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Simple Lie algebras having extremal elements

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ABSTRACT

Let L be a simple finite-dimensional Lie algebra of characteristic distinct from 2 and from 3. Suppose that L contains an extremal element that is not a sandwich, that is, an element x such that $[x, [x, L]]$ is equal to the linear span of x in L. In this paper we prove that, with a single exception, L is generated by extremal elements. The result is known, at least for most characteristics, but the proofs in the literature are involved. The current proof closes a gap in a geometric proof that every simple Lie algebra containing no sandwiches (that is, ad-nilpotent elements of order 2) is in fact of classical type.

1. INTRODUCTION

Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic distinct from 2. An element $x \in L$ is said to be *extremal* if $[x, [x, L]] \subseteq \mathbb{F}x$. If $[x, [x, L]] = 0$ we say x is a *sandwich*. By Premet [8,9], every finite-dimensional simple Lie algebra over an algebraically closed field of characteristic distinct from 2 and 3 is known to have an extremal element; see [13] for a self-contained proof in case $p > 5$. If a simple Lie algebra is finite-dimensional and generated by extremal elements, then it is of classical type. This fact follows from the classification of finite-dimensional simple Lie algebras as described in [10–12], but can also be derived from geometric arguments using the theory of buildings, cf. [4,5], up to small rank cases and the verification that the building determines a unique Lie algebra generated by extremal elements up

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to isomorphism – a subject of ongoing work. None of these extremal elements are sandwiches; see [6, Remark 9.9].

In order to use these two results for a revision of the classification of simple Lie algebras of classical type, the gap between the two has to be filled. In other words, an elementary proof would be needed of the fact that a simple Lie algebra over an algebraically closed field of characteristic distinct from 2 and 3 having an extremal element that is not a sandwich is generated by extremal elements.

Using powerful methods, Benkart [1, Theorem 3.2] shows that if a simple Lie algebra over an algebraically closed field of characteristic $p \geqslant 7$ or $p = 0$ contains a nilpotent element of order at most $p - 1$ and no sandwiches, then it is of classical type. Together with the abovementioned results of Premet, this gives that any simple Lie algebra over an algebraically closed field of characteristic 0 or greater than 5 without sandwiches is of classical type. Since Benkart's methods are rather involved, this paper is devoted to a self-contained proof of the observed gap and an extension to the case of characteristic 5. The field need not be algebraically closed. Our extension allows for one more example of a simple Lie algebra having a non-sandwich extremal element, namely the 5-dimensional Witt algebra $W_{1,1}(5)$ over a field of characteristic 5 (see Example 3.1 for an explicit description of this Lie algebra). It is a counterexample in that it only contains one such element up to scalar multiples.

Theorem 1.1. *Let $\mathbb{F}$ be a field of characteristic distinct from 2 and 3, and let L be a simple Lie algebra over $\mathbb{F}$. Suppose that L contains an extremal element that is not a sandwich. Then either $\mathbb{F}$ has characteristic 5 and L is isomorphic to $W_{1,1}(5)$ or L is generated by extremal elements.*

The counterexample was known to Alexander Premet. We are grateful for fruitful discussions with him about our work. We would also like to thank Helmut Strade for the insight he provided us into the classification of modular Lie algebras.

We briefly outline the paper. In Section 2 we find that a Lie algebra containing an extremal element that is not a sandwich either has more extremal elements or is defined over a field of characteristic 5 and has a particular Lie subalgebra. Elementary proofs for most of the statements in Section 2 were known before 1977; see [1] and references therein. We include proofs here for the sake of completeness. We gratefully acknowledge David Wales' contribution in the guise of Proposition 2.1. In Section 3, we pin down the exceptional case in characteristic 5, and in Section 4 we show that, in the absence of the exceptional case, there are many more extremal elements and finish the proof of the main theorem. The proofs of Lemma 2.2 and Proposition 3.2 were found by experiments with the GAP computer system package GBNP; see [3].

To finish the introduction, we fix some notation of use throughout the paper.

Notation 1.2. Throughout this paper, $\mathbb{F}$ will be a field whose characteristic is denoted by p, and L will be a Lie algebra over $\mathbb{F}$.

An element $x \in L$ is said to be *extremal on* M if $[x,[x,M]] \subseteq \mathbb{F}x$. If x is extremal on L and no confusion is imminent, we call x *extremal*. We write $\mathcal{E}_L(M)$ for the set of elements extremal on M and abbreviate $\mathcal{E}_L(L)$ to $\mathcal{E}_L$. Furthermore, we write $\mathcal{E}_L(y)$ for $\mathcal{E}_L(\{y\})$.

