Simple Lie Algebras having Extremal Elements

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Contents

Introduction – Lie Algebras – Example

Basefield is algebraically closed, char $\neq 2, 3$.

Definition ($\mathfrak{sl}_2$ (type A_1))

The 2×2 matrices of trace 0, a basis is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and multiplication given by

$$[x, y] := xy - yx.$$

Observe: $h = [e, f]$; $[e, [e, f]] = -2e$; $[f, [f, e]] = -2f$.

Introduction – Extremal Elements

$$A_1 : h = [e,f];$$
$$[e,[e,e]] = 0; \quad [e,[e,f]] = -2e; \quad [e,[e,h]] = -2[e,e] = 0;$$
$$[f,[f,e]] = -2f; \quad [f,[f,f]] = 0; \quad [f,[f,h]] = 2[f,f] = 0;$$

Notation: $\mathrm{ad}_x = [x, \cdot]$.

Definition (Extremal Elements)

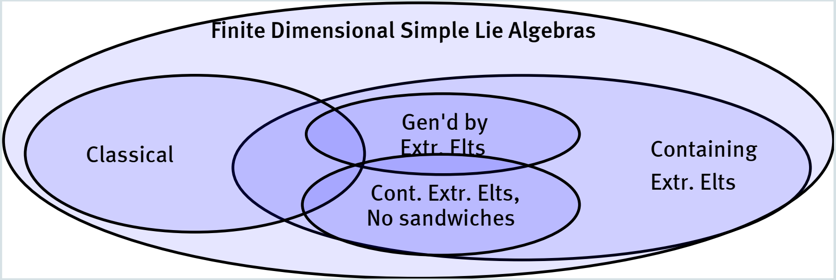
$x \in L$ is called *extremal* if $[x,[x,L]] \subseteq \mathbb{F}x$, i.e. $\mathrm{ad}_x^2(L) \subseteq \mathbb{F}x$.
$x \in L$ is called a *sandwich* if $\mathrm{ad}_x^2(L) = 0$.

Observe, by bilinearity, if x is extremal but not a sandwich,

$$\mathrm{ad}_x^2(L) = \mathbb{F}x.$$

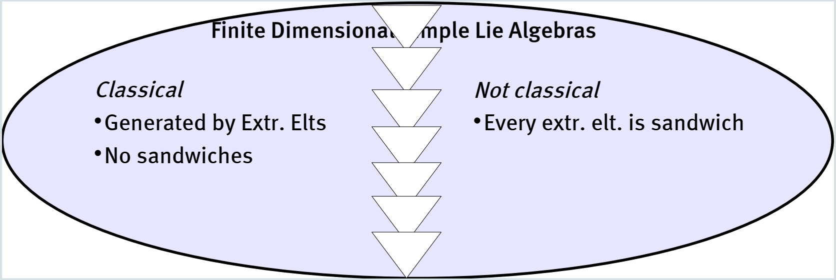
Previous results (1): Starting point

(Field is algebraically closed and not of characteristic 2, 3)



Previous results (2): Classification

(Field is algebraically closed and not of characteristic 2, 3, 5)



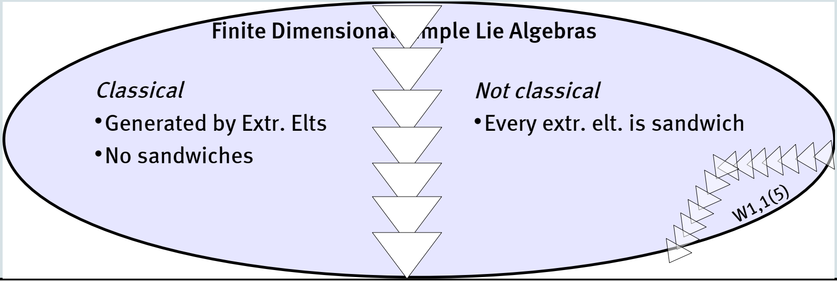
Due to Premet, Strade, Benkart, Block, Kostrikin, et al.

