Mutations of Laurent Polynomials and Lattice Polytopes

Mohammad Ehtisham Akhtar

Department of Mathematics
Imperial College London

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Declaration of Originality

I declare that any portion of this thesis which is not my own work has been properly acknowledged.

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Mohammad Ehtisham Akhtar
Imperial College London
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Abstract

It has been conjectured that Fano manifolds correspond to certain Laurent polynomials under Mirror Symmetry. This correspondence predicts that the regularized quantum period of a Fano manifold coincides with the classical period of a Laurent polynomial mirror. This correspondence is not one-to-one, as many different Laurent polynomials can have the same classical period; it should become one-to-one after imposing the correct equivalence relation on Laurent polynomials. In this thesis we introduce what we believe to be the correct notion of equivalence: this is *algebraic mutation* of Laurent polynomials. We also consider *combinatorial mutation*, which is the transformation of lattice polytopes induced by algebraic mutation of Laurent polynomials supported on them. We establish the basic properties of algebraic and combinatorial mutations and give applications to algebraic geometry, most notably to the classification of Fano manifolds up to deformation. Our main focus is on the surface case, where the theory is particularly rich.

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Chapter 1

Introduction

1.1 Background and Motivation

This thesis is part of a circle of ideas originating in the work of V. Golyshev [15] and later refined into the programme outlined in [7]. The programme aims to develop a classification theory for Fano manifolds, i.e. smooth projective varieties over $\mathbb{C}$ whose anti-canonical sheaf is ample. A central concept in this theory is as follows: a Fano manifold X of dimension $n \in \mathbb{Z}_{\geq 0}$ is *mirror dual* to a given Laurent polynomial in n variables, $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, if the regularized quantum period of X — a generating function for certain Gromov–Witten invariants of X — is equal to π_f, the *classical period* of f, where:

$$\pi_f(t) := \left(\frac{1}{2\pi i}\right)^n \int_{|x_n|=1} \cdots \int_{|x_1|=1} \frac{1}{1-tf} \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n} \quad ; \quad t \in \mathbb{C}, |t| \ll \infty.$$

If such an equality holds, then f is called a *Laurent polynomial mirror* for X.

It is expected that if f is a Laurent polynomial mirror to X then there exists a toric degeneration of X to the possibly-singular toric variety defined by the spanning fan of the Newton polytope of f. Indeed, an optimist might expect that every toric degeneration of a Fano manifold arises this way. In other words, if X admits a toric degeneration to the toric variety X_P defined by the spanning fan of a polytope P, then there is a Laurent polynomial mirror f to X with $\mathrm{Newt}(f) = P$. The coefficients of f are also expected to carry geometric meaning: informally they are expected to be certain holomorphic disc counts [5], although this will not be discussed in the present document. In the spirit of [7], we have:

Conjecture 1.1. Every Fano manifold has a Laurent polynomial mirror.

This conjecture has been proven in [8] for Fano manifolds of dimension ≤ 3.

We now explain how these ideas can be applied to the classical problem of classifying Fano manifolds up to deformation. Gromov–Witten invariants are invariant under deformations, so the regularized quantum period is constant on the deformation class $[X]$ of a Fano manifold X. It therefore makes sense to talk about mirror duals to deformation classes of Fano manifolds. We expect that Fano manifolds up to deformation correspond to a class of Laurent polynomials up to an appropriate notion of equivalence. In summary, the works [7, 8] suggest:

Conjecture 1.2. For each integer $n \geq 1$, there exists a class of Laurent polynomials $\mathcal{L}_n \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and an equivalence relation $\sim$ on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, such that elements of $\mathcal{L}_n/\sim$ are in one-to-one correspondence with deformation classes of n-dimensional Fano varieties. If the deformation class $[X]$ corresponds to the equivalence class $[f]$, then any $X \in [X]$ admits a $\mathbb{Q}$-Gorenstein toric degeneration to the toric variety defined by the Newton polytope of any $g \in [f]$. Furthermore, the following *mirror principle* is satisfied: The regularized quantum period $\hat{G}_X$ should be equal to the classical period of f.

In light of the above discussion, the equivalence relation $\sim$ on Laurent polynomials must preserve the classical period.

In Section 1.3, we introduce the notion of *algebraic mutation* of Laurent polynomials [2, 13]. We believe this to be the correct notion of the equivalence $\sim$ in Conjecture 1.2. Algebraic mutations are cluster-type transformations (birational changes of coordinates on $(\mathbb{C}^*)^n$) which preserve the classical period. We conjecture that any two Laurent polynomial mirrors to the same deformation class $[X]$ are related by algebraic mutation.

A precise description of the sets $\mathcal{L}_n$ is not currently known. At present they can only be detected indirectly, for instance by assuming the mirror principle of Conjecture 1.2 and then appealing to known classifications of Fano manifolds to determine the list of classical periods which are expected to occur. Although we briefly speculate on the definition of $\mathcal{L}_2$ in Section 4.1, the present document will focus on a different aspect of mirror duality:

Let f, g be two Laurent polynomial mirrors to a given Fano manifold X. We expect that X admits toric degenerations to the toric varieties X_f, defined by the spanning fan of $\mathrm{Newt}(f)$, and X_g, defined by the spanning fan of $\mathrm{Newt}(g)$. If one now forgets about X, a natural question to ask is whether the toric varieties X_f and X_g are related in some meaningful way. Our response to this question will be to study the relationship between the underlying polytopes $\mathrm{Newt}(f)$ and

Newt(g). Since f and g are conjectured to be related by algebraic mutation of Laurent polynomials, we ask if there exists an analogous theory of *combinatorial mutations* of lattice polytopes, which captures the transformation on Newton polytopes induced by algebraic mutations of Laurent polynomials supported on them. In particular, if $g = \varphi^* f$ for some algebraic mutation φ, then we expect Newt(g) to be a combinatorial mutation of Newt(f).

The existence of such a theory of combinatorial mutations was established in the joint work [2] of the author with T. Coates, S. Galkin and A. Kasprzyk. This is the content of Chapter 2, in which combinatorial mutations are defined and their basic properties are established. Chapter 3 summarizes the joint works [3, 4] of the author with A. Kasprzyk. This chapter studies first examples of combinatorial mutations: the case of surfaces with particular focus on the projective plane $\mathbb{P}^2$. In Chapter 4, we conclude by discussing some instances where combinatorial mutations arise in the classification programme discussed in this section.

1.2 Highlights and Main Results

The notion of *algebraic mutation* is introduced in Section 1.3 and two Laurent polynomials related by an algebraic mutation are shown to have the same classical period (Proposition 1.3). Thus, algebraic mutations are a good candidate for the equivalence relation $\sim$ discussed in Conjecture 1.2.

Chapter 2 studies the transformation of lattice polytopes induced by algebraic mutations of Laurent polynomials supported on them. The precise definition of these *combinatorial mutations* of lattice polytopes is given in Section 2.3 and basic properties of combinatorial mutations are established in Section 2.5. Most notably, this construction preserves the property of a lattice polytope being *Fano* (Proposition 2.18) and can be interpreted dually (and perhaps more naturally) as a transformation which is linear on each chamber of a wall-and-chamber decomposition of a certain dual vector space (Corollary 2.20). This decomposition arises naturally from the initial data of combinatorial mutations. The dual description is used in Section 2.6 to show that Ehrhart series of dual polytopes are preserved under combinatorial mutations. This implies that the anti-canonical degree of the toric variety defined by the spanning fan of a Fano polytope is also an invariant of combinatorial mutation.

Chapter 3 focuses on combinatorial mutations in the two-dimensional setting.

The results of Section 3.2 completely describe (one-step) combinatorial mutations between Fano triangles, generalizing recent work of Hacking–Prokhorov [16]. This is again a direct application of the dual description of combinatorial mutations obtained in the previous chapter. In Section 3.3 the *residue* of a surface cyclic quotient singularity is defined by means of an explicit formula. This allows us, in the same section, to define the *singularity content* of a Fano polygon which is an invariant of two-dimensional combinatorial mutations (Proposition 3.30). A formula relating the singularity content of a Fano polygon to the anti-canonical degree of the toric variety defined by its spanning fan is established (Proposition 3.34) and surface cyclic quotient singularities with empty residue are classified (Corollary 3.29): these are precisely the *T-singularities* appearing in the work [21] of Kollár–Shepherd-Barron.

In Section 4.1, we speculate on the definition of the set $\mathcal{L}_2$ of Conjecture 1.2 from the viewpoint of combinatorial mutations. Finally, in Section 4.2, we discuss some deformation-theoretic results related to combinatorial mutations.

1.3 Algebraic Mutations

Fix a positive integer n and, given $\mathsf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, let x^{a} denote the monomial $x_1^{a_1} \ldots x_n^{a_n} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. A birational map $(\mathbb{C}^*)^n \dashrightarrow (\mathbb{C}^*)^n$ is called an *algebraic mutation* [2, 13] if it is a composition: $\gamma \circ \varphi_A \circ \eta$ or $\gamma \circ (\varphi_A)^{-1} \circ \eta$. Here, $\gamma, \eta : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ are morphisms of the form $\mathsf{x} = (x_1, \ldots, x_n) \mapsto \mathsf{x}^M := (\mathsf{x}^{\mathsf{m}_1}, \ldots, \mathsf{x}^{\mathsf{m}_n})$, with $\mathsf{m}_1, \ldots, \mathsf{m}_n$ the rows of some $M \in GL_n(\mathbb{Z})$ and inverse given by $\mathsf{x} \mapsto \mathsf{x}^{M^{-1}}$. Furthermore, $\varphi_A : (\mathbb{C}^*)^n \dashrightarrow (\mathbb{C}^*)^n$ is the birational map:

$$(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, A(x_1, \ldots, x_{n-1})x_n), \tag{1.1}$$

corresponding to some $A \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$. We say two Laurent polynomials f, g, in the same number of variables, are *related by algebraic mutations* if there exists a map φ, which is a composition of algebraic mutations, such that $f = \varphi^* g := g \circ \varphi$. This is an equivalence relation on Laurent polynomials which satisfies the following key property (cf. Section 1.1):

Proposition 1.3 ([2, Lemma 1])**.** *If Laurent polynomials f, g, in n variables, are related by algebraic mutations, then their classical periods coincide: $\pi_f = \pi_g$.*

Proof. Suppose that $g = \varphi^* f$, where $\varphi : (\mathbb{C}^*)^n \dashrightarrow (\mathbb{C}^*)^n$ is defined by (1.1), for some fixed $A \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$. Let $(x_1, \ldots, x_n)$ (resp. $(y_1, \ldots, y_n)$) denote coordinate functions on the domain (resp. target) of φ. Note that φ is biregular when restricted to $U := (\mathbb{C}^*)^n \backslash Z$, where $Z := \{(x_1, \dots, x_n) \in (\mathbb{C}^*)^n \mid A(x_1, \dots, x_{n-1}) = 0\}$. We make two observations. Firstly:

$$\begin{aligned}
\varphi|_U^*\left(\frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n}\right) &= \bigwedge_{i=1}^{n}\varphi|_U^*\left(\frac{dy_i}{y_i}\right) = \left(\frac{dx_1}{x_1}\cdots\frac{dx_{n-1}}{x_{n-1}}\right)\wedge\left(\frac{d(Ax_n)}{Ax_n}\right)\\
&= \left(\frac{dx_1}{x_1}\cdots\frac{dx_{n-1}}{x_{n-1}}\right)\wedge\left(\frac{dx_n}{x_n}+\sum_{j=1}^{n-1}\left(\frac{\partial A}{\partial x_j}\right)\frac{x_n dx_j}{Ax_n}\right),
\end{aligned}$$

and so $\varphi|_U^*\left(\frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n}\right) = \frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$, because $dx_j \wedge dx_j = 0$ if $j \in \{1, \dots, n-1\}$. Secondly, there is a vector $\mathsf{r} = (r_1, \dots, r_n)$, whose entries are positive real numbers, such that the locus $C_{\mathsf{r}} := \{(x_1, \dots, x_n) \in (\mathbb{C}^*)^n \mid |x_i| = r_i, i = 1, \dots, n\}$ lies entirely in U. Indeed, consider

$$\mathfrak{A} := \mathrm{Log}(Z) = \{(\log|x_1|, \dots, \log|x_n|) \mid (x_1, \dots, x_n) \in Z\} \subseteq \mathbb{R}^n,$$

(the amoeba of Z). $\mathfrak{A}$ is a proper subset of $\mathbb{R}^n$, by [14, Ch. 6, Corollary 1.8]. So there exists a vector $\mathsf{a} := (a_1, \dots, a_n) \in \mathbb{R}^n \backslash \mathfrak{A}$. Let $r_i := \exp(a_i)$ for all $i \in \{1, \dots, n\}$. Then the preimage of a under Log is C_{r}, which lies entirely in U by construction. Now C_{r} and $C_{(1,\dots,1)}$ are homologous cycles: their difference is the boundary of a cycle Γ contained in Δ, a product of annuli. By continuity, both $|f|$ and $|g|$ attain maximum values, F and G, on Δ. Thus, for any $t \in \mathbb{C}$ such that $|t| \cdot \max\{F, G\} < 1$, we have that:

$$\begin{aligned}
(2\pi i)^n \pi_g(t) &= \int_{C_{(1,\dots,1)}} \frac{1}{1-tg}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} = \int_{C_{\mathsf{r}}} \varphi|_U^*\left(\frac{1}{1-tf}\frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n}\right)\\
&= \int_{\varphi(C_{\mathsf{r}})} \frac{1}{1-tf}\frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n}.
\end{aligned}$$

These equalities follow from Stokes' Theorem and the change of variables formula, which can be applied because the choice of t ensures both $1/(1-tf)$ and $1/(1-tg)$ are holomorphic on Γ and its boundary. Now since $H_n((\mathbb{C}^*)^n; \mathbb{Z})$ is freely generated by $[C_{(1,\dots,1)}]$, it follows that $[\varphi(C_{\mathsf{r}})] = k[C_{(1,\dots,1)}]$ for some integer k. But $k = 1$, by the following calculation:

$$\begin{aligned}
(2\pi i)^n &= \int_{C_{(1,\dots,1)}} \frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} = \int_{\varphi(C_{\mathsf{r}})} \frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n} = k\cdot\int_{C_{(1,\dots,1)}} \frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n}\\
&= (2\pi i)^n k. \qquad (1.2)
\end{aligned}$$

We conclude that for any $t \in \mathbb{C}$ satisfying $|t| \cdot \max\{F, G\} < 1$:

$$\begin{aligned}(2\pi i)^n \pi_g(t) &= \int_{\varphi(C_r)} \frac{1}{1-tf} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} = k \cdot \int_{C_{(1,\ldots,1)}} \frac{1}{1-tf} \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n} \\ &= (2\pi i)^n \pi_f(t).\end{aligned}$$

