# EQUIVALENCE CLASSES FOR SMOOTH FANO POLYTOPES

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ABSTRACT. Let $\mathcal{F}_n$ denote the set of smooth Fano polytopes of dimension $n$, up to unimodular equivalence. We describe all of the F- and I-equivalence classes for $n = 5$, 6, and 7. We also give a complete characterisation of F- and I-equivalence classes for smooth Fano $n$-polytopes with $n + 3$ vertices, and describe a family of I-isolated smooth Fano polytopes in every dimension. We conclude by considering a possible generalisation of F-equivalence to reflexive 3-polytopes.

## 1. INTRODUCTION

Let $N \cong \mathbb{Z}^n$ be a lattice of rank $n$, $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ the dual lattice of $N$, and let $P \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^n$ be an $n$-dimensional lattice polytope; that is, $P$ is a convex polytope whose vertices $\mathcal{V}(P)$ belong to the underlying lattice $N$. If the origin lies in the (strict) interior of $P$ – that is, if $0 \in \mathrm{int}(P)$ – and if the vertices of $P$ are primitive – that is, if for each $v \in \mathcal{V}(P)$ the line segment connecting $v$ and 0 contains no lattice points other than the two end-points – then we call $P$ *Fano*. We say that two polytopes $P$ and $Q$ are *unimodular-equivalent* if there exists an affine map $\varphi : N_{\mathbb{Q}} \to N_{\mathbb{Q}}$ such that $\varphi(N) = N$ and $\varphi(P) = Q$. Throughout we regard a Fano polytope as being defined only up to unimodular equivalence.

Fano polytopes are of significant importance in toric geometry: there is a one-to-one correspondence between Fano polytopes (up to unimodular equivalence) and toric Fano varieties (up to isomorphism). Given a Fano polytope $P$ one can construct its spanning fan $\Sigma_P$ whose cones $\sigma \in \Sigma_P$ span the faces of $P$, and from $\Sigma_P$ one obtains a projective toric variety $X$. The anticanonical divisor $-K_X$ of $X$ is an ample $\mathbb{Q}$-Cartier divisor; hence $X$ is a *Fano* variety. Much of the geometry of $X$ is encoded in the combinatorics of $\Sigma_P$. For details of toric geometry see [9, 11], and for an overview of Fano polytopes see [13].

A Fano polytope $P$ said to be *reflexive* if the dual polytope

$$P^* = \{u \in M_{\mathbb{Q}} \mid \langle u, v\rangle \geq -1 \text{ for every } v \in P\}$$

is also a lattice polytope, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing. A Fano polytope $P$ is reflexive if and only if $P^*$ is reflexive. Reflexive polytopes correspond to Gorenstein toric Fano varieties, and were introduced by Batryrev [3]. Their classification is known up to dimension four [16], where there are 473 800 776 unimodular equivalence classes.

A simplicial Fano polytope (that is, one whose facets are simplices) corresponds to a $\mathbb{Q}$-factorial toric Fano variety. A Fano polytope $P$ is said to be *smooth* if the vertices of each facet of $P$ forms a $\mathbb{Z}$-basis of the underlying lattice $N$. In particular, a smooth Fano polytope is both simplicial and reflexive. The smooth Fano polytopes correspond to the nonsingular toric Fano varieties. Because of this important correspondence, smooth Fano polytopes have been the focus of a great deal of research. Of particular interest is their classification in each dimension. In dimension two the classification is well-known: there are exactly five cases, corresponding to the five smooth toric del Pezzo surfaces ($\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and the blow-up of $\mathbb{P}^2$ in at most three points). In dimension three the classification is due to Batyrev [1] and Watanabe–Watanabe [22] (there are 18 cases) and in dimension four by Batyrev [4] and Sato [21] (124 cases). The five dimensional classification, obtained by Kreuzer–Nill [15] by "unprojecting" the four-dimensional reflexive classification, has 866 unimodular equivalence classes. Recently, an efficient algorithm to classify all $n$-dimensional smooth Fano polytopes was described by Øbro [17, 18]. This has allowed the classification up to dimension nine; see Table 1.

Let $\mathcal{F}_n$ denote the set of unimodular equivalence classes of $n$-dimensional smooth Fano polytopes. We shall consider the structure of $\mathcal{F}_n$ with respect to two equivalence relations: F-equivalence and I-equivalence.

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Table 1. The number $|\mathcal{F}_n|$ of smooth Fano $n$-polytopes, $n \leq 9$. Classifications at [6, 20].

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\|\mathcal{F}_n\|$ | 1 | 5 | 18 | 124 | 866 | 7 622 | 72 256 | 749 892 | 8 229 721 |

**Definition 1.1** (c.f. [21, Definitions 1.1 and 6.1], [19, Definition 1.1])**.** Two smooth Fano polytopes $P, Q \in \mathcal{F}_n$ are *F-equivalent*, and we write $P \overset{F}{\sim} Q$, if there exists a finite sequence $P_0, P_1, \ldots, P_k \in \mathcal{F}_n$ of smooth Fano polytopes satisfying:

(i) $P$ and $Q$ are unimodular-equivalent to $P_0$ and $P_k$, respectively;
(ii) For each $1 \leq i \leq k$ we have either that $\mathcal{V}(P_i) = \mathcal{V}(P_{i-1}) \cup \{w\}$, where $w \notin \mathcal{V}(P_{i-1})$, or that $\mathcal{V}(P_{i-1}) = \mathcal{V}(P_i) \cup \{w\}$, where $w \notin \mathcal{V}(P_i)$;
(iii) If $w \in \mathcal{V}(P_i) \setminus \mathcal{V}(P_{i-1})$ then there exists a proper face $F$ of $P_{i-1}$ such that

$$w = \sum_{v \in \mathcal{V}(F)} v$$

and the set of facets of $P_i$ containing $w$ is equal to

$$\{\mathrm{conv}\left(\{w\} \cup \mathcal{V}(F') \setminus \{v\}\right) \mid F' \text{ is a facet of } P_{i-1}, F \subset F', v \in \mathcal{V}(F)\}.$$

In other words, $P_i$ is obtained by taking a stellar subdivision of $P_{i-1}$ with $w$. Similarly, if $w \in \mathcal{V}(P_{i-1}) \setminus \mathcal{V}(P_i)$ then $P_{i-1}$ is given by a stellar subdivision of $P_i$ with $w$.

**Definition 1.2** (c.f. [12, §4], [19, §1])**.** Two smooth Fano polytopes $P, Q \in \mathcal{F}_n$ are *I-equivalent*, and we write $P \overset{I}{\sim} Q$, if there exists a finite sequence of smooth Fano polytopes satisfying conditions (i) and (ii) in Definition 1.1.

**Remark 1.3.** We make the following observations on the definitions of F- and I-equivalence.

(i) Clearly if $P \overset{F}{\sim} Q$ then $P \overset{I}{\sim} Q$.
(ii) These definitions can be naturally expressed in terms of fans.
(iii) Condition (iii) in Definition 1.1 should be interpreted as stating that the two nonsingular toric Fano varieties $X_{P_{i-1}}$ and $X_{P_i}$ obtained from $\Sigma_{P_{i-1}}$ and $\Sigma_{P_i}$, respectively, are related via an equivariant blow-up.
(iv) I-equivalence does not necessarily correspond to an equivariant blow-up.