New result

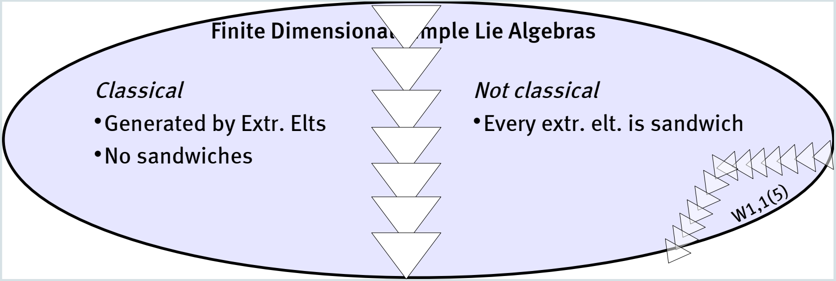
Theorem [Cohen, Ivanyos, R.; 2007]

L a simple finite dimensional Lie algebra, $\mathrm{char}(\mathbb{F}) \neq 2, 3$, L has an extremal element that is not a sandwich. Then

- Either L is generated by extremal elements,
- Or $\mathrm{char}(\mathbb{F}) = 5$ and $L \cong W_{1,1}(5)$.



New result



New:

- $\mathbb{F}$ is not necessarily algebraically closed,
- Characteristic 5 is included,
- Elementary proof.

A lemma (1)

Lemma

Suppose $S = \langle x, y, [x, y] \rangle$ *is an* $\mathfrak{sl}_2$*-triple in* L*. If* x *is extremal, then* y *acts quadratically on* L/S*, i.e.* $\mathrm{ad}_y^2(L/S) = 0$.

- We consider L/S as an S-module,
- and use the GAP package GBNP (*Gröbner Bases for Non-commutative Polynomials*) to find a proof.

A lemma (2)

- write X, Y for the action of $\mathrm{ad}_x, \mathrm{ad}_y$ on $\mathrm{End}(L/S)$,
- $[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_x\mathrm{ad}_y - \mathrm{ad}_y\mathrm{ad}_x$, so $[X, Y] = XY - YX$.

Calculate in $\mathrm{End}(L/S)$:

$$[x,[x,y]] = -2x \quad \Rightarrow \quad (\mathrm{R1})\ X^2Y - 2XYX + YX^2 + 2X = 0$$

$$[y,[y,x]] = -2y \quad \Rightarrow \quad (\mathrm{R2})\ -XY^2 + 2YXY - Y^2X - 2Y = 0$$

$$\mathrm{ad}_x^2(L) \subseteq \mathbb{F}x \subseteq S \quad \Rightarrow \quad \text{so } \mathrm{ad}_x^2(L) = 0 \text{ in } L/S\ (\mathrm{R3})\ X^2 = 0,$$

Use GBNP to compute (traced) GB, fiddle around, and find

$$(\mathrm{R1}), (\mathrm{R3}) \quad \Rightarrow \quad (\mathrm{R4})\ XYX - X = 0,$$

$$(\mathrm{R3}), (\mathrm{R4}) \quad \Rightarrow \quad (\mathrm{R5})\ XY^2X = 0,$$

A lemma (3)

Denote by R_2 the left hand side of (R2). GBNP gives us:

$$\begin{aligned}
0 &= YR_2YX - YXYR_2 + 2Y^2XR_2 - R_2YXY + XYR_2Y - 3YR_2 \\
&\quad -2YXR_2Y + 3R_2Y - 2YXR_2Y - 6R_2Y + 2XR_2Y^2 \\
&\overset{(R3)}{=} 12Y^2 - 3XY^3 + 7YXY^2 - 5Y^2XY + Y^3X + 3XYXY^3 \\
&\quad -7YXYXY^2 + 5Y^2XYXY - Y^3XYX \\
&\overset{(R3)}{=} 12Y^2 - 3XY^3 + 7YXY^2 - 5Y^2XY + Y^3X + 3XY^3 \\
&\quad -7YXY^2 + 5Y^2xY - Y^3X \\
&\overset{(R2)}{=} 12Y^2,
\end{aligned}$$

so that Y^2 is 0 if $12 \neq 0$, so $\mathrm{ad}_y^2(L/S) = 0$.