A similar argument can be made in the case when $\varphi : (\mathbb{C}^*)^n \dashrightarrow (\mathbb{C}^*)^n$ is the inverse of (1.1), or of the form $\mathsf{x} \mapsto \mathsf{x}^M$ for some $M \in GL_n(\mathbb{Z})$. In the latter case:

$$\varphi|_U^* \left(\frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n} \right) = (\det M) \cdot \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\det M$ is either $+1$ or -1, depending on whether M preserves or reverses orientation. In the latter case the minus sign is canceled because $k = -1$, by a similar calculation to (1.2). □

Period Sequence of a Laurent Polynomial

Let $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial. On its disc of convergence, the classical period π_f defines a holomorphic function of $t \in \mathbb{C}$, and so can be expressed as a power series: $\pi_f(t) = \sum_{k \geq 0} c_k t^k$. For $k \geq 0$:

$$\begin{aligned}c_k &= \frac{1}{k!} \frac{d^k}{dt^k}\bigg|_{t=0} \pi_f(t) \\ &= \frac{1}{k!} \left(\frac{1}{2\pi i} \right)^n \int_{|x_n|=1} \cdots \int_{|x_1|=1} \frac{k! f(x_1, \ldots, x_n)^k}{(1 - tf(x_1, \ldots, x_n))^{(k+1)}}\bigg|_{t=0} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.\end{aligned}$$

By evaluating at $t = 0$ and then applying the one-variable Cauchy residue theorem n times, one identifies c_k with the constant term of f^k, denoted $\mathrm{coeff}_{\mathbb{1}}(f^k)$. Following [7], we refer to $(c_k)_{k \geq 0}$ as the *period sequence* of the Laurent polynomial f. The following is an immediate consequence of Proposition 1.3:

Corollary 1.4. *If Laurent polynomials f, g, in n variables, are related by algebraic mutations, then their period sequences coincide.*

Example 1.5. We determine the period sequence of $f = x_1 + \ldots + x_n + (x_1 \cdot \ldots \cdot x_n)^{-1}$. A general term in the expansion of f^k is: $m(k; e_1, \ldots, e_{n+1}) \prod_1^n {x_j}^{e_j - e_{n+1}}$, where $\sum e_i = k$ and $m(k\,;\, e_1, \ldots, e_{n+1})$ is a multinomial coefficient. Such a term is constant if and only if $e_1 = \ldots = e_n = e_{n+1}$. Therefore, $\mathrm{coeff}_{\mathbb{1}}(f^k)$ is 0 if $n + 1$ does not divide k, and equals $m(s(n+1)\,;\, s, \ldots, s) = (s(n+1))!/(s!)^{n+1}$ when $k = s(n+1)$ for some integer $s \geq 0$.

Now let $f_a := x_1 + \ldots + x_n + a(x_1 \cdot \ldots \cdot x_n)^{-1}$ for any $a \in \mathbb{C}$ fixed, so that f_1 coincides with f from the previous paragraph. The same argument as above shows that $\mathrm{coeff}_{\mathbb{1}}(f_a^k)$ is 0 if $n+1$ does not divide k, and equals $a^s \cdot m(s(n+1)\,;\, s, \ldots, s) = a^s(s(n+1))!/(s!)^{n+1}$ when $k = s(n+1)$ for some integer $s \geq 0$. In particular, if $a \neq b$ then f_a, f_b are not related by algebraic mutations.

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This work is dedicated to my parents.

Chapter 2

Combinatorial Mutations

2.1 Preliminary Definitions and Notation

Throughout this chapter, we fix a lattice $N \cong \mathbb{Z}^n$, of rank n, with dual lattice $M := \mathrm{Hom}(N, \mathbb{Z})$. A *lattice polytope* will mean a convex polytope $P \subset N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying the following conditions:

(1) $\mathrm{verts}(P) \subset N$;
(2) $\mathbf{0} \in \mathrm{int}(P)$;
(3) $\dim(P) = \mathrm{rank}(N)$.

We say two lattice polytopes $P, Q \subset N_{\mathbb{Q}}$ are *isomorphic*, and write $P \cong Q$, if Q is the image of P under a $GL(N)$-transformation. Fixing an isomorphism of N with $\mathbb{Z}^n$ identifies $GL(N)$ with $GL_n(\mathbb{Z})$.

The *dual polytope* of a lattice polytope $P \subset N_{\mathbb{Q}}$ is:

$$P^\vee := \{u \in M_{\mathbb{Q}} \mid \langle u, x\rangle \geq -1 \text{ for all } x \in P\} \subset M_Q,$$

where $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$, $\langle u, x\rangle := u(x)$, is the natural pairing. By [27, Theorem 2.11], condition (2) ensures that $P^\vee$ is a (not necessarily lattice) polytope satisfying $\mathbf{0} \in \mathrm{int}(P^\vee)$, and that $(P^\vee)^\vee = P$.

The theory of *combinatorial mutations*, developed in this chapter, applies to all lattice polytopes. However its main application will be to Fano polytopes, as defined in [19, 23] (cf. Chapter 3. See also [1]). A *Fano polytope* is a lattice polytope P which satisfies the additional condition that every vertex v of P is a *primitive* lattice vector (i.e. $\mathrm{conv}(\mathbf{0}, v) \cap N = \{0, v\}$). A two-dimensional lattice (resp. Fano) polytope is called a lattice (resp. Fano) *polygon*.

The *spanning fan* of a polytope $P \subset N_{\mathbb{Q}}$, satisfying conditions (2) and (3), is the complete fan: $\{\mathrm{cone}(\tau) \subset N_{\mathbb{Q}} \mid \tau \text{ is a proper face of } P\}$. Here, $\mathrm{cone}(\tau)$ is the strictly convex rational polyhedral cone generated by the vertices of τ. Let $P \subset N_{\mathbb{Q}}$ be a polytope satisfying condition (1) (but not necessarily (2) or (3)). The *inner normal fan of* P is the fan in $M_{\mathbb{Q}}$ whose maximal cones, $\sigma_q \subseteq M_{\mathbb{Q}}$, consist of those linear functions on N whose minimal value on P is attained at a given vertex, $q \in \mathrm{verts}(P)$:

$$\sigma_q := \{u \in M_{\mathbb{Q}} \mid \langle u, q \rangle = \inf\{\langle u, x \rangle \mid x \in P\}\}.$$

Note that an inner normal fan is always complete[1], and is thus determined by the collection of its maximal cones. A well-known fact in toric geometry [10] is that the inner normal fan of a lattice polytope is the spanning fan of its dual. If P does not satisfy condition (3), then the cones of its inner normal fan are not necessarily strictly convex.

Example 2.1. Consider the polytope $F \subset \mathbb{Q}^2$ with vertex set $\{(0,0),(a,b)\}$. The inner normal fan of F, Σ in $M_{\mathbb{Q}}$, has $|\mathrm{verts}(F)| = 2$ maximal cones. The maximal cone $\sigma_{(a,b)} \subset M_{\mathbb{Q}}$ is generated by $(b,-a)^t, (-b,a)^t$ and $(-a,-b)^t$, while $\sigma_{(0,0)} \subset M_{\mathbb{Q}}$ is generated by $(b,-a)^t, (-b,a)^t$ and $(a,b)^t$. Listing all faces of both $\sigma_{(a,b)}$ and $\sigma_{(0,0)}$ shows that Σ has two 1-dimensional cones, generated by $(-b,a)^t$ and $(b,-a)^t$ respectively, and a single 0-dimensional cone, namely $\{(0,0)^t\}$.

The *Minkowski sum* of two polytopes $P, Q \subset N_{\mathbb{Q}}$ is:

$$P + Q := \{p + q \mid p \in P, q \in Q\}, \tag{2.1}$$

where we adopt the convention that $P + \varnothing := \varnothing$. The symbol $+$ relating two polytopes will always mean Minkowski sum. For any rational number $k \geq 0$, $kP := \{kp \mid p \in P\}$. The *multiplicity* of a lattice polytope $P \subset N_{\mathbb{Q}}$ is $\mathrm{mult}(P) := [N : L]$, where L is the sublattice of N spanned by the vertices of P.

2.2 Width Vectors and Factors

A *width vector* is a primitive lattice vector, $w \in M$. Primitivity ensures that there always exists a vector $x \in N$ such that $\langle w, x \rangle = 1$. Thus, any width

[1] Since $P \subset N_{\mathbb{Q}}$ is convex, any $u \in M_{\mathbb{Q}}$ will always attain its minimum value on at least one vertex of P.

vector can be thought of as an integer-valued height function on N, canonically extending to a height function $N_{\mathbb{Q}} \to \mathbb{Q}$. To emphasize that we are thinking of a width vector w as a function on $N_{\mathbb{Q}}$, we will write $w(x)$ instead of $\langle w, x\rangle$ for all $x \in N_{\mathbb{Q}}$. A subset $S \subset N_{\mathbb{Q}}$ is said to lie *at height* $h \in \mathbb{Q}$ with respect to w if $w(S) := \{w(s) \mid s \in S\} = \{h\}$. In this case we write $w(S) = h$.

The set of all points in $N_{\mathbb{Q}}$ lying at height $h \in \mathbb{Q}$ with respect to a given width vector w is the affine hyperplane $H_{w,h} := \{x \in N_{\mathbb{Q}} \mid w(x) = h\}$. If $P \subset N_{\mathbb{Q}}$ is a lattice polytope, then

$$P_{w,h} := \operatorname{conv}(H_{w,h} \cap P \cap N) \subset N_{\mathbb{Q}}$$

is the possibly empty convex hull of all lattice points in P at height h. We let

$$h_{\min} := \inf\{w(x) \mid x \in P\} \text{ and } h_{\max} := \sup\{w(x) \mid x \in P\}.$$

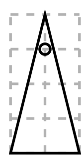
The condition $\mathbf{0} \in \operatorname{int}(P)$ ensures that $h_{\min} \in \mathbb{Z}_{<0}$ and $h_{\max} \in \mathbb{Z}_{>0}$. We define $P_{\min}$ (resp. $P_{\max}$) to be $P_{w,h_{\min}}$ (resp. $P_{w,h_{\max}}$).

Definition 2.2. Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope and fix a width vector $w \in M$. A *factor* of P with respect to w is a polytope $F \subset N_{\mathbb{Q}}$ such that $w(F) = 0$, $\operatorname{verts}(F) \subset N$ and for every integer h satisfying the inequality $h_{\min} \leq h < 0$, there exists a possibly empty polytope $G_h \subset N_{\mathbb{Q}}$ such that $w(G_h) = h$, $\operatorname{verts}(G_h) \subset N$ and:

$$H_{w,h} \cap \operatorname{verts}(P) \subseteq G_h + (-h)F \subseteq P_{w,h}. \tag{2.2}$$

A factor F is said to be *trivial* if $\dim F = 0$. Unless otherwise stated, *factor* will always mean *non-trivial factor*, that is, a factor F with $\dim F \geq 1$.

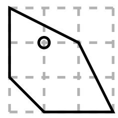
Example 2.3. If $P \subset N_{\mathbb{Q}}$ is a lattice polytope, then a (non-trivial) factor of P with respect to a chosen width vector w may not exist. For instance, let $N = \mathbb{Z}^2$ and let $P \subset \mathbb{Q}^2$ be the lattice polygon with vertex set $\{(0,1), (1,-3), (-1,-3)\}$.



There does not exist a factor of P with respect to $w = (0,1)^t \in (\mathbb{Z}^2)^\vee$. If a factor existed then, for dimension reasons, it would be a line segment and in particular, the smallest line segment $F = \operatorname{conv}((0,0),(1,0)) \subset N_{\mathbb{Q}}$ would also be a factor of

P with respect to w. But for this choice of F, there does not exist a polytope G_h satisfying (2.2) for $h = -3$ $(= h_{\min})$. Contradiction.

Example 2.4. The collection $\{G_h\}$ in Definition 2.2 is not unique in general. For instance, let $P \subset \mathbb{Q}^2$ be the lattice polygon with vertex set $\{(1,0),(2,-2),$ $(0,-2),(-1,-1),(-1,1)\}$.



Let $w = (0,1)^t \in (\mathbb{Z}^2)^\vee$ and let $F := \mathrm{conv}(\mathbf{0},(1,0)) \subset \mathbb{Q}^2$. Then F is a factor of P with respect to w. This can be shown by setting $G_{-2} := \{(0,-2)\} \subset \mathbb{Q}^2$ and taking $G_{-1} \subset \mathbb{Q}^2$ to be either the singleton $\{(-1,-1)\}$ or the line segment $\mathrm{conv}((-1,-1),(0,-1))$.

2.3 Construction of Combinatorial Mutations

Let $P \subset N_\mathbb{Q}$ be a lattice polytope and fix a width vector $w \in M$. If there exists a (trivial or non-trivial) factor F of P, with respect to w and the collection $\{G_h\}$, then [2, Definition 5] we may define the *combinatorial mutation* of P, with respect to the data $(w, F, \{G_h\})$, to be the following *lattice* polytope[2]:

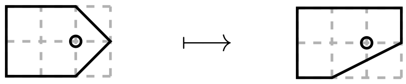
$$\mathrm{mut}_w(P,F;\{G_h\}) := \mathrm{conv}\left(\bigcup_{h=h_{\min}}^{-1} G_h \cup \bigcup_{h=0}^{h_{\max}} (P_{w,h} + hF)\right) \subset N_\mathbb{Q},$$

where the notation follows Section 2.2. A combinatorial mutation of P is called *trivial* if it is isomorphic to P. Note that if $v \in N$ satisfies $w(v) = 0$, then the translate $v + F$ is also a factor of P with respect to w and $\{G_h + hv\}$. So the lattice polytopes $\mathrm{mut}_w(P,F;\{G_h + hv\})$ and $\mathrm{mut}_w(P,F;\{G_h\})$ are isomorphic, related by a shear transformation. In particular, when constructing combinatorial mutations, it suffices to consider factors up to translation by elements of $H_{w,0} \cap N$.