**Example 1.4.** Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the unit coordinate vectors of $N \cong \mathbb{Z}^n$ and let

$$T^n = \mathrm{conv}(\{\mathbf{e}_1, \ldots, \mathbf{e}_n, -(\mathbf{e}_1 + \cdots + \mathbf{e}_n)\}).$$

Then $T^n$ is a smooth Fano polytope, with corresponding toric variety $X_{T^n}$ equal to $n$-dimensional projective space $\mathbb{P}^n$. Note that this is the unique smooth Fano $n$-polytope with $n+1$ vertices (up to unimodular equivalence). For a given positive integer $k$, let

$$V^{2k} = \mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{2k}, \pm(\mathbf{e}_1 + \cdots + \mathbf{e}_{2k})\}), \qquad \widetilde{V}^{2k} = \mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{2k}, \mathbf{e}_1 + \cdots + \mathbf{e}_{2k})\}.$$

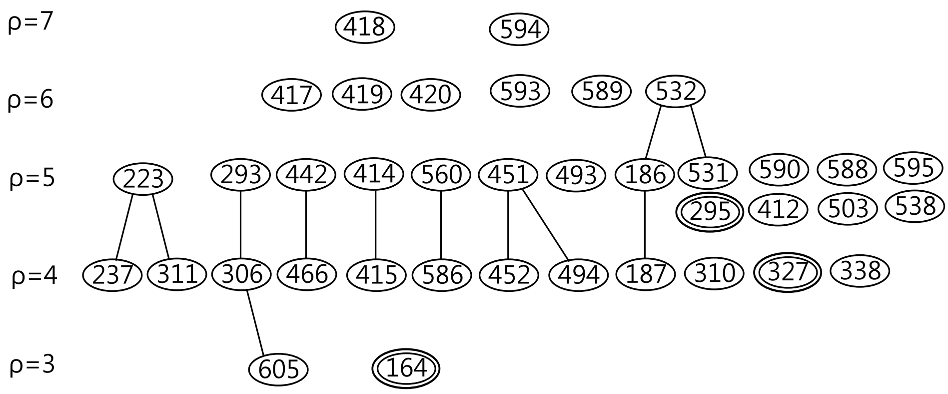
These are smooth Fano $2k$-polytopes, and $V^{2k} \overset{I}{\sim} \widetilde{V}^{2k} \overset{I}{\sim} T^{2k}$, but $V^{2k} \overset{F}{\not\sim} \widetilde{V}^{2k}$ when $k \geq 2$.

Recall that a polytope $P$ is *pseudo-symmetric* if $P$ contains a facet $F$ such that $-F$ is also a facet of $P$. By Ewald [10] we know that every pseudo-symmetric smooth Fano polytope is unimodular-equivalent to

$$\bigoplus_{i=1}^{p} V^{2l_i} \oplus \bigoplus_{j=1}^{q} \widetilde{V}^{2m_j} \oplus \bigoplus_{k=1}^{r} T^{n_k},$$

where for two reflexive polytopes $P \subset N_{\mathbb{Q}}$ and $Q \subset N'_{\mathbb{Q}}$, $P \oplus Q = \mathrm{conv}(P \times \{0\} \cup \{0\} \times Q) \subset (N + N')_{\mathbb{Q}}$ denotes the *free sum* of $P$ and $Q$. The free sum of two smooth Fano polytopes corresponds to the direct product of two nonsingular toric Fano varieties.

Figure 1. The F-equivalence relations between $P \in \mathcal{F}_5$, $P \overset{F}{\not\sim} T^5$. See Remark 2.1 for details.




**Dimensions 2, 3, and 4.** Sato [21] investigated the F-equivalence classes for $\mathcal{F}_n$ when $n \leq 4$. In two dimensions, every $P \in \mathcal{F}_2$ is F-equivalent to $T^2$, and in three dimensions every $P \in \mathcal{F}_3$ is F-equivalent to $T^3$. In four dimensions $\mathcal{F}_4$ consists of three F-equivalence classes, the largest of which – consisting of 122 smooth Fano 4-polytopes – contains $T^4$. The two remaining F-equivalence classes correspond to those $P \overset{F}{\sim} V^4$ and to those $P \overset{F}{\sim} \widetilde{V}^4$.

**Example 1.5** (Øbro's example)**.** Sato conjectured that any smooth Fano $n$-polytope is either F-equivalent to $T^n$ or is pseudo-symmetric [21, Conjecture 1.3 and 6.3]. This holds when $n \leq 4$, however Øbro [19] provides a counterexample in dimension five: a smooth Fano polytope

$$P = \operatorname{conv}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \pm\mathbf{e}_5, (-1,-1,-1,0,3), (0,1,1,-1,-1)\} \in \mathcal{F}_5$$

with 8 vertices which is neither pseudo-symmetric nor I-equivalent to any other smooth Fano 5-polytope.

In this paper we continue the investigation of F- and I-equivalence classes for $\mathcal{F}_n$. In §2 we begin by determining all F- and I-equivalence classes for $\mathcal{F}_n$, $5 \leq n \leq 7$. In §4 we introduce *F-* and *I-isolated* smooth Fano polytopes and, in Theorem 4.2, characterise those F- and I-isolated polytopes with $n+3$ vertices. In Theorem 4.3 and Corollary 4.6 we generalise the example of Øbro by constructing a family of I-isolated smooth Fano $n$-polytopes with $n+\rho$ vertices for all $n \geq 5$ and $3 \leq \rho \leq n$. Finally, we conclude in §5 by considering a generalisation of F-equivalence to reflexive 3-polytopes.

## 2. Equivalence classes for smooth Fano polytopes in dimensions 5, 6, and 7

We now describe all F- and I-equivalence classes for $\mathcal{F}_5$, and enumerate the F- and I-components for $\mathcal{F}_6$ and $\mathcal{F}_7$. The results are summarised in Table 2 (the cases $2 \leq n \leq 4$ are due to Sato [21]). Figure 1 shows all of the smooth Fano 5-polytopes which are *not* F-equivalent to $T^5$.

Table 2. The number of F- and I-equivalence classes for $\mathcal{F}_n$, $n \leq 7$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| F-equivalence | 1 | 1 | 1 | 3 | 27 | 441 | 6132 |
| I-equivalence | 1 | 1 | 1 | 1 | 4 | 43 | 436 |

**Remark 2.1.** Each circled number in Figure 1 corresponds to a smooth Fano 5-polytope $P \in \mathcal{F}_5$ with $5+\rho$ vertices, $3 \leq \rho \leq 7$; the number is the ID of $P$ in [6]. Two circles are connected by a line if the corresponding polytopes are F-equivalent. A double circle indicates that $P$ is I-isolated: that is $P \overset{I}{\not\sim} Q$ for any $Q \in \mathcal{F}_5 \setminus \{P\}$. The polytope in Example 1.5 has ID 164, and is I-isolated. Figure 1 has 26 F-connected components; there are 38 polytopes which are not F-equivalent to $T^5$. The polytopes with a single circle are I-equivalent to $T^5$; only 3 polytopes are not I-equivalent to $T^5$.

The number of F- and I-equivalence classes for $\mathcal{F}_6$ and $\mathcal{F}_7$ are listed in Tables 3 and 4. These were generated by computer[1] using the computational algebra system Magma [5]. These tables do *not* contain

[1]Computer code and output listing the elements in each equivalence class are available from [6].

the equivalence class generated by $T^n$. There are 282 F- and 34 I-isolated smooth Fano 6-polytopes; there are 3839 F- and 323 I-isolated smooth Fano 7-polytopes. The number of F- and I-equivalence classes for $\mathcal{F}_n$ is given by the sum of the second row in the corresponding table, plus 1 (the excluded class of $T^n$).

TABLE 3. The number of F- and I-components for $\mathcal{F}_6$, excluding the class containing $T^6$

| #representatives | 1 | 2 | 3 | 4 | 5 | 6 | 9 | 10 | 11 | 15 | 16 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| F-components | 282 | 80 | 47 | 14 | 8 | 3 | 1 | 1 | 1 | 1 | 1 | 1 |

| #representatives | 1 | 2 | 3 | 5 |
|---|---|---|---|---|
| I-components | 34 | 5 | 2 | 1 |

TABLE 4. The number of F- and I-components for $\mathcal{F}_7$, excluding the class containing $T^7$

| #representatives | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| F-components | 3839 | 1163 | 545 | 215 | 118 | 60 | 29 | 31 | 30 | 13 | 14 | 8 | 5 | 1 | 7 | 5 | 6 | 6 |

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 19 | 20 | 21 | 22 | 23 | 25 | 26 | 27 | 28 | 30 | 32 | 35 | 37 | 39 | 40 | 41 | 42 | 52 | 69 | 73 | 89 | 109 |
| 1 | 2 | 6 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 1 | 3 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 2 | 1 | 1 |

| #representatives | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| I-components | 323 | 67 | 13 | 11 | 9 | 4 | 4 | 2 | 1 | 1 |

## 3. Primitive collections

For a smooth Fano polytope $P \subset N_{\mathbb{Q}}$ let $\Sigma_P = \{\mathrm{cone}(F) \subset N_{\mathbb{Q}} \mid F \text{ is a face of } P\}$ denote the spanning fan of $P$. Then $\Sigma_P$ is a complete nonsingular fan in $N$. For any complete nonsingular fan $\Sigma$ in $N$, let $G(\Sigma)$ denote the set of primitive vectors in $N$ corresponding to the rays of $\Sigma$. We recall the following key definitions from Batyrev [2].

