{special}}$ such that $Cone(T \to Id) = [2]$.*

Let $\mathcal{C}_0$ be a category of comodules corresponding to $\mathcal{C}_{\mathrm{special}}$ and T. There is a functor $\mathcal{C}_{\mathrm{special}} \to \mathcal{C}_0$. We define $\mathcal{C}_{\mathrm{gen}} = \mathcal{C}_0 / \mathcal{C}_{\mathrm{special}}$. In the situation above there exists a stability condition on $\mathcal{C}_{\mathrm{gen}}$ such that its central charge and its phase are lift from a central charge and a phase on $\mathcal{C}_{\mathrm{special}}$.

Applied to our situation the above theorem suggests the following:

   (i) Stability conditions of Fano varieties can be obtained from stability conditions of (singular) toric varieties. Indeed, the latter have exceptional collections and moduli spaces of stability conditions are easier to understand.
   (ii) Stability conditions of Calabi–Yau varieties can be obtained from stability conditions of Fano manifolds via Tyurin degenerations.

Combining these facts with the finding of the present paper suggests:

Conjecture 1.7. *The moduli spaces of stability conditions (defined by Bridgeland) for directed graphs of Fano varieties in Theorem 1.2 are contained in each other with the top moduli spaces of stability conditions contained in the ones obtained by projections.*

The above observations suggest that obtaining, via degenerations, stability conditions for one of three-dimensional Fano varieties leads to computing stability conditions for all of them. In a similar fashion we propose that Apery constants (defined in [34]) for all these Fano varieties are connected with each other. We expect that similar behaviour of Fano varieties extends to high dimensions.

The paper is organized as follows. In §2 we recall results from [15] and define toric basic links relating Fano threefolds. In §3 we define the toric Landau–Ginzburg models associated with toric Fano threefolds. Toric basic links can be interpreted as their transformations. In §4 we define mutations between toric Landau–Ginzburg models; that is, relative birational transformations between them. They correspond to deformations of toric degenerations of given Fano threefolds and they can be implemented to the toric basic links graph. In §5 we study toric degenerations of Fano threefolds. In §6 we describe the directed graph of reflexive polytopes. In §7 we describe the algorithm we use to compute the projections graph. In §8 we compute the directed (sub)graph of projections relating smooth Fano threefolds; roots of the graphs are several particular Fano threefolds. Finally in the Appendices we present the data which is the output of our construction. In Appendix A we present the appropriate projection directed (sub)graphs. In Appendix B we present Gorenstein toric degenerations of Fano threefolds.

1.1. **Notation.** Smooth del Pezzo threefolds, that is, smooth Fano threefolds of index two, we denote by V_d, where d is the degree with respect to a generator of the Picard group; the single exception is the quadric, which we denote by Q. Fano threefold of Picard rank one, index one, and degree d, we denote by X_d. The remaining Fano threefolds we denote by X_{k-n}, where k is the Picard rank and n is its number according to [46]. When $k = 4$ the numbers n differ from the identifiers in Mori–Mukai's original classification [58] due to the 'missing' rank four Fano X_{4-2} [59], which has been placed in the appropriate position within the list.

To any reflexive polytope $P \subset N_{\mathbb{Q}}$ we associate the Gorenstein toric Fano variety X_P whose fan Σ in the lattice N is generated by taking the cones over the faces of P. We call Σ the *spanning fan* of P. The moment polytope is denoted by either $P^* \subset M_{\mathbb{Q}}$ or by $\Delta \subset M_{\mathbb{Q}}$, and is dual to P. We identify Gorenstein toric Fano varieties by the corresponding reflexive polytope P, numbered from 1 to 4319 by the *Reflexive ID* as given by the online Graded Ring Database [14]. This agrees with the order of the output from the software PALP [56], developed by Kreuzer–Skarke for their classification [54], and with the databases used by the computational algebra systems MAGMA and SAGE, the only complication being whether the numbering starts from 1 (as we do here) or from 0 (as done in, for example, SAGE).

The Laurent polynomial associated to a Fano threefold of Picard rank k and number m we denote by f_{k-m}. The toric variety whose spanning fan is generated by the Newton polytope of f_{k-m} we denote by F_{k-m}. When appropriate, we may refer to the period period sequence $\pi_f(t)$ of a Laurent polynomial f by its *Minkowski ID*, an integer from 1 to 165. The Minkowski IDs are defined in [2, Appendix A], used in [20], and can be looked-up online at [14].

1.2. **The use of computer algebra and databases.** Because of the large number of 3-dimensional Gorenstein toric Fano varieties, several results are derived with the help of computational algebra software and databases of classifications such as [14]. Computer-assisted rigorous proofs play an increasingly important role as we move from surfaces to threefolds, and will become an essential mathematical technique if we ever hope to progress to the systematic study of fourfolds and Kreuzer–Skarke's massive classification [55] of $473\,800\,776$ reflexive polytopes in dimension four.

We highlight our use of computer algebra. In §5, step (i), we make use of MAGMA [13] in order to compute additional toric models, besides those arising from the Minkowski polynomials as classified in [2]. In §5, step (iii), we use MACAULAY2 [35] to compute the dimension of the tangent space for each of the Gorenstein toric Fano threefolds. The software TOPCOM [70] is used in §5.2 to search for reflexive polytopes with appropriate boundary triangulations. Finally, in §8 we make use of several computer programs that rely on PALP [56] in order to build and manipulate the relevant projection graphs, as well as to further explore the effects of using different combinations of allowed projections and mutations. We emphasise that although any particular example can be worked by hand, the number of cases under consideration means that the only practical way to ensure accuracy is to employ the use of a computer.

2. Basic links

Definition 2.1. A Fano variety X is called *Gorenstein* if its anticanonical class $-K_X$ is a Cartier divisor. Such a variety is called *canonical* (or is said to have *canonical singularities*) if for any resolution $\pi\colon \widetilde{X} \to X$ the relative canonical class $K_X - \pi^* K_{\widetilde{X}}$ is an effective divisor.

Definition 2.2. A threefold singularity P (not necessary isolated) is called *cDV* if its general transversal section has a du Val singularity at P.

The only canonical Gorenstein Fano curve is $\mathbb{P}^1$. Canonical Gorenstein Fano surfaces are called del Pezzo surfaces; they are given by the quadric surface, $\mathbb{P}^2$, and the blow-up of $\mathbb{P}^2$ in at most eight points in general position, along with the degenerations of these smooth surfaces. Canonical Gorenstein Fano threefolds are not yet classified, although partial results can be found in [16, 47, 48, 60, 64]. Smooth Fano threefolds were classified by Iskovskikh [43, 44] and Mori–Mukai [58, 59]: there are 105 deformation classes, of which 98 have very ample anticanonical divisor $-K_X$.

Let $\varphi_{|-K_X|} \colon X \to \mathbb{P}^{g+1}$ be a map given by $|-K_X|$. Then one of the following occurs:

(i) $\varphi_{|-K_X|}$ is not a morphism, that is $\mathrm{Bs}\,|-K_X| \neq \varnothing$, and all such X are found in [47];

(ii) $\varphi_{|-K_X|}$ is a morphism but not an embedding, the threefold X is called *hyperelliptic*, and all such X are found in [16];

(iii) $\varphi_{|-K_X|}$ is an embedding and $\varphi_{|-K_X|}(X)$ is not an intersection of quadrics — then the threefold X is called *trigonal*, and all such X are found in [16];

(iv) $\varphi_{|-K_X|}(X)$ is an intersection of quadrics.

In particular an anticanonically embedded Fano variety is either trigonal or an intersection of quadrics. The varieties that cannot be anticanonically embedded are classified: only few of them can be smoothed. In this paper we focus on anticanonically embedded threefolds.

2.1. **The del Pezzo surfaces.** In the 1880s Pasquale del Pezzo considered the surfaces of degree n in $\mathbb{P}^n$; in other words, surfaces with a very ample anticanonical class. The modern definition of *del Pezzo surfaces* as ones with ample anticanonical class adds blow-ups of $\mathbb{P}^2$ in seven and eight general points to del Pezzo's initial list of surfaces.

When classifying Fano varieties, we in fact classify their moduli spaces or components of their deformation spaces. This means that it is natural to consider degenerations of Fano varieties. Given one point on each moduli space together with its deformation information one can reconstruct a classification of Fano varieties. For del Pezzo surfaces this means that we allow birational transformations related to points in both general and non-general position. Of course, in this case we may get singular surfaces. Following the work of del Pezzo, let us consider anticanonically embedded surfaces. A blow-up of a general point is simply a projection from this point. A similar transformation related to a non-general point is a blow-up of this point followed by taking the anticanonical model. The anticanonical map contracts -2-curves that can appear after the blow-up, i.e. strict transforms of the lines passing through the original point. Thus these transformations are nothing but projections in the anticanonical embedding.

Projections from smooth points relate (possibly non-general) anticanonically embedded del Pezzo surfaces, starting at either $\mathbb{P}^2 = S_9 \subset \mathbb{P}^9$ or the quadric Q, and finishing with the cubic $S_3 \subset \mathbb{P}^3$:

$$
\begin{array}{c}
Q \\
| \\
| \, \pi_{8'} \\
\downarrow \\
\mathbb{P}^2 = S_9 \xrightarrow{\ \pi_9\ } S_8 \xrightarrow{\ \pi_8\ } S_7 \xrightarrow{\ \pi_7\ } S_6 \xrightarrow{\ \pi_6\ } S_5 \xrightarrow{\ \pi_5\ } S_4 \xrightarrow{\ \pi_4\ } S_3.
\end{array}
$$

Our projections terminated on a cubic: further projections are non-birational. The reason for this is that cubic surface is trigonal, and blowing up a point produces a hyperelliptic surface whose anticanonical map is a double covering. Thus we want to consider birational transformations

$$
\begin{array}{ccc}
 & \widetilde{X} & \\
{\scriptstyle \alpha}\swarrow & & \searrow{\scriptstyle \varphi_{|-K_{\widetilde{X}}|}} \\
X \dashrightarrow[\pi_i] & & X',
\end{array}
$$

where α is a blow-up, such that:

(i) the map $\varphi_{|-K_{\widetilde{X}}|}$ is birational;

(ii) the variety X' is Fano.

Condition (i) is satisfied by being X an intersection of quadrics; in this case, $\widetilde{X}$ is at most trigonal. Via a careful choice of centres of the blow-ups, condition (ii) can also be satisfied. Considering projections in the surface case, we need to include smooth points in the set of admissible centres; *a posteriori* we see that this is sufficient.

In order to simplify the model surface picture, we make one final choice: the particular points of the deformation spaces of del Pezzo surfaces that we want to relate. We wish to make use of the toric del Pezzo surfaces. Our main motivation for this choice comes from our desire to exploit the relation, via toric Landau–Ginzburg models, with mirror symmetry; the relative ease of working with toric varieties; and the classifications of toric varieties. In this case, we insist that the centres of blow-ups should also be toric points. Thus we obtain to the following definition:

Definition 2.3. A diagram

$$
\begin{array}{ccc}
 & \widetilde{S}_i & \\
\alpha_i \swarrow & & \searrow \varphi_{|-K_{\widetilde{S}_i}|} \\
S_i \dashrightarrow_{\pi_i} & & S_{i-1},
\end{array}
$$

where α_i is a blow-up of a smooth point and $i \geq 4$ is called a *basic link* between del Pezzo surfaces. If the varieties and the centre of the blow-up are toric then the basic link is called *toric*.

There are 16 toric del Pezzo surfaces, corresponding to the 16 reflexive polygons [7,69]. All possible toric basic links between them are drawn in Figure 1. Any chain of toric basic links from $\mathbb{P}^2$ to the toric cubic, plus an appendix with the quadric, gives us the classification of del Pezzo surfaces in the sense discussed above.

2.2. **Threefold case.** The situation described in §2.1 changes dramatically when moving to three dimensions. Iskovskih [43,44] and Mori–Mukai [58,59] classified the smooth Fano threefolds around 1980, a century after del Pezzo's work in 2-dimensions. It had already observed by Iskovskih–Manin [45] that, unlike the 2-dimensional case, not all of the smooth Fano threefolds are rational. Thus, if the basic links considered are required to be birational, there is no hope of producing a direct analogue of Figure 1 for smooth 3-dimensional Fano varieties. This can be rectified by considering not the smooth Fano variety X itself, but the toric degenerations of X (and corresponding basic links). When X is very ample, these degenerations are toric Fano threefolds with Gorenstein singularities. Reid has shown [71, Corollary 3.6] that Gorenstein toric varieties have at worst canonical singularities, and Batyrev has shown [6] that the Gorenstein toric Fano varieties of dimension n are equivalent, in a precise sense arising from the combinatorics of toric geometry, to the n-dimensional reflexive polytopes.

Definition 2.4. Let $N \cong \mathbb{Z}^n$ be a lattice of rank n, and let $P \subset N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ be a convex lattice polytope of maximum dimension. That is, the vertices $\mathrm{vert}(P)$ of P are points in N, and the dimension of the smallest affine subspace containing P is equal to the rank of N. We say that P is *reflexive* if the *dual* (or *polar*) polyhedron

$$P^* := \{u \in M_{\mathbb{Q}} \mid u(v) \geq -1 \text{ for all } v \in P\}, \qquad \text{where } M := \mathrm{Hom}(N, \mathbb{Z}),$$

is a lattice polytope in M.

The 3-dimensional reflexive polytopes were classified by Kreuzer–Skarke [54]; up to $\mathrm{GL}_3(\mathbb{Z})$-equivalence there are 4319 cases. As noted in §1.1 we will blur the distinction between a reflexive polytope $P \subset N_{\mathbb{Q}}$ and the corresponding toric Fano threefold X_P, and both will often be referred to by their Reflexive ID [14].

We now need to produce a generalisation of the 2-dimensional basic links in Definition 2.3 that would naturally connect the Gorenstein toric Fano threefolds. Let X be such a threefold. Following [15, §2.5], let Z be one of:

  (i) a smooth point of X;
  (ii) a terminal cDV point of X;
  (iii) a line on X not passing through any non-cDV points.

Let $\alpha : \tilde{X} \to X$ be the blow-up of the ideal sheaf of the subvariety $Z \subset X$, and let $\beta : \tilde{X} \to X'$ be the morphism defined by $|-K_{\tilde{X}}|$, making β an embedding. Then (see [15, Lemma 2.2]):

Proposition 2.5. *The morphism β is birational and X' is a Fano threefold with Gorenstein singularities.*
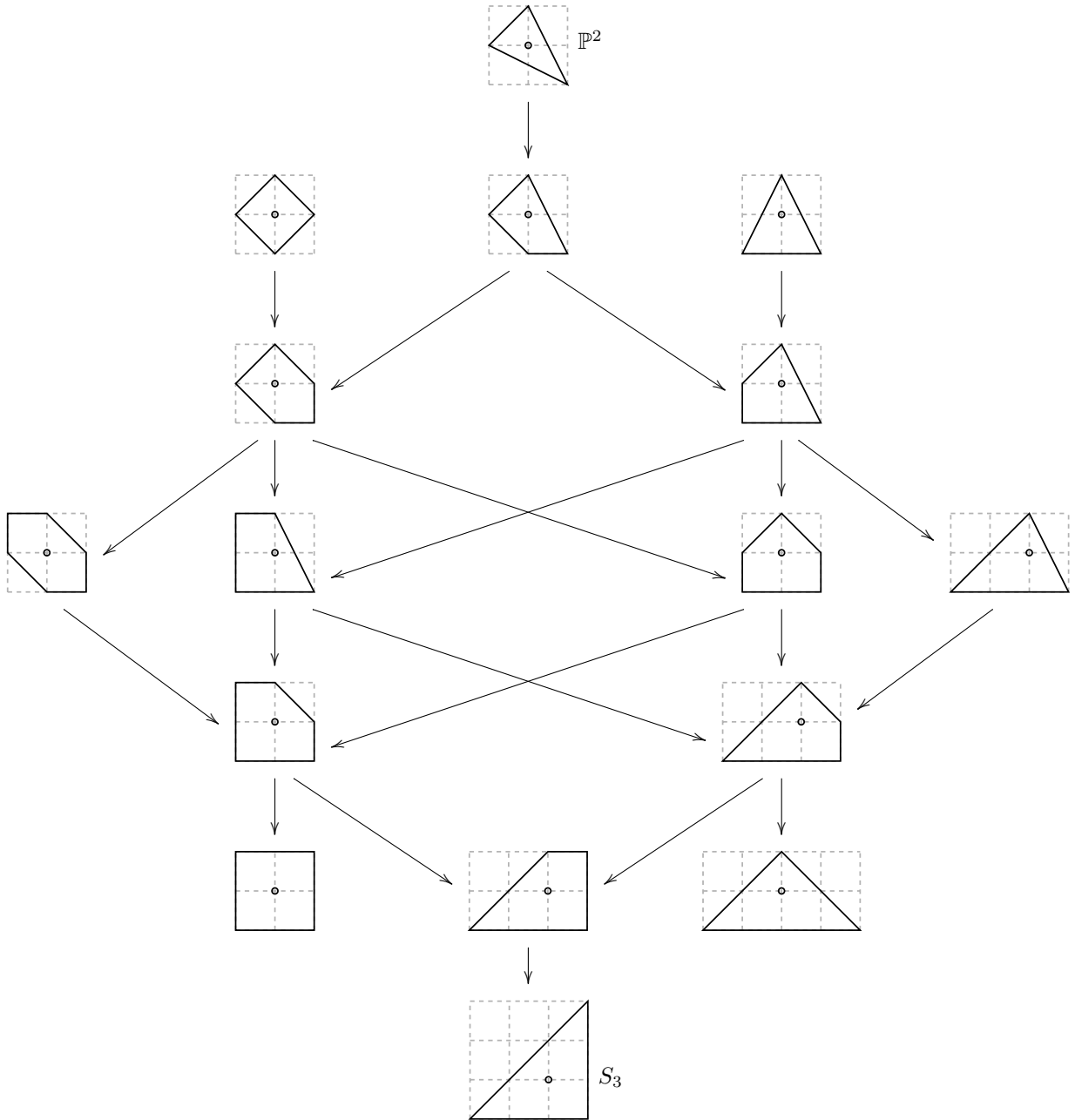
FIGURE 1. The toric del Pezzo directed graph, starting with $\mathbb{P}^2$ at the top of the diagram, and working down through basic links to the toric cubic S_3 at the bottom of the diagram.

Definition 2.6. Using the morphisms α and β as defined above, consider the commutative diagram:

$$
\begin{array}{ccc}
 & \widetilde{X} & \\
{\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\
X \dashrightarrow[\pi] & & X'
\end{array}
$$

We call $\pi : X \dashrightarrow X'$ a *basic link* between the threefolds X and X'. We denote individual types of basic links as:

  (i) Π_{p} if Z is a smooth point;
  (ii) Π_{dp} (or Π_{o}, or Π_{cDV}) if Z is a double point (or, respectively, an ordinary double point, or a non-ordinary double point);

(iii) Π_l if Z is a line.

If all the varieties in question are toric, we call π a *toric basic link*.

In all of these cases, π can be naturally seen as a projection of X: if Z is a smooth point, then π is the projection from the projective tangent space of X at Z, and in all other cases it is the projection of X from Z itself. We call X the *root* of the projection π. Under certain additional assumptions, it is possible to similarly define basic links for Z being a higher-degree curve. For example, we let Π_c denote the basic link in the case when $Z \subset X$ is a conic curve; see [15] for the definition. Since these more-general basic links are less natural, we deliberately try to avoid them in our main calculation. We will refer to projections from points and lines as the *allowed* projections.