The Witt algebra $W_{1,1}(5)$ and $\widetilde{W_{1,1}}(5)$ (1)

Definition ($W_{1,1}(5)$)

A Lie algebra over $\mathbb{F}_5$ with basis elements $\partial_z, z\partial_z, z^2\partial_z, z^3\partial_z, z^4\partial_z$, and multiplication for example

$$\begin{aligned}[z^1\partial_z, z^3\partial_z] &:= z^1\partial_z(z^3\partial_z) - z^3\partial_z(z^1\partial_z) && (1)\\ &= 3z^{1+2}\partial_z - 1z^{3+0}\partial_z = 2z^3\partial_z, && (2)\end{aligned}$$

where $z^i\partial_z := 0$ if $i \notin \{0, 1, 2, 3, 4\}$.

And its central extension:

Definition ($\widetilde{W_{1,1}}(5)$ (Block, 1966))

A Lie algebra over $\mathbb{F}_5$ with basis elements $\partial_z, z\partial_z, z^2\partial_z, z^3\partial_z, z^4\partial_z$, $z^6\partial_z$, with the same multiplication.

The Witt algebra $W_{1,1}(5)$ and $\widetilde{W_{1,1}(5)}$ (2)

Definition ($\widetilde{W_{1,1}(5)}$) (Block, 1966))

A Lie algebra over $\mathbb{F}_5$ with basis elements $\partial_z, z\partial_z, z^2\partial_z, z^3\partial_z, z^4\partial_z, z^6\partial_z$, with the same multiplication.

Observe:

$$\begin{aligned}
[-z^2\partial_z,[-z^2\partial_z,\partial_z]] &= [z^2\partial_z,(-2)z\partial_z] = 2z^2\partial_z,\\
[-z^2\partial_z,[-z^2\partial_z,z\partial_z]] &= [z^2\partial_z,(-1)z^2\partial_z] = 0,\\
[-z^2\partial_z,[-z^2\partial_z,z^2\partial_z]] &= 0,\\
[-z^2\partial_z,[-z^2\partial_z,z^3\partial_z]] &= [z^2\partial_z,z^4\partial_z] = 0,\\
[-z^2\partial_z,[-z^2\partial_z,z^4\partial_z]] &= 0,\\
[-z^2\partial_z,[-z^2\partial_z,z^6\partial_z]] &= 0,
\end{aligned}$$

The Witt algebra $W_{1,1}(5)$ and $\widetilde{W_{1,1}(5)}$ (3)

Definition ($\widetilde{W_{1,1}(5)}$ (Block, 1966))

A Lie algebra over $\mathbb{F}_5$ with basis elements $\partial_z, z\partial_z, z^2\partial_z, z^3\partial_z, z^4\partial_z, z^6\partial_z$, with the same multiplication.

Observe: $-z^2\partial_z$ is extremal, $\langle -z^2\partial_z, \partial_z, 2z\partial_z\rangle$ is an $\mathfrak{sl}_2$-triple, and $[\partial_z, [\partial_z, 2z^4\partial_z]] = -z^2\partial_z$, so ∂_z is not extremal.

Proof sketch – General case (1)

1. x is an extremal element of L,
2. Find $\mathfrak{sl}_2 : x, y, h$, (adapted Jacobson-Morozov),
3. Show that ad_h induces a *grading* of L:

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

i.e.

- $v \in L_i \Rightarrow [v, h] = iv$,
- $[L_i, L_j] \subseteq L_{i+j}$,

and $x \in L_{-2}$ and $y \in L_2$.

Proof sketch – General case (2)

3. Show that ad_h induces a *grading* of L:

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

4. Show that y is extremal (unless char. 5 case),
5. Define the ideal $I = \langle x, y, L_1 \rangle$, by simplicity $I = L$,
6. Find for every $z \in L_{-1}$ an extremal element $u \in L_1$ such that $z \in \langle x, y, u \rangle$,
7. Conclude that L is generated by extremal elements.