(i) A subset $\varnothing \neq A \subset G(\Sigma)$ is called a *primitive collection of* $\Sigma$ if $\mathrm{cone}(A) \notin \Sigma$ but $\mathrm{cone}(A \setminus \{x\}) \in \Sigma$ for every $x \in A$. The set of all primitive collections of $\Sigma$ is denoted by $\mathrm{PC}(\Sigma)$.
(ii) Let $A = \{x_1, \ldots, x_l\} \in \mathrm{PC}(\Sigma)$. If $x_1 + \cdots + x_l \neq 0$ then there is a unique $\mathrm{cone}(y_1, \ldots, y_m) \in \Sigma$ such that $x_1 + \cdots + x_l = a_1y_1 + \cdots + a_my_m$, where $a_i \in \mathbb{Z}_{>0}$. The relation $x_1 + \cdots + x_l = 0$ or $x_1 + \cdots + x_l = a_1y_1 + \cdots + a_my_m$ is called the *primitive relation for* $A$.
(iii) Let $A \in \mathrm{PC}(\Sigma)$ have primitive relation $x_1 + \cdots + x_l = a_1y_1 + \cdots + a_my_m$, The integer $\deg(A) = l - (a_1 + \cdots + a_m)$ is called the *degree of* $A$.

In the case of a smooth Fano polytope $P$ we will write $\mathrm{PC}(P)$ in place of $\mathrm{PC}(\Sigma_P)$.

**Proposition 3.1** ([4], [21, Theorem 3.10]). *Let $\Sigma$ be a complete nonsingular fan. Then $\deg(A) > 0$ for all $A \in \mathrm{PC}(\Sigma)$ if and only if there exists a smooth Fano polytope $P$ such that $\Sigma = \Sigma_P$.*

The primitive relations for smooth Fano $n$-polytopes with $n+2$ or $n+3$ vertices are completely characterised:

**Proposition 3.2** ([14, Theorem 1]). *Let $P$ be a smooth Fano $n$-polytope with $n+2$ vertices $v_1, \ldots, v_{n+2}$. Then, up to relabelling of the vertices, the primitive relations are of the form*

$$v_1 + \cdots + v_k = 0 \quad \text{for some} \quad 2 \leq k \leq n,$$
$$v_{k+1} + \cdots + v_{n+2} = a_1v_1 + \cdots + a_kv_k \quad \text{with } a_i \geq 0 \text{ and } n+2-k > a_1 + \cdots + a_k.$$

**Proposition 3.3** ([2, Theorem 6.6]). *Let $P$ be a smooth Fano $n$-polytope with $n+3$ vertices. Then one of the following holds:*

(i) *$P$ consists of three disjoint primitive collections, i.e.* $\mathrm{PC}(P) = \{A_1, A_2, A_3\}$ *with $A_i \cap A_j = \varnothing$ for each $i \neq j$;*

*(ii)* $|\mathrm{PC}(P)| = 5$ *and there exist* $p_0, p_1, p_2, p_3, p_4 \in \mathbb{Z}_{>0}$ *with* $p_0 + \cdots + p_4 = n+3$ *such that the primitive relations are of the form*

$$\begin{aligned} v_1 + \cdots + v_{p_0} + y_1 + \cdots + y_{p_1} &= c_2 z_2 + \cdots + c_{p_2} z_{p_2} + (d_1+1)t_1 + \cdots + (d_{p_3}+1)t_{p_3}, \\ y_1 + \cdots + y_{p_1} + z_1 + \cdots + z_{p_2} &= u_1 + \cdots + u_{p_4}, \\ z_1 + \cdots + z_{p_2} + t_1 + \cdots + t_{p_3} &= 0, \\ t_1 + \cdots + t_{p_3} + u_1 + \cdots + u_{p_4} &= y_1 + \cdots + y_{p_1}, \\ u_1 + \cdots + u_{p_4} + v_1 + \cdots + v_{p_0} &= c_2 z_2 + \cdots + c_{p_2} z_{p_2} + d_1 t_1 + \cdots + d_{p_3} t_{p_3}, \end{aligned} \tag{3.1}$$

*where* $\mathcal{V}(P) = \{v_1, \ldots, v_{p_0}, y_1, \ldots, y_{p_1}, z_1, \ldots, z_{p_2}, t_1, \ldots, t_{p_3}, u_1, \ldots, u_{p_4}\}$, $c_2, \ldots, c_{p_2}, d_1, \ldots, d_{p_3} \in \mathbb{Z}_{\geq 0}$, *and the degree of each of these primitive relations is positive.*

We also recall the following useful lemma.

**Lemma 3.4** ([7, Corollary 4.4], [19, Lemma 3.1])**.** *Let*

$$v_1 + \cdots + v_k = a_1 w_1 + \cdots + a_m w_m \tag{3.2}$$

*be a linear relation between some vertices of a smooth Fano polytope* $P$ *such that* $a_i \in \mathbb{Z}_{>0}$ *and* $\{v_1, \ldots, v_k\} \cap \{w_1, \ldots, w_m\} = \varnothing$. *Suppose that* $k - (a_1 + \cdots + a_m) = 1$ *and* $\mathrm{conv}(w_1, \ldots, w_m)$ *is a face of* $P$. *Then* (3.2) *is a primitive relation. Moreover, for each face* $F$ *of* $P$ *with* $\{w_1, \ldots, w_m\} \subset \mathcal{V}(F)$, $\mathrm{conv}(\mathcal{V}(F) \cup \{v_1, \ldots, v_k\} \setminus \{v_i\})$ *is also a face of* $P$ *for every* $1 \leq i \leq k$.

## 4. I-isolated smooth Fano polytopes

**Definition 4.1.** Let $P$ be a smooth Fano $n$-polytope. We say that $P$ is *F-isolated* (respectively, *I-isolated*) if $P \overset{F}{\not\sim} Q$ (respectively, $P \overset{I}{\not\sim} Q$) for any $Q \in \mathcal{F}_n \setminus \{P\}$. Clearly if $P$ is I-isolated then $P$ is F-isolated.

First we characterise the primitive relations for an I-isolated smooth Fano $n$-polytope with $n+3$ vertices. Note that such a polytope is of dimension at least five.

**Theorem 4.2.** *Let* $P$ *be a smooth Fano* $n$*-polytope with* $n+3$ *vertices. The following three conditions are equivalent:*

- (i) $P$ *is F-isolated;*
- (ii) $P$ *is I-isolated;*
- (iii) *The primitive relations in* $P$ *are of the form*

$$\begin{aligned} v_1 + \cdots + v_a + y_1 + \cdots + y_b &= (a+b-1)t, \\ y_1 + \cdots + y_b + z &= u_1 + \cdots + u_b, \\ z + t &= 0, \\ t + u_1 + \cdots + u_b &= y_1 + \cdots + y_b, \\ u_1 + \cdots + u_b + v_1 + \cdots + v_a &= (a+b-2)t, \end{aligned} \tag{4.1}$$

*where* $\mathcal{V}(P) = \{v_1, \ldots, v_a, y_1, \ldots, y_b, z, t, u_1, \ldots, u_b\}$, $a \geq 2$, $b \geq 2$, *and* $n = a + 2b - 1$.

*Proof.* (i) $\Rightarrow$ (iii): Assume that $P$ is F-isolated. Since $|\mathcal{V}(P)| = n+3$, from Proposition 3.3 we know that the primitive relations are of the form (3.1). In what follows we will consider the complete nonsingular fan obtained by introducing a new ray $\mathrm{cone}(w)$, $w = \sum_{v \in G(\sigma)} v$, for some $\sigma \in \Sigma_P$ and taking the stellar subdivision. By considering the primitive relations, we will prove that if there does not exist a smooth Fano polytope $Q \neq P$ with $P \overset{F}{\sim} Q$, then the primitive relations in $P$ are of the form (4.1).

For a face $F \subset P$, let $\Sigma_F^*(P)$ denote the complete nonsingular fan obtained by taking a stellar subdivision of $\Sigma_P$ with the ray $\mathrm{cone}(w)$, where $w = \sum_{v \in \mathcal{V}(F)} v = \sum_{v \in G(\mathrm{cone}(F))} v$. Suppose first that $p_1 = 1$ or $p_4 = 1$. Then by [21, Proposition 8.3] we obtain a new smooth Fano $n$-polytope with $n+2$ vertices which is F-equivalent to $P$, a contradiction. Hence $p_1 \geq 2$ and $p_4 \geq 2$.