Example 2.7 ([15, §2.6]). Similarly to the 2-dimensional case, we can begin with $\mathbb{P}^3$ (or Q^3) and start applying toric basic links to it and to the varieties we get as a result. We would expect to obtain, up to degeneration, all (or almost all) the very ample smooth Fano threefolds. Since the basic links can be seen as projections, we can formulate this as a directed graph with vertices corresponding to smooth Fano threefolds, up to degeneration, and arrows corresponding to the projections from a degeneration of one variety to that of another. For example, one can see a small piece of such a graph in Figure 2.
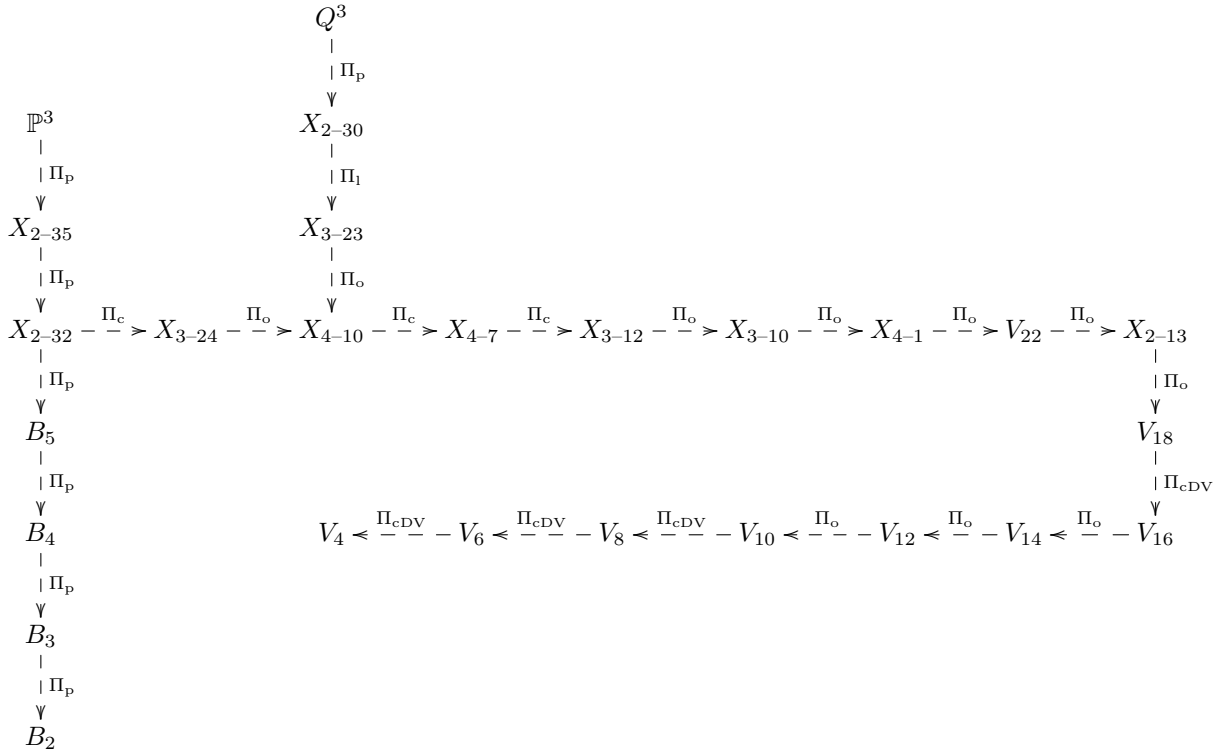
FIGURE 2. The Fano snake

3. TORIC LANDAU–GINZBURG MODELS

In this section we define a toric Landau–Ginzburg model, the main object of our study. For details and examples see [5, 8, 19, 20, 40, 65, 66, 68] and references therein.

Let X be a smooth Fano variety of dimension n and Picard rank ρ. Fix a basis $\{H_1, \ldots, H_\rho\}$ in $H^2(X)$ so that for any $i \in [\rho]$ and any curve β in the Kähler cone K of X one has $H_i \cdot \beta \geq 0$. Introduce formal variables $q_i := q^{\tau_i}$ for each $i \in [\rho]$, and for any $\beta \in H_2(X)$ define

$$q^\beta := q^{\sum \tau_i (H_i \cdot \beta)}.$$

Consider the Novikov ring $\mathbb{C}_q$, i.e. a group ring for $H_2(X)$. We treat it as a ring of polynomials over $\mathbb{C}$ in formal variables q^β, with relations $q^{\beta_1} q^{\beta_2} = q^{\beta_1 + \beta_2}$. Notice that for any $\beta \in K$ the monomial q^β has non-negative degrees in the q_i.

Let the number

$$\langle \tau_a \gamma \rangle_\beta, \qquad \text{where } a \in \mathbb{Z}_{\geq 0}, \gamma \in H^*(X), \beta \in K,$$

be a one-pointed Gromov–Witten invariant with descendants for X; see [57, VI-2.1]. Let $\mathbf{1}$ be the fundamental class of X. The series

$$I_0^X(q_1, \ldots, q_\rho) = 1 + \sum_{\beta \in K} \langle \tau_{-K_X \cdot \beta - 2} \mathbf{1} \rangle_{-K_X \cdot \beta} \cdot q^\beta$$

is called the *constant term of I-series* (or the *constant term of Givental's J-series*) for X, and the series

$$\widetilde{I}_0^X(q_1, \ldots, q_\rho) = 1 + \sum_{\beta \in K} (-K_X \cdot \beta)! \langle \tau_{-K_X \cdot \beta - 2} \mathbf{1} \rangle_{-K_X \cdot \beta} \cdot q^\beta$$

is called the *constant term of regularised I-series* for X. Given a divisor $H = \sum \alpha_i H_i$, one can restrict these series to a direction corresponding to the given divisor by setting $\tau_i = \alpha_i \tau$ and $t = q^\tau$. Thus one can define the restriction of the constant term of regularised I-series to the anticanonical direction, referred to as the *regularised quantum period*. This has the form

$$\widetilde{I}^X(t) = 1 + a_1 t + a_2 t^2 + \ldots.$$

Definition 3.1. A *toric Landau–Ginzburg model* is a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm}]$ which satisfies three conditions:

(i) *(Period condition)* The constant term of f^k is a_k, for each $k \in \mathbb{Z}_{>0}$;
(ii) *(Calabi–Yau condition)* Any fibre of $f \colon (\mathbb{C}^\times)^n \to \mathbb{C}$ has trivial dualising sheaf;
(iii) *(Toric condition)* There exists an embedded degeneration $X \rightsquigarrow T$ to a toric variety T whose fan is equal to the spanning fan of $\mathrm{Newt}(f)$, the Newton polytope of f.

Given a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the *classical period* of f is given by:

$$\pi_f(t) = \left(\frac{1}{2\pi i} \right)^n \int_{|x_1| = \ldots = |x_n| = 1} \frac{1}{1 - tf(x_1, \ldots, x_n)} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}, \qquad \text{for } t \in \mathbb{C}, \ |t| \ll \infty.$$

This gives rise to a solution of a GKZ hypergeometric differential system associated to the Newton polytope of f. Expanding this integral by the formal variable t one obtains a generating series for the constant terms of exponents of f, which we call the *period sequence* and also denote by π_f:

$$\pi_f(t) = 1 + \sum_{k \geq 1} \mathrm{coeff}_1(f^k) t^k.$$

For details see [65, Proposition 2.3] and [19, Theorem 3.2]. The period condition in Definition 3.1 tells us that we have an isomorphism between the Picard–Fuchs differential equation for a family of fibres of the map $f \colon (\mathbb{C}^\times)^n \to \mathbb{C}$, and the regularised quantum differential equation for X.

The Calabi–Yau condition is motivated by the following (see, for example, [66, Principle 32]):

Principle 3.2 (Compactification principle). The relative compactification of the family of fibres of a "good" toric Landau–Ginzburg model (defined up to flops) satisfies the (B-side of the) Homological Mirror Symmetry conjecture.

From the point of view of Homological Mirror Symmetry, the fibres of the Landau–Ginzburg model for a Fano variety are Calabi–Yau varieties. The Calabi–Yau condition is designed to remove any obstructions for the compactification principle in this case. Finally, the toric condition is a generalisation of Batyrev's principle for small toric degenerations.

Claim 3.3. All toric Landau–Ginzburg models associated with the same toric degeneration of X have the same support. In other words, given a toric degeneration $X \rightsquigarrow T$, by varying a symplectic form on X one can vary the coefficients of the Laurent polynomial f_X whilst keeping the Newton polytope fixed.

Conjecture 3.4 (Strong version of Mirror Symmetry for variation of Hodge structures). *Any smooth Fano variety has a toric Landau–Ginzburg model.*

Some progress has been made towards proving Conjecture 3.4 in the case of threefolds; [66, Proposition 9 and Theorem 14] and [42, Theorem 2.2 and Theorem 3.1] tell us the following:

Theorem 3.5. *Conjecture 3.4 holds for Picard rank 1 Fano threefolds and for complete intersections.*

Moreover, Period condition for all Fano threefolds holds by [19], and Calabi–Yau condition holds for them by [67]. Thus, Theorem 5.1 implies the following.

Corollary 3.6. *Conjecture 3.4 holds for smooth Fano threefolds.*

The compactification principle requires that the fibres of a Calabi–Yau compactification of a toric Landau–Ginzburg model for X are mirror dual to anticanonical sections of X. In the threefold case, this duality is called the *Dolgachev–Nikulin–Pinkham duality* [27, 61], and can be formulated in terms of orthogonal Picard lattices. In [29, 68] the uniqueness of compactified toric Landau–Ginzburg models satisfying these conditions is proved for rank one Fano threefolds, and so the theorem holds for all Dolgachev–Nikulin toric Landau–Ginzburg models.

4. MUTATIONS

Mutations are a special class of birational transformations which act on Laurent polynomials and arise naturally in the context of mirror symmetry for Fano manifolds. As discussed in §3, an n-dimensional Fano manifold X is expected to correspond to a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, with the period sequence π_f of f agreeing with the regularised quantum period of X. This correspondence is far from unique: typically there will be infinitely many Laurent polynomials corresponding to a given Fano manifold, and it is expected that these Laurent polynomials are related via mutation [2, 32, 36, 51].

We recall the definition of mutation as given in [2]. Write f in the form

$$f = \sum_{h=h_{\min}}^{h_{\max}} C_h(x_1, \ldots, x_{n-1}) x_n^h, \qquad \text{for some } h_{\min} < 0 \text{ and } h_{\max} > 0,$$

where each C_h is a Laurent polynomial in $n-1$ variables, and let $F \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$ be a Laurent polynomial such that C_h is divisible by $F^{|h|}$ for each $h \in \{h_{\min}, h_{\min} + 1, \ldots, -1\}$. We call F a *factor*. Define a birational transformation $\varphi : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ by

$$(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, F(x_1, \ldots, x_{n-1}) x_n).$$

The pullback of f by φ gives a Laurent polynomial

$$g := \varphi^*(f) = \sum_{h=h_{\min}}^{h_{\max}} F(x_1, \ldots, x_{n-1})^h C_h(x_1, \ldots, x_{n-1}) x_n^h.$$

Notice that the requirement that $F^{|h|}$ divides C_h for each $h \in \{h_{\min}, h_{\min} + 1, \ldots, -1\}$ is essential: it is precisely this condition that ensures that g a Laurent polynomial.

Definition 4.1. A *mutation* (or *symplectomorphism of cluster type*) is the birational transformation φ, possibly pre- and post-composed with a monomial change of basis. We say that f and g are *related by the mutation φ*. If there exists a finite sequence of Laurent polynomials $f = f_0, f_1, \ldots, f_k = g$, where each f_i and f_{i+1} are related by mutation, then we call f and g *mutation equivalent*.

One important property is that mutations preserve the classical period:

Lemma 4.2 ([2, Lemma 1]). *If the Laurent polynomials f and g are mutation equivalent then $\pi_f = \pi_g$.*

Example 4.3 (cf. [51, Example 2.6]). Consider the Laurent polynomial

$$f_1 := \frac{(1 + x_1 + x_2)^3}{\prod_{i=1}^n x_i} + x_3 + \ldots + x_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],$$

where $n \geq 3$. This is a weak Landau–Ginzburg model for the n-dimensional cubic as described in, for example, [42, §2.1]. By, for example, [20, Corollary D.5] it has period sequence:

$$\pi(t) = \sum_{k=0}^{\infty} \frac{(3k)!((n-1)k)!}{(k!)^{n+2}} t^{(n-1)k}.$$

Set $F := 1 + x_1 + x_2$ and define the map

$$\varphi_1 : (x_1, x_2, x_3, x_4, \ldots, x_n) \mapsto (x_1, x_2, F x_3, x_4, \ldots, x_n).$$

Then φ_1 defines a mutation of f_1 (in this case x_3 is playing the role of x_n in the definition above):

$$f_2 := \varphi_1^*(f_1) = \frac{(1 + x_1 + x_2)^2}{\prod_{i=1}^n x_i} + (1 + x_1 + x_2)x_3 + x_4 + \ldots + x_n.$$

If $n \geq 4$, we can define

$$\varphi_2 : (x_1, x_2, x_3, x_4, x_5, \ldots, x_n) \mapsto (x_1, x_2, x_3, Fx_4, x_5, \ldots, x_n),$$

giving us a second mutation:

$$f_3 := \varphi_2^*(f_2) = \frac{1 + x_1 + x_2}{\prod_{i=1}^n x_i} + (1 + x_2 + x_2)(x_3 + x_4) + x_5 + \ldots + x_n.$$

We could attempt to continue this process. If $n \geq 5$ we define the map

$$\varphi_3 : (x_1, \ldots, x_4, x_5, x_6, \ldots, x_n) \mapsto (x_1, \ldots, x_4, Fx_5, x_6, \ldots, x_n).$$

This gives a mutation

$$f_4 := \varphi_3^*(f_3) = \frac{1}{\prod_{i=1}^n x_i} + (1 + x_1 + x_2)(x_3 + x_4 + x_5) + x_6 + \ldots + x_n.$$

However, $f_3 \cong f_4$ via the obvious monomial change of basis, and so we regard these two weak Landau–Ginzburg models as being essentially the same.

Suppose that f is mirror to an n-dimensional Fano manifold X, and consider the Newton polytope $P := \mathrm{Newt}(f) \subset N_{\mathbb{Q}}$ of f. We may assume without loss of generality that P is of maximum dimension and that it contains the origin strictly in its interior. The *spanning fan* of P – the fan whose cones in $N_{\mathbb{Q}}$ are generated by the faces of P – gives rise to a toric variety X_P, which we call a *toric model* for X. In general X_P will be singular. However, it is expected to admit a smoothing with general fibre X. Since a smooth Fano manifold can degenerate to many different singular toric varieties, we expect many different mirrors for X. This is reflected in that fact that f is mutation equivalent to many different Laurent polynomials, all of which have the same classical period by Lemma 4.2.

Example 4.4. The Laurent polynomial $f = x + y + \frac{1}{xy} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ has Newton polytope $P = \mathrm{conv}\{(1, 0), (0, 1), (-1, -1)\} \subset N_{\mathbb{Q}}$ and corresponding toric variety $X_P = \mathbb{P}^2$. Via mutation, we obtain $g = y + 2x^2y^2 + x^4y^3 + \frac{1}{xy}$, giving the toric model $X_Q = \mathbb{P}(1, 1, 4)$. Notice that the singular point $\frac{1}{4}(1, 1)$ is a T-singularity and hence admits a $\mathbb{Q}$-Gorenstein (qG) one-parameter smoothing [53, 73]. We can continue mutating, resulting in a directed graph of toric models of the form $X_{(a,b,c)} := \mathbb{P}(a^2, b^2, c^2)$, where $(a, b, c) \in \mathbb{Z}_{>0}^3$ is a solution to the Markov equation $3abc = a^2 + b^2 + c^2$ [4, 37]. There are infinitely many positive solutions to the Markov equation, obtainable via cluster-style mutations of the form $(a, b, c) \mapsto (b, c, 3bc - a)$, and so we have infinitely many toric models $X_{(a,b,c)}$ for $\mathbb{P}^2$. In each case the weighted projective space $X_{(a,b,c)}$ has only T-singularities and so admits a qG-smoothing.

A mutation between two Laurent polynomials induces a mutation of the corresponding Newton polytopes P and Q. The corresponding toric varieties X_P and X_Q are deformation equivalent in the following precise sense:

Lemma 4.5 ([41, Theorem 1.3]). *Let X_P and X_Q be related by mutation. Then there exists a flat family $\mathcal{X} \to \mathbb{P}^1$ such that $\mathcal{X}_0 \cong X_P$ and $\mathcal{X}_\infty \cong X_Q$.*

Very little is known about the behaviour of mutations in general. The two-dimensional setting has been studied in [1, 3, 4, 32, 50, 62]. In particular, the toric surface X_P is qG-smoothable to X [1, Theorem 3], and, for each del Pezzo surface X, the set of all polygons P such that X_P is qG-smoothable to X forms a single mutation-equivalence class [50, Theorem 1.2].

In dimension three a special class of Laurent polynomials called *Minkowski polynomials* were introduced in [2]. These are Laurent polynomials f in three variables whose Newton polytope is a reflexive polytope, with f satisfying certain additional conditions. There are 3747 Minkowski polynomials (up to monomial change of basis), and together they generate 165 periods. Furthermore, any two Minkowski polynomials have the same period sequence if and only if they are mutation equivalent. Of these periods, 98 are of so-called *manifold type*, with the remaining 67 being of *orbifold type* (these are properties of the associated Picard–Fuchs differential equations; see [19, §7] for the definitions). In [20], the period sequences of

manifold type were shown to correspond under mirror symmetry to the 98 deformation families of three-dimensional Fano manifolds X with very ample anticanonical bundle $-K_X$.

Proposition 4.6. *Let $P \subset N_{\mathbb{Q}}$ be a reflexive polytope, with associated Gorenstein toric Fano variety X_P. Assume that X_P has a smooth deformation space, and is a degeneration of a generic smooth Fano variety X. Let Q be any mutation of P, with associated toric variety X_Q. Then X_Q is also a degeneration of X.*

Proof. By Lemma 4.5 there is a flat projective family over $\mathbb{P}^1$ with X_P and X_Q as special fibres. Since $[X_P] \in \mathcal{H}_X$ lies on a single irreducible component, $[X_Q]$ must also lie on this component. A general point of this component is a smooth Fano threefold deformation equivalent to X, so X degenerates to X_Q. $\square$

By applying the above proposition, we may find many more toric degenerations of a given Fano threefold. Indeed, suppose that we have a smooth Fano threefold X which degenerates to a Gorenstein toric Fano variety X_P having a smooth deformation space, as described in §§B.2, B.3, B.4, B.5, and B.6. We can use mutation to construct other Gorenstein toric Fano varieties to which X degenerates. By consulting the calculations of §B.1 and [49, Table B.2] we can determine which of these have smooth deformation spaces, and then iterate using these new examples. We record the resulting degenerations in §B.7.