Proof sketch – Characteristic 5 case (1)

ad_h induces a grading of L:

$$\begin{array}{ccccccccccc} L & = & L_{-2} & \oplus & L_{-1} & \oplus & L_0 & \oplus & L_1 & \oplus & L_2 \\ & & = \mathbb{F}x & & & & \ni h & & & & = \mathbb{F}y \end{array}$$

$$\longleftarrow \mathrm{ad}_x$$
$$\mathrm{ad}_y \longrightarrow$$

Now suppose y is not extremal, i.e. $[y, [y, L]] \not\subseteq \mathbb{F}y$. Then:

- $[y, L_1] \neq 0$, but $[y, L_1] \subseteq L_3$, so $p = 5$ and $[y, L_1] = L_{-2} = \mathbb{F}x$,
- It follows that $[y, [y, L_{-1}]] = \mathbb{F}x$.

Proof sketch – Characteristic 5 case (2)

It follows that $[y, [y, L_{-1}]] = \mathbb{F}x$, so there exists a $v \in L_{-1}$ such that

$$[y, [y, v]] = x.$$

- Define W to be the linear span in L of $\{x, y, h, v, [v, y], [v, [v, y]]\}$.
- Calculate all products, by hand,
- Prove that there is a surjective morphism $\varphi : \widetilde{W_{1,1}(5)} \to W$.

Proof sketch – Characteristic 5 case (3)

Prove that there is a surjective morphism $\varphi : \widetilde{W_{1,1}(5)} \to W$.

- It remains to prove that $L = W$, since then $L \cong W_{1,1}(5)$ (since $\widetilde{W_{1,1}(5)}$ is not simple).

Calculate in $\mathrm{End}(L/W)$:

$$\begin{array}{lcl} & & \text{(R6) } Y^2 = 0 \\ [y,[y,v]] = x & \Rightarrow & \text{(R7) } Y^2V - 2YVY + VY^2 - X = 0 \\ [x,[v,y]] = -v & \Rightarrow & \text{(R8) } XVY - XYV - VXY + YVX + V = 0 \\ \text{(R6), (R7)} & \Rightarrow & \text{(R9) } X + 2YVY = 0, \\ \text{(R6), (R2)} & \Rightarrow & \text{(R10) } Y - YXY = 0, \end{array}$$

Proof sketch – Characteristic 5 case (4)

Denote by R_9, R_{10} the left hand side of (R9), (R10), respectively; GBNP gives us:

$$\begin{aligned} 0 &= R_9(1 - XY) - 2YVR_{10} \\ &= (X + 2YVY)(1 - XY) - 2YV(Y - YXY) \\ &= X + 2YVY - X^2Y - 2YVYXY - 2YVY + 2YVYXY \\ &= X, \end{aligned}$$

it follows that $Y = 0$ and $V = 0$.

Proof sketch – Characteristic 5 case (5)

- Started with W is linear span of $\{x, y, h, v, [v, y], [v, [v, y]]\}$,
- We proved $X = Y = V = 0$.

Conclusion

- So the images of ad_w $(w \in W)$ in $\mathrm{End}(L/W)$ are trivial, so W is an ideal of L.
- But L is simple and W is nontrivial, so $L \cong W$.
- Recall $\varphi : \widetilde{W_{1,1}}(5) \to W$, that $W_{1,1}(5)$ is simple, and that $\widetilde{W_{1,1}}(5)$ is nonsimple.
- So $[v, [v, y]] = 0$ and $L \cong W_{1,1}(5)$.

Conclusion

Theorem [Cohen, Ivanyos, R.; 2007]

Let L be a simple finite dimensional Lie algebra over a field $\mathbb{F}$ of characteristic not $2, 3$. Suppose L has an extremal element that is not a sandwich. Then

- Either L is generated by extremal elements,
- Or $\mathrm{char}(\mathbb{F}) = 5$ and $L \cong W_{1,1}(5)$.

With an elegant, constructive proof brought to you by GAP and GBNP!