Now suppose that $p_0 = 1$. Since $\{v_1, z_1\}$ is not contained in $\mathrm{PC}(P)$, so $\mathrm{conv}(v_1, z_1)$ is a face of $P$. We consider a stellar subdivision of $\Sigma_P$ with the (ray through the) lattice point $w = v_1 + z_1$ and the complete nonsingular fan $\Sigma^*_{\mathrm{conv}(v_1,z_1)}(P)$. The new primitive relations involving $w$ are (see [21, Theorem 4.3]):

$$w + y_1 + \cdots + y_{p_1} = (c_2 - 1)z_2 + \cdots + (c_{p_2} - 1)z_{p_2} + d_1 t_1 + \cdots + d_{p_3} t_{p_3},$$
$$w + z_2 + \cdots + z_{p_2} + t_1 + \cdots + t_{p_3} = v_1,$$
$$u_1 + \cdots + u_{p_4} + w = (c_2 - 1)z_2 + \cdots + (c_{p_2} - 1)z_{p_2} + (d_1 - 1)t_1 + \cdots + (d_{p_3} - 1)t_{p_3}.$$

Since the degrees of these primitive relations are positive, by Proposition 3.1 there exists a smooth Fano polytope $Q$ with $\Sigma^*_{\mathrm{conv}(v_1,z_1)}(P) = \Sigma_Q$. In particular $P \overset{F}{\sim} Q$, a contradiction. Hence $p_0 \geq 2$.

Suppose that $p_1 + p_2 - p_4 \geq 2$. Then we see that the complete nonsingular fan $\Sigma^*_{\mathrm{conv}(y_1,z_1)}(P)$ is the spanning fan of a smooth Fano polytope, a contradiction. Thus $p_1 + p_2 - p_4 \leq 1$. On the other hand, since $\{y_1, \ldots, y_{p_1}, z_1, \ldots, z_{p_2}\}$ is a primitive collection of $P$ and its degree is equal to $p_1 + p_2 - p_4$, we obtain that $p_1 + p_2 - p_4 \geq 1$ by Proposition 3.1. We conclude that $p_1 + p_2 - p_4 = 1$. Similarly, suppose that $p_3 + p_4 - p_1 \geq 2$. Then we see that the complete nonsingular fan $\Sigma^*_{\mathrm{conv}(t_1,u_1)}(P)$ is the spanning fan of a smooth Fano polytope, a contradiction. Thus $p_3 + p_4 - p_1 \leq 1$. Moreover, since $\{t_1, \ldots, t_{p_3}, u_1, \ldots, u_{p_4}\}$ is a primitive collection of $P$ with degree $p_3 + p_4 - p_1$, we have that $p_3 + p_4 - p_1 \geq 1$. Hence $p_3 + p_4 - p_1 = 1$. Since $p_1 + p_2 - p_4 = p_3 + p_4 - p_1 = 1$, we see that $p_2 = p_3 = 1$ and $p_1 = p_4$.

In a similar fashion, suppose that $p_0 + p_1 - (d_1 + 1) \geq 2$. Since the complete nonsingular fan $\Sigma^*_{\mathrm{conv}(v_1,v_2)}(P)$ comes from a smooth Fano polytope we have a contradiction. Thus $p_0 + p_1 - (d_1 + 1) \leq 1$. On the other hand, since $\{v_1, \ldots, v_{p_0}, y_1, \ldots, y_{p_1}\}$ is a primitive collection of $P$ of degree $p_0 + p_1 - (d_1 + 1)$, we obtain that $p_0 + p_1 - (d_1 + 1) \geq 1$. Hence $d_1 = p_0 + p_1 - 2$. We obtain the desired primitive relations (4.1).

(iii) $\Rightarrow$ (ii): Let

$$\begin{aligned}
v_i &= \mathbf{e}_i,\ i = 1, \ldots, a-1, & v_a &= -\mathbf{e}_1 - \cdots - \mathbf{e}_{a+2b-2} + (a+b-1)\mathbf{e}_{a+2b-1},\\
y_j &= \mathbf{e}_{a-1+j},\ j = 1, \ldots, b-1, & y_b &= \mathbf{e}_{a+b-1} + \cdots + \mathbf{e}_{a+2b-2},\\
z &= -\mathbf{e}_{a+2b-1}, & t &= \mathbf{e}_{a+2b-1},\\
u_j &= \mathbf{e}_{a+b-1+j},\ j = 1, \ldots, b-1, & u_b &= \mathbf{e}_a + \cdots + \mathbf{e}_{a+b-1} - \mathbf{e}_{a+2b-1},
\end{aligned}$$

and set $V = \{v_1, \ldots, v_a\}$, $Y = \{y_1, \ldots, y_b\}$, and $U = \{u_1, \ldots, u_b\}$. Define $P = \mathrm{conv}(V \cup Y \cup \{z, t\} \cup U)$. By construction the primitive relations in $P$ are of the form (4.1). We will show that $P$ is I-isolated.

First, we prove there is no smooth Fano $n$-polytope $Q$ with $n+2$ vertices such that $\mathcal{V}(Q) \subset \mathcal{V}(P)$. Suppose for a contradiction that such a $Q$ exists. Write $\mathcal{V}(Q) = \mathcal{V}(P) \setminus \{w\}$. Assume that $w \in \{z, t\}$. In this case, although there must be a primitive collection $A \in \mathrm{PC}(Q)$ with $\sum_{v \in A} v = 0$ by Proposition 3.2, no non-empty subset of $\mathcal{V}(Q)$ adds to 0, a contradiction. Assume instead that $w \in V$. Then $w = v_i$ for some $1 \leq i \leq a$. However, for each $1 \leq j \leq a-1$ (respectively, $j = a$), the $j$-th coefficient of each $\mathcal{V}(P) \setminus \{v_j\}$ is non-positive (respectively, non-negative). Thus $Q$ cannot contain the origin in its interior, a contradiction. Finally, assume that $w \in Y \cup U$. Let $A$ be a primitive collection of $Q$ with $\sum_{v \in A} v = 0$. Then $A = \{z, t\}$. By Proposition 3.2, $\sum_{x \in (V \cup Y \cup U) \setminus \{w\}} x$ should be expressed as a linear combination of $z$ and $t$, which is a contradiction.

Second, we prove there exists is no smooth Fano $n$-polytope $R$ with $n+4$ vertices such that $\mathcal{V}(P) \subset \mathcal{V}(R)$. Suppose for a contradiction that such an $R$ exists, and let $\mathcal{V}(R) = \mathcal{V}(P) \cup \{w\}$ for some $w \in N$. Since $t$ is a vertex of $R$, by Lemma 3.4 the relation

$$v_1 + \cdots + v_a + y_1 + \cdots + y_b = (a+b-1)t \tag{4.2}$$

is a primitive relation in $R$. Hence $\mathrm{conv}((V \setminus \{v\}) \cup Y \cup \{t\})$ is a face of $R$ for every $v \in V$. Since we have the relation

$$t + u_1 + \cdots + u_b = y_1 + \cdots + y_b, \tag{4.3}$$

we obtain that (4.3) is also a primitive relation in $R$, and

$$\mathrm{conv}((V \setminus \{v\}) \cup Y \cup \{t\} \cup (U \setminus \{u\})) \qquad \text{and} \qquad \mathrm{conv}((V \setminus \{v\}) \cup Y \cup U)$$

are facets of $R$ for every $v \in V$ and $u \in U$. Moreover, since $\mathrm{conv}((V \setminus \{v\}) \cup Y \cup \{t\} \cup (U \setminus \{u\}))$ is a face of $R$, we obtain that

$$\mathrm{conv}(V \cup (Y \setminus \{y\}) \cup \{t\} \cup (U \setminus \{u\}))$$

is a facet of $R$ for every $y \in Y$ and $u \in U$ by (4.2). In addition, from the relation

$$y_1 + \cdots + y_b + z = u_1 + \cdots + u_b, \tag{4.4}$$

since $\mathrm{conv}((V\setminus\{v\})\cup Y\cup U)$ is a face of $R$ we obtain that (4.4) is also a primitive relation in $R$ and hence

$$\mathrm{conv}((V\setminus\{v\})\cup(Y\setminus\{y\})\cup\{z\}\cup U)$$

is also a facet of $R$ for every $v\in V$ and $y\in Y$. Therefore $R$ contains the following four kinds of facets:

(i) $\mathrm{conv}((V\setminus\{v\})\cup Y\cup\{t\}\cup(U\setminus\{u\}))$, where $v\in V, u\in U$;
(ii) $\mathrm{conv}((V\setminus\{v\})\cup Y\cup U)$, where $v\in V$;
(iii) $\mathrm{conv}(V\cup(Y\setminus\{y\})\cup\{t\}\cup(U\setminus\{u\}))$, where $y\in Y, u\in U$;
(iv) $\mathrm{conv}((V\setminus\{v\})\cup(Y\setminus\{y\})\cup\{z\}\cup U)$, where $v\in V, y\in Y$.