5. Toric degenerations

Let X be a smooth Fano threefold with a very ample anticanonical divisor, and consider its anticanonical embedding $V \hookrightarrow \mathbb{P}^n$. As explained in §2.2, we are interested in finding embedded degenerations of X to Gorenstein toric Fano varieties. The main result of this section is the following:

Theorem 5.1. *Let X be a generic smooth Fano threefold with a very ample anticanonical divisor. Then X has an embedded degeneration to a Gorenstein toric Fano variety X' with smooth deformation space.*

Our proof makes use of the classification of smooth Fano threefolds [43,44,58,59] and consists of several steps:

(i) We use several techniques, described in §§5.1–5.5 below, to construct at least one degeneration X' for each smooth Fano X. The vast majority of these come from the models constructed in [20] of smooth Fano threefolds as complete intersections in toric varieties; see §5.4. Note that after we constructed these degenerations the papers [22, 28, 29] appeared; they provide further tools for systematically constructing toric degenerations, however we do not make use of them here.

(ii) For any Fano variety X with very ample anticanonical divisor, let $\mathcal{H}_X$ denote the Hilbert scheme parameterising projective schemes with the same Hilbert polynomial as X in its anticanonical embedding. If X is smooth, then it corresponds to a smooth point on an irreducible component of $\mathcal{H}_X$. For each smooth Fano X, we compute the dimension of this component using [18, Proposition 3.1].

(iii) Using the comparison theorem of Kleppe [52, Theorem 3.6] we compute the tangent space dimensions for all Gorenstein toric Fano threefolds, viewed as points in relevant Hilbert schemes.

(iv) Using steps (ii) and (iii) we check that all special fibres X' of the degenerations in step (i) have tangent space dimension equal to the dimension of the Hilbert scheme component of X, and hence correspond to smooth points in the Hilbert scheme. Since the forgetful functor to the deformation space of X' is smooth [18, §2.1], each such X' has smooth deformation space.

None of our results claim to be exhaustive; for example, it is certainly possible for there to exist degenerations which do not appear in our search at step (i).

5.1. **Previously known degenerations.** We make use of a number of known toric degenerations. Galkin [31] classified all degenerations of smooth Fano threefolds to Gorenstein toric Fano varieties with at worst terminal singularities, which are called *small* toric degenerations. These are recorded in §B.2. Notice that these include the 18 smooth toric Fano threefolds classified by Batyrev [6] and Watanabe–Watanabe [74]. The embedded degenerations of smooth Fano threefolds with a very ample anticanonical divisor to Gorenstein toric Fano varieties of degree at most twelve were classified in [18]. These are recorded in §B.3. Since writing this paper, the preprint [63] has appeared; here Prince uses the methods of [22] to construct Gorenstein toric degenerations of Fano threefolds.

5.2. **Triangulations of the moment polytope.** We use the techniques developed in [17] to construct degenerations of rank one, index one Fano threefolds. Consider the two-dimensional simplicial complexes T_i, $i \in \{4, 5, 6, 7, 8, 9, 10, 11\}$, defined as follows: when $i = 4$ we define T_4 to be the boundary of the three-simplex; when $i = 5$ we define T_5 to be the bipyramid over the boundary complex of a two-simplex; when $6 \leq i \leq 10$, let T_i be the unique triangulation of the sphere with i vertices having valencies four and five; when $i = 11$, let T_{11} be the unique triangulation of the sphere having valencies $4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 6$.

Theorem 5.2 ([17, §3]). *For $4 \leq i \leq 11$, let $d := 2i - 4$ and let X_d be a general rank one, index one, degree d smooth Fano threefold. Consider a three-dimensional reflexive polytope $\Delta \subset M_{\mathbb{Q}}$ whose boundary admits a regular unimodular triangulation of the form T_i, and let $X' := X(\Delta)$ be the associated Gorenstein toric Fano variety with fan normal to Δ. If $i < 11$, or if $i = 11$ and $h^0(X', \mathcal{N}_{X'/\mathbb{P}^i}) = 153$, then X_d has an embedded degeneration to X'. Furthermore, the point corresponding to X' in $\mathcal{H}_{X_d}$ is smooth.*

The resulting degenerations are recorded in §B.4.

5.3. **Products with del Pezzo surfaces.** A number of Fano threefolds are products of del Pezzo surfaces with $\mathbb{P}^1$: $X_{5-3}, X_{6-1}, X_{7-1}, X_{8-1}$, and X_{9-1}, of which the first three have very ample anticanonical divisor. Toric degenerations of these threefolds may be found by degenerating the corresponding del Pezzo surfaces. In many cases these degenerations are well understood; see for example the work of Hacking–Prokhorov [37]. The threefold degenerations constructed this way are recorded in §B.5.

(i)   (ii)
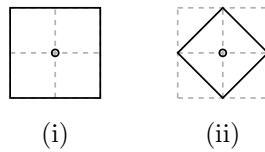
FIGURE 3. The spanning polytope (i) $P \subset N_{\mathbb{Q}}$ and the moment polytope (ii) $P^* \subset M_{\mathbb{Q}}$ of a toric degeneration of the del Pezzo surface of degree four.

Example 5.3. Consider the del Pezzo surface S_4 of degree four given by the blow-up of $\mathbb{P}^2$ in five points. This degenerates to the toric variety whose spanning polytope is picture in Figure 3(i). The Fano threefold X_{7-1} is $S_4 \times \mathbb{P}^1$, and thus degenerates to the toric Fano threefold X_P where, up to $\mathrm{GL}_3(\mathbb{Z})$-action, $P := \mathrm{conv}\{\pm(1, 1, 0), \pm(1, -1, 0), \pm(0, 0, 1)\} \subset N_{\mathbb{Q}}$. This has Reflexive ID 510.

5.4. **Complete Intersections in toric varieties.** It appears to be common for smooth Fano varieties to arise as complete intersections in the homogeneous coordinate ring of a smooth toric Fano variety. Eight of the del Pezzo surfaces can be realised this way; so can at least 78 of the 105 smooth Fano threefolds [20], and at least 738 of the smooth Fano fourfolds [21]. This leads to the following natural construction of degenerations.

Let W be a smooth complete toric variety, with I the set of invariant prime divisors. Its homogeneous coordinate ring $R = \mathbb{C}[x_i \mid i \in I]$ is a polynomial ring graded by $\mathrm{Pic}(W)$. Then W is the quotient of $U = \mathrm{Spec}\,R \setminus Z$ by the Picard torus $T = \mathrm{Spec}\,\mathbb{C}[\mathrm{Pic}(W)]$, where Z is the so-called irrelevant, or exceptional, set. Suppose that X is a complete intersection in W of Cartier divisors $D_1, \ldots, D_k$; let $\overline{D}_i$ be the class of D_i in $\mathrm{Pic}(W)$. Each divisor D_i may be encoded by a homogeneous degree $\overline{D}_i$ polynomial f_i in R. Then X arises as the quotient

$$X = (U \cap V(f_1, \ldots, f_k)) /\!/ T.$$

In order to degenerate X, we may degenerate the polynomials f_i to polynomials g_i of the same multidegree. As long as the degenerate polynomials g_i still form a regular sequence in R, we get a degeneration from X to $X' = (U \cap V(g_1, \ldots, g_k)) /\!/ T$. If $V(g_1, \ldots, g_k) \cap U \hookrightarrow U$ is an equivariant embedding of toric varieties, then the resulting quotient X' is toric as well.

To construct a toric X', we may choose the g_i to be binomials. If M is the character lattice of the torus of W, we have an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^I \xrightarrow{\ \pi\ } \mathrm{Pic}(W) \longrightarrow 0,$$

and each binomial g_i determines a rank-one sublattice L_i of M. Let O be the positive orthant of $\mathbb{Z}^I \otimes \mathbb{Q}$, and let O' be its image in $\left(\mathbb{Z}^I / \sum L_i\right) \otimes \mathbb{Q}$. If $M' := \mathbb{Z}^I / \sum L_i$ is torsion free and the natural map $R \to \mathbb{C}[O' \cap M']$ is surjective with kernel generated by the g_i, then $V(g_1, \ldots, g_k)$ is toric, with $U \cap V(g_1, \ldots, g_k) \hookrightarrow U$ an equivariant embedding. In order to determine the quotient in this situation, let A be any ample class in $\mathrm{Pic}(W)$. Then the quotient X' of $U \cap V(g_1, \ldots, g_k)$ is the toric variety whose moment polytope Δ is the image of $\pi^{-1}(A) \cap O$ in $M' \otimes \mathbb{Q}$.

Example 5.4. The smooth Fano threefold $X = X_{2\text{-}7}$ is a codimension two complete intersection in the toric variety $W = \mathbb{P}^1 \times \mathbb{P}^4$. The homogeneous coordinate ring $R = \mathbb{C}[x_0, x_1, y_0, \ldots, y_4]$ of W has a $\mathrm{Pic}(W) = \mathbb{Z}^2$-grading given by the columns of the weight data

x_0	x_1	y_0	y_1	y_2	y_3	y_4	
1	1	0	0	0	0	0	A
0	0	1	1	1	1	1	B

with ample cone $\overline{\mathrm{Amp}}\, W = \langle A, B \rangle$. The threefold X is the intersection of divisors $(A + 2B) \cap (2B)$. Consider the $\mathbb{Z}^2$-homogeneous polynomials

$$g_1 = x_0 y_0 y_1 - x_1 y_2 y_3$$
$$g_2 = y_3^2 - y_2 y_4.$$

Then $V(g_1, g_2)$ is toric, and the resulting quotient by the Picard torus gives the Gorenstein toric Fano variety X' with Reflexive ID 3813 given by the moment polytope

$$\Delta = \mathrm{conv}\{(1,0,0),(1,1,0),(1,0,1),(1,1,1),(0,0,1),(0,-1,1),(-1,0,-1),(-1,1,-1)\}.$$

In the above construction, the resulting toric variety X' need not be Fano. However, in the case when X' is a weak Fano with Gorenstein singularities we can construct a degeneration from X to the anticanonical model X'' of X'. Indeed, let $\pi : \mathcal{X} \to S$ be the total space of the degeneration of X to X'. Since all fibres are Cohen-Macaulay, $-K_{\mathcal{X}/S}|_F \cong -K_F$ for any fibre F [23, Theorem 3.5.1]. Furthermore, since $-K_{X'}$ is Cartier and nef (and X' is toric), all higher cohomology of $-K_F$ vanishes for every fibre F. By cohomology and base change, the restriction map $H^0(\mathcal{X}, -K_{\mathcal{X}/S}) \to H^0(F, -K_F)$ is surjective. Hence, taking Proj of the section ring of $-K_{\mathcal{X}/S}$ gives a flat family over S with X'' as the special fibre and X as the general fibre.

We apply the above discussion to find toric degenerations of many of the smooth Fano threefolds, using their descriptions in [20] as complete intersections in toric varieties. These degenerations are recorded in §B.6. For each degeneration, we also record the corresponding regular sequence g_i, using the same ordering on the homogeneous coordinates as in [20]. For a similar approach to constructing degenerations, see also [39].

5.5. **The exceptional case:** $X_{2\text{-}14}$. The previous methods fail to construct toric degenerations for the smooth Fano threefold $X_{2\text{-}14}$. We now discuss a modification of §5.4 which does produce a toric degeneration.

The smooth Fano threefold $X = X_{2\text{-}14}$ may be realised as a divisor of bidegree $(1,1)$ in $V_5 \times \mathbb{P}^1$, where B_5 is a codimension-three linear section of the Grassmannian $\mathrm{Gr}(2,5)$ in its Plücker embedding [20]. The anticanonical embedding places V_5 in $\mathbb{P}^6$ as the intersection of five quadrics. The approach of §5.2 can be applied to find degenerations of V_5. In particular, it degenerates to the Gorenstein toric Fano with Reflexive ID 68.

We can realize X as the intersection in the toric variety $\mathbb{P}^6 \times \mathbb{P}^1$ of $V_5 \times \mathbb{P}^1$ and a divisor of bidegree $(1,1)$, defined by a bihomogeneous polynomial f. Simultaneously degenerating the quadrics of V_5 and the polynomial f leads to a degeneration of X. More precisely, we may degenerate the quadrics to

$$x_1 x_6 - x_2 x_5, \qquad x_1 x_6 - x_3 x_4, \qquad x_0 x_3 - x_1 x_2,$$
$$x_0 x_5 - x_1 x_4, \qquad x_0 x_6 - x_2 x_4,$$

and the polynomial f to $g = x_3 x_7 - x_4 x_8$. The resulting variety is the Gorenstein toric Fano threefold with Reflexive ID 2353, generated by the polytope

$$P := \mathrm{conv}\{(1,0,0), (0,1,0), \pm (0,0,1), \pm(1,-1,0),$$

$$\pm(1,0,-1), \pm (0,1,-1), (-1,-1,0), (-1,-1,1)\} \subset N_{\mathbb{Q}}.$$

6. THE GRAPH OF REFLEXIVE POLYTOPES

In [72] Sato investigated the concept of F-equivalence classes of a smooth Fano polytope. Recall that a smooth Fano polytope $P \subset N_{\mathbb{Q}}$ corresponds to a smooth toric Fano variety X_P via its spanning fan. Smooth Fano polytopes are necessarily reflexive, and a reflexive polytope $P \subset N_{\mathbb{Q}}$ is smooth if, for each facet F of P, the vertices $\mathrm{vert}(F)$ give a $\mathbb{Z}$-basis for the underlying lattice N (and hence, in particular, P needs be simplicial).

Definition 6.1 (F-equivalence of smooth Fano polytopes)**.** Two smooth Fano polytopes $P, Q \in N_{\mathbb{Q}}$ are F-*equivalent*, and we write $P \overset{F}{\sim} Q$, if there exists a finite sequence $P_0, P_1, \ldots, P_k \subset N_{\mathbb{Q}}$ of smooth Fano polytopes satisfying:

  (i) P and Q are $\mathrm{GL}_n(\mathbb{Z})$-equivalent to P_0 and P_k, respectively;
 (ii) for each $1 \le i \le k$ we have either that $\mathrm{vert}(P_i) = \mathrm{vert}(P_{i-1}) \cup \{w\}$, where $w \notin \mathrm{vert}(P_{i-1})$, or that $\mathrm{vert}(P_{i-1}) = \mathrm{vert}(P_i) \cup \{w\}$, where $w \notin \mathrm{vert}(P_i)$;
(iii) if $w \in \mathrm{vert}(P_i) \setminus \mathrm{vert}(P_{i-1})$ then there exists a proper face F of P_{i-1} such that

$$w = \sum_{v \in \mathrm{vert}(F)} v$$

and the set of facets of P_i containing w is equal to

$$\{\mathrm{conv}(\{w\} \cup \mathrm{vert}(F') \setminus \{v\}) \mid F' \text{ is a facet of } P_{i_1}, F \subset F', v \in \mathrm{vert}(F)\}.$$

In other words, P_i is obtained by taking a stellar subdivision of P_{i-1} with w. Similarly, if $w \in \mathrm{vert}(P_{i-1}) \setminus \mathrm{vert}(P_i)$ then P_{i-1} is given by a stellar subdivision of P_i with w.

Notice that condition (iii) means that if $P \overset{F}{\sim} Q$ then the corresponding smooth toric Fano varieties X_P and X_Q are related via a sequence of equivariant blow-ups or blow-downs. Little is known about F-equivalence in general, although it is known that all smooth Fano polytopes are F-equivalent in dimensions ≤ 4, and that there exists non-F-equivalence polytopes in all dimensions ≥ 5.

We wish to generalise the notion of F-equivalence to encompass projections between reflexive polytopes. Our focus here is on three-dimensions, although what follows can readily be generalised to higher dimensions. Fix a three-dimensional reflexive polytope $P \subset N_{\mathbb{Q}}$ and consider the corresponding Gorenstein toric Fano variety $X = X_P$. Toric points and curves on X correspond to, respectively, facets and edges of P. Projections from these points (or curves) correspond to blowing-up the cone generated by the relevant facet (or edge). Since we are restricting ourselves to the class of reflexive polytopes, it is clear that the original facet (or edge) only needs to be considered if it has no (relative) interior points: any interior point of that face will become an interior point of the resulting polytope, preventing it from being reflexive.

Proposition 6.2. *Let $P \subset N_{\mathbb{Q}}$ be a three-dimensional reflexive polytope, and let F be a facet or edge of P such that F contains no (relative) interior points. Up to $\mathrm{GL}_3(\mathbb{Z})$-equivalence, F is one of the four possibilities shown in Table 1.*

Proof. First consider the case when F is a facet of P. Since P is reflexive, there exists some primitive lattice point $u_F \in M$ such that $u_F(v) = 1$ for each $v \in F$. In particular, by applying any change of basis sending u_F to e_1^* we can insist that F is contained in the two-dimensional affine subspace $\Gamma := \{(1, a, b) \mid a, b \in \mathbb{Q}\}$. Pick an edge E_1 of F such that $|E_1 \cap N|$ is as large as possible. Let $v \in \mathrm{vert}(E_1)$ be a vertex of E_1 (and hence of F), and let $E_2 \ne E_1$ be the second edge of F such that $E_1 \cap E_2 = \{v\}$. Let $v_i \in E_i \cap N$ be such that $v_i - v$ is primitive. Since $F^\circ = \varnothing$ we have that $\Delta := \mathrm{conv}\{v, v_1, v_2\}$ is a lattice triangle with $|\Delta \cap N| = 3$; that is, Δ is an empty triangle. Up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence, the empty triangle is unique. Hence we can apply a change of basis to the affine lattice $\Gamma \cap N$ such that $v = (1,0,0)$, $v_1 = (1,1,0)$, and $v_2 = (1,0,1)$. By considering the possible

	Face	Points added
1	$(1,0,1)$ $(1,0,0)$ $(1,1,0)$	$(3,1,1)$
2	$(1,0,1)$ $(1,a,1)$ $(1,0,0)$ $(1,a,0)$ $(1,a+b,0)$	$(2,1,1), (2,2a+b-1,1)$
3	$(1,0,2)$ $(1,0,1)$ $(1,1,1)$ $(1,0,0)$ $(1,1,0)$ $(1,2,0)$	$(3,2,2)$
4	$(1,0,0)$ —— $(1,1,0)$	$(2,1,0)$

TABLE 1. The possible choices of faces to be blown-up, and corresponding points to be added, when considering projections between Gorenstein toric Fano threefolds.

lengths of the edges E_1 and E_2, and remembering that $(1,1,1)$ cannot be an interior point, we find the first three cases in Table 1. In the case when F is an edge, it must have length one and hence (up to $\mathrm{GL}_3(\mathbb{Z})$-equivalence) $F = \mathrm{conv}\{(1,0,0),(1,1,0)\}$, the fourth case in Table 1. $\qquad\square$

Corollary 6.3. *Let $P,Q \subset N_{\mathbb{Q}}$ be two three-dimensional reflexive polytopes such that X_Q is obtained from X_P via a projection. Then the corresponding blow-up of the face F of P introduces new vertices as given in Table 1.*

Proof. We prove this only in case 2 in Table 1. The remaining cases are similar. We refer to Dais' survey article [24] for background, and for the combinatorial interpretation. Let

$$C = \mathrm{cone}\{(1,0,0),(1,a+b,0),(1,a,1),(1,0,1)\} \subset N_{\mathbb{Q}},$$

where $a,b \in \mathbb{Z}_{>0}$. This is defined by the intersection of four half-spaces of the form $\{v \in N_{\mathbb{Q}} \mid u_i(v) \geq 0\}$, where the u_i are given by $(0,0,1),(0,1,0),(1,0,-1),(a+b,-1,-b) \in M$. Moving these half-spaces in by one, we obtain the polyhedron:

$$\bigcap_{i=1}^{4}\{v \in N_{\mathbb{Q}} \mid u_i(v) \geq 1\} = \mathrm{conv}\{(2,1,1),(2,2a+b-1,1)\} + C.$$

Hence the blow-up is given by the subdivision of C into four cones generated by inserting the rays $(2,1,1)$ and $(2,2a+b-1,1)$. $\qquad\square$

Of course case 1 in Table 1 is a specialisations of case 2. However, since the point added is what is important, we list it separately. When $a = 1$, $b = 0$, or when $a = 0$, $b = 2$ in case 2, the coordinates of the two points to be added coincide, so we add the single point $(2,1,1)$.