Similarly, if $[x,[x,M]] = 0$ we say x is a *sandwich on* M, and if $[x,[x,L]] = 0$ we simply call x a *sandwich*. We write $\mathcal{S}_L(M)$ for the set of sandwiches on M. Again, we write $\mathcal{S}_L(y)$ for $\mathcal{S}_L(\{y\})$.

By linearity of the expression $[x,[x,m]]$ in m, we have $\mathcal{E}_L(M) = \mathcal{E}_L(\langle M\rangle)$, where $\langle M\rangle$ denotes the linear subspace of L spanned by M. Hence, when writing $\mathcal{E}_L(M)$, we may assume that M is a linear subspace of L, and similarly for $\mathcal{S}_L(M)$. For $x \in \mathcal{E}_L(M)$ and $m \in M$, we define $f_x(m) \in \mathbb{F}$ to be such that $[x,[x,m]] = f_x(m)x$.

2. JACOBSON-MOROZOV TYPE RESULTS

For extremal elements we present a slightly better version of the well-known theorem by Jacobson and Morozov. The original result, ascribed to Morozov in [7, p. 98], is adapted by David Wales to extremal elements and works for characteristic at least 5.

Proposition 2.1. *Suppose that p is distinct from 2 and 3 and that L contains an extremal element x. If w is an element for which $f_x(w) = -2$, then, with $h = [x,w]$, there is $y \in L$ for which*

$$(1) \qquad [x,y] = h, \qquad [h,x] = 2x \quad \text{and} \quad [h,y] = -2y.$$

The three elements x, y, h are the usual generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ of 2×2 matrices of trace 0 over $\mathbb{F}$. A triple satisfying the relations (1) is called an $\mathfrak{sl}_2$-triple.

Proof of Proposition 2.1. Let $X = \mathrm{ad}_x$. Let $h = [x,w]$ and $H = \mathrm{ad}_h$. The hypothesis $f_x(w) = -2$ means $[x,[x,w]] = -2x$. In particular $[h,x] = -[x,[x,w]] = 2x$ as required.

We know X is nilpotent as $[x,[x,[x,y]]] = [x, f_x(y)x] = 0$ and so $X^3 = 0$.

Let $C_L(x) = \{u \in L \mid [u,x] = 0\} = \operatorname{Ker} X$. The following computation shows that $[w,h] - 2w \in C_L(x)$:

$$\begin{aligned}\big[[w,h] - 2w, x\big] &= \big[[w,h],x\big] - 2[w,x] = \big[[w,x],h\big] + \big[w,[h,x]\big] - 2(-h)\\ &= [-h,h] + [w,2x] + 2h = 2(-h) + 2h\\ &= 0.\end{aligned}$$

Consequently, $[w,h] = 2w + x_1$ where $x_1 \in C_L(x)$. We claim $C_L(x)$ is H-invariant. To see this notice $[X,H] = -2X$ and so $XH - HX = -2X$. Suppose $u \in C_L(x)$. Then $XHu = HXu - 2Xu = 0$ and so $X(Hu) = 0$, proving $Hu \in C_L(x)$ as claimed.

Next we consider the action of H on $C_L(x)$. Let $u \in C_L(x)$. Now, with $W = \mathrm{ad}_w$,

$$Hu = [X, W]u = XWu - WXu = XWu \in XL$$

and

$$\begin{aligned}\left[X^2, W\right] &= X^2W - WX^2 = X[X, W] + [X, W]X \\ &= XH + HX = HX - 2X + HX = 2(H-1)X,\end{aligned}$$

so

$$2(H-1)Hu = 2(H-1)XWu = \left(X^2W - WX^2\right)Wu.$$

But $XW = [X, W] + WX = H + WX$ and so $X^2Wu = XHu + XWXu = 0$. In particular, $2(H-1)Hu = X^2W^2u$. As x is extremal, for any $z \in L$ we have $X^2z = f_x(z)x$, and so $(H-2)X^2z = (H-2)f_x(z)x = 0$ as $Hx = 2x$. Now

$$2(H-2)(H-1)Hu = (H-2)X^2W^2u = 0.$$

This means the eigenvalues of H acting on $C_L(x)$ are $0, 1, 2$ and as the characteristic is at least 5 we see -2 is not an eigenvalue. In particular $H+2$ is nonsingular on $C_L(x)$. Pick $w_1 \in C_L(x)$ for which $(H+2)w_1 = x_1$ and so $[h, w_1] = x_1 - 2w_1$. Set $y = w + w_1$. Now $[x, y] = [x, w + w_1] = h + 0 = h$ and so $[x, y] = h$. Also $[h, y] = [h, w + w_1] = (-2w - x_1) + (x_1 - 2w_1) = -2(w + w_1) = -2y$. This completes the proof of the proposition. □