But these are also facets of $P$, thus for any these facets $F$, $w$ is not contained in $\mathrm{cone}(F)$. Hence $w$ must be contained in the cone generated by the remaining facet of $P$,

$$w\in\mathrm{cone}(V\cup(Y\setminus\{y\})\cup\{z\}\cup(U\setminus\{u\}))$$

for some $y\in Y$ and $u\in U$. Without loss of generality we may assume that $w\in\mathrm{cone}(V\cup(Y\setminus\{y_b\})\cup\{z\}\cup(U\setminus\{u_b\}))$. Let

$$w=c_1v_1+\cdots+c_{a-1}v_{a-1}+c_ay_1+\cdots+c_{a+b-2}y_{b-1}+c_{a+b-1}v_a+ c_{a+b}u_1+\cdots+c_{a+2b-2}u_{b-1}+c_{a+2b-1}z,$$

where $c_i\in\mathbb{Z}_{\geq 0}$. Let $F=\mathrm{conv}(V\cup Y'\cup\{t\}\cup U')$, where $Y'=Y\setminus\{y_b\}$ and $U'=U\setminus\{u_b\}$. Then $F$ is a facet of $R$. Let $F'$ be the unique facet of $R$ such that $F\cap F'$ is a ridge (i.e. a face of dimension $n-2$) of $R$ with $\mathcal{V}(F)\setminus\mathcal{V}(F')=\{t\}$. Then $\mathcal{V}(F')\setminus\mathcal{V}(F)=\{w\}$. In fact, for each $w'\in\{y_b,u_b\}$, $\mathrm{conv}(V\cup Y'\cup U'\cup\{w'\})$ cannot be a face of $R$ because $\mathrm{conv}(V\cup Y)$ and $\mathrm{conv}(V\cup U)$ cannot be a face, and for $z$, $\mathrm{conv}(V\cup Y'\cup\{z\}\cup U')$ is not a face by assumption. Hence by [19, Lemma 2.1], $t+w$ lies in the linear subspace spanned by $V\cup Y'\cup U'$. Therefore, from the relation $z+t=0$, we have that $c_{a+2b-1}=1$. Moreover it follows from [19, Lemma 2.1] that

$$1>\langle\mathbf{a}_F,w\rangle=\sum_{i=1}^{a+2b-2}c_i-1>\langle\mathbf{a}_F,z\rangle=-1,$$

where $\mathbf{a}_F$ is the lattice vector defining $F$, i.e. $F=\{\alpha\in P\mid\langle\mathbf{a}_F,\alpha\rangle=1\}$. Thus $\sum_{i=1}^{a+2b-2}c_i=1$. Hence $w$ can be written as $w=z+w''$ for some $w''\in V\cup Y'\cup U'$. Let $G=\mathrm{conv}((V\setminus\{v\})\cup Y\cup U)$. Since $G$ is a facet of $R$ for each $v\in V$ and the relation $y_1+\cdots+y_b+w=u_1+\cdots+u_b+w''$ holds, we see that $w$ is also contained in $G$ – that is, $R$ is not simplicial, which is a contradiction.

(ii) $\Rightarrow$ (i): This is obvious. $\square$

Next, we describe a family of I-isolated smooth Fano polytopes.

**Theorem 4.3.** *Let $a\geq 2$, $b\geq 1$, $k\geq 1$, and $l_j\geq 1$ for $j=1,\dots,k$. Then there exists an I-isolated (and hence F-isolated) smooth Fano polytope $P$ of dimension $n=a+2b-1+\sum_{j=1}^k l_j$ with $n+k+3$ vertices whose primitive relations are of the form*

$$\begin{aligned}
v_1+\cdots+v_a+y_1+\cdots+y_b&=(a+b-1)t,\\
y_1+\cdots+y_b+z&=u_1+\cdots+u_b,\\
z+t&=0,\\
t+u_1+\cdots+u_b&=y_1+\cdots+y_b,\\
u_1+\cdots+u_b+v_1+\cdots+v_a&=(a+b-2)t,\\
w_{1,1}+\cdots+w_{l_1+1,1}&=\alpha_{1,1}+\cdots+\alpha_{l_1,1},\\
w_{1,2}+\cdots+w_{l_2+1,2}&=\alpha_{1,2}+\cdots+\alpha_{l_2,2},\\
&\vdots\\
w_{1,k}+\cdots+w_{l_k+1,k}&=\alpha_{1,k}+\cdots+\alpha_{l_k,k},
\end{aligned}\tag{4.5}$$

*where $\mathcal{V}(P)=\{v_1,\dots,v_a,y_1,\dots,y_b,z,t,u_1,\dots,u_b\}\cup\bigcup_{j=1}^k\{w_{1,j},\dots,w_{l_j+1,j}\}$ and $\{\alpha_{i,j}\mid 1\leq i\leq l_j, 1\leq j\leq k\}=\{y_q,u_q\mid 1\leq q\leq b\}$.*

*Proof.* Let

$$\begin{aligned} v_i &= \mathbf{e}_i,\ i = 1, \ldots, a-1, & v_a &= -\mathbf{e}_1 - \cdots - \mathbf{e}_{a+2b-2} + (a+b-1)\mathbf{e}_{a+2b-1}, \\ y_j &= \mathbf{e}_{a-1+j},\ j = 1, \ldots, b-1, & y_b &= \mathbf{e}_{a+b-1} + \cdots + \mathbf{e}_{a+2b-2}, \\ z &= -\mathbf{e}_{a+2b-1}, & t &= \mathbf{e}_{a+2b-1}, \\ u_j &= \mathbf{e}_{a+b-1+j},\ j = 1, \ldots, b-1, & u_b &= \mathbf{e}_a + \cdots + \mathbf{e}_{a+b-1} - \mathbf{e}_{a+2b-1}, \\ w_{i,j} &= \mathbf{e}_{a+2b-1+\sum_{q=1}^{j-1} l_q + i},\ i = 1, \ldots, l_j,\ j = 1, \ldots, k, \end{aligned}$$

$$w_{l_j+1,j} = -\mathbf{e}_{a+2b-1+\sum_{q=1}^{j-1} l_q + 1} - \cdots - \mathbf{e}_{a+2b-1+\sum_{q=1}^{j} l_q} + \sum_{i=1}^{l_j} \alpha_{i,j},\ j = 1, \ldots, k,$$

and set $V = \{v_1, \ldots, v_a\}$, $Y = \{y_1, \ldots, y_b\}$, $U = \{u_1, \ldots, u_b\}$, and $W_j = \{w_{i,j} \mid 1 \le i \le l_j + 1\}$ for each $1 \le j \le k$. Define $P = \mathrm{conv}(V \cup Y \cup \{z, t\} \cup U \cup W)$, where $W = \bigcup_{j=1}^k W_j$. By construction $P$ is a smooth Fano polytope of dimension $n = a + 2b - 1 + \sum_{i=1}^k l_j$ with $n + k + 3$ vertices, and its primitive relations are of the form (4.5). By Lemmas 4.4 and 4.5 below we see that $P$ is I-isolated. □

**Lemma 4.4.** *Let $P$ be the polytope given in the proof of Theorem 4.3. There does not exist a smooth Fano $n$-polytope $Q$ with $n+k+2$ vertices such that $\mathcal{V}(Q) \subset \mathcal{V}(P)$.*

*Proof.* We follow the notation of the proof of Theorem 4.3. Suppose that there is a smooth Fano $n$-polytope $Q$ with $n+k+2$ vertices such that $\mathcal{V}(Q) = \mathcal{V}(P) \setminus \{x\}$, where $x \in \mathcal{V}(P) = V \cup Y \cup \{z, t\} \cup U \cup W$. Assume that $x \in \{z, t\}$. In this case, although there must be a primitive collection $A \in \mathrm{PC}(Q)$ with $\sum_{v \in A} v = 0$ by Proposition 3.2, no non-empty subset of $\mathcal{V}(P) \setminus \{x\}$ sums to 0. This is a contradiction.