Since the choice of the basic links we are using relies on distinguishing curves of small degrees on a given toric variety, it is necessary to be able to easily calculate the anticanonical degree of a given curve. This can be done as follows:

Lemma 6.4. *Let T be a three-dimensional $\mathbb{Q}$-factorial projective toric variety with simplicial fan Δ. Let c_1 and c_2 be rays in $\Delta^{(1)}$ generating a two-dimensional cone in $\Delta^{(2)}$, with corresponding torus-invariant curve C. Here $\Delta^{(n)}$ denotes the set of n-dimensional cones in the fan Δ. Let a_1 and a_2 be rays such that $F_i = \mathrm{cone}\{c_1,c_2,a_i\} \in \Delta^{(3)}$, $F_1 \neq F_2$, and let $e_1,\dots,e_r$ denote the remaining rays of Δ, so that*

$\Delta^{(1)} = \{c_1, c_2, a_1, a_2, e_1, \ldots, e_r\}$. Let $\mathrm{Vol}(F_i) = |\Gamma : N|$ be the index of the sublattice Γ_i in N generated by c_1, c_2, a_i, where by a standard abuse of notation we confuse a ray with its primitive lattice generator in N; equivalently, $\mathrm{Vol}(F_i)$ is equal to the lattice-normalised volume of the tetrahedron $\mathrm{conv}\{\mathbf{0}, c_1, c_2, a_i\}$.

Denote the boundary divisor corresponding to c_i, a_i, or e_i by C_i, A_i, or E_i respectively. Let L_1 and L_2 be linear forms such that L_1 vanishes on c_2 but not on c_1, and L_2 vanishes on c_1 but not on c_2. Then the anticanonical degree of C is equal to

$$\deg C = \left(1 - \frac{L_1(a_1)}{L_1(c_1)} - \frac{L_2(a_1)}{L_2(c_2)}\right) \frac{1}{\mathrm{Vol}(F_1)} + \left(1 - \frac{L_1(a_2)}{L_1(c_1)} - \frac{L_2(a_2)}{L_2(c_2)}\right) \frac{1}{\mathrm{Vol}(F_2)}.$$

Proof. Recall that the anticanonical degree of C is the sum of its intersection with all boundary divisors

$$\deg C = C \cdot \left(C_1 + C - 2 + A_1 + A_2 + \sum_{i=1}^{r} E_i\right) = C \cdot C_1 + C \cdot C_2 + C \cdot A_1 + C \cdot A_2,$$

with intersection numbers given by

$$C \cdot A_1 = 1/\mathrm{Vol}(F_1), \qquad C \cdot A_2 = 1/\mathrm{Vol}(F_2).$$

Given the linear forms L_1, L_2 as above, let D_i be the principal divisor corresponding to the form L_i. Then:

$$0 \equiv D_i \equiv L_i(c_1)C_1 + L_i(c_2)C_2 + L_i(a_1)A_1 + L_i(a_2)A_2 + \sum_{j=1}^{r} L_i(e_j)E_j.$$

This gives:

$$0 = C \cdot D_i = L_i(c_i)C \cdot C_i + L_i(a_1)C \cdot A_1 + L_i(a_2)C \cdot A_2.$$

Hence:

$$C \cdot C_i = -\frac{L_i(a_1)}{L_i(c_i)}\frac{1}{\mathrm{Vol}\,(F_1)} - \frac{L_i(a_2)}{L_i(c_i)}\frac{1}{\mathrm{Vol}\,(F_2)}. \qquad \square$$

Corollary 6.5. *Let X be a toric Fano threefold and let P the corresponding three-dimensional Fano polytope. Let E be an edge of P corresponding to a curve C on X, and let F_1 and F_2 be the two facets of P meeting at E. Let c_1 and c_2 be the two vertices of P lying on E, and let a_1 and a_2 be vertices of P lying on $F_1 \setminus E$ and $F_2 \setminus E$, respectively. Then:*

$$\deg C = \frac{1}{|a_1 \cdot (c_1 \times c_2)|} + \left(1 + \frac{(c_1 - c_2) \cdot (a_1 \times a_2)}{a_1 \cdot (c_1 \times c_2)}\right) \frac{1}{|a_2 \cdot (c_1 \times c_2)|}.$$

Proof. If the point $p_1 \in X$ corresponding to the face F_1 is singular, then one can take a small resolution X' of X at this point and calculate the degree of C via that resolution. This is done by choosing a triangulation of F_1, with the result being independent of the choice. So, by picking a triangulation containing the triangle (c_1, c_2, a_1), one can assume that p_1 is a smooth point of X.

Similarly, the point $p_2 \in X$ corresponding to the face F_2 of P can also be assumed to be smooth. Therefore, X satisfies the conditions of Lemma 6.4.

Take:

$$L_1(x) = x \cdot (c_2 \times a_1), \ L_2(x) = x \cdot (a_1 \times c_1).$$

Since $\mathrm{Vol}\,(F_i) = |a_i \cdot (c_1 \times c_2)|$, have:

$$\deg C = \frac{1}{|a_1 \cdot (c_1 \times c_2)|} + \left(1 - \frac{a_2 \cdot (c_2 \times a_1)}{c_1 \cdot (c_2 \times a_1)} - \frac{a_2 \cdot (a_1 \times c_1)}{c_2 \cdot (a_1 \times c_1)}\right) \frac{1}{|a_2 \cdot (c_1 \times c_2)|},$$

which simplifies to the form above. $\square$

Definition 6.6 (F-equivalence of reflexive polytopes)**.** Two three-dimensional reflexive polytopes $P, Q \in N_{\mathbb{Q}}$ are F-equivalent, and we write $P \overset{F}{\sim} Q$, if there exists a finite sequence $P_0, P_1, \ldots, P_k \subset N_{\mathbb{Q}}$ of reflexive polytopes satisfying:

(i) P and Q are $\mathrm{GL}_3(\mathbb{Z})$-equivalent to P_0 and P_k, respectively;

(ii) For each $1 \leq i \leq k$ we have that either $\mathrm{vert}(P_i) \subsetneq \mathrm{vert}(P_{i-1})$ or $\mathrm{vert}(P_{i-1}) \subsetneq \mathrm{vert}(P_i)$.

(iii) If $\mathrm{vert}(P_i) \subsetneq \mathrm{vert}(P_{i-1})$ then there exists a face F of P_i and $\varphi \in \mathrm{GL}_3(\mathbb{Z})$ such that $\varphi(F)$ is one of the seven faces in Table 1. Furthermore, the points $\varphi(\mathrm{vert}(P_{i-1}) \setminus \mathrm{vert}(P_i))$ are equal to the corresponding points in Table 1, and $\partial F \subset \partial P_{i-1}$. The case when $\mathrm{vert}(P_{i-1}) \subsetneq \mathrm{vert}(P_i)$ is similar, but with the roles of P_{i-1} and P_i exchanged.
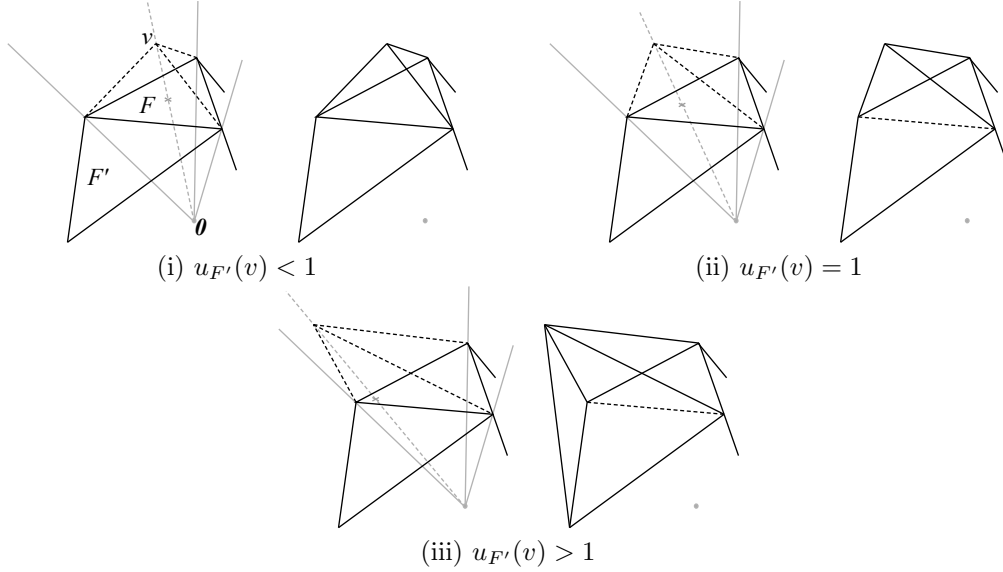
FIGURE 4. The three possible ways a facet F' adjacent to F can be modified when adding a new vertex v. See Remark 6.7 for an explanation.

If $P \overset{F}{\sim} Q$ then the corresponding Gorenstein toric Fano threefolds X_P and X_Q are related via a sequence of projections.

Remark 6.7. The requirement that $\partial F \subset \partial P_{i-1}$ in Definition 6.6(iii) perhaps needs a little explanation. Consider the case when F is a facet. Adding the new vertices can affect a facet F' adjacent to F, with common edge E, in one of three ways. Let $u_{F'} \in M$ be the primitive dual lattice vector defining the hyperplane at height one containing F'. Let $v_1, \ldots, v_s \in N$ be the points to be added according to Table 1, so that $P_{i-1} = \mathrm{conv}(P_i \cup \{v_1, \ldots, v_s\})$.

(i) If $u_{F'}(v_i) < 1$ for $1 \le i \le s$ then F' is unchanged by the addition of the new vertices, and hence F' is also a facet of P_{i-1}.

(ii) Suppose that $u_{F'}(v_i) = 1$ for $1 \le i \le m$, and $u_{F'}(v_i) < 1$ for $m+1 \le i \le s$, for some $1 \le m \le s$. Then the facet F' is transformed to the facet $F'' := \mathrm{conv}(F' \cup \{v_i \mid 1 \le i \le m\})$ in P_{i-1}. Notice that $F' \subset F''$, but that E is no-longer an edge of F''. This is equivalent to a blowup, followed by a contraction of the curve corresponding to E.

(iii) The final possibility is that there exists one (or more) of the v_i such that $u_{F'}(v_i) > 1$. When we pass to P_{i-1} we see that F' is no-longer contained in the boundary; in particular, $E^\circ \subset P_{i-1}^\circ$ and so $\partial F \not\subset \partial P_{i-1}$. This case is excluded since it does not correspond to a projection between the two toric varieties.

These three possibilities are illustrated in Figure 4.

Corollary 6.8 (Reflexive polytopes are F-connected)**.**

(i) *Let $P, Q \subset N_{\mathbb{Q}}$ be any two three-dimensional reflexive polytopes. Then $P \overset{F}{\sim} Q$.*

(ii) *Let $\mathcal{G}_F$ be the directed graph whose vertices are given by the three-dimensional reflexive polytopes, with an edge $P \to Q$ if and only if there exists a projection from X_P to X_Q (that is, if and only if $\mathrm{vert}(P) \subset \mathrm{vert}(Q)$, $P \overset{F}{\sim} Q$, and $k = 1$ in Definition 6.6). Then $\mathcal{G}_F$ has 16 roots and 16 sinks, and these are related via duality. The Reflexive IDs of the roots are*

$$1, \ 2, \ 3, \ 4, \ 5, \ 8, \ 9, \ 10, \ 11, \ 16, \ 18, \ 31, \ 45, \ 89, \ 102, \ and \ 105.$$

The corresponding sinks have Reflexive IDs

$$4312, 4282, 4318, 4284, 4287, 4310, 3314, 4313, 4315, 4299, 4238, 4251, 4303, 4319, 4309, \ and \ 4317.$$

Proof. This is a simple computer calculation using the classification [54] and Table 1. □

Corollary 6.8 is somewhat surprising. Although we know of no reason to expect the reflexive polytopes to be F-connected, nor would we have expected the roots and sinks to be related via duality, we can make a small observation. Let $P, Q \subset N_{\mathbb{Q}}$ be two reflexive polytopes such that there exists a finite sequence $P_0, P_1, \ldots, P_k \subset N_{\mathbb{Q}}$ of reflexive polytopes satisfying conditions (i) and (ii) in Definition 6.1 (that is, P_{i-1} and P_i are obtained via the addition or subtraction of a vertex). Then we say that P and Q are *I-equivalent*. It is well-known that the three-dimensional reflexive polytopes are I-connected (although, once more, this is an experimental rather than theoretical fact). Furthermore, if one constructs a directed graph $\mathcal{G}_I$ in an analogous way to $\mathcal{G}_F$ in Corollary 6.8(ii), one finds the exact same list of 16 roots and 16 sinks (although the two graphs are different, clearly $\mathcal{G}_F$ can be included in $\mathcal{G}_I$ after possibly factoring edges). Here this duality is less mysterious: if P is minimal with respect to the removal of vertices (that is, if $\mathrm{conv}(\mathrm{vert}(P) \setminus \{v\})$ is not a three-dimensional reflexive polytope for any $v \in \mathrm{vert}(P)$) then $Q = P^*$ is maximal with respect to the addition of vertices (that is, $\mathrm{conv}(\mathrm{vert}(Q) \cup \{v\})$ is not a reflexive polytope for any $v \notin Q$).

7. COMPUTING PROJECTIONS

In order to do the associated computations, it is in most cases better to represent the toric degenerations of a smooth Fano variety by reflexive lattice polytopes. Given a toric degeneration, the associated reflexive Newton polytope is computed via the standard methods. However, it is worth noting that given a smooth Fano threefold F, it is sometimes possible to produce several different toric degenerations for F, resulting is different Newton polytopes, which give rise to different basic links. However, it is usually possible to connect these degenerations by a sequence of mutations (see [2]). One can choose to consider the set of degenerations of F either as a whole (in order to concentrate on F itself, using mutations to move between different degenerations) or as a collection of individual degenerations (to concentrate on the projections, avoiding the use of mutations).

Given a reflexive polytope, it is usually harder to determine which smooth Fano threefold it originated from. However, it is now possible due to [2, 20], where all the 3-dimensional reflexive polytopes have been listed and the possibilities for the corresponding smooth Fano threefolds have been given. It is worth noting that some of the polytopes do not correspond to any smooth Fano threefold even on the level of period sequences. In this paper such polytopes are disregarded, and the projections with them as intermediate steps are avoided.

In this representation, it is also possible to compute the basic links (projections) by manipulating the Newton polytopes directly.

We can take $X = \mathbb{P}^3$ and start applying the allowed toric basic links to it. This will give us birational maps between 2868 toric Fano threefolds with canonical Gorenstein singularities (containing representatives of 107 different period sequences and 74 different smooth Fano threefolds). Similarly, we can take X to be any other suitable toric Fano threefold and apply the same process, getting maps between a number of toric Fano varieties. Since there are only finitely many such Fano threefolds, there must be some minimal set of projection roots that allows us to obtain all the other ones in this way. Clearly, this minimal set of roots will contain all the varieties represented by the minimal polytopes (with respect to the removal of vertices). However, it may (depending on what projections are used) include some further toric varieties (for an example, see Remark 8.2).

We can also look at the situation in a different way: we can consider the toric Fano threefolds with canonical Gorenstein singularities primarilly as degenerations of smooth Fano threefolds. From this point of view, the mutations (see §4) give us a second set of basic links, connecting pairs of degenerations of the same smooth Fano threefold. With this in mind, we can repeat the above process, aiming to represent all the smooth Fano threefolds (via their degenerations). One can do this with or without allowing the use of mutations. Since there are only finitely many toric Fano threefolds with canonical Gorenstein singularities, it is possible to complete all the constructions described above.

Given a starting reflexive polytope (a toric Fano threefold with canonical Gorenstein singularities), the program considers all its faces (torus-invariant points) and edges (torus-invariant curves). It selects the relevant ones (according to the rules of the "allowed" projections) and projects from them, obtaining a number of new polytopes (discarding those that turn out not to be reflexive). These polytopes are then added to the processing queue, taking care to avoid duplicates to the polynomials that have been found

previously (two explicitly given three-dimensional polytopes are considered to be the same if one can be mapped to the other by an action of the orthogonal group on the underlying $\mathbb{Z}^3$ lattice). Such a pair of duplicates is merely an indication that there are several projection paths between a pair of varieties.

For example, there is a pair of paths between polytope 232 (corresponding to variety B_3) and polytope 1969 (corresp. to variety X_{2-8}) – such a path can go either through polytope 428 (variety B_2) or through polytope 1599 (variety X_{2-15}). These maps correspond to taking a pair of points (a smooth point and a cDV point) on the degeneration of B_3 and projectiong from them, the intermediate polytope is defined by choosing the order of projections.

Aside from obtaining new polytopes via projections from previously known ones, the program (when set to do so) also builds them by considering the possible antiprojections (i.e. inverses of projections) from the known polytopes. This serves a dual purpose: on the one hand, it makes it easier to explore ways of connecting several different polytopes (or the corresponding smooth Fano threefolds); on the other hand, since the procedures for the projections and the antiprojections have been written independently, this serves as an error-checking technique (if a projection $P_1 \rightsquigarrow P_2$ is found, not finding an antiprojection $P_2 \rightsquigarrow P_1$ of the same type would indicate an error. This is checked automatically to make sure the calculations are accurate).

Once the projection graph obtained this way is built, the program identifies all the obtained polytopes according to their ID numbers from the Graded Ring Database [14]. This also makes it possible to read off other information about the polytopes, including their period sequence (bucket) numbers, which (if any) smooth Fano threefolds they are degenerations of and what mutations can be made from them.

The data gathered this way was fed into a second program for analysis. The program considered projections between reflexive polytopes and calculated the pairs of smooth Fano threefolds that these projections link (via toric degenerations). Having found such links, the program also determined the "root" smooth Fano threefolds and built the corresponding graph. In order to help justify the graph, it also produced examples of each of the links (trying to pick "good" degenerations for each threefold).

8. Projection directed graphs

We are interest in constructing a three-dimensional analogue of Figure 1. As such, we wish to restrict our attention to those reflexive polytopes P whose corresponding Gorenstein Fano variety X_P is a toric degeneration of a smooth Fano threefold X. As a first approximation, it is reasonable to restrict our attention to those P such that $\mathrm{Hilb}(X_P, -K_{X_P}) = \mathrm{Hilb}(X, -K_X)$; this condition on the Hilbert series is satisfied by 4310 of the reflexive polytopes. Even if we only allow projections passing through this subset, the result is F-connected.

The calculations described above have been performed, yielding the following results. Given a toric variety T let us call a projection in an anticanonical embedding from tangent space to invariant smooth point, invariant cDV point, or an invariant smooth line (see Table 1) F-projection.