Example 2.5. Consider the lattice (in fact Fano) polygon $P \subset \mathbb{Q}^2$ with vertex set $\{(0,1),(1,0),\ (0,-1),(-2,-1),(-2,1)\}$. Choose the width vector $w = (0,1)^t \in (\mathbb{Z}^2)^\vee$, and let F be the line segment $\mathrm{conv}(\mathbf{0},(1,0)) \subset \mathbb{Q}^2$. Taking $G_{-1} = \mathrm{conv}((-2,-1),(-1,-1)) \subset \mathbb{Q}^2$ shows that F is a factor of P with respect to w. The combinatorial mutation $Q := \mathrm{mut}_w(P,F;\{G_{-1}\})$ is the lattice (Fano)

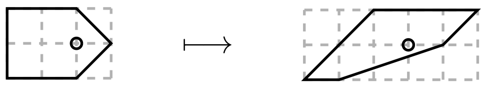
[2]Note that $\mathbf{0} \in \mathrm{int}(\mathrm{mut}_w(P,F;\{G_h\}))$, and that $\dim(\mathrm{mut}_w(P,F;\{G_h\})) = \mathrm{rank}(N)$.

polygon with vertex set $\{(1,1),(1,0),(-1,-1),(-2,-1),(-2,1)\}$.



Note that the construction can be reversed, and P realized as a combinatorial mutation of Q, by choosing $w = (0,-1)^t \in (\mathbb{Z}^2)^\vee$, F as before and taking[3] $G_{-1} := \operatorname{conv}((-2,1),(0,1))$. The reader may find it fruitful to compare this example with the results obtained later in the chapter.

Example 2.6. Let P, F and G_{-1} be as in Example 2.5 and let $w = (0,1)^t \in (\mathbb{Z}^2)^\vee$. The line segment $F' := \operatorname{conv}((1,0),(2,0)) \subset \mathbb{Q}^2$ is a translate of F by the vector $v = (1,0) \in H_{w,0} \cap N$; $F' = F + v$. Taking $G'_{-1} = G_{-1} + (-1)v = \operatorname{conv}((-3,-1),(-2,-1)) \subset \mathbb{Q}^2$ shows that F' is also a factor of P with respect to w, and that $\operatorname{mut}_w\big(P, F'; \{G'_{-1}\}\big)$ is the lattice (Fano) polygon with vertex set $\{(2,1),(1,0),(-2,-1),(-3,-1),(-1,1)\}$.



In particular, $\operatorname{mut}_w\big(P, F'; \{G'_{-1}\}\big)$ is the image of $\operatorname{mut}_w(P, F; \{G_{-1}\})$ under the shear automorphism of $\mathbb{Q}^2$ defined by $(1,0) \mapsto (1,0)$ and $(0,1) \mapsto (1,1)$.

Remark 2.7. Note that $H_{w,0} \cap N$ is precisely the set of trivial factors of P with respect to w. If $f \in H_{w,0} \cap N$, then $\operatorname{mut}_w(P, f; \{P_{w,h} + hf\}) \cong \operatorname{mut}_w(P, \mathbf{0}; \{P_{w,h}\}) = P$. Thus, a trivial (zero-dimensional) factor constructs a trivial combinatorial mutation; all non-trivial combinatorial mutations of a lattice polytope P must be constructed using non-trivial (positive-dimensional) factors $F \subset N_{\mathbb{Q}}$.

2.4 Comparison With Algebraic Mutations

As discussed in Section 1.1, the primary motivation for introducing a theory of combinatorial mutations is to describe the transformation on Newton polytopes induced by the operation of pulling back Laurent polynomials under algebraic mutations. The notion of combinatorial mutation should have the following property: if $f, g \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ are Laurent polynomials related by $g = \varphi^* f$ for some algebraic mutation φ, then $\operatorname{Newt}(g)$ must be a combinatorial mutation of

[3]Note that in this reverse construction, the height function on $N_{\mathbb{Q}}$ is induced by $w = (0,-1)^t$.

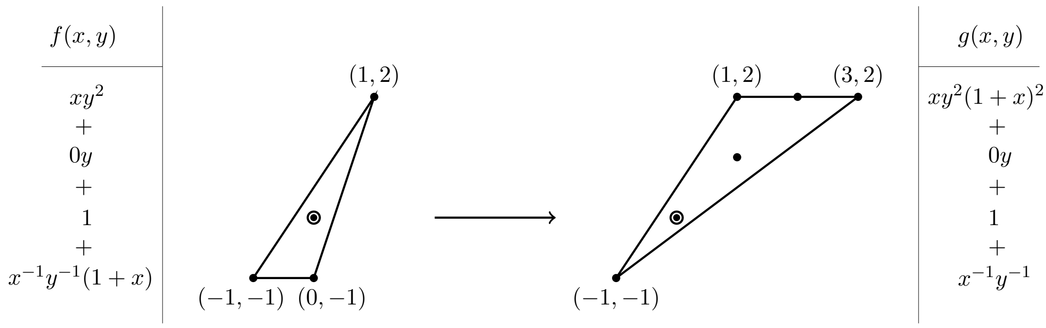


Figure 2.1: The Newton polygons of f and g from Example 2.8.

Newt(f). The aim of this section is to show that the construction of Section 2.3 fulfills this requirement. We begin by illustrating the key ideas in Example 2.8, followed by a more general discussion.

Example 2.8. Consider the following Laurent polynomials:

$$f(x,y) = xy^2 + 1 + \frac{1+x}{xy}, \qquad g(x,y) = xy^2(1+x)^2 + 1 + \frac{1}{xy}.$$

The Newton polygons of f and g are illustrated in Figure 2.1. A direct check shows that $g = \varphi_A^* f$, where φ_A is the algebraic mutation

$$(x,y) \mapsto (x, (1+x)y).$$

Formally, g is obtained from f by the following procedure, which is also illustrated in Figure 2.1. First we partition the terms of f according to the power of y that they contain; combinatorially, this corresponds to choosing the width vector $w = (0,1)^t \in (\mathbb{Z}^2)^\vee$. Then we choose the Laurent polynomial $A(x) = 1 + x$; combinatorially this corresponds to choosing the factor $F = \mathrm{Newt}(A) = \mathrm{conv}(\mathbf{0}, (1,0)) \subset \mathbb{Q}^2$. The statement that $g = \varphi_A^* f$ is a Laurent polynomial is captured by the combinatorial condition (2.2): A divides $C_{-1} = \frac{(1+x)}{x}$, and we define G_{-1} to be the Newton polytope of the quotient $\frac{C_{-1}}{Ay} = \frac{1}{xy}$. The operation of pullback φ_A^* replaces every instance of y by $(1+x)y$. Therefore the term $\frac{1+x}{xy}$ in f becomes $\frac{1}{xy}$ in g, and the term xy^2 in f becomes $xy^2(1+x)^2$ in g; combinatorially, the line segment $\mathrm{conv}((-1,-1),(0,-1))$ in Newt(F) is contracted by a single copy of F to the point $(-1,-1)$ in Newt(g) and the point $(1,2)$ in Newt(f) is extended by two copies of F to the line segment $\mathrm{conv}((1,2),(3,2))$ in Newt(g). This is precisely the operation described in Section 2.3. We see that $P = \mathrm{Newt}(f)$ and $Q = \mathrm{Newt}(g)$ satisfy $Q = \mathrm{mut}_w(P, F, \{G_{-1}\})$.

The discussion of Example 2.8 translates to higher dimensions with minimal changes: Suppose that $f, g \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ are related by a single algebraic mutation: $g = \varphi^* f$ where, in the notation of Section 1.3, φ equals either $\gamma \circ \varphi_A \circ \eta$ or $\gamma \circ (\varphi_A)^{-1} \circ \eta$ for some fixed $A \in \mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$. Since $(\gamma \circ \varphi_A \circ \eta)^* = \gamma^* \circ \varphi_A^* \circ \eta^*$, it suffices to understand the combinatorics of each pullback separately.

If φ equals either γ or η, then it is of the form $\mathsf{x} \mapsto \mathsf{x}^M$ for all $x \in (\mathbb{C}^*)^n$, where $M \in GL_n(\mathbb{Z})$ is fixed. If $p \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a Laurent polynomial then $\mathrm{Newt}(\varphi^* p)$ is the image of $\mathrm{Newt}(p) \subset \mathbb{Q}^n$ under the linear automorphism of $\mathbb{Q}^n$ which maps the k^{th} basis vector to the k^{th} row of M.

Now consider a Laurent polynomial, $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Collect powers of x_n in f so that:

$$f = \sum_{h=h_{\min}}^{-1} C_h \cdot x_n^h + \sum_{h=0}^{h_{\max}} C_h \cdot x_n^h \quad ; \quad C_h \in \mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}].$$

For a given $A \in \mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$, the pullback $\varphi_A^* f$ is a Laurent polynomial if and only if A^{-h} divides C_h for all negative values of h. Equivalently, for all h satisfying $h_{\min} \le h < 0$, there exists $R_h \in \mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ such that $R_h \cdot A^{-h} = C_h$. If this condition holds, we may construct the Laurent polynomial

$$g := \varphi_A^* f = \sum_{h=h_{\min}}^{-1} R_h \cdot x_n^h + \sum_{h=0}^{h_{\max}} C_h \cdot A^h \cdot x_n^h. \tag{2.3}$$

The combinatorial interpretation of the above construction is as follows: start with $P := \mathrm{Newt}(f) \subset \mathbb{Q}^2$ and choose the width vector $w = (0, \dots, 0, 1)^t \in (\mathbb{Z}^2)^\vee$, so that:

$$P = \mathrm{conv}\left(\bigcup_{h=h_{\min}}^{-1} P_{w,h} \cup \bigcup_{h=0}^{h_{\max}} P_{w,h} \right) \quad ; \quad P_{w,h} := \mathrm{Newt}(C_h).$$

For a given $F := \mathrm{Newt}(A) \subset \mathbb{Q}^2$, the requirement: A^{-h} divides C_h for all h satisfying $h_{\min} \le h < 0$, implies that F is a factor of P with respect to w, by taking $\{G_h\} := \{\mathrm{Newt}(R_h)\}_{h=h_{\min}}^{-1}$. Thus we may construct the combinatorial mutation:

$$\mathrm{mut}_w(P, F) := \mathrm{conv}\left(\bigcup_{h=h_{\min}}^{-1} G_h \cup \bigcup_{h=0}^{h_{\max}} (P_{w,h} + hF) \right) \subset N_{\mathbb{Q}},$$

and this coincides with $\mathrm{Newt}(g)$, by comparison with (2.3).

2.5 Properties of Combinatorial Mutations

This section forms the technical heart of this document. We will study properties of the construction presented in Section 2.3. The reader wishing to reach examples as soon as possible may read Summary 2.9, then skip to Section 2.6, stopping only to collect the definition of one-step mutations on page 24.

Summary 2.9. The basic properties of combinatorial mutations are as follows:

(1) If $F \subset N_{\mathbb{Q}}$ is a factor of $P \subset N_{\mathbb{Q}}$ with respect to the width vector $w \in M$ and both $\{G_h\}, \{G'_h\}$, then $\mathrm{mut}_w(P, F; \{G_h\}) = \mathrm{mut}_w(P, F; \{G'_h\})$ (Proposition 2.13). Thus, the data (w, F) unambiguously determines the combinatorial mutation, and we may write $\mathrm{mut}_w(P, F)$ instead of $\mathrm{mut}_w(P, F; \{G_h\})$.

(2) If $Q := \mathrm{mut}_w(P, F)$ then $P = \mathrm{mut}_{-w}(Q, F)$ (Lemma 2.12). Thus, the combinatorial mutation construction is reversible and being related by combinatorial mutations is an equivalence relation on lattice polytopes (Remark 2.14).

(3) Up to isomorphism, there are only finitely many lattice polytopes that can be obtained from a given lattice polytope P by a one-step mutation (Proposition 2.15). Thus, the mutation graph of a lattice polytope (defined on page 35) is locally finite.

(4) The property of being Fano is preserved: $P \subset N_{\mathbb{Q}}$ is a Fano polytope if and only if $\mathrm{mut}_w(P, F)$ is a Fano polytope (Proposition 2.18).

(5) The construction of combinatorial mutations has the following dual description: if $Q := \mathrm{mut}_w(P, F) \subset N_{\mathbb{Q}}$ exists then there is a map $\varphi := \varphi(w, F) : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$, which is linear on each maximal cone of the inner normal fan of F in $M_{\mathbb{Q}}$, such that $\varphi(P^\vee) = Q^\vee$ (Corollary 2.20). In particular, the toric varieties defined by the spanning fans of P and Q have the same anti-canonical degree (Corollary 2.21) and the Ehrhart series of the dual polytopes $P^\vee$ and $Q^\vee$ coincide (Corollary 2.22).

In the remainder of this section, we will prove the statements from Summary 2.9. We make essential use of the statement (and proof) of:

Lemma 2.10. *Let $Q := \mathrm{mut}_w(P, F; \{G_h\})$ be a combinatorial mutation of $P \subset N_{\mathbb{Q}}$. If $u \in \mathrm{verts}(Q)$ satisfies $w(u) = h$, then $u = v + hf$ for some $v \in P_{w,h} \cap N$ and $f \in \mathrm{verts}(F)$.*

Proof. If $u \in Q \cap N$, with $w(u) = h \geq 0$, and if $u \notin P_{w,h} + hF$, then u must lie in the interior of a line segment in $N_{\mathbb{Q}}$ joining two distinct (not necessarily lattice) points of Q. In particular, u cannot be a vertex of Q. So if $u \in \mathrm{verts}(Q)$ and $w(u) = h \geq 0$ then $u \in P_{w,h} + hF$ and so $u \in \mathrm{verts}(P_{w,h} + hF)$. Since the vertices of a Minkowski sum of polytopes are contained in the set of sums of vertices of the Minkowski summands, we conclude that:

$$u \in \mathrm{verts}(P_{w,h} + hF) \subseteq \mathrm{verts}(P_{w,h}) + h \cdot \mathrm{verts}(F),$$

and in particular $u = v + hf$ for some $v \in \mathrm{verts}(P_{w,h}) \subseteq P_{w,h} \cap N$ and $f \in \mathrm{verts}(F)$.

If $u \in \mathrm{verts}(Q)$ and $w(u) = h < 0$, then a similar argument shows that $u \in \mathrm{verts}(G_h)$. Since $Q = \mathrm{mut}_w(P, F; \{G_h\})$ exists, F must be a factor of P with respect to w and $\{G_h\}$. In particular, the following inclusion must hold:

$$G_h + (-h)F \subseteq P_{w,h}.$$

So for any $f \in \mathrm{verts}(F)$ there exists a $v \in P_{w,h} \cap N$ such that $u + (-h)f = v$. $\square$

Remark 2.11. Note that if $u \in \mathrm{verts}(Q)$ satisfies $w(u) = h \geq 0$, then the proof of Lemma 2.10 shows the slightly stronger fact that $u = v + hf$, where $f \in \mathrm{verts}(F)$ and $v \in \mathrm{verts}(P_{w,h})$. In fact, we must have $v \in \mathrm{verts}(P)$ for the following reason: if $v \notin \mathrm{verts}(P)$, then v must lie in the strict interior of a line segment $L \subset P$, which joins two (not necessarily lattice) points of P. But then the line segment $\{p + w(p)f \mid p \in L\} \subset Q$ joins two (not necessarily lattice) points of Q and contains $u = v + hf$ in its strict interior. This gives the contradiction $u \notin \mathrm{verts}(Q)$.

