A NOTE ON PALINDROMIC δ-VECTORS FOR CERTAIN RATIONAL POLYTOPES

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ABSTRACT. Let P be a convex polytope containing the origin, whose dual is a lattice polytope. Hibi's Palindromic Theorem tells us that if P is also a lattice polytope then the Ehrhart δ-vector of P is palindromic. Perhaps less well-known is that a similar result holds when P is rational. We present an elementary lattice-point proof of this fact.

1. INTRODUCTION

A *rational polytope* $P \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{Q}^n$. We shall assume that P is of maximum dimension, so that $\dim P = n$. Throughout let k denote the smallest positive integer for which the dilation kP of P is a *lattice polytope* (i.e. the vertices of kP lie in $\mathbb{Z}^n$).

A *quasi-polynomial* is a function defined on $\mathbb{Z}$ of the form:

$$q(m) = c_n(m)m^n + c_{n-1}(m)m^{n-1} + \ldots + c_0(m),$$

where the c_i are periodic coefficient functions in m. It is known ([Ehr62]) that for a rational polytope P, the number of lattice points in mP, where $m \in \mathbb{Z}_{\geq 0}$, is given by a quasi-polynomial of degree $n = \dim P$ called the *Ehrhart quasi-polynomial*; we denote this by $L_P(m) := |mP \cap \mathbb{Z}^n|$. The minimum period common to the cyclic coefficients c_i of L_P divides k (for further details see [BSW08]).

Stanley proved in [Sta80] that the generating function for L_P can be written as a rational function:

$$\mathrm{Ehr}_P(t) := \sum_{m \geq 0} L_P(m)t^m = \frac{\delta_0 + \delta_1 t + \ldots + \delta_{k(n+1)-1}t^{k(n+1)-1}}{(1-t^k)^{n+1}},$$

whose coefficients δ_i are non-negative. For an elementary proof of this and other relevant results, see [BS07] and [BR07]. We call $(\delta_0, \delta_1, \ldots, \delta_{k(n+1)-1})$ the *(Ehrhart) δ-vector* of P.

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The *dual polyhedron* of P is given by $P^\vee := \{u \in \mathbb{R}^n \mid \langle u, v\rangle \le 1 \text{ for all } v \in P\}$. If the origin lies in the interior of P then $P^\vee$ is a rational polytope containing the origin, and $P = (P^\vee)^\vee$. We restrict our attention to those P containing the origin for which $P^\vee$ is a lattice polytope.

We give an elementary lattice-point proof that, with the above restriction, the δ-vector is palindromic (i.e. $\delta_i = \delta_{k(n+1)-1-i}$). When P is *reflexive*, meaning that P is also a lattice polytope (equivalently, $k = 1$), this result is known as *Hibi's Palindromic Theorem* [Hib91]. It can be regarded as a consequence of a theorem of Stanley's concerning the more general theory of Gorenstein rings; see [Sta78].

2. The main result

Let P be a rational polytope and consider the Ehrhart quasi-polynomial L_P. There exist k polynomials $L_{P,r}$ of degree n in l such that when $m = lk + r$ (where $l, r \in \mathbb{Z}_{\ge 0}$ and $0 \le r < k$) we have that $L_P(m) = L_{P,r}(l)$. The generating function for each $L_{P,r}$ is given by:

$$\mathrm{Ehr}_{P,r}(t) := \sum_{l \ge 0} L_{P,r}(l) t^l = \frac{\delta_{0,r} + \delta_{1,r} t + \ldots + \delta_{n,r} t^n}{(1-t)^{n+1}}, \tag{2.1}$$

for some $\delta_{i,r} \in \mathbb{Z}$.

Theorem 2.1. *Let P be a rational n-tope containing the origin, whose dual $P^\vee$ is a lattice polytope. Let k be the smallest positive integer such that kP is a lattice polytope. Then:*

$$\delta_{i,r} = \delta_{n-i,k-r-1}.$$

Proof. By Ehrhart–Macdonald reciprocity ([Ehr67, Mac71]) we have that:

$$L_P(-lk - r) = (-1)^n L_{P^\circ}(lk + r),$$

where L_{P° enumerates lattice points in the strict interior of dilations of P. The left-hand side equals $L_P(-(l+1)k + (k-r)) = L_{P,k-r}(-(l+1))$. We shall show that the right-hand side is equal to $(-1)^n L_P(lk + r - 1) = (-1)^n L_{P,r-1}(l)$.

Let $H_u := \{v \in \mathbb{R}^n \mid \langle u, v\rangle = 1\}$ be a bounding hyperplane of P, where $u \in \operatorname{vert} P^\vee$. By assumption, $u \in \mathbb{Z}^n$ and so the lattice points in $\mathbb{Z}^n$ lie at integer heights relative to H_u; i.e. given $u' \in \mathbb{Z}^n$ there exists some $c \in \mathbb{Z}$ such that $u' \in \{v \in \mathbb{R}^n \mid \langle u, v\rangle = c\}$. In particular, there do not exist lattice points at non-integral heights. Since:

$$P = \bigcap_{u \in \operatorname{vert} P^\vee} H_u^-,$$

where H_u^- is the half-space defined by H_u and the origin, we see that $(mP^\circ) \cap \mathbb{Z}^n = ((m-1)P) \cap \mathbb{Z}^n$. This gives us the desired equality.

We have that $L_{P,k-r}(-(l+1)) = (-1)^n L_{P,r-1}(l)$. By considering the expansion of (2.1) we obtain:

$$\begin{aligned}\sum_{i=0}^{n} \delta_{i,k-r} \binom{-(l+1)+n-i}{n} &= L_{P,k-r}(-(l+1)) \\ &= (-1)^n L_{P,r-1}(l) = (-1)^n \sum_{i=0}^{n} \delta_{i,r-1} \binom{l+n-i}{n}.\end{aligned}$$

But $\binom{-(l+1)+n-i}{n} = (-1)^n \binom{l+i}{n}$, and since $\binom{l}{n}, \binom{l+1}{n}, \ldots, \binom{l+n}{n}$ form a basis for the vector space of polynomials in l of degree at most n, we have that $\delta_{i,k-r} = \delta_{n-i,r-1}$. $\square$

Corollary 2.2. *The δ-vector of P is palindromic.*

Proof. This is immediate once we observe that:

$$\mathrm{Ehr}_P(t) = \mathrm{Ehr}_{P,0}(t^k) + t\mathrm{Ehr}_{P,1}(t^k) + \ldots + t^{k-1}\mathrm{Ehr}_{P,k-1}(t^k).$$

$\square$

3. Concluding remarks

The crucial observation in the proof of Theorem 2.1 is that $(mP^\circ) \cap \mathbb{Z}^n = ((m-1)P) \cap \mathbb{Z}^n$. In fact, a consequence of Ehrhart–Macdonald reciprocity and a result of Hibi [Hib92] tells us that this property holds if and only if $P^\vee$ is a lattice polytope. Hence rational convex polytopes whose duals are lattice polytopes are characterised by having palindromic δ-vectors. This can also be derived from Stanley's work [Sta78] on Gorenstein rings.

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