Theorem 8.1. *Given any smooth Fano threefold X, there exists a Gorenstein toric degeneration of X that can be obtained by a sequence of mutations and F-projections from a toric degeneration of one of 15 smooth Fano threefolds (from now on referred to as the projections roots, see Table 2). The directed (sub)graph connecting all Fano varieties via the projections and mutations is presented in Table 3. Each of toric degenerations we use can be equipped with a toric Landau–Ginzburg model.*

Proof. The list of such paths that minimize the number of projections used form the graph that can be seen in Appendix A. The vertices represent the Fano varieties (one vertex can represent several different degenerations of the same variety), and the arrows represent projections between degenerations of different varieties. The existence of all the arrows is shown in Appendix A, where for each arrow an example of a relevant pair of explicit degenerations is given (according to their polytope ID from the Graded Ring Database). □

Remark 8.2. Not all the projection roots correspond to polytopes that are minimal with respect to the removal of vertices: for example, none of the polytopes corresponding to variety X_{2-33} are minimal. The appearance of such roots depends on the choice of projections used – such roots can be eliminating by allowing the use of additional projection types, like projections from curves of higher degree. In fact, a

Variety	Degree	Eliminated by projections from:
$\mathbb{P}^3$	64	
X_{2-36}	62	
Q^3	54	
X_{3-27}	48	
X_{2-33}	54	quartics
X_{3-29}	50	conics
X_{3-28}	48	conics
X_{4-11}	42	conics
X_{2-28}	40	conics
X_{3-22}	40	conics
X_{5-3}	36	conics
X_{6-1}	30	conics
X_{3-9}	26	conics
X_{7-1}	24	conics
X_{8-1}	18	conics

TABLE 2. Projection roots

degeneration of X_{2-33} can be obtained by projecting $\mathbb{P}^3$ (polytope ID 1, minimal) from a curve of degree 4 (obtaining polytope ID 7). However, as discussed above, such projections do not represent natural geometric operations on Fano varieties, and hence are not being considered.

One can note that the graph above is not connected – in fact, it has 3 connected components. This is due to the choice of the types of projections used in the graph: using only projections from points and lines (as discussed above), the graph splits into the main component and two one-variety components (containing varieties X_{7-1} and X_{8-1}). Variety X_{7-1} (polytope ID 506) can be reached from the main component by a projection of X_{6-1} (polytope ID 357) from a conic, and X_{8-1} (polytope ID 769) can be obtained by projecting X_{7-1} (polytope ID 506) from a conic.

Theorem 8.3. *(i) For any smooth Fano threefold there is its toric degeneration such that all these degenerations are connected by sequences of projections with toric centres which are either cDV points or smooth lines. The directed graph of such projections containing toric degenerations of all smooth Fano threefolds with very ample anticanonical class can be chosen as a union of 15 trees with roots shown in Table 2. The directed (sub)graph connecting all Fano varieties via the projections and mutations is presented in Figure 5. Each of the toric degenerations can be equipped with a toric Landau–Ginzburg model.*

(ii) For any smooth Fano threefold with very ample anticanonical class there is its Gorenstein toric degeneration such that all these degenerations are connected by sequences of projections with toric centres which are either cDV points, smooth lines, or smooth conics. The directed (Sub)graph of such projections can be chosen to have five roots which are: $\mathbb{P}^3$, X_{2-36}, Q^3, X_{3-27}, and X_{2-33}. The directed graph connecting all Fano varieties with very ample anticanonical class via the projections and mutations is presented in Figure 6. Each of the toric degenerations can be equipped with a toric Landau–Ginzburg model.

Proof. The algorithm described in §7 was implemented to find all possible projections between degenerations of smooth Fano threefolds. After that, one can choose a toric degeneration for each of the Fano varieties as described in the statement of the theorem. An example of a choice of toric degenerations that create such graphs can be seen in Table 3. For each variety, it gives the corresponding Minkowski ID, as well as the Reflexive ID of the chosen degenerations (for the graphs in part (i) and (ii) of the theorem). If a variety has a terminal degeneration, its Minkowski ID is written in bold; if the degeneration used is terminal, its Reflexive ID is also made bold.

Note that if the choice of polytopes from part (i) is used in a graph that includes projections with centres in conics, one gets a directed graph with 6 roots (the additional one being X_{3-28}). To rectify that, a different choice of degeneration needed to be made for 4 varieties in part (ii). □

TABLE 3. Degeneration choice

Var.	ID	(i)	(ii)	Var.	ID	(i)	(ii)	Var.	ID	(i)	(ii)
$\mathbb{P}^3$	**1**	**1**		X_{3-27}	**45**	**31**		X_{4-2}	110	668	
X_{2-33}	**2**	**7**		B_5	**46**	**68**		X_{4-1}	**111**	**1530**	
Q^3	**3**	**4**		X_{4-11}	**48**	**85**		X_{3-8}	112	1082	
X_{2-30}	**4**	**23**		X_{3-21}	**49**	**214**		V_{22}	**113**	**1943**	
X_{2-28}	**5**	**69**		X_{4-9}	**54**	**217**		X_{3-6}	117	1501	
X_{2-36}	**6**	**8**		X_{4-8}	**57**	**425**		X_{2-12}	**118**	**2356**	
X_{2-35}	**7**	**6**		X_{2-26}	**58**	175★		X_{2-13}	119	1924	
X_{3-29}	**8**	**27**		X_{5-2}	**64**	**220**		X_{2-11}	120	1701	
X_{2-34}	**10**	**24**		X_{4-7}	**65**	**740**		X_{2-14}	122	2353	
X_{3-30}	**11**	**29**		X_{3-15}	**67**	**420**		V_{18}	124	2703	
X_{3-26}	**12**	**26**		X_{4-5}	**68**	**427**		X_{3-3}	135	2678	
X_{3-22}	**13**	**76**		X_{2-22}	**69**	373★		X_{7-1}	136	506	
X_{3-31}	**14**	**28**		X_{3-13}	**70**	**737**		X_{3-5}	138	1367	
X_{2-31}	**15**	**70**		X_{3-11}	**72**	**732**		X_{2-9}	139	3136	
X_{3-25}	**16**	**74**		X_{2-18}	74	1090		B_2	140	428	
X_{3-23}	**17**	**205**		B_4	75	154★		X_{3-4}	142	2222	
X_{3-19}	**18**	**206**		X_{5-3}	**76**	**219**		V_{16}	143	2482	
X_{2-27}	**19**	**201**		X_{2-23}	**78**	**411**		X_{2-8}	144	1969	
X_{3-14}	**21**	**203**		X_{4-6}	**81**	**426**		X_{2-10}	145	3036	
X_{3-9}	22	374		X_{4-4}	**83**	**741**		V_{14}	147	3283	
X_{2-32}	**24**	**22**		X_{2-21}	**84**	**731**		X_{2-7}	148	3239	
X_{3-28}	**28**	**30**	**81**	X_{3-12}	**85**	723★		X_{2-6}	149	3319	
X_{4-13}	**29**	**84**		X_{2-19}	**86**	**1109**		V_{12}	150	3966	
X_{3-24}	**31**	**78**		X_{2-20}	**87**	**1110**		X_{3-1}	154	3350	
X_{4-12}	**34**	**83**	190★	X_{4-3}	**88**	**735**		X_{8-1}	155	769	
X_{2-29}	**35**	**72**	**204**	X_{3-10}	**99**	**1113**		X_{3-2}	157	2791	
X_{4-10}	**37**	**215**		X_{5-1}	100	673		X_{2-5}	158	3453	
X_{3-20}	**38**	**80**		X_{2-17}	**101**	**1528**		V_{10}	160	4132	
X_{3-17}	**39**	**210**		X_{3-7}	**103**	**1529**		X_{2-4}	161	4031	
X_{3-18}	**41**	**212**	**419**	X_{2-16}	104	1485		V_8	163	4205	
X_{3-16}	**42**	**418**		B_3	106	232		V_6	164	4286	
X_{2-25}	**43**	**410**		X_{6-1}	107	357		V_4	165	4312	
X_{2-24}	**44**	**412**		X_{2-15}	109	1599					

Remark 8.4. These graphs are not the only ones that satisfy the statement of the theorem. For many smooth Fano threefolds, a choice of degeneration needed to be made. In this case, the choice was made according to the following priorities:

  (i) to minimize the number of the graphs' roots;
  (ii) to minimize the number of the connected components of the graphs;
  (iii) where possible, to use a terminal degeneration of the variety.

In the choices above, the numbers of roots and connected components are indeed minimal, and a terminal degeneration was used where available except in 4 cases (5 cases if projections with centres in smooth conics were used), marked by ★, where this would lead to getting additional roots or connected components. The degenerations in question are those for varieties B_4, X_{2-22}, X_{2-26}, X_{3-12} (and X_{4-12} if projections with centres at conics were used).

APPENDIX A. PROJECTION–MINIMIZING GRAPH

Table 4 contains a choice of degenerations (X_1, X_2) for every arrow in the graph of projections discussed in Theorem 8.1. The "From" column gives the starting variety of the projection along with the variety's degree and Minkowski ID. The "To" column lists all the possible destination varieties, and, for each of them, an example of the corresponding pair of degenerations (X_1, X_2) (in terms of their Reflexive ID's).

TABLE 4. Projection graph

Level	Var.	P.S.	Deg.	To:								
0	1.1	1	64	2.35: (	1	6	)					
0	2.36	6	62	2.34: (	8	24	)					
0	1.2	3	54	3.31: (	2	20	)	2.30: (	2	14	)	
0	2.33	2	54	3.30: (	7	29	)	3.26: (	7	26	)	
0	3.29	8	50	3.26: (	27	73	)	3.24: (	27	78	)	
0	3.27	45	48	1.7: (	18	43	)					
0	3.28	28	48	4.12: (	52	181	)	2.29: (	52	106	)	
0	4.11	48	42	3.21: (	62	184	)	4.9: (	191	291	)	2.26: (62 163)
0	2.28	5	40	2.27: (	34	305	)	3.14: (	34	143	)	
0	3.22	13	40	3.19: (	139	270	)	3.17: (	76	210	)	3.15: (139 316)
0	5.3	76	36	4.6: (	195	409	)	2.21: (	114	238	)	
0	6.1	107	30	4.2: (	284	602	)					
0	3.9	22	26	2.18: (	447	1999)						
0	7.1	136	24									
0	8.1	155	18									
1	2.35	7	56	2.32: (	6	13	)					
1	2.34	10	54	2.31: (	5	21	)					
1	3.31	14	52	3.25: (	20	47	)					
1	3.30	11	50	4.13: (	29	60	)					
1	2.30	4	46	3.25: (	14	47	)	3.23: (	23	77	)	
1	3.26	12	46	3.23: (	26	77	)	3.20: (	26	44	)	
1	4.12	34	44	4.10: (	309	638	)	3.18: (	309	639	)	
1	3.24	31	42	4.10: (	169	390	)	3.20: (	169	259	)	
1	1.7	46	40	1.6: (	221	429	)					
1	2.29	35	40	3.20: (	106	259	)	3.18: (	19	63	)	2.25: (106 251)
1	2.27	19	38	3.18: (	71	212	)	3.16: (	71	213	)	2.24: (157 322)
1	3.19	18	38	3.16: (	75	213	)	3.13: (	35	93	)	
1	3.21	49	38	4.8: (	488	1350)		2.23: (	488	765	)	
1	4.9	54	38	4.8: (	291	580	)	4.7: (	67	185	)	2.22: (291 547)
1	3.17	39	36	3.16: (	130	323	)					
1	2.26	58	34	2.22: (	1263	1749)		2.23: (	481	765	)	2.19: (957 1230)
1	3.14	21	32	2.24: (	143	631	)	3.11: (	308	656	)	3.7: (143 263)
1	3.15	67	32	2.22: (	2125	2450)		3.13: (	564	977	)	4.4: (598 1355)
				3.11: (	564	985	)	3.12: (	954	1435)		2.17: (564 931)
1	4.6	81	32	4.4: (	1682	2179)		3.12: (	1682	2110)		4.3: (1682 1976)
				2.17: (	1708	2182)						
1	2.21	84	28	2.19: (	123	338	)	2.20: (	238	1320)		3.10: (123 267)
				2.17: (	1753	2252)		2.12: (	2729	3077)		
1	4.2	110	26	3.8: (	602	1867)		1.17: (	602	930	)	3.6: (1048 1437)
1	2.18	74	24	2.16: (	449	1998)		2.13: (	1441	1824)		

Level	From: Var.	Deg.	P.S.	Projection to							
2	2.32	24	48								
2	2.31	15	46								
2	4.13	29	46								
2	3.25	16	44								
2	3.23	17	42								
2	4.10	37	40	5.2: (180 408)							
2	3.20	38	38	5.2: (44 194)							
2	3.18	41	36	4.5: (63 151)							
2	4.8	57	36								
2	3.16	42	34	4.5: (615 981)							
2	4.7	65	34								
2	1.6	75	32	1.5: (433 742)							
2	2.25	43	32								
2	2.22	69	30								
2	2.23	78	30	2.15: (1165 1599)							
2	2.24	44	30								
2	3.13	70	30	5.1: (93 285)							
2	4.4	83	30								
2	3.11	72	28								
2	3.12	85	28								
2	4.3	88	28								
2	2.19	86	26	2.15: (2066 2397)		2.11: (1392 1701)					
2	2.20	87	26	1.16: (1186 1559)							
2	3.10	99	26	4.1: (267 489)		1.16: (3571 3764)					
2	2.17	101	24	2.9: (2182 2462)							
2	3.7	103	24	2.14: (1374 1721)							
2	3.8	112	24	2.14: (2541 2958)		3.5: (984 1367)					
2	1.17	113	22	2.14: (1141 1659) 1.14: (3772 3891)		3.5: (930 1367)		1.16: (4037 4085)			
2	2.16	104	22	1.16: (2960 3233)		2.11: (1233 1701)					
2	3.6	117	22	2.14: (3123 3921) 3.4: (2166 2394)		1.16: (2166 2494)		3.3: (3123 3445)			
2	2.12	118	20	1.16: (1548 2500) 2.6: (3077 3319)		2.11: (1194 1701)		2.9: (2311 2606)			
2	2.13	119	20	1.16: (3557 4085) 1.15: (1717 2024)		3.3: (1662 2070) 2.9: (2770 3136)		3.4: (2114 2544)			

		From:		Projection to								
Level	Var.	Deg.	P.S.									
3	5.2	64	36									
3	4.5	68	32									
3	5.1	100	28									
3	1.5	106	24	1.4:	(1952	2364)						
3	4.1	111	24									
3	2.15	109	22	2.8:	(1599	1969)						
3	2.14	122	20	2.10:	(3921	4002)						
3	3.5	138	20									
3	1.16	124	18	2.10:	(3361	3617)						
3	2.11	120	18	2.8:	(1701	1969)						
3	3.3	135	18	2.7:	(3445	3592)						
3	3.4	142	18	2.10:	(2394	2746)	3.2:	(2544	2791)			
3	1.15	143	16	2.7:	(3373	3592)	3.2:	(2482	2791)	1.13:	(4050	4119)
3	2.9	139	16									
3	1.14	147	14	1.13:	(3887	4119)	2.5:	(3587	3736)	1.12:	(4171	4200)
3	2.6	149	12	1.12:	(3319	4007)						
4	1.4	140	16									
4	2.10	145	16									
4	2.7	148	14	3.1:	(3102	3329)						
4	2.8	144	14									
4	3.2	157	14									
4	1.13	150	12	2.4:	(3966	4031)						
4	2.5	158	12	2.4:	(3736	4031)						
4	1.12	160	10	1.11:	(3051	3314)						
5	3.1	154	12									
5	2.4	161	10									
5	1.11	163	8									

FIGURE 5. Connecting varieties by projections from points and lines

The Fano threefolds are denoted by their Minkowski ID's, each arrow signify a projection between them. See Table 3, (i) for the explicit choice of degenerations for each of the varieties.
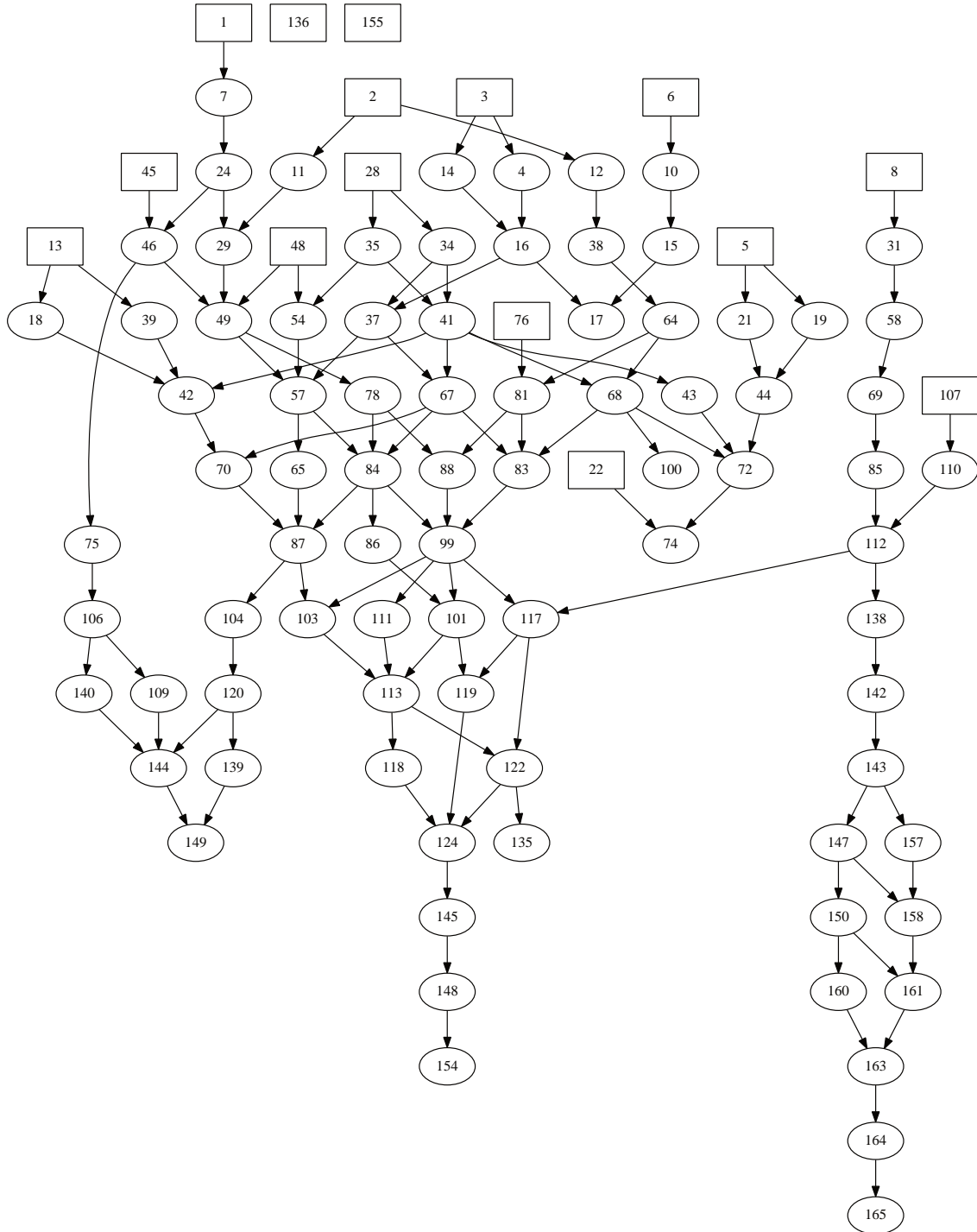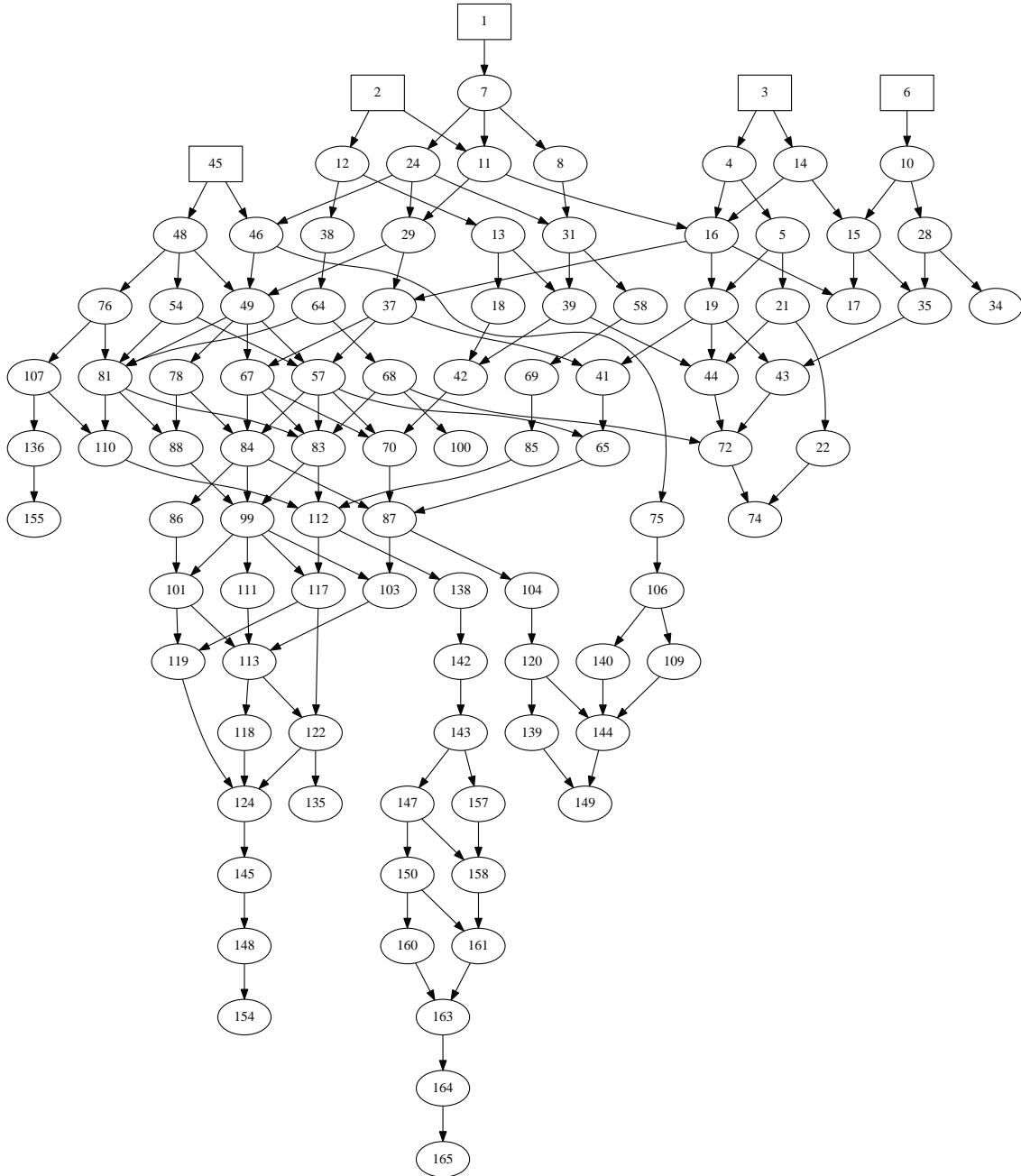
FIGURE 6. Connecting varieties by projections from points, lines, and conics