In the remainder of this section we suppose that $x, y, h \in L$ are an $\mathfrak{sl}_2$-triple. In fact, the triple is determined by the pair x, y and the relations

$$\left[[x, y], x\right] = 2x \quad \text{and} \quad \left[[x, y], y\right] = -2y, \tag{2}$$

as $h = [x, y]$. Such a pair will be called an $\mathfrak{sl}_2$-pair. Note that, if x is extremal, this implies $f_x(y) = -2$.

Lemma 2.2. *Suppose that $\mathbb{F}$ is of characteristic $p \neq 2, 3$ and that x and y are an $\mathfrak{sl}_2$-pair in L. Set $S = \langle x, y, [x, y]\rangle$. If $x \in \mathcal{E}_L$, then y acts quadratically on L/S, i.e., $\mathrm{ad}_y^2(L/S) = 0$.*

Proof. S is a Lie subalgebra of L isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. Consider L as a module on which S acts. Obviously S is an invariant subspace, so L/S is an S-module. Write X, Y for the action of ad_x, ad_y, respectively, on L/S. As $\mathrm{ad}_x^2(L) \subseteq \mathbb{F}x \subseteq S$, we have $X^2 = 0$. We list the relations (2) in terms of X and Y, and the quadraticity of X that we just found.

$$X^2Y - 2XYX + YX^2 + 2X = 0, \tag{R1}$$
$$-XY^2 + 2YXY - Y^2X - 2Y = 0, \tag{R2}$$
$$X^2 = 0. \tag{R3}$$

The relations (R1) and (R3) immediately imply

(R4) $\quad XYX - X = 0.$

Multiplying (R2) from the left by X gives

$$-X^2Y^2 + 2XYXY - XY^2X - 2XY = 0,$$

which, after application of (R3) and (R4), gives

(R5) $\quad XY^2X = 0.$

Denote by R_2 the left-hand side of (R2). Then, by (R3),

$$\begin{aligned}
0 &= YR_2YX - YXYR_2 + 2Y^2XR_2 - R_2YXY + XYR_2Y - 3YR_2 \\
&\quad - 2YXR_2Y + 3R_2Y - 2YXR_2Y - 6R_2Y + 2XR_2Y^2 \\
&= 12Y^2 - 3XY^3 + 7YXY^2 - 5Y^2XY + Y^3X + 3XYXY^3 \\
&\quad - 7YXYXY^2 + 5Y^2XYXY - Y^3XYX.
\end{aligned}$$

Replacing XYX by X and X^2 by 0, using (R4) and (R3), we find

$$\begin{aligned}
0 &= 12Y^2 - 3XY^3 + 7YXY^2 - 5Y^2XY + Y^3X + 3XY^3 \\
&\quad - 7YXY^2 + 5Y^2XY - Y^3X \\
&= 12Y^2.
\end{aligned}$$

As $p \neq 2, 3$, we conclude that $Y^2 = 0$. $\square$

For $a \in \mathrm{End}(L)$ and $\lambda \in \mathbb{F}$, we denote by $L_\lambda(a)$ the λ-eigenspace of a in L.

Theorem 2.3. *Suppose that $\mathbb{F}$ is a field of characteristic $p \neq 2, 3$, that L is a Lie algebra over $\mathbb{F}$, and that x is an extremal element of L that is not a sandwich. Then there are $y, h \in L$ such that x, y, h is an $\mathfrak{sl}_2$-triple. Moreover, for each such a triple, ad_h is diagonizable with eigenvalues $0, \pm 1, \pm 2$ and satisfies $L_{-2}(-\mathrm{ad}_h) = \mathbb{F}x$ and $L_2(-\mathrm{ad}_h) = \mathbb{F}y$.*