Assume now that $x \in V$. Then $x = v_i$ for some $1 \le i \le a$. Notice that for each $1 \le j \le a-1$ (respectively, $j = a$) the $j$-th entry of each vertex $\mathcal{V}(P) \setminus \{v_j\}$ is non-positive (respectively, non-negative). Thus $Q$ cannot contain the origin in its interior, a contradiction. Similarly if $x \in W$.

Assume that $x \in Y$. Without loss of generality we assume $x = y_b$. For each $1 \le j \le k$, set $m_j = |\{i \mid \alpha_{i,j} = y_b, 1 \le i \le l_j\}|$ and $m'_j = |\{i \mid \alpha_{i,j} = u_b, 1 \le i \le l_j\}|$. Let $r$ be an index attaining $\max\{(m_j - 1)/(m'_j + 1) \mid 1 \le j \le k\}$, and consider

$$F = \mathrm{conv}\big((V \setminus \{v_a\}) \cup (Y \setminus \{y_b\}) \cup \{z\} \cup (U \setminus \{u_b\}) \cup (W \setminus \{w_{l_j+1,j} \mid 1 \le j \le k, j \ne r\})\big).$$

Then $F$ is a face of $Q$. In fact, let

$$\mathbf{a} = \sum_{\substack{1 \le q \le n \\ q \ne a+b-1}} \mathbf{e}_q + \left(\frac{1 + m'_r}{m_r + m'_r} - b + 2\right) \mathbf{e}_{a+b-1}.$$

We make the following observations.

(i) We have: $\langle \mathbf{a}, v_i \rangle = 1$ for each $1 \le i \le a-1$; $\langle \mathbf{a}, t \rangle = 1$; $\langle \mathbf{a}, y_j \rangle = \langle \mathbf{a}, u_j \rangle = 1$ for each $1 \le j \le b-1$; and $\langle \mathbf{a}, w_{i,j} \rangle = 1$ for each $1 \le i \le l_j$, $1 \le j \le k$.

(ii) Since $w_{l_r+1,r} = \sum_{j=1}^{l_r} (\alpha_{j,r} - w_{j,r})$ we have:

$$\begin{aligned} \langle \mathbf{a}, w_{l_r+1,r} \rangle &= \sum_{j=1}^{l_r} \langle \mathbf{a}, \alpha_{j,r} \rangle - \sum_{j=1}^{l_r} \langle \mathbf{a}, w_{j,r} \rangle \\ &= (m_r \langle \mathbf{a}, y_b \rangle + m'_r \langle \mathbf{a}, u_b \rangle + (l_r - m_r - m'_r)) - l_r \\ &= m_r \left(1 + \frac{1 + m'_r}{m_r + m'_r}\right) + m'_r \frac{1 + m'_r}{m_r + m'_r} - m_r - m'_r = 1. \end{aligned}$$

(iii) We have $\langle \mathbf{a}, v_a \rangle = -(1 + m'_r)/(m_r + m'_r) < 1$ and $\langle \mathbf{a}, z \rangle = -1 < 1$.

(iv) Moreover, we have $\langle \mathbf{a}, u_b \rangle = (1 + m'_r)/(m_r + m'_r)$. Thus $\langle \mathbf{a}, u_b \rangle = 1$ if $m_r = 1$, otherwise $\langle \mathbf{a}, u_b \rangle < 1$.

(v) In addition, for each $1 \leq j \leq k$, we also have:

$$\begin{aligned}\langle \mathbf{a}, w_{l_j+1,j}\rangle &= \sum_{i=1}^{l_j}\langle \mathbf{a}, \alpha_{i,j}\rangle - \sum_{i=1}^{l_j}\langle \mathbf{a}, w_{i,j}\rangle \\ &= m_j\left(1+\frac{1+m'_r}{m_r+m'_r}\right) + m'_j\frac{1+m'_r}{m_r+m'_r} + (l_j - m_j - m'_j) - l_j \\ &= \frac{m_j(m'_r+1) - m'_j(m_r-1)}{m_r+m'_r} \leq \frac{m_r+m'_r}{m_r+m'_r} = 1.\end{aligned}$$

Here the inequality $m_j(m'_r+1) - m'_j(m_r-1) \leq m_r + m'_r$ follows from the the maximality of $(m_r-1)/(m'_r+1)$, i.e. this inequality is equivalent to $(m_r-1)/(m'_r+1) \geq (m_j-1)/(m'_j+1)$.

If $m_r = 1$ then $F$ contains at least $n+1$ vertices, since $u_b \in F$. Hence $Q$ is not simplicial, a contradiction. If $m_r > 1$ then $(1+m'_r)/(m_r+m'_r)$ is not an integer, and so $Q$ is not reflexive, which is also a contradiction.

Assume that $x \in U$. Without loss of generality we assume $x = u_b$. For each $1 \leq j \leq k$, let $m_j$ and $m'_j$ be as above. Let $s$ be the index attaining $\max\{(m'_j-1)/(m_j+1) \mid 1 \leq j \leq k\}$ and consider

$$G = \operatorname{conv}\big((V\setminus\{v_a\}) \cup (Y\setminus\{y_b\}) \cup \{z\} \cup (U\setminus\{u_b\}) \cup (W\setminus\{w_{l_j+1,j} \mid 1 \leq j \leq k, j \neq s\})\big).$$

Then $G$ is a face of $Q$. In fact, let

$$\mathbf{b} = \sum_{\substack{1\leq q\leq n\\ q\neq a+b-1\\ q\neq a+2b-1}} \mathbf{e}_q + \left(\frac{1+m_s}{m_s+m'_s} - b + 1\right)\mathbf{e}_{a+b-1} - \mathbf{e}_{a+2b-1}.$$

We make the following observations.

(i) We have: $\langle \mathbf{b}, v_i\rangle = 1$ for each $1 \leq i \leq a-1$; $\langle \mathbf{b}, z\rangle = 1$; $\langle \mathbf{b}, y_j\rangle = \langle \mathbf{b}, u_j\rangle = 1$ for each $1 \leq j \leq b-1$; and $\langle \mathbf{b}, w_{i,j}\rangle = 1$ for each $1 \leq i \leq l_j$, $1 \leq j \leq k$.

(ii) We have:

$$\langle \mathbf{b}, w_{l_s+1,s}\rangle = \sum_{j=1}^{l_s}\langle \mathbf{b}, \alpha_{j,s}\rangle - \sum_{j=1}^{l_s}\langle \mathbf{b}, w_{j,s}\rangle = 1.$$

(iii) We have $\langle \mathbf{b}, v_a\rangle < -2a-2b+3 < 1$ and $\langle \mathbf{b}, t\rangle = -1 < 1$.

(iv) Moreover, we have $\langle \mathbf{b}, y_b\rangle = (1+m_s)/(m_s+m'_s)$. Thus $\langle \mathbf{b}, y_b\rangle = 1$ if $m'_s = 1$, otherwise $\langle \mathbf{b}, y_b\rangle < 1$.

(v) In addition, for each $1 \leq j \leq k$, we also have:

$$\langle \mathbf{b}, w_{l_j+1,j}\rangle = \sum_{i=1}^{l_j}\langle \mathbf{b}, \alpha_{i,j}\rangle - \sum_{i=1}^{l_j}\langle \mathbf{b}, w_{i,j}\rangle \leq 1.$$

As before we obtain a contradiction. ◻

**Lemma 4.5.** *Let $P$ be the polytope given in the proof of Theorem 4.3. There does not exist a smooth Fano $n$-polytope $R$ with $n+k+4$ vertices such that $\mathcal{V}(P) \subset \mathcal{V}(R)$.*