The Fano threefolds are denoted by their Minkowski ID's, each arrow signify a projection between them. See Table 3, (ii) for the explicit choice of degenerations for each of the varieties.

APPENDIX B. TABLES FOR TORIC DEGENERATIONS

B.1. **Hilbert scheme component dimensions for smooth Fano threefolds.** In the following table, we compute $h^0(V, \mathcal{N}_{V/\mathbb{P}^n})$ for each smooth Fano threefold with very ample anticanonical divisor. Here $V \hookrightarrow \mathbb{P}^n$ denotes the anticanonical embedding. This value equals the dimension of the Hilbert scheme component of $Hilb_V$ corresponding to V.

We record the tangent space dimension $h^0(\mathcal{N})$ of the Hilbert scheme points corresponding to antica-nonically-embedded Gorenstein toric Fano threefolds in [49, Table B.2].

Name	b_2	$b_3/2$	$(-K_V)^3$	$h^0(V, \mathcal{N}_{V/\mathbb{P}^n})$
V_4	1	30	4	69
V_6	1	20	6	69
V_8	1	14	8	75
V_{10}	1	10	10	85
V_{12}	1	7	12	98
V_{14}	1	5	14	114
V_{16}	1	3	16	132
V_{18}	1	2	18	153
V_{22}	1	0	22	201
B_2	1	10	16	139
B_3	1	5	24	234
B_4	1	2	32	363
B_5	1	0	40	525
Q^3	1	0	54	889
$\mathbb{P}^3$	1	0	64	1209
2–4	2	10	10	84
2–5	2	6	12	96
2–6	2	9	12	99
2–7	2	5	14	113
2–8	2	9	14	117
2–9	2	5	16	133
2–10	2	3	16	131
2–11	2	5	18	155
2–12	2	3	20	177
2–13	2	2	20	176
2–14	2	1	20	175
2–15	2	4	22	204
2–16	2	2	22	202
2–17	2	1	24	229
2–18	2	2	24	230
2–19	2	2	26	260
2–20	2	0	26	258

Name	b_2	$b_3/2$	$(-K_V)^3$	$h^0(V, \mathcal{N}_{V/\mathbb{P}^n})$
2–21	2	0	28	290
2–22	2	0	30	324
2–23	2	1	30	325
2–24	2	0	30	324
2–25	2	1	32	361
2–26	2	0	34	398
2–27	2	0	38	480
2–28	2	1	40	525
2–29	2	0	40	524
2–30	2	0	46	668
2–31	2	0	46	668
2–32	2	0	48	720
2–33	2	0	54	888
2–34	2	0	54	888
2–35	2	0	56	948
2–36	2	0	62	1140
3–1	3	8	12	97
3–2	3	3	14	110
3–3	3	3	18	152
3–4	3	2	18	151
3–5	3	0	20	173
3–6	3	1	22	200
3–7	3	1	24	228
3–8	3	0	24	227
3–9	3	3	26	260
3–10	3	0	26	257
3–11	3	1	28	290
3–12	3	0	28	289
3–13	3	0	30	323
3–14	3	1	32	360
3–15	3	0	32	359
3–16	3	0	34	397
3–17	3	0	36	437
3–18	3	0	36	437
3–19	3	0	38	479
3–20	3	0	38	479
3–21	3	0	38	479

Name	b_2	$b_3/2$	$(-K_V)^3$	$h^0(V, \mathcal{N}_{V/\mathbb{P}^n})$
3–22	3	0	40	523
3–23	3	0	42	569
3–24	3	0	42	569
3–25	3	0	44	617
3–26	3	0	46	667
3–27	3	0	48	719
3–28	3	0	48	719
3–29	3	0	50	773
3–30	3	0	50	773
3–31	3	0	52	829
4–1	4	1	24	227
4–2	4	0	26	256
4–3	4	1	28	289
4–4	4	0	30	322
4–5	4	0	32	358
4–6	4	0	32	358
4–7	4	0	34	396
4–8	4	0	36	436
4–9	4	0	38	478
4–10	4	0	40	522
4–11	4	0	42	568
4–12	4	0	44	616
4–13	4	0	46	666
5–1	5	0	28	287
5–2	5	0	36	435
5–3	5	0	36	435
6–1	6	0	30	320
7–1	7	0	24	223
8–1	8	0	18	144

B.2. **Small toric degenerations for Fano threefolds.** Below, we record all degenerations of smooth Fano threefolds to Gorenstein toric Fano varieties with at most terminal singularities [31].

Name	Reflexive IDs
V_{22}	1943
B_4	198
B_5	68
Q^3	4
$\mathbb{P}^3$	1
2–12	2356
2–17	1528
2–19	1109
2–20	1110, 1112

2–31	21, 70				4–1	1530
2–32	22	3–18	212, 419		4–3	735
2–33	7	3–19	75, 206		4–4	741
2–34	5, 24	3–20	80, 208, 211, 416		4–5	427
2–35	6	3–21	214		4–6	426
2–36	8	3–22	76		4–7	424, 740
3–7	1529	3–23	77, 205		4–8	216, 218, 425
3–10	1113	3–24	78, 79, 207		4–9	217
3–11	730, 732	3–25	25, 74		4–10	82, 215
3–12	738, 1114	3–26	26, 73		4–11	85
3–13	421, 737, 739	3–27	31		4–12	83
3–14	203	3–28	30, 81		4–13	84
3–15	420, 422, 423, 736	3–29	27		5–2	220
3–16	213, 417, 418	3–30	29		5–3	219
3–17	209, 210, 415	3–31	28			

B.3. **Select toric degenerations for low degree Fano threefolds.** Below we record degenerations of products of low degree Fano threefolds as in [18]. We only list those degenerations with smooth deformation space.

Name	Reflexive IDs
V_4	4312
V_6	4282, 4284, 4286, 4287, 4297, 4310, 4318
V_8	3314, 4005, 4167, 4194, 4203, 4205, 4217, 4231, 4238, 4244, 4250, 4251, 4267, 4269, 4275, 4280, 4290, 4298, 4299, 4304, 4313, 4314, 4315
V_{10}	3051, 3792, 3903, 3922, 3928, 3965, 4007, 4023, 4032, 4044, 4074, 4118, 4131, 4132, 4135, 4143, 4161, 4168, 4181, 4186, 4190, 4200, 4202, 4214, 4216, 4219, 4225, 4228, 4245, 4252, 4255, 4258, 4261, 4268, 4281, 4291, 4292, 4295, 4301, 4303, 4307, 4311
V_{12}	2756, 3406, 3625, 3626, 3667, 3683, 3702, 3796, 3848, 3853, 3869, 3937, 3938, 3946, 3962, 3966, 3981, 3984, 3985, 3991, 4059, 4060, 4070, 4080, 4102, 4104, 4119, 4122, 4124, 4133, 4134, 4144, 4145, 4179, 4180, 4184, 4218, 4220, 4247, 4254, 4263, 4270, 4272, 4273, 4293, 4294
2–4	4031
2–5	3453, 3736, 3777
2–6	3319, 3349, 3416
3–1	3329, 3350

B.4. **Select toric degenerations for rank one index one Fano threefolds.** Below we record select degenerations of rank one index one Fano threefolds as described in §5.2.

Name	Reflexive IDs
V_{14}	2464, 3214, 3229, 3268, 3298, 3461, 3506, 3535, 3536, 3633, 3670, 3675, 3676, 3712, 3719, 3724, 3794, 3837, 3849, 3852, 3870, 3876, 3887, 3891, 3944, 3968, 3970, 3982, 3986, 3992, 4046, 4066, 4069, 4077, 4078, 4139, 4140, 4171, 4185, 4223, 4226, 4253, 4271
V_{16}	2024, 2498, 2650, 2894, 2899, 2903, 2987, 2990, 3026, 3027, 3034, 3112, 3162, 3210, 3277, 3280, 3294, 3300, 3310, 3373, 3374, 3396, 3464, 3513, 3515, 3519, 3546, 3642, 3648, 3674, 3685, 3700, 3709, 3710, 3711, 3721, 3725, 3742, 3797, 3798, 3816, 3858, 3864, 3872, 3954, 3975, 3990, 3993, 4050, 4063, 4065, 4109, 4121, 4146, 4150, 4172, 4227
V_{18}	1559, 1827, 2116, 2187, 2223, 2289, 2302, 2337, 2480, 2494, 2500, 2520, 2539, 2605, 2646, 2648, 2658, 2683

B.5. Select toric degenerations for product Fano threefolds. Below we record degenerations of products of del Pezzo surfaces with $\mathbb{P}^1$.

Name	Reflexive ID
5–3	219
6–1	357
7–1	510

B.6. Select toric degenerations for complete intersections in toric varieties. We list degenerations constructed as in §5.4.

Name	Reflexive ID	Regular Sequence
B_2	1978	$x_2^2 x_4^2 - x_1 x_3 x_5$
	2364	$x_2^3 x_4 - x_1 x_3 x_5$
	3313	$x_2^4 - x_1 x_3 x_5$
B_3	232	$x_1 x_2 x_4 - x_3 x_5^2$
	742	$x_1 x_2 x_3 - x_5^3$
2–7	3813	$x_1 x_3 x_4 - x_2 x_5 x_6,\ x_6^2 - x_5 x_7$
2–8	1969	$x_1 x_2 x_3 x_4 x_5 - x_6^2$
2–9	2606	$x_2 x_5 - x_4 x_7,\ x_1^2 x_5 - x_2 x_3 x_6$
	3014	$x_2 x_5 - x_4 x_7,\ x_2^2 x_5 - x_1 x_3 x_6$
	3242	$x_2 x_5 - x_4 x_7,\ x_2 x_3 x_5 - x_1^2 x_6$
	3479	$x_2 x_5 - x_4 x_7,\ x_4^2 x_5 - x_1 x_3 x_6$
	3480	$x_2 x_5 - x_4 x_7,\ x_1 x_3 x_5 - x_2^2 x_6$
	1972	$x_5 x_6 - x_7^2,\ x_2 x_3 x_7 - x_4^2$
	2131	$x_2^2 x_3^2 - x_6 x_7,\ x_1^2 x_3^2 - x_4 x_5$
	2447	$x_2 x_3 x_6 - x_7^2,\ x_1 x_3 x_6 - x_4 x_5$
	2746	$x_2 x_3 x_6 - x_7^2,\ x_1^2 x_3^2 - x_4 x_5$
	2808	$x_2 x_3 x_6 - x_7^2,\ x_4 x_5 - x_6 x_7$
	2924	$x_5 x_6 - x_7^2,\ x_1 x_2 x_3^2 - x_4 x_7$
	2928	$x_2 x_3 x_5 - x_6 x_7,\ x_1^2 x_3^2 - x_4 x_5$
	2984	$x_2^2 x_3^2 - x_6 x_7,\ x_1 x_2 x_3^2 - x_4 x_5$
	3019	$x_2 x_3 x_5 - x_6 x_7,\ x_1 x_2 x_3^2 - x_4 x_5$
	3036	$x_4 x_5 - x_6 x_7,\ x_1 x_2 x_3^2 - x_6 x_7$

Name	Reflexive ID	Regular Sequence
2–10		
	3153	$x_2x_3x_6 - x_7^2,\ x_1x_3x_7 - x_4x_5$
	3362	$x_5x_6 - x_7^2,\ x_2^2x_3^2 - x_4x_7$
	3387	$x_5x_6 - x_7^2,\ x_2x_3x_4 - x_7^2$
	3481	$x_2x_3x_5 - x_6x_7,\ x_2^2x_3^2 - x_4x_5$
	3487	$x_2^2x_3^2 - x_6x_7,\ x_4x_5 - x_6x_7$
	3615	$x_2x_3x_6 - x_7^2,\ x_1x_2x_3^2 - x_4x_5$
	3617	$x_5x_6 - x_7^2,\ x_2x_3x_4 - x_6x_7$
	3789	$x_2x_3x_6 - x_7^2,\ x_2x_3x_7 - x_4x_5$
	4002	$x_5x_6 - x_7^2,\ x_2x_3x_4 - x_6^2$
	4021	$x_2x_3x_6 - x_7^2,\ x_2^2x_3^2 - x_4x_5$
2–11	1701	$x_1x_3x_4x_5 - x_2x_6^2$
	2203	$x_1x_3^2x_4^2 - x_2x_5x_6$
	2815	$x_3^3x_4^2 - x_2x_5x_6$
2–12	1497	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_8,\ x_4x_6 - x_3x_7$
	2311	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_8,\ x_4x_5 - x_3x_7$
	2350	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_6,\ x_3x_7 - x_2x_8$
	2356	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_6,\ x_2x_6 - x_3x_7$
	2996	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_8,\ x_2x_5 - x_3x_7$
	3029	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_6,\ x_3x_6 - x_2x_7$
2–13	1254	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_7^2 - x_5x_8$
	1419	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_4x_7 - x_8^2$
	1662	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_7^2 - x_4x_8$
	2236	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_5x_7 - x_8^2$
	2332	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_8,\ x_4x_7 - x_8^2$
	2343	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_4x_7 - x_5x_8$
	2354	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_8,\ x_4x_7 - x_6x_8$
	2448	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_5x_6 - x_7^2$
	2882	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_8,\ x_7^2 - x_4x_8$
	2931	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_6^2 - x_5x_7$
	2940	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_5^2 - x_4x_7$
	2995	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_6^2 - x_4x_7$
	3031	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_5x_6 - x_4x_7$
	3197	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_4x_7 - x_8^2$
	3616	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_4x_6 - x_7^2$
2–15	911	$x_2^2x_4x_5 - x_1x_3x_6$
	1573	$x_3^3x_5 - x_1x_2x_6$
	1599	$x_1x_2x_4x_5 - x_3^2x_6$
	810	$x_3x_6 - x_2x_7,\ x_1x_4x_7 - x_5^2$
	994	$x_3^2x_4 - x_2x_7,\ x_1^2x_4^2 - x_5x_6$
	1016	$x_1x_3x_4 - x_2x_7,\ x_5x_6 - x_7^2$
	1174	$x_3^2x_4 - x_2x_7,\ x_1x_4x_5 - x_6^2$
	1233	$x_1x_3x_4 - x_2x_7,\ x_5^2 - x_6x_7$
	1464	$x_3^2x_4 - x_2x_7,\ x_1x_4x_7 - x_5x_6$
	1485	$x_3x_6 - x_2x_7,\ x_1x_4x_5 - x_6x_7$
	1520	$x_1x_3x_4 - x_2x_7,\ x_3x_4x_7 - x_5x_6$
	1669	$x_3^2x_4 - x_2x_7,\ x_5x_6 - x_7^2$
	1712	$x_3x_6 - x_2x_7,\ x_1x_4x_5 - x_7^2$
	1764	$x_3x_6 - x_2x_7,\ x_1^2x_4^2 - x_5x_7$
	1883	$x_1x_3x_4 - x_2x_7,\ x_3^2x_4^2 - x_5x_6$