Proof. As x is not a sandwich and the characteristic of $\mathbb{F}$ is not 2, there is $w \in L$ with $f_x(w) = -2$. By Proposition 2.1 with $h = [x, w]$, there is $y \in L$ such that x, y, h are an $\mathfrak{sl}_2$-triple. They generate a Lie subalgebra S of L isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. Viewing L as an S-module as in the proof of Lemma 2.2, we see that S itself is an invariant submodule. Denote by X, Y, and H the actions of ad_x, ad_y, and ad_h, respectively, on the quotient module L/S. As $x \in \mathcal{E}_L$, we have $X^2 = 0$. By Lemma 2.2, also $Y^2 = 0$. It readily follows that the subalgebra of $\mathrm{End}(L/S)$ generated by X, Y, and H is linearly spanned by $1, X, Y, H, XY, XH$, and YH, and that the relation $H^3 = H$ is satisfied. In particular, H is diagonizable on L/S with eigenvalues 0, 1, and -1 only. Consequently, there are subspaces U and V of L such that $L = S + U + V$ is a direct sum of subspaces such that $(S + U)/S = \mathrm{Ker}(H^2 - 1)$

and $(S+V)/S = \operatorname{Ker} H$. Notice that ad_h has eigenvalues $-2, 0, 2$ on S, each with multiplicity 1. A small computation shows that actually $\operatorname{Ker} \mathrm{ad}_h^2 = \operatorname{Ker} \mathrm{ad}_h$, so that $-\mathrm{ad}_h$ is diagonizable with eigenspaces $L_i(-\mathrm{ad}_h)$ $(i = -2, -1, 0, 1, 2)$ satisfying $L_{-2}(-\mathrm{ad}_h) = \mathbb{F}x$ and $L_2(-\mathrm{ad}_h) = \mathbb{F}y$. □

To end this section, we exploit the ad_h-grading with five components. The following result is a slight variation of [4, Proposition 22].

Proposition 2.4. *Suppose $x \in \mathcal{E}_L$ and $y \in L$ are an $\mathfrak{sl}_2$-pair in a Lie algebra L of characteristic $p > 3$. Let $L_i = L_i(-\mathrm{ad}_h)$ $(i = -2, -1, 0, 1, 2)$ be the components of the $\mathbb{Z}_p$-grading by $h = [x, y]$. Then either $p = 5$ and $[y, [y, v]] = x$ for some $v \in L_{-1}$, or y is extremal in L, the components L_i $(i = -2, -1, 0, 1, 2)$ actually give a $\mathbb{Z}$-grading of L, with $L_{-2} = \mathbb{F}x$, $L_2 = \mathbb{F}y$, $[x, L_{-1}] = L_1$, and $[y, L_1] = L_{-1}$.*

Proof. Set $S = \mathbb{F}x + \mathbb{F}y + \mathbb{F}h$. By assumption, $S \cong \mathfrak{sl}_2(\mathbb{F})$. The identifications of L_{-2} and L_2 with $\mathbb{F}x$ and $\mathbb{F}y$, respectively, were established in Theorem 2.3. Suppose that y is not an extremal element. As $\mathrm{ad}_y^2 L_i \subseteq \mathbb{F}y$ for $i \neq \pm 1$ and $\mathrm{ad}_y L_{-1} \subseteq L_1$, this can only happen if $[y, L_1] \neq 0$. Then, by the grading properties, $[y, L_1] \subseteq L_3(-\mathrm{ad}_h)$ and so 3 is equal to a member i of $\{-2, -1, 0, 1, 2\}$ modulo p. As $p \geqslant 5$, this implies $p = 5$ and $i = -2$. Thus $[y, L_1] = \mathbb{F}x$. It follows that, for every $u \in L_1$, $\mathrm{ad}_x \mathrm{ad}_y u = 0$, whence $\mathrm{ad}_y \mathrm{ad}_x u = (\mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_h) u = -u$. Therefore $[y, [y, L_{-1}] \supseteq [y, [y, [x, L_1]]] = [y, L_1] = \mathbb{F}x$, and, by homogeneity, $[y, [y, L_{-1}]] \subseteq L_{-2} = \mathbb{F}x$, so the first case holds. To complete the proof, assume that both x and y are extremal. The argument above also shows that if $[y, L_1] \neq 0$ then $p = 5$ and $[y, L_1] = \mathbb{F}x$. It follows then that $[y, [y, L_1]] = \mathbb{F}h \not\subseteq \mathbb{F}y$, a contradiction to extremality of y. Thus $[y, L_1] = 0$ and, similarly, $[x, L_{-1}] = 0$. It follows that for every pair i, j from the interval $[-2, 2]$, we have $[L_i, L_j] = 0$ whenever the ordinary sum $i + j$ falls outside the interval $[-2, 2]$. Thus the grading is indeed a $\mathbb{Z}$-grading. To see the very last two equalities of the proposition just notice that for every $u \in L_1$ we have $\mathrm{ad}_y \mathrm{ad}_x u = -u$ for every $u \in L_1$ as observed above and, similarly, $\mathrm{ad}_x \mathrm{ad}_y v = -v$ for every $v \in L_{-1}$. □