*Proof.* Suppose for a contradiction that there exists a smooth Fano $n$-polytope $R$ with $n+k+4$ vertices such that $\mathcal{V}(R) = \mathcal{V}(P) \cup \{x\}$, where $x \in N$. Since the linear relation

$$v_1 + \cdots + v_a + y_1 + \cdots + y_b = (a+b-1)t \tag{4.6}$$

amongst the vertices of $R$ holds, and since $t$ is a vertex of $R$, by Lemma 3.4 this is a primitive relation in $R$ and $\operatorname{conv}(Y)$ is a face of $R$. Hence the relation

$$t + u_1 + \cdots + u_b = y_1 + \cdots + y_b \tag{4.7}$$

is also a primitive relation in $R$ and $\operatorname{conv}(U)$ is also a face of $R$. Hence the relation

$$y_1 + \cdots + y_b + z = u_1 + \cdots + u_b \tag{4.8}$$

is also a primitive relation in $R$. Moreover, $\operatorname{conv}(Y \cup U)$ is also a face of $R$, and so is $\operatorname{conv}(T)$ for each $T \subset Y \cup U$. Hence

$$w_{1,j} + \cdots + w_{l_j+1,j} = x_{1,j} + \cdots + x_{l_j,j} \tag{4.9}$$

is also a primitive relation in $R$ for each $1 \leq j \leq k$. Thus, by Lemma 3.4,

$$\operatorname{conv}\left(Y \cup U \cup \bigcup_{j=1}^{k} (W_j \setminus \{w^{(j)}\})\right)$$

is a face of $R$ for each $w^{(j)} \in W_j$, as is

$$\operatorname{conv}\left(Y \cup \{t\} \cup (U \setminus \{u\}) \cup \bigcup_{j=1}^{k} (W_j \setminus \{w^{(j)}\})\right)$$

for each $u \in U$ (by (4.7)). From (4.6) and Lemma 3.4 we obtain that

$$\operatorname{conv}\left((V \setminus \{v\}) \cup Y \cup \{t\} \cup (U \setminus \{u\}) \cup \bigcup_{j=1}^{k} (W_j \setminus \{w^{(j)}\})\right)$$

is a facet of $R$ for each $v \in V$, $u \in U$ and $w^{(j)} \in W_j$. By (4.7) and Lemma 3.4 we obtain that

$$\operatorname{conv}\left((V \setminus \{v\}) \cup Y \cup U \cup \bigcup_{j=1}^{k} (W_j \setminus \{w^{(j)}\})\right)$$

is also a facet of $R$. Similarly we have that

$$\operatorname{conv}\left(V \cup (Y \setminus \{y\}) \cup \{t\} \cup (U \setminus \{u\}) \cup \bigcup_{j=1}^{k} (W_j \setminus \{w^{(j)}\})\right) \qquad \text{and}$$

$$\operatorname{conv}\left((V \setminus \{v\}) \cup (Y \setminus \{y\}) \cup \{z\} \cup U \cup \bigcup_{j=1}^{k} (W_j \setminus \{w^{(j)}\})\right)$$

are also facets of $R$ (from (4.6), (4.7), (4.8) and Lemma 3.4).

These four kinds of facets of $R$ are also facets of $P$. In addition, $P$ contains one more kind of facet, which is of the form

$$\operatorname{conv}\left(V \cup (Y \setminus \{y\}) \cup \{z\} \cup (U \setminus \{u\}) \cup \bigcup_{j=1}^{k} (W_j \setminus \{w^{(j)}\})\right)$$

where $y \in Y$, $u \in U$, and $w^{(j)} \in W_j$. Therefore $x$ must belong to the cone generated by this facet, i.e.

$$x \in \operatorname{cone}\left(V \cup (Y \setminus \{y\}) \cup \{z\} \cup (U \setminus \{u\}) \cup \bigcup_{j=1}^{k} (W_j \setminus \{w^{(j)}\})\right). \tag{4.10}$$

Without loss of generality, we may assume that $y = y_b$, $u = u_b$, and $w^{(j)} = w_{l_j+1,j}$ for each $1 \leq j \leq k$. Let

$$x = c_1 v_1 + \cdots + c_{a-1} v_{a-1} + c_a y_1 + \cdots + c_{a+b-2} y_{b-1} + c_{a+b-1} v_a + \\ c_{a+b} u_1 + \cdots + c_{a+2b-2} u_{b-1} + c_{a+2b-1} z + c_{a+2b} w_{1,1} + c_{a+2b+1} w_{2,1} + \cdots + c_n w_{l_k,k}.$$

Let $F = \operatorname{conv}(V \cup Y' \cup \{t\} \cup U' \cup \bigcup_{j=1}^{k} W_j')$, where $Y' = Y \setminus \{y_b\}$, $U' = U \setminus \{u_b\}$, and $W_j' = W_j \setminus \{w_{l_j+1,j}\}$ for $1 \leq j \leq k$. Then $F$ is a facet of $R$. Let $F'$ be the unique facet of $R$ such that $F \cap F'$ is a ridge of $R$ with $\mathcal{V}(F) \setminus \mathcal{V}(F') = \{t\}$. Then it must be that $\mathcal{V}(F') \setminus \mathcal{V}(F) = \{x\}$. In fact, by considering the primitive relations (4.5) and the assumption (4.10), the vertices $y_b$, $z$, $u_b$, and $w_{l_j+1,j}$ cannot belong to $\mathcal{V}(F')$. Thus, by [19, Lemma 2.1], $x + t$ lies in the linear subspace spanned by $V \cup Y' \cup U' \cup \bigcup_{j=1}^{k} W_j'$. From the relation $z + t = 0$ we have $c_{a+2b-1} = 1$. Moreover, again from [19, Lemma 2.1],

$$1 > \langle \mathbf{a}_F, x \rangle = \sum_{\substack{1 \leq i \leq n \\ i \neq a+2b-1}} c_i - 1 > \langle \mathbf{a}_F, z \rangle = -1,$$

where $\mathbf{a}_F$ is the lattice vector defining $F$. Thus

$$\sum_{\substack{1 \leq i \leq n \\ i \neq a+2b-1}} c_i = 1.$$

Namely, $x$ can be written in the form $x = z + \gamma$ for some $\gamma \in V \cup Y' \cup U' \cup \bigcup_{j=1}^{k} W_j'$.

We have the relation $y_1 + \cdots + y_b + x = u_1 + \cdots + u_b + \gamma$. Moreover, for every $\gamma \in V \cup Y' \cup U' \cup \bigcup_{j=1}^{k} W_j'$ there exists a face $F''$ of $R$ containing $y_1, \ldots, y_b$, $u_1, \ldots, u_b$, and $\gamma$. Thus $x$ must be contained in $F''$. This is a contradiction, since $R$ is simplicial. □

As an immediate corollary of Theorem 4.3 we obtain:

**Corollary 4.6.** *There exists an I-isolated smooth Fano $n$-polytope with $n+\rho$ vertices for every $n \geq 5$ and $3 \leq \rho \leq n$.*

*Proof.* If $\rho = 3$ set $a = n-3$ and $b = 2$. By Theorem 4.2 there exists an I-isolated smooth Fano $n$-polytope with $n+3$ vertices. Now suppose that $\rho \geq 4$.

(i) If $\rho = 4$ set $a = n-3$, $b = 1$, $k = 1$, and $l_1 = 2$.
(ii) If $\rho \geq 5$ set $a = n - \rho + 2$, $b = 1$, $k = \rho - 3$, and $l_j = 1$ for each $1 \leq j \leq k$.

By Theorem 4.3 there exists an I-isolated smooth Fano $n$-polytope with $n+\rho$ vertices. □

**Example 4.7.** Consider the polytope

$$\operatorname{conv}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \pm\mathbf{e}_7, \mathbf{e}_7 - \mathbf{e}_1, \pm(\mathbf{e}_7 - \mathbf{e}_2), \mathbf{e}_6 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_6, \mathbf{e}_6 - \mathbf{e}_2 - \mathbf{e}_4, \mathbf{e}_2 - \mathbf{e}_5 - \mathbf{e}_7).$$

This is an I-isolated smooth Fano 7-polytope with 15 vertices, and is the smallest example of an I-isolated smooth Fano $n$-polytope with at least $2n+1$ vertices.

## 5. F-equivalence for reflexive polytopes

In this section we generalise the notion of F-equivalence to encompass projections between reflexive 3-polytopes. Fix a reflexive 3-polytope $P \subset N_{\mathbb{Q}}$ and consider the corresponding Gorenstein toric Fano variety $X_P$. Toric points and curves on $X_P$ correspond to, respectively, facets and edges of $P$. Projections from these points (or curves) correspond to blowing-up the cone generated by the relevant facet (or edge). Since we are restricting ourselves to the class of reflexive polytopes, it is clear that the original face only needs to be considered if it has no (relative) interior points: any interior point of that face will become an interior point of the resulting polytope, preventing it from being reflexive.