Name	Reflexive ID	Regular Sequence
2–16	1904	$x_3^2 x_4 - x_2 x_7,\ x_1 x_3 x_4^2 - x_5 x_6$
	1908	$x_3 x_6 - x_2 x_7,\ x_1 x_2 x_4^2 - x_5 x_7$
	1940	$x_1 x_3 x_4 - x_2 x_7,\ x_2 x_4 x_7 - x_5 x_6$
	1998	$x_3 x_6 - x_2 x_7,\ x_2 x_4 x_7 - x_5^2$
	2259	$x_3^2 x_4 - x_2 x_7,\ x_3 x_4 x_7 - x_5 x_6$
	2264	$x_3^2 x_4 - x_2 x_7,\ x_1 x_2 x_4^2 - x_5 x_6$
	2346	$x_1 x_3 x_4 - x_2 x_7,\ x_2 x_3 x_4^2 - x_5 x_6$
	2446	$x_1 x_3 x_4 - x_2 x_7,\ x_3 x_4 x_5 - x_6^2$
	2546	$x_3 x_6 - x_2 x_7,\ x_2^2 x_4^2 - x_5 x_7$
	2629	$x_3^2 x_4 - x_2 x_7,\ x_2 x_4 x_7 - x_5 x_6$
	2671	$x_1 x_3 x_4 - x_2 x_7,\ x_2^2 x_4^2 - x_5 x_6$
	2809	$x_3 x_6 - x_2 x_7,\ x_2 x_4 x_5 - x_7^2$
	2960	$x_3^2 x_4 - x_2 x_7,\ x_2 x_3 x_4^2 - x_5 x_6$
	3196	$x_3^2 x_4 - x_2 x_7,\ x_2^2 x_4^2 - x_5 x_6$
	3347	$x_3^2 x_4 - x_2 x_7,\ x_3 x_4 x_5 - x_6^2$
2–18	449	$x_1 x_2 x_4 x_5 - x_6^2$
	1033	$x_1 x_3 x_5^2 - x_2 x_4 x_6$
	1250	$x_2^2 x_4^2 - x_1 x_5 x_6$
3–2	2570	$x_1 x_2 x_4^3 x_5^2 - x_3 x_6 x_7$
	2791	$x x_2^2 x_3^3 x_5^2 - x_4 x_6 x_7$
3 − 3	2678	$x_2 x_4 x_5^2 - x_1 x_3 x_6 x_7$
3–4	2005	$x_2 x_3 x_4 x_5 x_6 - x_7^2$
	2314	$x_1 x_2 x_3^2 x_5^2 - x_4 x_6 x_7$
	2544	$x_2^2 x_3^2 x_5^2 - x_4 x_6 x_7$
	2807	$x_4^2 x_6^2 - x_2 x_3 x_5 x_7$
3–5	1326	$x_2 x_4^2 x_6 - x_5 x_8,\ x_1 x_4^2 x_6 - x_3 x_7$
	1367	$x_2 x_4^2 x_6 - x_5 x_8,\ x_3 x_7 - x_4 x_8$
	1820	$x_2 x_4^2 x_6 - x_5 x_8,\ x_1 x_4 x_5 x_6 - x_3 x_7$
	1837	$x_2 x_4^2 x_6 - x_5 x_8,\ x_3 x_7 - x_5 x_8$
	2128	$x_2 x_4^2 x_6 - x_5 x_8,\ x_1 x_5^2 x_6 - x_3 x_7$
	2218	$x_2 x_4^2 x_6 - x_5 x_8,\ x_2 x_4 x_5 x_6 - x_3 x_7$
	2492	$x_2 x_4^2 x_6 - x_5 x_8,\ x_2 x_5^2 x_6 - x_3 x_7$
3–6	1773	$x_2 x_3 x_5 x_7 - x_4^2 x_6$
	1938	$x_1 x_2 x_3^2 x_7 - x_4 x_5 x_6$
	2269	$x_2^2 x_3^2 x_7 - x_4 x_5 x_6$
3–8	1082	$x_1 x_3 x_6^2 - x_4 x_5 x_7$
	1776	$x_1 x_3 x_6 x_7 - x_4 x_5^2$
3–9	289	$x_1^4 x_4^2 - x_5 x_6$
	344	$x_1^3 x_3 x_4^2 - x_5 x_6$
	354	$x_1^2 x_3^2 x_4^2 - x_5 x_6$
	374	$x_1^2 x_2 x_3 x_4^2 - x_5 x_6$
	447	$x_1 x_3 x_4 x_5 - x_6^2$
4 − 2	1081	$x_1 x_4^2 x_5 x_7 - x_3 x_6 x_8$
5 − 1	1083	$x_3 x_5 x_6 x_7 x_8 - x_1 x_2 x_9$

B.7. Further toric degenerations for smooth Fano threefolds. In the following table, we list degenerations of smooth Fano threefolds to Gorenstein toric Fano threefolds obtained from those of §B.2,

B.3, B.4, B.5, B.6 by (possibly repeated) mutations. Those toric varieties corresponding to a smooth point of $\mathcal{H}_V$ appear in the column "Interior Points"; those corresponding to singular points of $\mathcal{H}_V$ appear in the column "Boundary Points".

Name	Interior Points	Boundary Points
V_4	4312	
V_6	4282, 4284, 4286, 4287, 4297, 4310, 4318	
V_8	3314, 4005, 4167, 4194, 4203, 4205, 4217, 4231, 4238, 4244, 4250, 4251, 4267, 4269, 4275, 4280, 4290, 4298, 4299, 4304, 4313, 4314, 4315	
V_{10}	3051, 3792, 3903, 3922, 3928, 3965, 4007, 4023, 4032, 4044, 4074, 4118, 4131, 4132, 4135, 4143, 4161, 4168, 4181, 4186, 4190, 4200, 4202, 4214, 4216, 4219, 4225, 4228, 4245, 4252, 4255, 4258, 4261, 4268, 4281, 4291, 4292, 4295, 4301, 4303, 4307, 4311	3879, 3927, 3964, 4006, 4024, 4042
V_{12}	2756, 3406, 3625, 3626, 3667, 3683, 3702, 3796, 3848, 3853, 3869, 3937, 3938, 3946, 3962, 3966, 3981, 3984, 3985, 3991, 4059, 4060, 4070, 4080, 4102, 4104, 4119, 4122, 4124, 4133, 4134, 4144, 4145, 4179, 4180, 4184, 4218, 4220, 4247, 4254, 4263, 4270, 4272, 4273, 4293, 4294	3452, 3756, 3760, 3795, 3846, 3857, 3875, 3933, 3936, 3967, 3983, 4004, 4027, 4041, 4043, 4075, 4117, 4241, 4249
V_{14}	2464, 3082, 3214, 3219, 3229, 3268, 3283, 3298, 3461, 3471, 3506, 3535, 3536, 3587, 3633, 3637, 3640, 3670, 3675, 3676, 3712, 3719, 3724, 3794, 3837, 3849, 3852, 3870, 3876, 3887, 3891, 3932, 3944, 3968, 3970, 3982, 3986, 3992, 4046, 4066, 4069, 4077, 4078, 4139, 4140, 4171, 4185, 4198, 4223, 4226, 4253, 4271	2472, 2553, 2725, 2966, 3056, 3202, 3266, 3397, 3399, 3509, 3511, 3530, 3537, 3543, 3544, 3559, 3574, 3578, 3624, 3636, 3666, 3698, 3704, 3718, 3766, 3799, 3800, 3824, 3825, 3832, 3847, 3855, 3867, 3871, 3892, 3896, 3900, 3925, 3929, 3935, 3950, 3963, 3973, 3977, 4026, 4049, 4057, 4062, 4079, 4126, 4136, 4138, 4187, 4195, 4196, 4201, 4207, 4209, 4221

Name	Interior Points	Boundary Points
V_{16}	2024, 2291, 2437, 2482, 2498, 2650, 2894, 2899, 2903, 2987, 2990, 3026, 3027, 3034, 3112, 3162, 3210, 3277, 3280, 3294, 3300, 3310, 3373, 3374, 3396, 3464, 3513, 3515, 3519, 3546, 3642, 3648, 3674, 3685, 3700, 3709, 3710, 3711, 3721, 3725, 3742, 3797, 3798, 3816, 3858, 3864, 3872, 3954, 3975, 3990, 3993, 4050, 4063, 4065, 4109, 4121, 4146, 4150, 4172, 4227	2093, 2414, 2567, 2724, 2773, 2888, 2891, 2896, 2901, 2967, 2969, 2988, 3008, 3012, 3013, 3032, 3129, 3131, 3132, 3209, 3227, 3265, 3312, 3333, 3336, 3339, 3352, 3375, 3376, 3401, 3404, 3409, 3450, 3455, 3462, 3477, 3507, 3512, 3518, 3529, 3531, 3532, 3533, 3538, 3542, 3664, 3672, 3680, 3686, 3687, 3703, 3720, 3759, 3761, 3802, 3803, 3826, 3828, 3830, 3850, 3851, 3854, 3865, 3873, 3880, 3894, 3898, 3904, 3914, 3934, 3939, 3943, 3972, 3974, 3989, 4030, 4064, 4072, 4125, 4137, 4147, 4152, 4153, 4164, 4165, 4175, 4264, 4302
V_{18}	1559, 1827, 2116, 2187, 2223, 2289, 2302, 2337, 2480, 2494, 2500, 2520, 2539, 2605, 2646, 2648, 2658, 2683, 2684, 2699, 2701, 2703, 2832, 2840, 2858, 2902, 2911, 2982, 2985, 2989, 3015, 3023, 3025, 3035, 3061, 3099, 3160, 3180, 3221, 3230, 3233, 3235, 3269, 3274, 3287, 3290, 3291, 3297, 3299, 3305, 3307, 3361, 3415, 3457, 3470, 3486, 3496, 3514, 3516, 3522, 3526, 3541, 3545, 3596, 3603, 3605, 3681, 3682, 3692, 3699, 3714, 3717, 3764, 3823, 3831, 3859, 3862, 3866, 3912, 3940, 3951, 3958, 3959, 3988, 4068, 4073, 4085, 4114, 4123, 4174, 4178, 4265	1616, 1695, 1792, 2044, 2049, 2180, 2188, 2189, 2271, 2453, 2463, 2469, 2565, 2571, 2578, 2595, 2633, 2680, 2691, 2763, 2816, 2847, 2885, 2898, 2963, 2980, 3011, 3020, 3024, 3037, 3038, 3041, 3042, 3055, 3081, 3086, 3096, 3142, 3144, 3155, 3161, 3203, 3226, 3237, 3278, 3279, 3281, 3293, 3308, 3389, 3394, 3395, 3405, 3423, 3456, 3463, 3466, 3508, 3517, 3520, 3534, 3558, 3586, 3588, 3627, 3635, 3671, 3678, 3693, 3707, 3716, 3722, 3723, 3757, 3767, 3768, 3769, 3770, 3804, 3812, 3829, 3834, 3899, 3930, 3942, 3969, 3971, 3987, 4096, 4106, 4110, 4142, 4242

Name	Interior Points	Boundary Points
V_{22}	839, 842, 843, 930, 1022, 1028, 1141, 1428, 1482, 1484, 1700, 1722, 1816, 1855, 1886, 1887, 1889, 1898, 1899, 1901, 1929, 1930, 1937, 1942, 1943, 2115, 2212, 2214, 2235, 2248, 2309, 2495, 2503, 2523, 2585, 2592, 2655, 2657, 2661, 2662, 2664, 2665, 2694, 2696, 2702, 2761, 2842, 2852, 2934, 2937, 2953, 2999, 3004, 3030, 3094, 3247, 3248, 3249, 3306, 3366, 3431, 3483, 3488, 3494, 3499, 3527, 3589, 3695, 3772, 3843, 3960, 4037, 4054	1143, 1272, 1313, 1604, 1716, 1786, 1880, 2089, 2121, 2181, 2228, 2229, 2273, 2274, 2284, 2288, 2330, 2335, 2349, 2351, 2368, 2373, 2400, 2459, 2467, 2597, 2600, 2618, 2632, 2643, 2692, 2697, 2772, 2822, 2826, 2844, 2906, 2914, 2919, 2920, 2933, 2936, 2974, 2992, 3016, 3017, 3018, 3022, 3078, 3093, 3128, 3149, 3157, 3206, 3222, 3225, 3236, 3238, 3240, 3250, 3255, 3270, 3272, 3284, 3285, 3302, 3332, 3354, 3363, 3365, 3420, 3432, 3459, 3465, 3493, 3500, 3630, 3638, 3643, 3647, 3656, 3688, 3715, 3805, 3841, 3877, 3883, 3888, 3910, 4052, 4055, 4129, 4155
B_2	428, 1978, 2078, 2364, 3313, 3316, 3317	
B_3	87, 232, 459, 742, 773, 906, 1128, 1583, 1952, 1953, 2006, 2712, 3040, 3726	
B_4	9, 92, 120, 154, 198, 234, 429, 433, 434, 437, 472, 524, 607, 825, 1138, 1558, 1596, 2709	
B_5	43, 68, 221, 245, 246, 296	444, 460, 476, 514, 1123, 1152, 1159, 1162, 2362
Q^3	2, 4, 95	746
$\mathbb{P}^3$	1, 10, 11	
2–4	4031	4024, 4056
2–5	3453, 3736, 3777	3731, 3735, 3762
2–6	3319, 3349, 3416	3756, 3790, 3846, 3857, 3875, 3933, 4004, 4043, 4229, 4236, 4241, 4249
2–7	3102, 3133, 3215, 3239, 3484, 3592, 3641, 3813, 4101	2366, 2401, 3130, 3202, 3266, 3448, 3530, 3544, 3559, 3574, 3624, 3629, 3636, 3666, 3698, 3701, 3704, 3718, 3799, 3800, 3824, 3832, 3847, 3855, 3871, 3900, 3925, 3929, 3931, 3935, 3963, 3973, 3977, 4049, 4062, 4098, 4111, 4126, 4127, 4136, 4192, 4201, 4207, 4221
2–8	1969	3137, 3263, 3351, 3996, 4009

Name	Interior Points	Boundary Points
2–9	2462, 2606, 2922, 3014, 3100, 3136, 3217, 3242, 3479, 3480, 3782	1560, 2097, 2197, 2567, 2635, 2830, 2831, 2838, 2889, 2907, 2965, 2968, 2988, 3013, 3131, 3265, 3275, 3296, 3312, 3333, 3404, 3413, 3450, 3451, 3455, 3507, 3533, 3538, 3579, 3583, 3621, 3623, 3631, 3680, 3686, 3687, 3720, 3801, 3809, 3826, 3830, 3865, 3915, 3934, 3943, 3989, 4030, 4033, 4067, 4125, 4137, 4151, 4208
2–10	1972, 2131, 2447, 2746, 2808, 2924, 2928, 2984, 3019, 3036, 3153, 3362, 3387, 3481, 3487, 3615, 3617, 3789, 4002, 4021	1954, 2093, 2367, 2410, 2414, 2773, 2897, 2967, 2969, 3032, 3129, 3209, 3275, 3339, 3357, 3375, 3376, 3409, 3473, 3477, 3532, 3542, 3601, 3614, 3645, 3703, 3783, 3802, 3803, 3854, 3873, 3914, 3934, 3939, 3972, 3998, 4016, 4072, 4152, 4164, 4165, 4175, 4264, 4302
2–11	1701, 2203, 2815	1212, 2191, 2327, 2561, 2679, 2900, 2910, 2963, 3101, 3148, 3205, 3224, 3394, 3425, 3671, 3774, 3942
2–12	1194, 1197, 1281, 1284, 1497, 1548, 1684, 1885, 2311, 2350, 2356, 2540, 2580, 2743, 2996, 3029, 3077	858, 1181, 1348, 1546, 1552, 1787, 1888, 2011, 2082, 2092, 2225, 2278, 2279, 2300, 2328, 2412, 2549, 2555, 2644, 2654, 2682, 2689, 2698, 2726, 2748, 2749, 2762, 2820, 2890, 2909, 2975, 2976, 3010, 3090, 3134, 3154, 3201, 3271, 3303, 3403, 3495, 3501, 3525, 3540, 3552, 3553, 3554, 3669, 3705, 3758, 3861, 3893, 3980, 4010, 4177, 4188

Name	Interior Points	Boundary Points
2–13	1254, 1393, 1415, 1419, 1430, 1662, 1680, 1717, 1757, 1824, 1834, 1874, 1884, 1924, 1932, 2113, 2114, 2213, 2236, 2253, 2295, 2304, 2332, 2333, 2341, 2343, 2347, 2354, 2434, 2448, 2489, 2564, 2604, 2636, 2642, 2649, 2667, 2670, 2672, 2770, 2855, 2859, 2882, 2893, 2931, 2938, 2939, 2940, 2954, 2994, 2995, 3002, 3031, 3158, 3164, 3197, 3256, 3444, 3485, 3557, 3616, 3649, 3654	1136, 1180, 1584, 1694, 1821, 2082, 2092, 2152, 2221, 2278, 2290, 2292, 2297, 2300, 2336, 2411, 2412, 2478, 2483, 2484, 2485, 2519, 2549, 2569, 2645, 2653, 2656, 2681, 2688, 2690, 2698, 2700, 2704, 2721, 2744, 2760, 2762, 2774, 2781, 2821, 2886, 2904, 2916, 2932, 2955, 2972, 2975, 2977, 2979, 2981, 2986, 3010, 3021, 3033, 3090, 3092, 3134, 3135, 3139, 3159, 3176, 3201, 3207, 3208, 3251, 3271, 3286, 3288, 3289, 3295, 3301, 3303, 3311, 3325, 3337, 3353, 3368, 3414, 3422, 3454, 3458, 3468, 3469, 3495, 3501, 3510, 3521, 3524, 3528, 3539, 3540, 3552, 3591, 3600, 3604, 3634, 3650, 3658, 3662, 3669, 3673, 3677, 3690, 3706, 3773, 3807, 3810, 3821, 3838, 3842, 3861, 3893, 3952, 3978, 4011, 4035, 4051, 4108, 4113, 4128, 4212
2–14	1193, 1322, 1416, 1659, 1721, 1835, 1878, 2151, 2233, 2234, 2263, 2312, 2342, 2353, 2524, 2598, 2626, 2693, 2930, 2958, 2978, 3028, 3124, 3252, 3261, 3273, 3388, 3417, 3503, 3663, 3817, 3921	1282, 1323, 1790, 1794, 1821, 2183, 2221, 2292, 2297, 2336, 2484, 2485, 2593, 2631, 2645, 2656, 2681, 2690, 2700, 2704, 2774, 2829, 2904, 2912, 2932, 2972, 2976, 2977, 2979, 2981, 2986, 3021, 3033, 3159, 3174, 3176, 3207, 3208, 3211, 3251, 3253, 3286, 3288, 3289, 3295, 3301, 3311, 3421, 3422, 3454, 3468, 3472, 3482, 3510, 3521, 3528, 3639, 3650, 3658, 3662, 3673, 3677, 3694, 3705, 3706, 3708, 3821, 3838, 3842, 3941, 3957, 3978, 4035, 4051, 4108, 4212
2–15	229, 837, 911, 1240, 1394, 1573, 1599, 1617, 2397	475, 1279, 1386, 1698, 1871, 2083, 2272, 2557, 2560, 2771, 2780, 2908, 3128, 3398, 3597, 4008
2–16	810, 994, 1016, 1174, 1233, 1464, 1485, 1520, 1669, 1712, 1764, 1883, 1904, 1908, 1940, 1998, 2259, 2264, 2346, 2446, 2546, 2629, 2671, 2809, 2960, 3196, 3347	782, 1135, 1272, 1420, 1427, 1603, 1785, 1818, 1832, 1907, 1926, 2219, 2273, 2274, 2275, 2284, 2313, 2331, 2349, 2373, 2374, 2375, 2419, 2426, 2459, 2486, 2590, 2647, 2686, 2853, 2892, 2914, 2918, 2926, 2974, 2992, 3017, 3204, 3225, 3234, 3245, 3270, 3334, 3363, 3365, 3426, 3594, 3630, 3907, 3910, 4155