Reversibility

Example 2.5 suggests that if P is a lattice polytope for which a combinatorial mutation, $Q := \mathrm{mut}_w(P, F; \{G_h\})$, exists, then there also exists data $(w', F'; \{G'_h\})$, with respect to which $P = \mathrm{mut}_{w'}(Q, F'; \{G'_h\})$. In other words: the construction of a combinatorial mutation should be reversible. This is true, by the following:

Lemma 2.12 ([2, Lemma 2]). *Let $Q := \mathrm{mut}_w(P, F; \{G_h\})$ be a combinatorial mutation of $P \subset N_{\mathbb{Q}}$. Then $\mathrm{mut}_{-w}(Q, F; \{P_{w,h}\})$ exists, and is equal to P as a subset of $N_{\mathbb{Q}}$.*

Proof. To avoid confusion between the two width vectors $w, -w \in M$, all heights in the current proof will be computed using w. To show that $\mathrm{mut}_{-w}(Q, F; \{P_{w,h}\})$

exists, it is sufficient to show that F is a factor of Q with respect to $-w$ and $\{P_{w,h}\}_{h>0}$. By assumption, F is a factor of P with respect to w and $\{G_h\}_{h<0}$. In particular, $w(F) = 0$, and so $-w(F) = 0$. It remains to verify that the following sequence of inclusions holds for every integer $h \in \{1, \ldots, h_{\max}\}$:

$$H_{w,h} \cap \text{verts}(Q) \subseteq P_{w,h} + hF \subseteq Q_{w,h}.$$

The right-most inclusion follows from the definition of Q as a combinatorial mutation of P. The left-most inclusion holds by Lemma 2.10. Therefore $P' := \text{mut}_{-w}(Q, F; \{P_{w,h}\}) \subset N_{\mathbb{Q}}$ exists.

To prove the inclusion $P \subseteq P'$, it suffices to show that $\text{verts}(P) \subset P'$. This is because P' is convex and P is, by definition, the smallest convex set (with respect to inclusion) containing $\text{verts}(P)$. Suppose $v \in \text{verts}(P)$ and $w(v) = h$, so that $v \in H_{w,h} \cap \text{verts}(P)$. If $h \geq 0$ then by the definition of P' we must have:

$$P' \supset P_{w,h} \supset H_{w,h} \cap \text{verts}(P).$$

On the other hand, if $h < 0$, then by the definitions of P' and Q we must have:

$$P' \supset Q_{w,h} + (-h)F \supset G_h + (-h)F \supset H_{w,h} \cap \text{verts}(P),$$

where the right-most inclusion holds because F is a factor of P with respect to w and $\{G_h\}$. Finally, to show that $P' \subseteq P$, let $v \in \text{verts}(P')$ and suppose $w(v) = h$ so that $v \in H_{w,h} \cap \text{verts}(P')$. If $h \geq 0$ then the same argument as the negative height case in the proof of Lemma 2.10 shows that $v \in \text{verts}(P_{w,h})$, and in particular, $v \in P$. On the other hand, if $h < 0$ then the same argument as the positive height case in the proof of Lemma 2.10 shows that $v \in \text{verts}(Q_{w,h} + (-h)F) \subseteq \text{verts}(Q_{w,h}) + (-h) \cdot \text{verts}(F)$. The negative height case in the proof of Lemma 2.10, applied a second time, shows that $\text{verts}(Q_{w,h}) \subseteq \text{verts}(G_h)$. Therefore:

$$v \in \text{verts}(G_h) + (-h) \cdot \text{verts}(F) \subseteq G_h + (-h)F \subseteq P_{w,h} \subseteq P,$$

where $G_h + (-h)F \subseteq P_{w,h}$ as F is a factor of P with respect to w and $\{G_h\}$. $\square$

Independence of Choices

Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope, $w \in M$ a width vector and let $F \subset N_{\mathbb{Q}}$ be a polytope satisfying $w(F) = 0$ and $\text{verts}(F) \subset N$. In order for F to be a factor of

P with respect to w, there must exist a collection of polytopes $\{G_h\}$ satisfying the conditions of Definition 2.2. Such a collection need not be unique (Example 2.4), and if $\{G'_h\}$ is any other collection with the same properties, a natural question to ask is whether the combinatorial mutations $Q := \mathrm{mut}_w(P, F; \{G_h\})$ and $Q' := \mathrm{mut}_w(P, F; \{G'_h\})$ differ in any essential way. Proposition 2.13 below shows that Q and Q' are in fact equal, as subsets of $N_{\mathbb{Q}}$. In particular, since the resulting combinatorial mutation of P is independent of the choice of collection $\{G_h\}$, we shall henceforth write $\mathrm{mut}_w(P, F)$ instead of $\mathrm{mut}_w(P, F; \{G_h\})$.

Proposition 2.13 ([2, Proposition 1])**.** *Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope. Fix $w \in M$ and suppose that $F \subset N_{\mathbb{Q}}$ is a factor of P with respect to w and both $\{G_h\}$, $\{G'_h\}$. Then the lattice polytopes* $\mathrm{mut}_w(P, F; \{G_h\})$ *and* $\mathrm{mut}_w(P, F; \{G'_h\})$ *are equal as subsets of $N_{\mathbb{Q}}$.*

Proof. Let $Q := \mathrm{mut}_w(P, F, \{G_h\})$ and $Q' := \mathrm{mut}_w(P, F; \{G'_h\})$, and suppose $Q \neq Q'$. Then (possibly after interchanging Q and Q'), there exists a vertex $q' \in \mathrm{verts}(Q')$ such that $q' \notin Q$. In particular, there exists an $s \in M$ and $k \in \mathbb{Z}_{>0}$ such that $H_{s,k}$ is a supporting hyperplane of Q and also separates Q and q', i.e. $s(x) \leq k$ for all $x \in Q$ and $s(q') > k$. By construction, Q and Q' are identical at non-negative heights (with respect to w), so we must have $w(q') < 0$.

By Lemma 2.12, $P = \mathrm{mut}_{-w}(Q, F; \{P_{w,h}\})$, so if $u \in \mathrm{verts}(P)$ then we may, by Lemma 2.10, write $u = x + w(u)f$, for some $x \in Q_{w,w(u)}$ and $f \in \mathrm{verts}(F)$. Thus:

$$s(u) = s(x) + w(u)s(f) \leq \begin{cases} k - w(u)s_{\min}, & w(u) \geq 0, \\ k - w(u)s_{\max}, & w(u) < 0, \end{cases}$$

where $s_{\min} := \min\{s(f) \mid f \in \mathrm{verts}(F)\}$ and $s_{\max} := \max\{s(f) \mid f \in \mathrm{verts}(F)\}$. But now, since P is also equal to $\mathrm{mut}_{-w}(Q', F; \{P_{w,h}\})$, we have that $q' - w(q')F \subset P$. By definition, there exists some $f \in \mathrm{verts}(F)$ such that $s(f) = s_{\max}$. For this choice of f, we must have that $q' - w(q')f \in \mathrm{verts}\big(P_{w,w(q')}\big)$. By Remark 2.11, it follows that $q' - w(q')f \in \mathrm{verts}(P)$. But then:

$$s(q' - w(q')f) = s(q') - w(q')s_{\max} > k - w(q')s_{\max},$$

which is a contradiction. Therefore, we must have $Q = Q'$. □

Remark 2.14 (An Equivalence Relation on Lattice Polytopes)**.** We say that two lattice polytopes $P, Q \subset N_{\mathbb{Q}}$ are *related by (a sequence of) combinatorial mutations*, and write $P \sim Q$, if and only if there exists a sequence of lattice polytopes: $P_0 := P, \ldots, P_m := Q$ ($P_i \subset N_{\mathbb{Q}}$ for $i \in \{0, \ldots, m\}$) such that either $P_{i+1} \cong P_i$ or

$P_{i+1} = \mathrm{mut}_{w_i}(P_i, F_i)$ for suitable data (w_i, F_i). By Lemma 2.12, this is an equivalence relation on the set of all lattice polytopes in $N_{\mathbb{Q}}$. It is the combinatorial analogue of the equivalence relation which identifies Laurent polynomials related by a sequence of algebraic mutations (see Sections 1.3 and 2.4).

Finiteness of One-Step Mutations

Let $P, Q \subset N_{\mathbb{Q}}$ be lattice polytopes. We say that Q is obtained from P by a *one-step mutation* if there exists a width vector $w \in M$ and a factor $F \subset N_{\mathbb{Q}}$ of P with respect to w, satisfying $\dim(F) \geq 1$, such that $\mathrm{mut}_w(P, F)$ exists and $Q \cong \mathrm{mut}_w(P, F)$.

Proposition 2.15 ([2, Proposition 3])**.** *Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope. Up to isomorphism, there are only finitely many lattice polytopes that can be obtained from P by a one-step mutation.*

Proof. By Remark 2.7, it suffices to restrict attention to positive-dimensional factors. For a given width vector $w \in M$, there are (up to translation) at most a finite number of positive-dimensional factors[4] of P with respect to w. Thus, it suffices to show that there are only finitely many width vectors $w \in M$ for which there exists a factor F of P, with $\dim(F) \geq 1$.

For a given face Q of P, let $\mathcal{L}_Q$ denote the set of all positive integers l for which there exist lattice polytopes $F, G \subset N_{\mathbb{Q}}$, with $F \neq \varnothing$ and $\dim(F) \geq 1$, such that $Q = lF + G$. Since Q is bounded in $N_{\mathbb{Q}}$, $\mathcal{L}_Q$ is a finite set. Let $l_Q := \max\{l \mid l \in \mathcal{L}_Q\}$ and let $l_P := \max\{l_Q \mid Q \text{ is a face of } P\}$.

Let $w \in M$ be a fixed width vector for which there exists a factor F of P with $\dim(F) \geq 1$. Let Q be the face of P at minimum height with respect to w and let $h := w(Q) < 0$. Then w lies on the boundary of $(-h)P^\vee \subset M_{\mathbb{Q}}$. Furthermore, $-h \in \mathcal{L}_Q$ by assumption, so $-h \leq l_Q \leq l_P$. Thus, $(-h)P^\vee \subset l_P P^\vee$, which implies that w lies in the finite set $l_P P^\vee \cap M$ (which depends only on P). □

Remark 2.16 (The Two-Dimensional Case)**.** In the notation of Proposition 2.15, let $n := \dim(P)$, and let Σ in $M_{\mathbb{Q}}$ denote the inner normal fan of $P \subset N_{\mathbb{Q}}$. Since $\mathbf{0} \in \mathrm{int}(P^\vee)$, any width vector $w \in M$ lies in at least one cone $\sigma \in \Sigma$, and uniquely

[4]The reason for this is as follows: Let F be a positive-dimensional factor of P with respect to w. Without loss of generality (Section 2.3), we may translate F so that one of its vertices is the origin. For any integer h satisfying $h_{\min} \leq h < 0$, condition (2.2) then implies that $G_h \subseteq P_{w,h}$. But $P_{w,h}$ is compact, so there are only finitely many choices for G_h. By condition (2.2), we must also have $G_h + (-h)F \subseteq P_{w,h}$, forcing the number of choices for F to be finite.

determines a cone $\sigma(w) \in \Sigma$, with the property that

$$\dim(\sigma(w)) = \min\{\dim(\sigma) \mid \sigma \in \Sigma, w \in \sigma\}.$$

If $w \in M$ is such that $\dim(\sigma(w)) = \dim(P)$ (so that $\sigma(w)$ is a maximal cone of Σ) then the face Q of P corresponding to w is zero-dimensional (by the definition of inner normal fan). Hence any factor F of P with respect to w is zero-dimensional. This fact, together with the proof of Proposition 2.15 shows that the width vectors for which there exists a factor F of P with $\dim F \geq 1$ in fact lie in the set:

$$\Sigma^{(n-1)} \cap l_P P^\vee \cap M,$$

(which still depends only on P), where $\Sigma^{(n-1)}$ denotes the subset of Σ consisting only of cones of dimension at most $n-1$. In particular , when $n = 2$, we have:

Corollary 2.17. *If $P \subset N_\mathbb{Q}$ is a lattice polygon and $w \in M$ is a width vector, for which there exists a non-trivial factor of P, then*

$$w \in \{\overline{u} \in M \mid u \in \operatorname{verts}(P^\vee)\}, \tag{2.4}$$

where $\overline{u}$ denotes the primitive lattice vector on the ray in $M_\mathbb{Q}$ through u and $\mathbf{0}$. In particular, there are at most $|\operatorname{verts}(P^\vee)|$ choices for w and at most $|\partial P \cap N|$ distinct non-trivial combinatorial mutations of P.

Proof. The first statement is immediate from Remark 2.16. The set (2.4) contains precisely $|\operatorname{verts}(P^\vee)|$ elements. In two dimensions, all positive-dimensional factors are line segments, and there are at most $|\partial P \cap N|$ of these, up to translation. □

Combinatorial Mutations of Fano Polytopes

Proposition 2.18 ([2, Proposition 2])**.** *Let $P \subset N_\mathbb{Q}$ be a lattice polytope. Then $\operatorname{mut}_w(P, F)$ is a Fano polytope if and only if P is a Fano polytope.*

Proof. Suppose that $P \subset N_\mathbb{Q}$ is a Fano polytope and let $Q := \operatorname{mut}_w(P, F)$. The construction of Section 2.3 makes it clear that $\mathbf{0} \in \operatorname{int}(Q)$, and that $\dim(Q) = \operatorname{rank}(N)$. It remains to show that the vertices of Q are primitive. Let $u \in \operatorname{verts}(Q)$.

If $h := w(u) \geq 0$ then by Remark 2.11, we may write $u = v + hf$, where $v \in \operatorname{verts}(P)$ and $f \in \operatorname{verts}(F)$. Since v is primitive by assumption, u must also be primitive. To see this, fix an isomorphism of lattices $N \cong \mathbb{Z}^{n+1}$, so that $w = (0, \ldots, 0, 1)^t$. In this basis: $f = (f_1, \ldots, f_n, 0) \in \mathbb{Z}^{n+1}$ and $v =$

$(v_1, \ldots, v_n, h) \in \mathbb{Z}^{n+1}$. The primitivity of v forces $\gcd\{v_1, \ldots, v_n, h\} = 1$. Hence $\gcd\{v_1 + hf_1, \ldots, v_n + hf_n, h\} = 1$, which implies that $u = v + hf$ is primitive.