3. THE CHARACTERISTIC 5 CASE

Suppose that $p = 5$, and that x is an extremal element of L that is not a sandwich. By Proposition 2.1 there are $y, h \in L$ such that x and y are an $\mathfrak{sl}_2$-triple. By Theorem 2.3, ad_h is diagonizable and there exists a grading of L by $-\mathrm{ad}_h$ eigenspaces L_i $(i = -2, -1, 0, 1, 2)$. In this section we consider the case where y is not an extremal element. By Proposition 2.4 there exists an element $v \in L_{-1}$ such that $[y, [y, v]] = x$.

Example 3.1. Before we proceed, we show that this case actually occurs. The 5-dimensional Witt algebra $W_{1,1}(5)$ can be defined as follows. Let $\mathbb{F}$ be a field of characteristic $p = 5$ and take the vector space over $\mathbb{F}$ with basis $z^i \partial_z$, for $i =$

$0, \dots, 4$. The Lie bracket is defined on two of these elements by

$$\begin{aligned}[z^i \partial_z, z^j \partial_z] &:= z^i \partial_z(z^j \partial_z) - z^j \partial_z(z^i \partial_z) = j z^i z^{j-1} \partial_z - i z^j z^{i-1} \partial_z \\ &= (j-i) z^{i+j-1} \partial_z,\end{aligned}$$

with the convention that

(3) $\qquad z^i := 0 \quad \text{whenever } i \notin \{0, \dots, 4\}.$

The Lie bracket extends bilinearly to a multiplication on $W_{1,1}(5)$. It is antisymmetric and satisfies the Jacobi identity, so that $W_{1,1}(5)$ is indeed a Lie algebra of dimension 5 over $\mathbb{F}$.

Now we construct an extension $\widetilde{W_{1,1}(5)}$ of $W_{1,1}(5)$: Add one basis element, namely $z^6\partial_z$, and adapt (3):

(4) $\qquad z^i := 0 \quad \text{whenever } i \notin \{0, 1, 2, 3, 4, 6\}.$

The only entry of the multiplication table that differs between $\widetilde{W_{1,1}(5)}$ and $W_{1,1}(5)$ is $[z^3\partial_z, z^4\partial_z]$: This is 0 in $W_{1,1}(5)$ and $z^6\partial_z$ in $\widetilde{W_{1,1}(5)}$. Furthermore, $z^6\partial_z$ commutes with all other elements. So $\widetilde{W_{1,1}(5)}$ is indeed an extension of $W_{1,1}(5)$ by a one-dimensional center. This extension was constructed in [2]. The analog over the complex numbers of $\widetilde{W_{1,1}(5)}$ is also known as the Virasoro algebra.

Now let W be either $W_{1,1}(5)$ or $\widetilde{W_{1,1}(5)}$. Then $x = -z^2\partial_z$ is readily seen to be extremal in W. Together with $y = \partial_z$ and $h = 2z\partial_z$ it forms an $\mathfrak{sl}_2$-triple in W. Moreover, setting $v = 2z^4\partial_z$, we find $[v, y] = 2z^3\partial_z$ and $[v, [v, y]] = z^6\partial_z$, so W is generated by x, y, v. But $[y, [y, v]] = x$, so y is not extremal in W.

The following result characterizes the simple Lie algebra of this example.