**Lemma 5.1.** *Let $P \subset N_{\mathbb{Q}}$ be a reflexive 3-polytope and let $F$ be a facet or edge of $P$ such that $F$ contains no interior points. Up to $\mathrm{GL}_3(\mathbb{Z})$-equivalence, $F$ is one of the four possibilities shown in Table 5.*

**Lemma 5.2.** *Let $P, Q \subset N_{\mathbb{Q}}$ be two reflexive 3-polytopes such that $X_Q$ is obtained from $X_P$ via a projection. Then the corresponding blow-up of the face $F$ of $P$ introduces new vertices as given in Table 5.*

*Proof.* We prove this only in case 2 in Table 5. The remaining cases are similar. We refer to Dais' survey article [8] for background, and for the combinatorial interpretation. Let $C = \operatorname{cone}((1,0,0),(1,a+b,0),(1,a,1),(1,0,1))$, where $a, b \in \mathbb{Z}_{>0}$. This is defined by the intersection of four half-spaces of the form $\{v \in N_{\mathbb{Q}} \mid \langle u_i, v\rangle \geq 0\}$, where the $u_i$ are given by $(0,0,1),(0,1,0),(1,0,-1),(a+b,-1,-b) \in M$. Moving these half-spaces in by one, we obtain the polyhedron:

$$\bigcap_{i=1}^{4} \{v \in N_{\mathbb{Q}} \mid \langle u_i, v\rangle \geq 1\} = \operatorname{cone}((2,1,1),(2,2a+b-1,1)) + C.$$

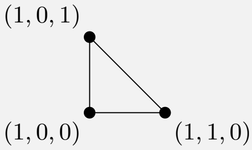
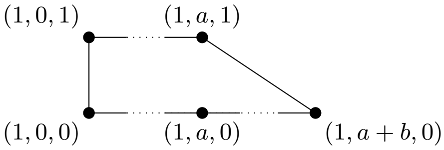
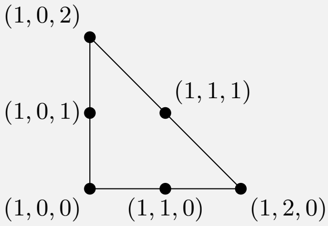
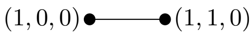
Hence the blow-up is given by the subdivision of $C$ into four cones generated by inserting the rays $(2,1,1)$ and $(2,2a+b-1,1)$. □

Of course case 1 in Table 5 is a specialisation of case 2. However, since the point added is what is important, we list it separately. When $a=1$, $b=0$, or when $a=0$, $b=2$ in case 2, the coordinates of the two points to be added coincide, so we add the single point $(2,1,1)$.

**Definition 5.3.** Two reflexive 3-polytopes $P, Q \subset N_{\mathbb{Q}}$ are *F-equivalent*, and we write $P \overset{F}{\sim} Q$, if there exists a finite sequence $P_0, P_1, \ldots, P_k \subset N_{\mathbb{Q}}$ of reflexive polytopes satisfying:

(i) $P$ and $Q$ are unimodularly equivalent to $P_0$ and $P_k$, respectively;
(ii) For each $1 \leq i \leq k$ we have that either $\mathcal{V}(P_i) \subsetneq \mathcal{V}(P_{i-1})$ or $\mathcal{V}(P_{i-1}) \subsetneq \mathcal{V}(P_i)$.

Table 5. The possible choices of faces to be blown-up, and corresponding points to be added, when considering projections between Gorenstein toric Fano threefolds.

| | Face | Points added |
|---|---|---|
| 1 |   | $(3,1,1)$ |
| 2 |   | $(2,1,1)$, $(2,2a+b-1,1)$ |
| 3 |   | $(3,2,2)$ |
| 4 |   | $(2,1,0)$ |

(iii) If $\mathcal{V}(P_i) \subsetneq \mathcal{V}(P_{i-1})$ then there exists a face $F$ of $P_i$ and $\varphi \in \mathrm{GL}_3(\mathbb{Z})$ such that $\varphi(F)$ is one of the seven faces in Table 5. Furthermore, the points $\varphi(\mathcal{V}(P_{i-1}) \setminus \mathcal{V}(P_i))$ are equal to the corresponding points in Table 5, and $\partial F \subset \partial P_{i-1}$. The case when $\mathcal{V}(P_{i-1}) \subsetneq \mathcal{V}(P_i)$ is similar, but with the roles of $P_{i-1}$ and $P_i$ exchanged.

If $P \overset{F}{\sim} Q$ then $X_P$ and $X_Q$ are related via a sequence of projections.

**Remark 5.4.** The requirement that $\partial F \subset \partial P_{i-1}$ in Definition 5.3(iii) perhaps needs a little explanation. Consider the case when $F$ is a facet. Adding the new vertices can affect a facet $F'$ adjacent to $F$, with common edge $E$, in one of three ways. Let $u_{F'} \in M$ be the primitive dual lattice vector defining the hyperplane at height one containing $F'$. Let $v_1, \ldots, v_s \in N$ be the points to be added according to Table 5, so that $P_{i-1} = \mathrm{conv}(P_i \cup \{v_1, \ldots, v_s\})$.

(i) If $\langle u_{F'}, v_i \rangle < 1$ for $1 \le i \le s$ then $F'$ is unchanged by the addition of the new vertices, and hence $F'$ is also a facet of $P_{i-1}$.

(ii) Suppose that $\langle u_{F'}, v_i \rangle = 1$ for $1 \le i \le m$, and $\langle u_{F'}, v_i \rangle < 1$ for $m+1 \le i \le s$, for some $1 \le m \le s$. Then the facet $F'$ is transformed to the facet $F'' := \mathrm{conv}(F' \cup \{v_1, \ldots, v_m\})$ in $P_{i-1}$. Notice that $F' \subset F''$, but that $E$ is no-longer an edge of $F''$. This is equivalent to a blow-up, followed by a contraction of the curve corresponding to $E$.

(iii) The final possibility is that there exists one (or more) of the $v_i$ such that $\langle u_{F'}, v_i \rangle > 1$. When we pass to $P_{i-1}$ we see that $F'$ is no-longer contained in the boundary; in particular, $\mathrm{int}(E) \subset \mathrm{int}(P_{i-1})$ and so $\partial F \not\subset \partial P_{i-1}$. This case is excluded since it does not correspond to a projection between the two toric varieties.

These three possibilities are illustrated in Figure 2.

By a simple computer calculation we obtain:

**Corollary 5.5.** *Every reflexive 3-polytope is F-equivalent to* $T^3$.

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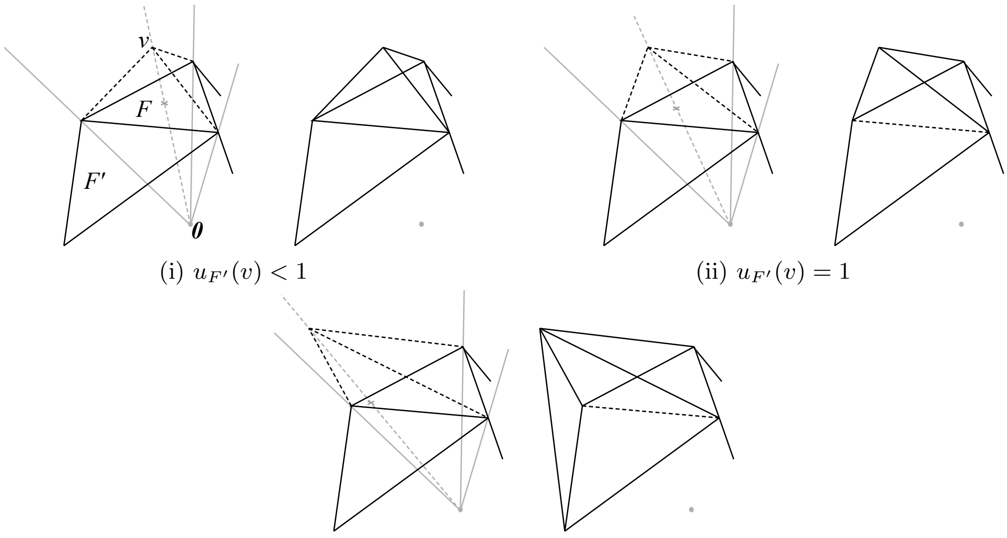
Figure 2. The three possible ways a facet $F'$ adjacent to $F$ can be modified when adding a new vertex $v$. See Remark 5.4 for an explanation.




(i) $u_{F'}(v) < 1$

(ii) $u_{F'}(v) = 1$

(iii) $u_{F'}(v) > 1$

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