Now suppose $h := w(u) < 0$. Since $Q = \mathrm{mut}_w(P, F)$ exists, there is at least one collection of polytopes $\{G_h\}$ such that F is a factor of P with respect to w and $\{G_h\}$. Arguing as in the proof of Lemma 2.10 shows that $u \in \mathrm{verts}(G_h)$. Replace G_h by the smallest polytope (with respect to inclusion) satisfying the following condition:

$$H_{w,h} \cap \mathrm{verts}(P) \subseteq G_h + (-h)F \subseteq P_{w,h}. \tag{2.5}$$

By Proposition 2.13, this leaves Q unchanged. Suppose that u is not primitive. Then for any $f \in \mathrm{verts}(F)$, the lattice vector $u - hf$ is not primitive (by the same argument as in the $h \geq 0$ case). Since the vertices of P are primitive by assumption, this implies that $u - hf \notin H_{w,h} \cap \mathrm{verts}(P)$. But now we may take $G'_h = \mathrm{conv}(G_h \cap N \backslash \{u\})$, which is strictly smaller than G_h and still satisfies (2.5), contradicting the minimality of G_h. So u must be primitive.

We conclude that $Q = \mathrm{mut}_w(P, F)$ is Fano. Conversely, if $\mathrm{mut}_w(P, F)$ is Fano, then the above argument, together with Lemma 2.12, shows that P is Fano. □

Dual Description

Throughout this section, let $P \subset N_\mathbb{Q}$ be a lattice polygon. Choose a width vector $w \in M$ and let $Q := \mathrm{mut}_w(P, F) \subset N_\mathbb{Q}$ for some factor F. We associate the following function to the pair (w, F):

$$\varphi(w, F) : M \to M \quad ; \quad u \mapsto u - u_{\min} w,$$

where $u_{\min} := \min\{u(f) \mid f \in \mathrm{verts}(F)\}$. Since $w(F) = 0$, $\varphi(w, F)$ is a bijection, with inverse given by $\varphi(-w, F)$ (cf. Lemma 2.12). The map $\varphi(w, F)$ canonically extends to a map $M_\mathbb{Q} \to M_\mathbb{Q}$ which (by construction) is linear on each maximal cone of the inner normal fan of F.

Proposition 2.19. *For any positive integer k, we have $\varphi(w, F)(\partial(kP^\vee)) = \partial(kQ^\vee)$.*

Proof. Let $u \in \partial(kP^\vee)$. Then u defines the supporting hyperplane $H_{u,-k} \subset N_\mathbb{Q}$ of P. We will show that $u - u_{\min} w \in \partial(kQ^\vee)$, i.e. $H_{u - u_{\min} w, -k}$ is a supporting hyperplane for Q. Let $q \in \mathrm{verts}(Q)$. If $w(q) \geq 0$, then by Remark 2.11 we may have $q = v + w(q)f$ for some $v \in \mathrm{verts}(P)$ and $f \in \mathrm{verts}(F)$. In particular, since $w(v) = w(q) \geq 0$, we have that

$$(u - u_{\min} w)(q) = (u - u_{\min} w)(v + w(q)f) = u(v) + w(q)(u(f) - u_{\min}) \geq -k,$$

where the last inequality holds because $v \in \mathrm{verts}(P)$ and $H_{u,-k}$ is a supporting hyperplane for P. If $w(q) < 0$ then for any $f \in \mathrm{verts}(F)$, we have that $q - w(q)f \in P$. Thus $u(q - w(q)f) \geq -k$, which implies that $u(q) \geq -k + u_{\min}w(q)$ and hence

$$(u - u_{\min}w)(q) = u(q) - u_{min}w(q) \geq -k + u_{\min}w(q) - u_{\min}w(q) = -k.$$

We deduce that every vertex of Q, and hence every point $x \in Q$, satisfies $(u - u_{\min}w)(x) \geq -k$. It remains to show that $u - u_{\min}w$ attains the value $-k$ at a point of Q. Now since $H_{u,-k}$ is a supporting hyperplane for P, there exists $v \in \mathrm{verts}(P)$ such that $u(v) = -k$. Suppose $w(v) \geq 0$. Choose $f \in \mathrm{verts}(F)$ such that $u(f) = u_{\min}$. The point $v + w(v)f$ lies in Q (by the definition of Q) and satisfies: $(u - u_{\min}w)(v + w(v)f) = -k$. Now suppose that $w(v) < 0$. Then, by Remark 2.11 and Lemma 2.12, there exists $q \in \mathrm{verts}(Q)$ and $f \in \mathrm{verts}(F)$ such that $v = q - w(v)f$. Suppose that $u(f) > u_{\min}$ and let $f' \in \mathrm{verts}(F)$ be such that $u(f') = u_{\min}$. Then $u(q - w(v)f') < u(q - w(v)f) = u(v) = -k$. This contradicts the fact that $q - w(v)f' \in P$. So $u(f) = u_{\min}$ and so $(u - u_{\min}w)(q) = (u - u_{\min}w)(v + w(v)f) = -k$. We conclude that $H_{u - u_{\min}w, -k}$ is a supporting hyperplane for Q.

The above argument shows that $\varphi(w, F)$ maps $\partial(kP^\vee)$ bijectively onto a subset of $\partial(kQ^\vee)$. By repeating the same argument for $\varphi(-w, F)$, we conclude that the image of $\partial(kP^\vee)$ under $\varphi(w, F)$ coincides with $\partial(kQ^\vee)$. □

Corollary 2.20. *If $P \subset N_{\mathbb{Q}}$ is a lattice polytope and $Q := \mathrm{mut}_w(P, F)$ then $\varphi(w, F)(P^\vee) = Q^\vee$. Thus applying $\varphi(w, F)$ is dual to constructing the combinatorial mutation $\mathrm{mut}_w(P, F)$.*

Proof. Let $\varphi := \varphi(w, F)$ and $\varphi^{-1} := \varphi(-w, F)$. Proposition 2.19 gives $\varphi(\partial P^\vee) = \partial Q^\vee$. Now:

$$\varphi(P^\vee) \subseteq \mathrm{conv}(\varphi(\partial P^\vee)) = \mathrm{conv}(\partial Q^\vee) = Q^\vee,$$

so $\varphi(P^\vee) \subseteq Q^\vee$. The application of φ^{-1} corresponds to the reverse construction of P as a combinatorial mutation of Q. Thus, a similar argument shows that $\varphi^{-1}(Q^\vee) \subseteq P^\vee$. We conclude that $Q^\vee = (\varphi \circ \varphi^{-1})(Q^\vee) \subseteq \varphi(P^\vee) \subseteq Q^\vee$, and so $\varphi(P^\vee) = Q^\vee$, as claimed. □

2.6 Invariants

Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope and let $F \subset N_{\mathbb{Q}}$ be a factor of P with respect to the width vector $w \in M$. Let Σ denote the inner normal fan of F in $M_{\mathbb{Q}}$. Fix an isomorphism $M \cong (\mathbb{Z}^n)^\vee$ such that $w = (0, \ldots, 0, 1)^t$. By definition, the map $\varphi(w, F)$ is linear on each maximal cone of Σ. Consider the maximal cone of Σ corresponding to the vertex $(f_1, \ldots, f_{n-1}, 0)$ of F. In this cone, with the current choice of basis, $\varphi(w, F) : M_\mathbb{Q} \to M_\mathbb{Q}$ acts by $u \mapsto Au$, where:

$$A = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ -f_1 & -f_2 & \ldots & -f_{n-1} & 1 \end{pmatrix}.$$

In particular, $\det A = 1$, so $\varphi(w, F) : M_\mathbb{Q} \to M_\mathbb{Q}$ is volume and orientation preserving in each maximal cone of Σ. We conclude:

Corollary 2.21. *Let $P \subset N_\mathbb{Q}$ be a Fano polytope and suppose that $Q := \mathrm{mut}_w(P, F)$ exists. Then the toric varieties X_P and X_Q defined by the spanning fans of P and Q have the same anti-canonical degree.*

Proof. Let $n := \dim(P) = \mathrm{rank}(N)$, let Σ denote the spanning fan of $P \subset N_\mathbb{Q}$ and let $D_0, \ldots, D_m$ denote the torus-invariant Weil divisors in X_P corresponding to the rays in Σ. Then $-K_{X_P} = D_0 + \ldots + D_m$ by [10, Theorem 8.2.3] and the construction described in [12, Section 3.4] shows that the polytope in $M_\mathbb{Q}$ corresponding to $-K_{X_P}$ is precisely the dual polytope[5], $P^\vee$. The first corollary in [12, Section 5.3] implies that $(-K_{X_P})^n = n!\mathrm{Vol}(P)$, where $\mathrm{Vol}(\cdot)$ denotes the standard Euclidean volume.

Corollary 2.20 states that $Q^\vee = \varphi(w, F)(P^\vee)$. Since $\varphi(w, F)$ is volume and orientation preserving, it follows that $\mathrm{Vol}(P^\vee) = \mathrm{Vol}(Q^\vee)$. Finally, Proposition 2.18 implies that Q is a Fano polytope, and hence $n!\mathrm{Vol}(Q^\vee) = (-K_{X_Q})^n$ by the same reasoning as for P. $\square$

If $Q \subset M_\mathbb{Q}$ is a polytope satisfying $\mathbf{0} \in \mathrm{int}(Q)$ and $\dim(Q) = \mathrm{rank}(M)$, with possibly rational vertices, then $\mathrm{Ehr}_Q(t) := \sum_{m \geq 0} |mQ \cap M| t^m$ is a formal power series with non-negative integer coefficients, called the *Ehrhart series* of Q. This series can be expressed as a rational function of the following type [26]:

$$\mathrm{Ehr}_Q(t) = \frac{\delta_0 + \delta_i t + \ldots + \delta_{r(d+1)-1} t^{r(d+1)-1}}{(1 - t^r)^{d+1}},$$

[5]In order to identify the polytope corresponding to $-K_{X_P}$ with $P^\vee$, one must ensure that the primitive lattice vectors in N determined by the rays of Σ coincide with the vertices of P. This is guaranteed by the assumption that P is a Fano polytope.

where the δ_i are non-negative integers, $d = \dim(Q)$ and r is the smallest positive integer such that rQ is a lattice polytope. The vector $\underline{\delta}_Q := (\delta_0, \delta_1, \ldots, \delta_{r(d+1)-1})$ is called the *Ehrhart δ-vector* of Q.

Corollary 2.22 ([2, Proposition 4])**.** *Let $P \subset N_\mathbb{Q}$ be a lattice polytope, and suppose that $Q := \mathrm{mut}_w(P, F)$ exists. Then $\mathrm{Ehr}_{P^\vee}(t) = \mathrm{Ehr}_{Q^\vee}(t)$. Equivalently, the Ehrhart δ-vector of the dual polytope is invariant under combinatorial mutations.*

Proof. We will show that $|(mP^\vee) \cap M| = |(mQ^\vee) \cap M|$ for all $m \geq 0$ by induction. If $m = 0$ then $mP^\vee = \{\mathbf{0}\} = mQ^\vee$, and we are done. Suppose that the desired equality holds for some $m \geq 0$. Since $(P^\vee)^\vee = P$ is a lattice polytope by assumption, the discussion of [11, Section 3] shows that

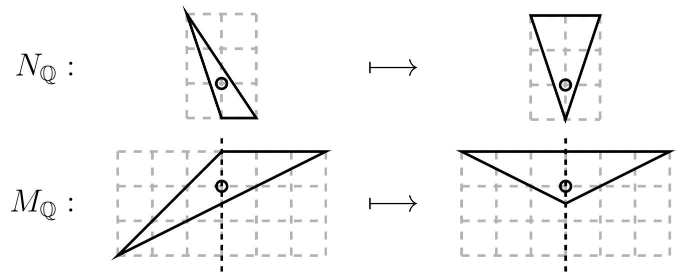
$$|((m+1)P^\vee) \cap M| = |(mP^\vee) \cap M| + |\partial(mP^\vee) \cap M|. \tag{2.6}$$

Here, $|(mP^\vee) \cap M| = |(mQ^\vee) \cap M|$ by the inductive hypothesis and $|\partial(mP^\vee)| = |\partial(mQ^\vee) \cap M|$ by Proposition 2.19 and the observation that the function $\varphi(w, F) : M_\mathbb{Q} \to M_\mathbb{Q}$ is piecewise $SL_n(\mathbb{Z})$-linear. Finally, since $(Q^\vee)^\vee = Q = \mathrm{mut}_w(P, F)$ is a lattice polytope by construction, an analogous formula to Equation (2.6) holds for $Q^\vee$. This shows that $|((m+1)P^\vee) \cap M| = |((m+1)Q^\vee) \cap M|$. $\square$

Note that Corollary 2.22 implies Corollary 2.21 when $P \subset N_\mathbb{Q}$ is a Fano polytope. In this case, Example 2.23 below shows that the Ehrhart δ-vector of the dual polytope is a strictly stronger invariant than the anti-canonical degree.

Example 2.23. Consider the Fano polytopes P and Q in $\mathbb{Q}^3$ with vertex sets $\{(1,0,1), (0,1,1), (0,-1,1), (-1,0,1), (1,0,-1), (0,1,-1), (0,-1,-1), (-1,0,-1)\}$ and $\{(1,0,1), (0,1,1), (-1,-1,1), (1,0,-1), (0,1,-1), (-1,0,-1), (0,-1,-1)\}$ respectively. The dual polytopes $P^\vee$ and $Q^\vee$ both have the same volume, and hence the toric varieties defined by the spanning fans of P and Q have the same anti-canonical degree. However, the Ehrhart δ-vector of $P^\vee$ is $(1, 7, 7, 1)$ while that of $Q^\vee$ is $(1, 10, 43, 113, 206, 275, 275, 206, 113, 43, 10, 1)$. It follows from Corollary 2.22 that P and Q are not related by a sequence of combinatorial mutations.

The following diagram, illustrating Example 2.24, is taken from [3]:



Example 2.24. This example illustrates some of the constructions and results discussed in this chapter. Let $P \subset N_{\mathbb{Q}} = \mathbb{Q}^2$ be the Fano polygon with vertex set $\{(1,-1),(-1,2),(0,-1)\}$. Choose the width vector[6] $w = (0,1)^t \in (\mathbb{Z}^2)^\vee$. The line segment $F := \operatorname{conv}(\mathbf{0},(1,0)) \subset \mathbb{Q}^2$ is a factor of P with respect to w. The combinatorial mutation $Q := \operatorname{mut}_w(P,F)$ is the lattice triangle in $\mathbb{Q}^2$ with vertex set $\{(1,2),(-1,2),(0,-1)\}$. Note that Q is also a Fano polytope. This construction can be reversed by choosing $w = (0,-1)^t \in (\mathbb{Z}^2)^\vee$ and F as above.