Proposition 3.2. *Suppose that L is a simple Lie algebra over the field $\mathbb{F}$ of characteristic $p = 5$ with an $\mathfrak{sl}_2$-triple x, y, h such that x is extremal, $-\mathrm{ad}_h$ is diagonizable with eigenspaces L_i ($i = -2, -1, 0, 1, 2$) and $[y, [y, L_{-1}]] \neq \{0\}$. Then L is isomorphic to the Witt algebra $W_{1,1}(5)$.*

Proof. As $[y, [y, L_{-1}]] \subseteq L_3 = L_{-2}$, we have $[y, [L_{-1}, y]] = \mathbb{F}x$. Let $v \in L_{-1}$ be such that $[y, [v, y]] = x$. Consider the linear span W in L of x, y, h, v, $[v, y]$, and $[v, [v, y]]$. The multiplication on these elements is fully determined:

$$\begin{aligned}
[x, y] &= h, \\
[x, h] &= -2x, \\
[x, v] &= 0 \quad \big(\text{for } [x, [x, v]] \in \mathbb{F}x \cap L_0 = \{0\}\big), \\
\big[x, [v, y]\big] &= \big[v, [x, y]\big] + \big[y, [v, x]\big] = [v, h] = -v, \\
\big[x, \big[v, [v, y]\big]\big] &= \big[v, \big[x, [v, y]\big]\big] = -[v, v] = 0, \\
[y, h] &= 2y,
\end{aligned}$$

$$
\begin{aligned}
[y,v] &= -[v,y], \\
\big[y,[v,y]\big] &= -\big[y,[y,v]\big] = -x \quad \text{(by definition)}, \\
\big[y,\big[v,[v,y]\big]\big] &= \big[v,\big[y,[v,y]\big]\big] + 0 = [v,x] = 0, \\
[h,v] &= v \quad \text{(implied by the grading)}, \\
\big[h,[v,y]\big] &= -[v,y] \quad \text{(implied by the grading)}, \\
\big[h,\big[v,[v,y]\big]\big] &= 0 \quad \text{(implied by the grading)}, \\
\big[v,[v,y]\big] &= \big[v,[v,y]\big], \\
\big[v,\big[v,[v,y]\big]\big] &= 0, \\
\big[[v,y],\big[v,[v,y]\big]\big] &= 0.
\end{aligned}
$$

Observe that $[v,[v,y]]$ is central and that the quotient with respect to the ideal it generates is simple of dimension 5. We claim that if $[v,[v,y]] = 0$ then W is isomorphic to the Witt algebra $W_{1,1}(5)$, and otherwise W is isomorphic to $\widetilde{W_{1,1}(5)}$, as defined in Example 3.1.

By comparison of the above multiplication rules for the spanning set x, y, h, v, $[v,y]$, $[v,[v,y]]$ of W and the basis $-z^2\partial_z$, ∂_z, $2z\partial_z$, $2z^4\partial_z$, $2z^3\partial_z$, and $z^6\partial_z$ of $\widetilde{W_{1,1}(5)}$ there is a surjective homomorphism $\varphi : \widetilde{W_{1,1}(5)} \to W$ of Lie algebras. By assumption $W \neq 0$. As $\widetilde{W_{1,1}(5)}$ only has one nontrivial proper ideal, which maps onto $\langle [v,[v,y]] \rangle$ under φ, it follows that φ is an isomorphism $\widetilde{W_{1,1}(5)} \to W$ if $[v,[v,y]] \neq 0$ and induces an isomorphism $W_{1,1}(5) \to W$ otherwise.

It remains to prove that L coincides with W, for then $L \cong W_{1,1}(5)$ as $\widetilde{W_{1,1}(5)}$ is not simple. To this end, suppose that L strictly contains W, and consider L as a module on which W acts. As in the proof of Lemma 2.2, we compute in the subalgebra $\mathrm{End}(L/W)$ generated by ad_W. Applying Lemma 2.2, we find that ad_x and ad_y act quadratically on $L/\langle x, y, [x,y]\rangle$ and hence on L/W, so we have relations (R1), ..., (R5) in $\mathrm{End}(L/W)$. Write X, Y, V for the action of ad_x, ad_y, ad_v, respectively, on L/W. Due to Lemma 2.2, and the multiplication rules $[y,[v,y]] = -x$ and $[x,[v,y]] = -v$ listed above, we have the following relations.

(R6) $$Y^2 = 0,$$

(R7) $$Y^2V - 2YVY + VY^2 - X = 0,$$

(R8) $$XVY - XYV - VYX + YVX + V = 0.$$

Applying (R6) to (R7) and to (R2), respectively, gives the following two relations.