Name	Interior Points	Boundary Points
2–17	527, 666, 836, 931, 942, 989, 1037, 1097, 1301, 1315, 1406, 1455, 1473, 1483, 1496, 1499, 1516, 1521, 1523, 1528, 1746, 1778, 1808, 1836, 1856, 1868, 1869, 1939, 2048, 2140, 2146, 2164, 2182, 2202, 2220, 2251, 2252, 2286, 2303, 2325, 2339, 2344, 2490, 2497, 2517, 2628, 2652, 2873, 2879, 2881, 2915, 3120, 3165, 3212, 3438, 3909	941, 1294, 1303, 1304, 1424, 1426, 1488, 1710, 1813, 1815, 1819, 1826, 1892, 1906, 1927, 2025, 2034, 2036, 2037, 2099, 2216, 2217, 2247, 2277, 2281, 2294, 2316, 2318, 2319, 2345, 2352, 2460, 2501, 2506, 2558, 2563, 2587, 2589, 2596, 2619, 2677, 2685, 2687, 2764, 2827, 2841, 2843, 2856, 2895, 2925, 2946, 2950, 2951, 2993, 2998, 3098, 3146, 3194, 3223, 3231, 3246, 3260, 3292, 3304, 3369, 3435, 3490, 3644, 3691, 3741, 3780, 3840, 3905, 3913, 3956
2–18	449, 451, 628, 702, 808, 1033, 1073, 1090, 1250, 1373, 1441, 1466, 1999	833, 909, 1288, 1478, 1779, 1783, 1876, 1923, 1955, 1987, 1988, 2020, 2085, 2111, 2112, 2280, 2433, 2511, 2574, 2641, 2828, 2857, 3095, 3097, 3108, 3367, 3743, 3779, 4100
2–19	338, 568, 572, 691, 882, 993, 1014, 1024, 1109, 1230, 1312, 1369, 1392, 1490, 1762, 1796, 2066, 2173, 2445	928, 934, 1306, 1345, 1414, 1476, 1590, 1740, 1782, 1817, 1875, 1877, 2104, 2109, 2141, 2195, 2475, 2515, 2562, 2651, 2867, 3145, 3169, 3358
2–20	619, 655, 701, 921, 1023, 1030, 1064, 1089, 1098, 1099, 1110, 1112, 1186, 1320, 1364, 1409, 1423, 1438, 1453, 1458, 1470, 1479, 1502, 1511, 1513, 1525, 1705, 1724, 1755, 1807, 1811, 1891, 1910, 1917, 2139, 2142, 2178, 2209, 2238, 2261, 2267, 2308, 2323, 2496, 2507, 2543, 2586, 2614, 2624, 2630, 3125, 3183, 3199	922, 940, 1018, 1298, 1310, 1487, 1495, 1634, 1703, 1795, 1800, 1804, 1831, 1872, 1881, 1895, 1896, 1897, 1905, 1928, 1936, 1941, 2122, 2133, 2215, 2227, 2276, 2306, 2421, 2435, 2518, 2581, 2582, 2584, 2588, 2612, 2616, 2651, 2668, 2673, 2674, 2676, 2695, 2846, 2850, 2851, 2854, 2869, 2913, 2943, 2945, 2957, 3005, 3103, 3106, 3145, 3173, 3185, 3188, 3259, 3377, 3491, 3492, 3502, 3590, 3653, 3655, 3661, 3776, 3822, 3885, 4040
2–21	123, 238, 295, 479, 626, 700, 703, 731, 733, 910, 918, 964, 1095, 1096, 1102, 1103, 1111, 1184, 1188, 1228, 1361, 1375, 1436, 1446, 1510, 1753, 1915, 1920, 2039, 2067, 2165, 2254, 2729	464, 513, 829, 1120, 1133, 1134, 1147, 1150, 1190, 1219, 1389, 1400, 1405, 1518, 1654, 1706, 1723, 1847, 1873, 1893, 1894, 1935, 1985, 2087, 2106, 2144, 2147, 2194, 2245, 2320, 2334, 2370, 2415, 2432, 2474, 2491, 2599, 2675, 2730, 2784, 2785, 2868, 2949, 2956, 3003, 3006, 3104, 3152, 3181, 3327, 3340, 3378, 3429, 3659, 3745, 3747, 3886

Name	Interior Points	Boundary Points
2–22	373, 389, 414, 547, 574, 662, 697, 714, 729, 870, 955, 1046, 1054, 1065, 1094, 1101, 1372, 1440, 1447, 1672, 1749, 1857, 2172, 2450	233, 785, 831, 886, 915, 926, 938, 1087, 1187, 1270, 1291, 1388, 1401, 1493, 1601, 1655, 1670, 1696, 1802, 1812, 1852, 1853, 1854, 1934, 2030, 2031, 2032, 2053, 2136, 2158, 2317, 2430, 2529, 2611, 2878, 3189, 3788
2–23	40, 266, 300, 304, 372, 411, 560, 690, 765, 806, 807, 933, 1165, 1229, 1581	521, 789, 857, 917, 927, 1036, 1341, 1346, 1413, 1475, 1621, 1625, 1781, 1849, 2057, 2123, 2162, 2510, 2572, 2782, 3070, 3372
2–24	322, 368, 412, 631, 642, 706, 973	618, 1015, 1035, 1085, 1086, 1206, 1296, 1343, 1410, 1411, 1492, 1814, 1846, 2161, 2239, 2242, 2243, 2505, 2531, 2848, 3191
2–25	108, 176, 199, 251, 384, 410, 483, 517, 549, 565, 627, 652, 685, 689, 879, 1163, 1170	461, 520, 830, 916, 1027, 1285, 1339, 1432, 1474, 1626, 1632, 1645, 2054, 2797
2–26	163, 175, 202, 387, 388, 413, 481, 554, 646, 711, 734, 957, 996, 1076, 1263, 1434, 1758	542, 616, 855, 913, 1031, 1034, 1336, 1387, 1407, 1570, 1643, 1649, 1738, 2138, 2231, 2526, 2740, 2804
2–27	71, 157, 165, 200, 201, 305, 321, 328, 375, 385, 814, 899	948, 1334, 2063
2–28	34, 55, 69	537, 1567
2–29	19, 56, 72, 106, 131, 171, 204, 253, 307, 367, 381, 503	1146, 2389
2–30	14, 23, 37, 160, 225, 273	241, 1125
2–31	21, 46, 70	535
2–32	13, 22	90, 104, 122, 156, 249
2–33	7, 50, 140	
2–34	5, 24	
2–35	6, 118	33, 42
2–36	8	
3–1	3329, 3350	3795, 3846, 3875, 3967, 4027, 4236, 4241, 4249
3–2	2570, 2791	

Name	Interior Points	Boundary Points
3–3	1307, 1726, 1805, 1833, 2070, 2120, 2299, 2594, 2602, 2638, 2678, 2941, 3001, 3166, 3178, 3445	1616, 1692, 2003, 2108, 2270, 2301, 2565, 2566, 2571, 2579, 2595, 2680, 2691, 2763, 2964, 2980, 3037, 3038, 3141, 3144, 3161, 3205, 3226, 3237, 3267, 3276, 3278, 3281, 3293, 3308, 3423, 3456, 3460, 3467, 3517, 3520, 3534, 3646, 3678, 3684, 3716, 3722, 3768, 3769, 3829, 3834, 3835, 3839, 3948, 3969, 3987, 4053, 4142
3–4	1619, 1725, 1823, 2005, 2222, 2224, 2298, 2314, 2394, 2493, 2544, 2603, 2640, 2807, 2927, 2942, 3179, 3348	1692, 1975, 2003, 2270, 2377, 2525, 2566, 2579, 2803, 2898, 2964, 3011, 3141, 3267, 3342, 3460, 3466, 3467, 3508, 3588, 3608, 3627, 3684, 3768, 3770, 3835, 3930, 3948, 4096, 4097, 4110, 4242
3–5	1326, 1367, 1820, 1837, 2128, 2218, 2492	2183, 2631, 2829, 2912, 3211, 3472, 3639
3–6	932, 997, 1318, 1357, 1429, 1437, 1501, 1720, 1773, 1829, 1861, 1900, 1911, 1931, 1938, 2166, 2167, 2269, 2296, 2321, 2326, 2479, 2542, 2620, 2622, 2625, 2627, 2929, 3123, 3195, 3241, 3424	1313, 1604, 1677, 1788, 1880, 1909, 2121, 2171, 2196, 2228, 2229, 2330, 2335, 2351, 2466, 2559, 2590, 2597, 2600, 2618, 2686, 2692, 2905, 2917, 2933, 2936, 3016, 3018, 3022, 3091, 3147, 3157, 3204, 3234, 3238, 3240, 3250, 3255, 3272, 3370, 3381, 3426, 3432, 3459, 3500, 3656, 3657, 3715, 3841, 3907, 3918, 4055, 4129
3–7	263, 630, 959, 1025, 1047, 1055, 1075, 1237, 1247, 1321, 1374, 1417, 1448, 1462, 1522, 1526, 1529, 1912, 1919, 1922, 2021, 2069, 2226, 2256, 2262, 2310, 2340, 2512, 2959	767, 1236, 1303, 1395, 1461, 1660, 1710, 1718, 1772, 1813, 1815, 1879, 1892, 1925, 1933, 1971, 2028, 2034, 2036, 2085, 2119, 2148, 2153, 2207, 2210, 2211, 2216, 2217, 2258, 2265, 2277, 2280, 2293, 2315, 2329, 2338, 2345, 2348, 2352, 2355, 2444, 2488, 2506, 2563, 2589, 2615, 2621, 2639, 2660, 2677, 2687, 2734, 2828, 2841, 2843, 2946, 2961, 2971, 2983, 2991, 2993, 2997, 2998, 3007, 3050, 3076, 3095, 3107, 3108, 3156, 3171, 3172, 3175, 3220, 3223, 3244, 3257, 3304, 3430, 3435, 3436, 3498, 3606, 3691, 3743, 3771, 3779, 3814, 3818, 4036
3–8	984, 1049, 1082, 1456, 1505, 1506, 1776, 1866, 1867, 1913, 2257, 2541	1395, 1425, 1772, 1830, 1925, 1933, 2105, 2119, 2149, 2150, 2207, 2265, 2315, 2583, 2587, 2621, 2639, 2660, 2685, 2845, 2997, 3171, 3186, 3244, 3246, 3369, 3430, 3490, 3956
3–9	289, 344, 354, 374, 447	1385, 1535, 1582, 2720, 2727, 3556, 3997

Name	Interior Points	Boundary Points
3–10	267, 336, 342, 632, 821, 890, 969, 976, 983, 1050, 1078, 1100, 1113, 1266, 1463, 1469, 1500, 1512, 1515, 1527, 1577, 1674, 1678, 1745, 1774, 1862, 1882, 1921, 2169, 2266, 2324, 2395, 2452, 2545, 2591, 2811, 2884, 3571	835, 922, 1298, 1486, 1542, 1555, 1608, 1634, 1771, 1795, 1804, 1881, 1895, 1896, 1897, 1902, 1903, 1928, 1936, 1941, 2088, 2122, 2170, 2206, 2215, 2420, 2435, 2552, 2568, 2584, 2588, 2616, 2659, 2668, 2673, 2674, 2676, 2695, 2767, 2776, 2851, 2869, 2944, 2945, 2957, 3005, 3109, 3150, 3169, 3173, 3185, 3259, 3491, 3492, 3497, 3502, 3561, 3562, 3590, 3661, 3787, 3822, 4040
3–11	401, 653, 656, 657, 724, 730, 732, 980, 985, 1005, 1009, 1079, 1088, 1452, 1465, 1489, 1754, 1761, 1769	567, 880, 925, 1069, 1295, 1297, 1391, 1402, 1404, 1459, 1477, 1480, 1517, 1519, 1602, 1624, 1675, 1733, 1752, 1803, 1848, 1850, 1865, 1918, 2143, 2168, 2193, 2230, 2285, 2491, 2513, 2536, 2547, 2623, 2877, 2947, 3170, 3380
3–12	592, 654, 665, 704, 723, 738, 1052, 1070, 1071, 1104, 1107, 1114, 1381, 1435, 1442, 1457, 1472, 1514, 1770, 1806, 1870, 1916, 2110, 2255, 2260, 2514, 2814	829, 925, 1017, 1074, 1147, 1405, 1421, 1422, 1477, 1480, 1481, 1494, 1517, 1706, 1714, 1723, 1733, 1756, 1799, 1801, 1825, 1850, 1865, 1893, 1894, 1935, 2029, 2118, 2144, 2147, 2193, 2208, 2230, 2244, 2283, 2285, 2305, 2320, 2522, 2532, 2576, 2599, 2607, 2609, 2623, 2675, 2766, 2784, 2785, 2786, 2836, 2861, 2866, 2868, 2870, 2874, 2877, 2947, 2948, 3006, 3170, 3232, 3384, 3429, 3584, 3607, 3747
3–13	93, 129, 370, 421, 556, 636, 737, 739, 977, 982, 1063, 1105, 1765	749, 757, 777, 802, 831, 937, 974, 1043, 1085, 1235, 1296, 1396, 1498, 1508, 1524, 1578, 1630, 1673, 1767, 1768, 1812, 1852, 1859, 1914, 1934, 1951, 2031, 2176, 2192, 2241, 2322, 2505, 2530, 2798, 2812, 2860, 3039, 3177, 3441
3–14	143, 186, 203, 308	522, 686, 881, 949, 1013, 1201, 1340, 1342, 1719, 2156, 2416, 2439
3–15	182, 316, 371, 402, 420, 422, 423, 561, 564, 598, 677, 698, 699, 712, 713, 717, 719, 736, 954, 1061, 1066, 1068, 1077, 1108, 1259, 1356, 1378, 1382, 1439, 1744, 1759, 1843, 2125	924, 1032, 1057, 1091, 1249, 1289, 1408, 1433, 1444, 1468, 1491, 1507, 1509, 1627, 1732, 1766, 1809, 1810, 1838, 1844, 1890, 2137, 2159, 2232, 2240, 2449, 2516, 2527, 2601, 2608, 2610, 2613, 2775, 2802, 2805, 2865, 2876, 3116, 3187, 3437, 3439, 3609, 3751

Name	Interior Points	Boundary Points
3–16	213, 323, 369, 392, 417, 418, 615, 640, 648, 708, 716, 783, 950, 1000, 1058, 1316, 1376, 1742, 2062	478, 573, 624, 643, 649, 696, 972, 1042, 1062, 1092, 1093, 1338, 1358, 1359, 1380, 1443, 1451, 1664, 1734, 1840, 1863, 2175, 2177, 2438, 2521, 2528, 2747, 2790, 2875, 3382
3–17	130, 161, 209, 210, 405, 415, 587	297, 470, 692, 707, 794, 828, 832, 853, 920, 946, 1056, 1222, 1403, 1737, 1990, 2038, 2064, 2806
3–18	63, 212, 326, 341, 383, 393, 419, 597, 639, 709, 970, 1053, 1245	552, 613, 641, 710, 1029, 1332, 1344, 1736, 1839, 2043, 2155, 2440
3–19	35, 57, 75, 164, 206, 270, 283, 339	303, 364, 471, 485, 536, 695, 967, 1041, 1258, 1532, 1566, 1702, 2391
3–20	44, 80, 174, 208, 211, 259, 404, 416, 667, 715, 888	255, 313, 548, 693, 818, 998, 1040, 1164, 1349, 1431, 1566, 1641, 2004, 2793
3–21	173, 184, 214, 382, 488, 558, 897	612, 647, 847, 947, 951, 1026, 1278, 1319, 1562, 1640, 2061, 2127, 2135, 3111
3–22	64, 76, 139	363, 611, 635, 943, 1210, 1743, 2065
3–23	77, 178, 205	127, 365, 585, 694, 850, 876
3–24	78, 79, 169, 207, 379	315, 377, 538, 584, 634, 919, 995, 1242
3–25	25, 47, 74, 136, 187	366, 462, 469, 486, 533, 589, 1563
3–26	26, 73	167, 306, 376, 621
3–27	18, 31, 133	104, 122, 156
3–28	30, 52, 81, 166	
3–29	27	132, 177
3–30	29	168, 276, 314
3–31	20, 28	240, 1118
4–1	489, 495, 578, 610, 1503, 1530, 1775, 2201, 2810	1191, 1236, 1276, 1461, 1660, 1987, 1991, 2023, 2041, 2153, 2211, 2258, 2338, 2348, 2355, 2619, 2734, 2880, 3007, 3257, 3490, 3814
4–2	602, 668, 1048, 1081, 1365, 1454, 1504, 1864	885, 936, 1486, 1574, 1663, 1668, 1771, 1800, 1902, 2206, 2306, 2537, 2581, 2659, 2944, 3106
4–3	109, 153, 603, 664, 684, 728, 735, 822, 1179, 1268, 1269, 1976	464, 513, 1120, 1133, 1134, 1149, 1172, 1213, 1389, 1400, 1460, 1518, 1654, 1658, 2087, 2132, 2396, 2415, 3340
4–4	350, 398, 622, 680, 722, 726, 741, 901, 1039, 1080, 1084, 1106, 1267, 1355, 1449, 1471, 1858, 1860, 2075, 2174, 2179	659, 802, 974, 1043, 1072, 1231, 1235, 1343, 1353, 1498, 1508, 1524, 1767, 1768, 1797, 1846, 1859, 1914, 2158, 2161, 2176, 2204, 2241, 2317, 2322, 2451, 2782, 2798, 2848, 2878, 2883, 3073, 3177, 3189, 3788

Name	Interior Points	Boundary Points
4–5	151, 196, 329, 427, 500, 591, 683, 720, 727, 820, 981, 1576	872, 924, 961, 988, 1044, 1249, 1377, 1450, 1491, 1509, 1627, 1648, 1766, 1838, 2159, 2449, 2794, 2797, 2876, 3564, 3565
4–6	406, 409, 426, 651, 661, 681, 721, 904, 1002, 1008, 1067, 1292, 1379, 1682, 1708	968, 1032, 1057, 1091, 1360, 1399, 1408, 1444, 1468, 1507, 1611, 1732, 1798, 1844, 2232, 2240, 2409, 2516, 2527, 2802, 3187
4–7	185, 352, 424, 669, 740, 1010	624, 871, 962, 972, 1060, 1092, 1264, 1358, 1359, 1443, 2175, 2231, 2521
4–8	216, 218, 391, 396, 397, 425, 580, 718, 953, 1350	675, 692, 725, 819, 828, 1056, 1059, 1243, 1445, 1467, 1629, 1667, 1841, 2064, 2154, 2399, 2538
4–9	67, 217, 291, 346, 395	255, 313, 548, 551, 693, 817, 945, 998, 1040, 1164, 1349, 1653
4–10	82, 180, 215, 320, 390, 394, 638	403, 623, 644, 958, 1001, 1331, 1362, 1607, 1731, 2422
4–11	62, 85, 191	315, 319, 377, 584, 634, 637, 995
4–12	83, 181, 190, 309	944, 1330
4–13	60, 84	269, 311, 504, 588
5–1	285, 359, 673, 903, 1007, 1083, 1354	1166, 1248, 1676, 1799, 2305, 2609, 3115
5–2	194, 220, 348, 408, 678, 898	596, 676, 1059, 1178, 1243, 1332, 1445, 1839, 2042, 2155
5–3	114, 150, 195, 219	484, 559, 675, 725, 794, 819, 853, 1966
6–1	284, 357	1231, 1353
7–1	454, 506, 510	1991, 2023, 2041
8–1	769	

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