The inner normal fan of F in $M_{\mathbb{Q}} = (\mathbb{Q}^2)^\vee$ has two maximal cones, one for each vertex of F. These are: $M^+ := \operatorname{cone}((1,0)^t,(0,1)^t,(0,-1)^t) \subset M_{\mathbb{Q}}$ and $M^- := \operatorname{cone}((-1,0)^t,(0,1)^t,(0,-1)^t) \subset M_{\mathbb{Q}}$. The data (w,F) induces the map $\varphi := \varphi(w,F) : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$, which acts by $u \mapsto Au$, for $u = (\alpha,\beta)^t \in M_{\mathbb{Q}}$, where:

$$A = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } u \in M^+ \text{ (i.e. } \alpha \geq 0), \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

The polygon $P^\vee \subset M_{\mathbb{Q}}$ has vertex set $\{(-3,-2),(0,1),(3,1)\}$, and $Q^\vee \subset M_{\mathbb{Q}}$ has vertex set $\{(0,-1/2),(-3,1),(3,1)\}$. Direct calculation shows that $\varphi(P^\vee) = Q^\vee$. The toric varieties defined by the spanning fans of P, Q are $\mathbb{P}^2$ and $\mathbb{P}(1,1,4)$ respectively. They both have the same anti-canonical degree, which equals 9 and the same Ehrhart δ-vector, which equals $(1,7,1)$. One-step combinatorial mutations between Fano triangles will be discussed in Section 3.2.

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[6]Note that the chosen width vector lies in the set (2.4) from Corollary 2.17.

Chapter 3

The Two-Dimensional Case

3.1 Features of the Two-Dimensional Case

We begin by collecting some observations from Chapter 2 and introducing concepts that are useful for the study of combinatorial mutations of two-dimensional lattice polytopes (*polygons*). We adopt the notation of Sections 2.1 to 2.3.

Width Vectors and Factors in Two Dimensions

We recall Corollary 2.17: in the study of combinatorial mutations of a given lattice polygon $P \subset N_{\mathbb{Q}} \cong \mathbb{Q}^2$, it suffices to restrict attention to width vectors contained in the finite set

$$\{\overline{u} \in M \mid u \in \operatorname{verts}(P^{\vee})\}, \tag{3.1}$$

where $\overline{u}$ denotes the primitive lattice vector on the ray in $M_{\mathbb{Q}}$ through u and $\mathbf{0}$. Any width vector for which there exists a non-trivial factor of P lies in (3.1), which is geometrically the set of those width vectors which define a supporting hyperplane for an edge of P. Note that the converse to Corollary 2.17 is false by Example 2.3: if $w \in M$ is a width vector contained in the set (3.1), then there need not exist a non-trivial factor of P with respect to w.

Factors in the two-dimensional setting are particularly simple. For dimension reasons, a non-trivial factor (if it exists) must be a line segment. For any width vector $w \in M$, it suffices to consider factors up to translation by elements of $H_{w,0} \cap N$ and so we may assume, without loss of generality, that any non-trivial factor F (if it exists) is of the form: $F = \operatorname{conv}(\mathbf{0}, f)$, for some $f \in N$. Moreover, given a Minkowski sum decomposition, $F = F_1 + F_2$, the observation:

$$\operatorname{mut}_w(P, F_1 + F_2) = \operatorname{mut}_w(\operatorname{mut}_w(P, F_1), F_2),$$

(valid in all dimensions[1]) suggests that every construction of a non-trivial combinatorial mutation in two dimensions can be seen as a finite sequence of successive constructions, each corresponding to the same data (w, F), where $F = \mathrm{conv}(\mathbf{0}, f)$ and $f \in N \backslash \{\mathbf{0}\}$ is *primitive.*

Cones. Width and Local Index

The *width* of a line segment $S := \mathrm{conv}(n_1, n_2) \subset N_{\mathbb{Q}}$ with endpoints $n_1, n_2 \in N$ is $\mathrm{width}(S) := |S \cap N| - 1$. Let $C \subset N_{\mathbb{Q}} \cong \mathbb{Q}^2$ be a (strictly convex, rational polyhedral) cone of dimension two and let u and v be the primitive lattice vectors in N defined by the rays of C. The *width* of C is $\mathrm{width}(C) := \mathrm{width}(E)$, where $E := \mathrm{conv}(u, v)$. The *local index*, ℓ_C, of C in N is the lattice height in N of E above the origin, i.e. $\ell_C = -\langle p, u \rangle$, where $p \in M$ is the unique primitive lattice vector satisfying both $\langle p, u - v \rangle = 0$ and $\langle p, u \rangle < 0$.

From the viewpoint of toric geometry [12, Section 2.2], C defines a surface cyclic quotient singularity i.e. $\mathrm{Spec}(\mathbb{C}[C^\vee \cap M])$ is an affine variety of the form $\mathrm{Spec}(\mathbb{C}[x, y]^{\mu_r})$, $r \in \mathbb{Z}_{>0}$, where the cyclic group $\mu_r = \langle \eta \rangle$ acts on $\mathbb{C}[x, y]$ by $\eta \cdot x := \eta^a x$, $\eta \cdot y := \eta^b y$ for some $a, b \in \mathbb{Z}$. $\mathbb{C}[x, y]^{\mu_r}$ is the subring of $\mathbb{C}[x, y]$ consisting of polynomials invariant under the given μ_r-action. A self-contained introduction to this topic, from the viewpoint of lattice theory, is given in Appendix A.1. In particular, the *singularity type* of C can be represented (non-uniquely) by a symbol of the form $\frac{1}{r}(a, b)$, which encodes the group action on $\mathbb{C}[x, y]$.

In practice, the width and local index of C can be determined from the singularity type of C using the following:

Lemma 3.1. *If C has singularity type $\frac{1}{r}(a, b)$ then width$(C) = \gcd\{r, a + b\}$ and $\ell_C = r/\gcd\{r, a + b\}$.*

Proof. Any other representative of the singularity type of C can be written in the form: $\frac{1}{r}(ca + \alpha r, cb + \beta r)$, for some $c, \alpha, \beta \in \mathbb{Z}$ with $\gcd\{r, c\} = 1$ (Lemma A.6). In particular, $\gcd\{r, c(a + b) + r(\alpha + \beta)\} = \gcd\{r, a + b\}$, so the formulae are independent of the chosen representative of singularity type.

Since width and local index are independent of basis, we may assume without loss of generality that $N = (1, 0)\mathbb{Z} + (0, 1)\mathbb{Z} + \frac{1}{r}(a, b)\mathbb{Z}$ and $C \subset N_{\mathbb{Q}} \cong \mathbb{Q}^2$ is the cone with primitive generators $(1, 0), (0, 1)$. Set $E := \mathrm{conv}((1, 0), (0, 1)) \subset \mathbb{Q}^2$. By considering the group $\mathbb{Q}^2/((1, 0)\mathbb{Z} \oplus (0, 1)\mathbb{Z})$, we see that the quantity $|E \cap N| - 1$

[1]Minkowski sum is associative, so we have $G_h + (-h)(F_1 + F_2) = (G_h + (-h)F_1) + (-h)F_2$ in condition (2.2).

is equal to the number of integers k in $\{0, \ldots, r-1\}$ for which $k \cdot \frac{1}{r}(a, b) \in E$ i.e. for which $(ka+kb)/r$ is an integer. Equivalently, width(C) equals the number of incongruent solutions modulo r to $k(a+b) \equiv 0 \bmod r$. Similarly, ℓ_C equals the smallest positive integer solution to the congruence $k(a+b) \equiv 0 \bmod r$. $\square$

A Test for Non-Trivial Factors

In order to show that a polytope is a factor with respect to a given width vector, one has to produce a collection $\{G_h\}$, satisfying the conditions of Definition 2.2. In two dimensions, the situation is much simpler and it suffices to exhibit a single such polygon G_h, for $h = h_{\min}$ (see Example 2.3). This is recorded in:

Lemma 3.2. *Let $P \subset N_{\mathbb{Q}}$ be a lattice polygon and let $w \in M$ be a width vector. Let $F \subset N_{\mathbb{Q}}$ be a line segment (not necessarily of unit length) satisfying $w(F) = 0$. Then the following statements are equivalent:*

(1) F is a factor of P with respect to w;

(2) width$(P_{\min}) \geq (-h_{\min})$width$(F)$;

(3) There exists a nonempty polytope $G_{\min}$ satisfying:

(a) $\dim(G_{\min}) \in \{0, 1\}$*;*

(b) $w(G_{\min}) = h_{\min}$*;*

(c) $\mathrm{verts}(G_{\min}) \subset N$*; and*

(d) $G_{\min} + (-h_{\min})F = P_{\min}$*.*

Proof. Statement (1) implies (2), by setting $h = h_{\min}$ in the right-most inclusion of (2.2). The equivalence of (2) and (3) is immediate, because $P_{\min}$ and F are line segments. Suppose (3) holds. We will exhibit a collection $\{G_h\}$ satisfying the conditions of Definition 2.2.

Let h be an integer in the range $h_{\min} \leq h < 0$. If $h = h_{\min}$ then set $G_h = G_{\min}$. If $P_{w,h}$ does not contain a vertex of P, then set $G_h = \varnothing$. Finally, if $P_{w,h}$ contains a vertex of P then width$(P_{\min}) \geq (-h_{\min})$width$(F)$, because (3) implies (2). Now since $\mathbf{0} \in \mathrm{int}(P)$, the triangle $\mathrm{conv}(\mathbf{0}, P_{\min})$ (which equals the union of all line segments $\mathbf{0}x$, with $x \in P_{\min}$) must lie entirely in P, by convexity. By scaling and the fact that $P_{w,h}$ contains a vertex of P, we must have:

$$\mathrm{width}(P_{w,h}) \geq \left\lfloor \frac{(-h)\mathrm{width}(P_{\min})}{-h_{\min}} \right\rfloor \geq \left\lfloor \frac{(-h)(-h_{\min}\mathrm{width}(F))}{-h_{\min}} \right\rfloor = (-h) \cdot \mathrm{width}(F),$$

where $\lfloor a \rfloor$ $(a \in \mathbb{Q})$ denotes the largest integer smaller than or equal to a. Since $P_{w,h}$ and F are line segments, it is possible to choose a polytope G_h (of dimension 0 or 1) satisfying the conditions of Definition 2.2. Thus (1) follows from (3). $\square$

Example 3.3. The analogue of Lemma 3.2 is false in higher dimensions. Consider the lattice polytope $P \subset \mathbb{Q}^3$ with vertex set $\{(0,0,1),(0,2,-1),(-1,-1,-1),(1,-1,-1),(1,1,-2),(-1,1,-2),(-1,-1,-2),(1,-1,-2)\}$. Choose the width vector $w = (0,0,1)^t \in (\mathbb{Z}^3)^\vee$. Then $P_{\min}$ lies at height -2 with respect to w and is a translate of the polytope $2F \subset \mathbb{Q}^3$ where F is the square with vertex set $\{(0,0,0),(0,1,0),(1,1,0),(1,0,0)\}$. If an analogue of Lemma 3.2 were true, then F would be a factor of P with respect to w. This is false because in this example, $P_{w,-1}$ is the triangle with vertex set $\{(0,2,-1),(-1,-1,-1),(1,-1,-1)\}$ which, by inspection, does not contain a translate of F. Thus, for any polytope $G_{-1} \subset \mathbb{Q}^2$ satisfying $w(G_{-1}) = -1$ it is impossible to simultaneously satisfy both inclusions of Equation (2.2) at height $h = -1$.

Edge Collapsing

Let $P \subset N_{\mathbb{Q}}$ be a lattice polygon and let $w \in M$ be a width vector. Let $F \subset N_{\mathbb{Q}}$ be a non-trivial factor of P with respect to w. By the discussion on page 31, we may assume that $F = \mathrm{conv}(\mathbf{0}, f)$ with $f \in N$ primitive (i.e. $\mathrm{width}(F) = 1$). Then $\mathrm{width}(P_{\min}) \geq -h_{\min}$, by Lemma 3.2, and there exist integers τ, ρ such that:

$$\mathrm{width}(P_{\min}) = \tau(-h_{\min}) + \rho \quad ; \quad 0 \leq \rho < -h_{\min} \text{ and } \tau > 0.$$

By repeated application of Lemma 3.2, we may thus construct a finite sequence of lattice polygons: $Q^0 := P, Q^1, \ldots, Q^\tau$, where $Q^{k+1} = \mathrm{mut}_w\big(Q^k, F\big)$, for $k = 0, \ldots, \tau - 1$. At each step of these successive constructions, the width of the face at minimum height with respect to w decreases by $-h_{\min}$. More precisely:

$$\mathrm{width}(Q^{k+1}_{\min}) = \mathrm{width}(P_{\min}) - (k+1)(-h_{\min}) \text{ for all } k \in \{0, \ldots, \tau - 1\}.$$

Notice that $Q^\tau_{\min}$ is a vertex (informally, the edge $P_{\min}$ can be *collapsed* by a mutation[2]) if and only if $h_{\min}$ divides $\mathrm{width}(P_{\min})$. For lattice *triangles*, we have:

Lemma 3.4. *Let $T \subset N_{\mathbb{Q}}$ be a lattice triangle, and let $w \in M$ be a width vector. There exists a non-trivial factor F of T with respect to w such that* $\mathrm{mut}_w(T, F)$ *is a lattice triangle if and only if $h_{\min}$ divides width$(T_{\min})$.*

Proof. Suppose there exists a non-trivial factor F of T with respect to w such that $T' := \mathrm{mut}_w(T, F)$ is a lattice triangle. Since F is non-trivial, the face $T_{\min}$ must be an edge. By definition of a combinatorial mutation, the face $T'_{\max} = T_{\max} + h_{\max}E$

[2] By choosing given width vector $w \in M$ and factor $\tau F \subset N_{\mathbb{Q}}$.

is a line segment (and hence contains two vertices of T'). Thus T' is a lattice triangle only if $T'_{\min}$ is a vertex[3]. This implies that $h_{\min}$ divides width($T_{\min}$), by the discussion earlier in this section.