(R9) $$X + 2YVY = 0,$$

(R10) $$Y - YXY = 0.$$

Now with R_9, R_{10} denoting the left-hand sides of (R9), (R10), respectively,

$$
\begin{aligned}
0 &= R_9(1 - XY) - 2YVR_{10}, \\
&= (X + 2YVY)(1 - XY) - 2YV(Y - YXY),
\end{aligned}
$$

$$= X + 2YVY - X^2Y - 2YVYXY - 2YVY + 2YVYXY,$$
$$= X.$$

(R10) immediately implies $Y = 0$, and then (R8) implies $V = 0$.

So the images of ad_w, for $w \in W$, in $\mathrm{End}(L/W)$ are trivial. This means that W is an ideal of L. Since L is simple and W is nontrivial, we find $L \cong W$, as required. □

4. THE GENERAL CASE

Having dealt with the exceptional case in the previous section, we can now proceed with the general case of Proposition 2.4.

Proposition 4.1. *Assume that L is a simple Lie algebra over the field $\mathbb{F}$ of characteristic $p \neq 2, 3$, having an $\mathfrak{sl}_2$-pair x, y of extremal elements. If L is not isomorphic to $W_{1,1}(5)$, then L is generated by extremal elements.*

Proof. Note that $[y, [y, L_{-1}]] = 0$ as y is extremal and so Proposition 2.4 gives that $h = [x, y]$ is diagonizable and the components $L_i = L_i(-\mathrm{ad}_h)$ $(i = -2, -1, 0, 1, 2)$ of the grading by h satisfy $L_{-2} = \mathbb{F}x$, $L_{-1} = [x, L_1]$, $L_2 = \mathbb{F}y$, and $L_1 = [y, L_{-1}]$.

Consider the subalgebra I of L generated by x, y, and L_1. As $L_{-1} = [x, L_1]$, the subalgebra I contains the linear subspace $J = L_{-2} + L_{-1} + L_1 + L_2$ of L. As $[J, L_0] \subseteq J$ and J generates I, we have $[I, L_0] \subseteq I$. This implies $[I, L] = I$. In other words, I is an ideal of L, and so, by simplicity of L, it coincides with L. Therefore, it suffices to show that for each $z \in L_1$ there exists an extremal element $u \in L$ such that z is in the subalgebra generated by x, y, and u.

To this end, let $z \in L_1$. Put $h = [x, y]$. The following relations hold in L, for some $\alpha \in \mathbb{F}$.

(5) $$[h, x] = 2x,$$
(6) $$[h, y] = -2y,$$
(7) $$[z, h] = z,$$
(8) $$[y, z] = 0,$$
(9) $$\big[x, [x, z]\big] = 0,$$
(10) $$\big[y, [x, z]\big] = z,$$
(11) $$\big[y, \big[z, [z, x]\big]\big] = 0,$$
(12) $$\big[x, \big[z, [z, x]\big]\big] = 0,$$
(13) $$\big[y, \big[z, \big[z, [z, x]\big]\big]\big] = 0,$$
(14) $$\big[x, \big[x, \big[z, \big[z, [z, x]\big]\big]\big]\big] = 0,$$
(15) $$\big[y, \big[x, \big[z, \big[z, [z, x]\big]\big]\big]\big] = \big[z, \big[z, [z, x]\big]\big],$$
(16) $$\big[z, \big[z, \big[z, [z, x]\big]\big]\big] = \alpha y.$$

We claim that the Lie subalgebra L' of L generated by x, y, and z is linearly spanned by the following set B of eight elements, where $h_1 = [[x, z], z]$:

$$x \in L_{-2}; \qquad [x, z], \big[[h_1, z], x\big] \in L_{-1}; \qquad h, \big[[x, z], z\big] \in L_0;$$
$$z, [h_1, z] \in L_1; \qquad y \in L_2.$$

To see that this is true, we verify that the images of the elements of B under ad_x, ad_y, and ad_z are scalar multiples of these. For ad_x and ad_y this is straightforward. As for ad_z, the statement is trivially verified for all elements of B but $[[h_1, z], x]$. As $[h_1, x] = 0$, we have

$$\begin{aligned}
\mathrm{ad}_z\big([[h_1, z], x]\big) &= \big[\mathrm{ad}_z([h_1, z]), x\big] + \big[[h_1, z], \mathrm{ad}_z(x)\big] \\
&= \alpha[y, x] + \big[[h_1, \mathrm{ad}_z(x)], z\big] + \big[h_1, [z, \mathrm{ad}_z(x)]\big] \\
&= -\alpha h - \mathrm{ad}_z\big([h_1, \mathrm{ad}_z(x)]\big) + [h_1, h_1] \\
&= -\alpha h - \mathrm{ad}_z\big([[h_1, z], x]\big),
\end{aligned}$$

so $\mathrm{ad}_z([[h_1, z], x]) = -\frac{\alpha}{2}h$. This establishes the claim that L' is linearly spanned by B.