Conversely, suppose that width$(T_{\min}) = \tau(-h_{\min})$ for some $\tau \in \mathbb{Z}_{>0}$. This implies that there exists a primitive line segment $L \subset N_{\mathbb{Q}}$ (i.e. width$(L) = 1$) with $w(L) = 0$ such that

$$T_{\min} = v + \tau(-h_{\min})L,$$

where $v \in \operatorname{verts}(T_{\min})$. Let $G_{\min} := v$ and $F := \tau L$. Then F is a factor of T with respect to w by Lemma 3.2. Thus $T' := \operatorname{mut}_w(T, F)$ exists. It follows immediately from the construction of combinatorial mutations that T' is a lattice triangle. □

The following Corollary is often useful, in conjunction with Lemma 3.1 (see Example 3.11):

Corollary 3.5. *Let $T \subset N_{\mathbb{Q}}$ be a Fano triangle and let $w \in M$ be a width vector contained in*[4] *the set* (3.1)*. Let $C \subset N_{\mathbb{Q}}$ be the cone over the edge $T_{\min}$. There exists a non-trivial factor F of T with respect to w such that* $\operatorname{mut}_w(T, F)$ *is a lattice triangle if and only if ℓ_C divides width*(C).

Proof. Since T is a Fano triangle, its vertices are primitive. In particular, the vertices of $T_{\min}$ are the primitive lattice points defined by the rays of C. Thus width(C) = width$(T_{\min})$ and $\ell_C = -h_{\min}$. The result follows by Lemma 3.4. □

Mutation Graph

Let $[\Delta]$ denote the isomorphism class (Section 2.1) of a lattice polytope $\Delta \subset N_{\mathbb{Q}}$.

Definition 3.6. The *mutation graph* of a lattice polytope $P \subset N_{\mathbb{Q}}$ is the graph, $\Gamma(P)$, whose vertex set is the set of isomorphism classes of lattice polytopes related to P by combinatorial mutations (see Remark 2.14). Two vertices $[Q_1], [Q_2]$ of $\Gamma(P)$ are joined by a unique edge if and only if there exist $R_1 \in [Q_1]$ and $R_2 \in [Q_2]$ such that R_2 is obtained from R_1 by a one-step mutation (see page 24).

Suppose $\dim(P) = 2$. For any integer $m \geq 1$, the (Picard) *rank m mutation graph* of P is the subgraph, $\Gamma_m(P)$, of $\Gamma(P)$ obtained by constructing the subgraph $\mathcal{G}$ of $\Gamma(P)$ whose vertices are (isomorphism classes of) lattice polygons with at most $m + \operatorname{rank}(N) = m + 2$ vertices and then taking the connected component of $\mathcal{G}$ which contains $[P]$.

[3]Informally: only if the edge $T_{\min}$ can be collapsed by a mutation.
[4]The fact that w lies in (3.1) implies that $T_{\min}$ is an edge.

Remark 3.7. Each vertex of $\Gamma(P)$ (and hence of $\Gamma_m(P)$) has finite valency, by Proposition 2.15. The definition of mutation graph does not allow multiple edges between vertices. Example 3.8 shows that self-loops may exist.

Example 3.8. Consider the Fano triangle $T \subset \mathbb{Q}^2$ with vertex set $\{(1,0),(0,1),(-2,-3)\}$, whose spanning fan defines the weighted projective plane $\mathbb{P}(1,2,3)$. The isomorphism class of the Fano triangle $T' \subset \mathbb{Q}^2$ with vertex set $\{(0,1),(3,4),(-2,-3)\}$ is joined to $[T]$ in $\Gamma(T)$ by a path of length 3. This path is obtained by successively constructing the combinatorial mutations with respect to the data: $w = (-1,1)^t \in (\mathbb{Z}^2)^\vee$ and $F := \mathrm{conv}(\mathbf{0},(1,1)) \subset \mathbb{Q}^2$. Moreover, T and T' are isomorphic, via the transformation $\mathbb{Z}^2 \to \mathbb{Z}^2$; $u \mapsto uA$, where:

$$A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} \in GL_2(\mathbb{Z}).$$

Since $\mathrm{mut}_w(T, F_1 + F_2) = \mathrm{mut}_w(\mathrm{mut}_w(T, F_1), F_2)$, this example also shows that $[T]$ is connected to itself by a self-loop in $\Gamma(T)$, arising from the one-step mutation corresponding to the data $w = (-1,1)^t \in (\mathbb{Z}^2)^\vee$ and $F := \mathrm{conv}(\mathbf{0},(3,3)) \subset \mathbb{Q}^2$. In the language of Remark 2.7, we see that a trivial combinatorial mutation can be constructed using a non-trivial factor. See also Example 3.13.

Example 3.18 will present two Fano triangles which have the same mutation graph, but different rank 1 mutation graphs.

3.2 Weighted Projective Planes

In this section, we study rank 1 mutation graphs of Fano triangles. If two Fano triangles are joined by an edge in a rank 1 mutation graph, then their weight vectors are related by a simple transformation. These transformations are closely related to 'arithmetic mutations' of integer solutions to Markov-type Diophantine equations. In particular, we show that the rank 1 mutation graph corresponding to the projective plane $\mathbb{P}^2$ is isomorphic to the arithmetic mutation graph of solutions to the Markov equation $3x_0x_1x_2 = x_0^2 + x_1^2 + x_2^2$.

Preliminaries

We follow the conventions of Section 2.1, with $n = 2$. Thus, a Fano triangle T is a lattice polygon in $N_\mathbb{Q} \cong \mathbb{Q}^2$ whose vertices v_0, v_1, v_2 are all primitive lattice vectors in N. Since $\mathbf{0} \in \mathrm{int}(T)$, there is a unique $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$ (called the *weight vector* of T) satisfying $\gcd\{\lambda_0, \lambda_1, \lambda_2\} = 1$ and $\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 = \mathbf{0}$. The multiplicity of T, denoted $\mathrm{mult}(T)$, is defined to be $[N : L]$ where L is the sublattice of N spanned by v_0, v_1 and v_2.

The projective toric surface X defined by the spanning fan of a Fano triangle T with weight vector $(\lambda_0, \lambda_1, \lambda_2)$ has Picard rank 1 and is called a *fake weighted projective plane* with weight vector $(\lambda_0, \lambda_1, \lambda_2)$. X is the quotient of the weighted projective plane $\mathbb{P}(\lambda_0, \lambda_1, \lambda_2)$ by the action of a finite group of order $\mathrm{mult}(T)$. We will often not distinguish a Fano triangle T satisfying $\mathrm{mult}(T) = 1$ and having weight vector $(\lambda_0, \lambda_1, \lambda_2)$ from the weighted projective plane $\mathbb{P}(\lambda_0, \lambda_1, \lambda_2)$ defined by its spanning fan. Since the vertices of a Fano triangle are primitive, it follows additionally that the weights are pairwise coprime: $\gcd\{\lambda_i, \lambda_j\} = 1$ whenever $i \neq j$ (i.e. the weight vector of a Fano triangle is *well-formed*).

Weight Vectors and One-Step Mutations

In this section, we describe how the weight vector $(\lambda_0, \lambda_1, \lambda_2)$ of a Fano triangle $T \subset N_{\mathbb{Q}}$ transforms under a one-step mutation to another Fano triangle. We will require the following fact (see, for example, [9, Lemma 5.3]): Let $T^\vee = \mathrm{conv}(u_0, u_1, u_2)$ be the triangle in $M_{\mathbb{Q}}$ dual to T. Then, possibly after relabeling the vertices of $T^\vee$: $\lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2 = \mathbf{0}$. Hence T and $T^\vee$ have the same weight vector, up to permuting the entries.

Proposition 3.9 ([3, Proposition 3.3])**.** *Let T_1 be a Fano triangle with weight vector $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$. Suppose there exists a Fano triangle T_2 which is obtained from T_1 by a one-step mutation. Then, up to relabelling, $\lambda_0 \mid (\lambda_1 + \lambda_2)^2$ and T_2 has the following well-formed weight vector:*

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0}\right).$$

Proof. Suppose there exists a width vector $w \in M$ and a factor $F \subset N_{\mathbb{Q}}$ of T_1 with respect to w such that $T_2 := \mathrm{mut}_w(T_1, F)$ is also a triangle (T_2 must be Fano, by Proposition 2.18). The weight vector of a Fano triangle is independent of the choice of basis and, by the opening discussion of Section 2.3, it suffices to consider F up to translation by elements of $H_{w,0} \cap N$. Therefore, we assume without loss of generality that $w = (0, 1)^t$ and that $F = \mathrm{conv}(\mathbf{0}, (a, 0))$ for some $a \in \mathbb{Z}_{>0}$.

By Section 2.5, there exists a piecewise linear map $\varphi := \varphi(w, F) : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ such that $\varphi({T_1}^\vee) = {T_2}^\vee$. In the current choice of basis, the inner normal fan of F subdivides $M_{\mathbb{Q}}$ into two chambers (maximal cones), namely $M^+ := \{(\alpha, \beta)^t \in$

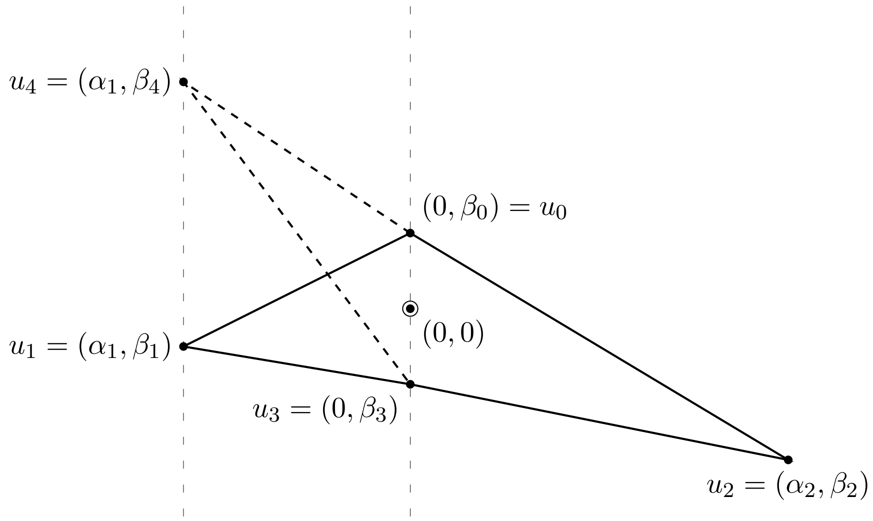


Figure 3.1: A one-step mutation, depicted in $M_{\mathbb{Q}}$, of the triangle $\operatorname{conv}(u_0, u_1, u_2)$ to the triangle $\operatorname{conv}(u_2, u_3, u_4)$.

$M_{\mathbb{Q}} \mid \alpha > 0\}$ and $M^- := \{(\alpha, \beta)^t \in M^- \mid \alpha < 0\}$, and φ acts by $u \mapsto Au$, where:

$$A = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } u \in M^+, \\ \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

Let $u_0, u_1, u_2 \in M_{\mathbb{Q}}$ be the (possibly rational) vertices of ${T_1}^\vee$, labeled so that $u_1 \in M^-$, $u_2 \in M^+$ and u_0 lies on the line $\langle w \rangle := \{\gamma w \in M_{\mathbb{Q}} \mid \gamma \in \mathbb{Q}\}$. The vertices of T_2 are then $u_2, u_3, u_4 \in M_{\mathbb{Q}}$ (possibly rational), where u_0 is contained in the line segment u_2u_4 joining u_2 and u_4, and u_3 is contained in the line segment u_1u_2. The situation is illustrated in Figure 3.1.

Since $\mathbf{0} \in \operatorname{int}({T_1}^\vee)$ there are unique weights $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$, $\gcd\{\lambda_0, \lambda_1, \lambda_2\} = 1$, such that:

$$\lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2 = \mathbf{0}. \tag{3.2}$$

Since $u_3 = (0, \beta_3) \in u_1u_2$, there exists a unique $\mu \in (0,1) \cap \mathbb{Q}$, such that $u_3 = \mu u_1 + (1-\mu)u_2$. Applying $(1,0)^t \in M_{\mathbb{Q}}$ to this relation shows that there is a unique $\mu \in (0,1) \cap \mathbb{Q}$ such that $\mu\alpha_1 + (1-\mu)\alpha_2 = 0$. But applying $(0,1)^t \in M_{\mathbb{Q}}$ to (3.2) shows that $\lambda_1\alpha_1 + \lambda_2\alpha_2 = 0$ and hence:

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}\alpha_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}\alpha_2 = 0,$$

where both coefficients lie in $(0,1) \cap \mathbb{Q}$. We must have $\mu = \lambda_1/(\lambda_1 + \lambda_2)$, by uniqueness, and the relation $u_3 = \mu u_1 + (1-\mu)u_2$ becomes:

$$u_3 = \frac{\lambda_1}{\lambda_1+\lambda_2}u_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}u_2. \tag{3.3}$$

Similarly, since $u_0 = (0,\beta_0) \in u_2u_4$, there exists a $\nu \in (0,1)\cap\mathbb{Q}$ such that $u_0 = \nu u_2 + (1-\nu)u_4$. Dividing by $1-\nu$ and rearranging gives:

$$u_4 = \frac{1}{1-\nu}u_0 - \frac{\nu}{1-\nu}u_2. \tag{3.4}$$

Applying $(1,0)^t \in M_{\mathbb{Q}}$ to (3.4) yields:

$$\alpha_1 = -\frac{\nu}{1-\nu}\alpha_2. \tag{3.5}$$

We also note that $u_4 = u_1 + \kappa u_0$ for some $\kappa > 0$. Dividing through (3.2) by $\lambda_1 \in \mathbb{Z}_{>0}$ and using the resulting expression to eliminate u_1 from $u_4 = u_1 + \kappa u_0$, we see that:

$$u_4 = \frac{\lambda_1\kappa - \lambda_0}{\lambda_1}u_0 - \frac{\lambda_2}{\lambda_1}u_2.$$

Apply $(1,0)^t \in M_{\mathbb{Q}}$ to this equation to deduce that:

$$\alpha_1 = -\frac{\lambda_2}{\lambda_1}\alpha_2. \tag{3.6}$$

Eliminating α_1/α_2 from (3.5) and (3.6) gives $\nu = \lambda_2/(\lambda_1+\lambda_2)$. Thus (3.4) becomes:

$$u_4 = \frac{\lambda_1+\lambda_2}{\lambda_1}u_0 - \frac{\lambda_2}{\lambda_1}u_2. \tag{3.7}$$

Note that, since both u_0 and u_3 are contained in $\langle w\rangle$, there exists some $\gamma > 0$ such that $-\gamma u_3 = u_0$. Substituting this into equation (3.7) we have

$$\frac{\lambda_2}{\lambda_1}u_2 + u_4 + \gamma' u_3 = \mathbf{0} \tag{3.8}$$

where $\gamma' = \gamma(\lambda_1+\lambda_2)/\lambda_1 > 0$. Substituting equation (3.3) into (3.8), we obtain:

$$\frac{\lambda_2}{\lambda_1}u_2 + u_4 + \frac{\gamma'\lambda_1}{\lambda_1+\lambda_2}u_1 + \frac{\gamma'\lambda_2}{\lambda_1+\lambda_2}u_2 = \mathbf{0}.$$