We exhibit an element $u \in L'$ as specified. Because of the grading induced by ad_h on L, the endomorphism ad_z on L is nilpotent of order at most 5 and $\exp(\mathrm{ad}_z)$ is a linear transformation of L (it is well defined as $p \neq 2, 3$). Put

$$u = \exp(\mathrm{ad}_z)x = x + \mathrm{ad}_z(x) + \frac{1}{2}\mathrm{ad}_z^2(x) + \frac{1}{6}\mathrm{ad}_z^3(x) + \frac{1}{24}\mathrm{ad}_z^4(x).$$

A straightforward computation in L' shows that y and u are an $\mathfrak{sl}_2$-pair in L. By (11), (13), (16), and (10) we find

$$\begin{aligned}
(17) \qquad [y, u] &= [y, x] + \big[y, [z, x]\big] + \frac{1}{2}\big[y, \mathrm{ad}_z^2(x)\big] \\
&\quad + \frac{1}{6}\big[y, \mathrm{ad}_z^3(x)\big] + \frac{1}{24}\big[y, \mathrm{ad}_z^4(x)\big] \\
&= [y, x] + \big[y, [z, x]\big] + 0 + 0 + [y, \alpha y] \\
&= -h - z
\end{aligned}$$

so, by (6), (8),

$$(18) \qquad \big[[y, u], y\big] = -[h, y] - [z, y] = 2y.$$

For $[[y, u], u]$ we compute, using (17), (5) and Proposition 2.4,

$$\begin{aligned}
\big[[y, u], x\big] &= -[h, x] - [z, x] = -2x - [z, x], \\
\big[[y, u], \mathrm{ad}_z(x)\big] &= -\mathrm{ad}_z(x) - \mathrm{ad}_z^2(x), \\
\big[[y, u], \mathrm{ad}_z^2(x)\big] &= 0 - \mathrm{ad}_z^3(x), \\
\big[[y, u], \mathrm{ad}_z^3(x)\big] &= \mathrm{ad}_z^3(x) - \mathrm{ad}_z^4(x), \\
\big[[y, u], \mathrm{ad}_z^4(x)\big] &= +2\mathrm{ad}_z^4(x) - \mathrm{ad}_z^5(x) = 2\mathrm{ad}_z^4(x),
\end{aligned}$$

so

$$(19)\qquad \big[[y,u],u\big] = -2x - [z,x] - \mathrm{ad}_z(x) - \mathrm{ad}_z^2(x) - \frac{1}{2}\mathrm{ad}_z^3(x) + \frac{1}{6}\big(\mathrm{ad}_z^3(x) - \mathrm{ad}_z^4(x)\big) + \frac{1}{24}\big(2\mathrm{ad}_z^4(x)\big) = -2u.$$

Now (18) and (19) show that y and u are an $\mathfrak{sl}_2$-pair in L.

By Propositions 2.4 and 3.2, and the assumption that L is not isomorphic to $W_{1,1}(5)$, this implies that u is extremal in L.

We verify that z lies in the subalgebra L'' of L generated by the three extremal elements x, y, and u. Observe that

$$\mathrm{ad}_z(x) + \frac{1}{2}\mathrm{ad}_z^2(x) + \frac{1}{6}\mathrm{ad}_z^3(x) = u - x - \frac{\alpha}{24}y \in L''.$$

Acting by ad_y and using (10), (11), (13), we find

$$z = -\mathrm{ad}_y\mathrm{ad}_z(x) - \frac{1}{2}\mathrm{ad}_y\mathrm{ad}_z^2(x) - \frac{1}{6}\mathrm{ad}_y\mathrm{ad}_z^3(x) \in \mathrm{ad}_y L'' \subseteq L''.$$

This proves that z belongs to L'' and so we are done. □

Proof of Theorem 1.1. Let L be as in the assumption. By Theorem 2.3, there is an $\mathfrak{sl}_2$-pair x, y in L with x extremal in L. Proposition 4.1 finishes the proof. □

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