Using equation (3.7) to rewrite the first two terms and clearing denominators gives:

$$(\lambda_1+\lambda_2)^2u_0 + \gamma'\lambda_1^2u_1 + \gamma'\lambda_1\lambda_2u_2 = \mathbf{0}. \tag{3.9}$$

Set $h := \lambda_0 + \lambda_1 + \lambda_2$ and divide (3.2) by h. Set $\Gamma := (\lambda_1 + \lambda_2)^2 + \gamma'\lambda_1^2 + \gamma'\lambda_1\lambda_2$ and divide (3.9) by Γ. This gives two sets of barycentric coordinates for ${T_1}^\vee$, both of which sum to 1. Uniqueness of barycentric coordinates now gives:

$$\begin{aligned} h(\lambda_1 + \lambda_2)^2 &= \Gamma\lambda_0, \\ h\gamma'\lambda_1^2 &= \Gamma\lambda_1, \\ h\gamma'\lambda_1\lambda_2 &= \Gamma\lambda_2. \end{aligned}$$

In particular:

$$\gamma' = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0\lambda_1}.$$

Substituting this expression for γ' back into (3.8) gives

$$\lambda_0\lambda_2 u_2 + (\lambda_1 + \lambda_2)^2 u_3 + \lambda_0\lambda_1 u_4 = \mathbf{0}. \tag{3.10}$$

Thus, the weight vector of ${T_2}^\vee$ (and hence of T_2) is:

$$\left(\frac{\lambda_0\lambda_1}{d}, \frac{\lambda_0\lambda_2}{d}, \frac{(\lambda_1 + \lambda_2)^2}{d}\right), \tag{3.11}$$

where $d := \gcd\{\lambda_0\lambda_1, \lambda_0\lambda_2, (\lambda_1 + \lambda_2)^2\}$. By assumption, T_1 is Fano. Hence T_2 is Fano by Proposition 2.18. This implies that (3.11) must be well-formed. In particular, we must have: $1 = \gcd\{(\lambda_0\lambda_1)/d, (\lambda_0\lambda_2)/d\} = \lambda_0/d$, i.e. $d = \lambda_0$. Thus, λ_0 divides $(\lambda_1 + \lambda_2)^2$ and the weight vector of T_2 is:

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0}\right). \tag{3.12}$$

It remains to show that (3.12) is well-formed. Since T_1 is Fano, its weight vector $(\lambda_0, \lambda_1, \lambda_2)$ is well-formed and hence $\gcd\{\lambda_1, \lambda_2\} = 1$. If there is a prime p such that

$$p \mid \lambda_1 \quad \text{and} \quad p \mid \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0}, \tag{3.13}$$

then p cannot divide λ_0, by well-formedness of $(\lambda_0, \lambda_1, \lambda_2)$. Thus p divides $(\lambda_1 + \lambda_2)^2$, which implies that $p \mid \lambda_2^2$ and so $p \mid \lambda_2$. But then $p \mid \gcd\{\lambda_1, \lambda_2\}$ and this contradicts the well-formedness of $(\lambda_0, \lambda_1, \lambda_2)$. Similarly, $\gcd\{\lambda_1, (\lambda_1 + \lambda_2)^2/\lambda_0\} = 1$. Hence (3.13) is the well-formed weight vector of T_2. $\square$

Example 3.10. The Fano triangle $T \subset \mathbb{Q}^2$ with vertex set $\{(-5,-7),(3,2),(0,1)\}$ has weight vector $(3,5,11)$ and multiplicity 1. No Fano triangle can be obtained from T by a one-step mutation, because $3 \nmid (5+11)^2$, $5 \nmid (3+11)^2$, and $11 \nmid (3+5)^2$.

Let T be a Fano triangle with weight vector $(\lambda_0, \lambda_1, \lambda_2)$. Proposition 3.9 states that (up to relabeling) the condition $\lambda_0 \mid (\lambda_1 + \lambda_2)^2$ is necessary for the existence of a Fano triangle, obtained from T by a one-step mutation. Example 3.11 shows that this condition is not sufficient:

Example 3.11. The Fano triangle $T \subset \mathbb{Q}^2$ with vertex set $\{(10,-7),(-5,2),(0,1)\}$ has weight vector $(1,2,3)$ and multiplicity 5. Notice that $1 \mid (2+3)^2$, $2 \mid (1+3)^2$ and $3 \mid (1+2)^2$. However, no Fano triangle can be obtained from T by a one-step mutation. This is because the cones C over the edges of T have singularity types $\frac{1}{5}(1,3)$, $\frac{1}{10}(1,3)$ and $\frac{1}{15}(1,11)$, and all three of these fail to satisfy the condition $\ell_C \mid \mathrm{width}(C)$ (see Corollary 3.5 and Lemma 3.1).

Remark 3.12. Let T be a Fano triangle with weight vector $(\lambda_0, \lambda_1, \lambda_2)$. If $\mathrm{mult}(T) = 1$ then the condition $\lambda_0 \mid (\lambda_1 + \lambda_2)^2$ in Proposition 3.9 is also sufficient (cf. Example 3.11), because in this case, up to relabeling, the cone C over an edge of T has singularity type $\frac{1}{\lambda_0}(\lambda_1, \lambda_2)$. So $\mathrm{width}(C) = \gcd\{\lambda_0, (\lambda_1 + \lambda_2)\}$ and $\ell_C = \lambda_0 / \gcd\{\lambda_0, (\lambda_1 + \lambda_2)\}$ by Lemma 3.1. Corollary 3.5 now implies that there exists a Fano triangle obtained from T by a one-step mutation if and only if $\lambda_0 \mid \gcd\{\lambda_0, (\lambda_1 + \lambda_2)\}^2$ i.e. if and only if $\lambda_0 \mid (\lambda_1 + \lambda_2)^2$. This partial converse to Proposition 3.9 will be needed in Examples 3.13 and 3.20.

Example 3.13. Let $a, b \in \mathbb{Z}_{>0}$ satisfy $\gcd\{a, b\} = 1$. The Fano triangle T whose spanning fan defines $\mathbb{P}(a, b, a+b)$ has well-formed weight vector $(a, b, a+b)$. Since $\mathrm{mult}(T) = 1$ and since $(a+b) \mid (a+b)^2$, T is obtained from itself by a one-step mutation (see Remark 3.12). This generalizes Example 3.8.

Markov-Type Diophantine Equations

The weight vector $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$ of a Fano triangle is well-formed: $\gcd\{\lambda_i, \lambda_j\} = 1$, whenever $i \neq j$. Any such vector determines a Markov-style Diophantine equation, and an integral solution to that equation. This is the content of Lemma 3.14, which appears in the proof of Theorem 4.1 of [16]. A slightly more general result, requiring only that $\gcd\{\lambda_0, \lambda_1, \lambda_2\} = 1$, can be found in [3, Lemma 3.11].

Lemma 3.14. *Let* $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$ *with* $\gcd\{\lambda_i, \lambda_j\} = 1$*, whenever* $i \neq j$*. Write:*

(1) $\lambda_i = c_i a_i^2$*, where* $a_i, c_i \in \mathbb{Z}_{>0}$ *and* c_i *is square-free, and*

(2) $(\lambda_0 + \lambda_1 + \lambda_2)^2/(\lambda_0\lambda_1\lambda_2) = m^2/(rk^2)$*;* $m, k, r \in \mathbb{Z}_{>0}$ *and* r *square-free.*

Then (a_0, a_1, a_2) *is a solution to the Diophantine equation*

$$mx_0x_1x_2 = k(c_0x_0^2 + c_1x_1^2 + c_2x_2^2). \tag{3.14}$$

Proof. By substituting the expressions (1) into (2) we obtain

$$(c_0c_1c_2)m^2(a_0a_1a_2)^2 = rk^2\left(c_0a_0^2 + c_1a_1^2 + c_2a_2^2\right)^2.$$

Comparing square-free parts, we conclude that $c_0c_1c_2 = r$. Canceling and taking square-roots on both sides establishes the result. □

Remark 3.15. Consider the Markov-type equation $E : mx_0x_1x_2 = c_0x_0^2 + c_1x_1^2 + c_2x_2^2$ ($m \in \mathbb{Q}_{>0}$, $c_0, c_1, c_2 \in \mathbb{Z}_{>0}$). Let $(a_0, a_1, a_2) \in \mathbb{Z}^3_{>0}$ be a solution to E. If all the c_i are square-free and $\gcd\{c_ia_i^2, c_ja_j^2\} = 1$ whenever $i \neq j$ then the tuple (E, a_0, a_1, a_2) can be recovered from the weighted projective plane $\mathbb{P}(c_0a_0^2, c_1a_1^2, c_2a_2^2)$ via the construction of Lemma 3.14.

Remark 3.16. The expression (2) appearing in Lemma 3.14 is geometrically significant: if $T \subset N_{\mathbb{Q}}$ is a Fano triangle with weight vector $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$ then the degree, K_X^2, of the corresponding fake weighted projective plane X is given by the following formula:

$$\mathrm{mult}(X) \cdot K_X^2 = \frac{(\lambda_0 + \lambda_1 + \lambda_2)^2}{\lambda_0\lambda_1\lambda_2},$$

where $\mathrm{mult}(X) := \mathrm{mult}(T)$. See Appendix A.2 for a proof of this formula.

Lemma 3.17 ([3, Proposition 3.12])**.** *Let* T_1 *be a Fano triangle and suppose the Fano triangle* T_2 *is obtained from* T_1 *by a one-step mutation. Then the weight vectors of* T_1 *and* T_2 *give solutions to the same Diophantine equation* (3.14)*. In particular,* $\mathrm{mult}(T_1) = \mathrm{mult}(T_2)$.

Proof. Let the weight vector of T_1 be $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$ and let m, k, r, c_i, a_i ($i = 0, 1, 2$) be as in Lemma 3.14. By Proposition 3.9 the weight vector of T_2 (up to relabeling the λ_i) is:

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0}\right) = \left(c_1a_1^2, c_2a_2^2, \frac{(c_1a_1^2 + c_2a_2^2)^2}{c_0a_0^2}\right),$$

where the final weight is an integer; in particular, it has square-free part c_0. Thus the quantities c_i are the same for T_1 and T_2. Furthermore:

$$\frac{\left(\lambda_1+\lambda_2+\frac{(\lambda_1+\lambda_2)^2}{\lambda_0}\right)^2}{\lambda_1\cdot\lambda_2\cdot\frac{(\lambda_1+\lambda_2)^2}{\lambda_0}}=\frac{(\lambda_0+\lambda_1+\lambda_2)^2}{\lambda_0\lambda_1\lambda_2}=\frac{m^2}{rk^2},$$

and so the ratio m/k is also the same for T_1 and T_2. Hence the weight vectors of T_1 and of T_2 both generate solutions to the same Diophantine equation (3.14).

Finally, let X_i denote the fake weighted plane defined by T_i $(i=1,2)$. By the above calculation and Remark 3.16:

$$\operatorname{mult}(T_1)\cdot K_{X_1}^2=\frac{m^2}{rk^2}=\operatorname{mult}(T_2)\cdot K_{X_2}^2.$$

But since T_1,T_2 are related by combinatorial mutations we have $K_{X_1}^2=K_{X_2}^2$, by Corollary 2.21. Thus, $\operatorname{mult}(T_1)=\operatorname{mult}(T_2)$, as claimed. □

Lemma 3.17 implies that Fano triangles with the same rank 1 mutation graph have the same multiplicity. Based on this, Example 3.18 presents two Fano triangles with the same mutation graph but different rank 1 mutation graphs.

Example 3.18. Consider the Fano triangles: $T\subset\mathbb{Q}^2$ with vertex set $\{(1,0),(0,1),(-3,-5)\}$ and $T'\subset\mathbb{Q}^2$ with vertex set $\{(-1,-1),(-3,-5),(21,25)\}$. T and T' are related by combinatorial mutations, as follows: starting with T, construct the quadrilateral $Q\subset\mathbb{Q}^2$ with vertex set $\{(-1,-1),(3,4),(1,0),(-3,-5)\}$ using the data $w=(2,-1)^t\in(\mathbb{Z}^2)^\vee$ and $F:=\operatorname{conv}(\mathbf{0},(1,2))\subset\mathbb{Q}^2$. Then, starting from Q, construct T' using the data $w=(5,-4)^t\in(\mathbb{Z}^2)^\vee$ and $F:=\operatorname{conv}(\mathbf{0},(4,5))\subset\mathbb{Q}^2$. This shows that T and T' have the same mutation graph, in which they are connected by a path of length 2.

The rank 1 mutation graphs $\Gamma_1(T),\Gamma_1(T')$ are different; if $\Gamma_1(T)=\Gamma_1(T')$ then, by connectedness, there must exist a finite sequence of Fano triangles $T_0:=T,T_1,\dots,T_m:=T'$, where T_{k+1} is obtained from T_k by a one-step mutation. But then, by Lemma 3.17, we must have: $1=\operatorname{mult}(T)=\operatorname{mult}(T')=2$. Contradiction.

Remark 3.19. In the notation of Lemma 3.14, let (a_0,a_1,a_2) (resp. (b_0,b_1,b_2)) be the solution to the Diophantine equation (3.14) determined by the weight vector of T_1 (resp. T_2). Substituting (a_0,a_1,a_2) into (3.14) and rearranging gives:

$$\left(\frac{m}{k}a_0a_1a_2-c_0a_0^2\right)^2=(c_1a_1^2+c_2a_2^2)^2=(\lambda_1+\lambda_2)^2.$$

Dividing both sides by $\lambda_0 = c_0 a_0^2$ allows us to deduce (by the construction of Lemma 3.14) that, up to permuting the entries, (b_0, b_1, b_2) is the image of (a_0, a_1, a_2) under the *arithmetic mutation*:

$$(a_0, a_1, a_2) \mapsto \left(a_1, a_2, \frac{m a_1 a_2}{k c_0} - a_0\right). \tag{3.15}$$

Special cases of (3.15) are found in the Diophantine approximation literature, for instance [25].

Example 3.20 (Rank 1 mutation graph of $\mathbb{P}^2$)**.** Consider the Fano triangle $T \subset \mathbb{Q}^2$ with vertex set $\{(1,0),(0,1),(-1,-1)\}$. The weight vector of T is $(1,1,1) \in \mathbb{Z}^3_{>0}$ and $\mathrm{mult}(T) = 1$. The weighted projective plane defined by the spanning fan of T is the projective plane $\mathbb{P}^2 = \mathbb{P}(1,1,1)$.

By the construction of Lemma 3.14, the weight vector $(1,1,1)$ determines the *Markov equation*:

$$3x_0x_1x_2 = x_0^2 + x_1^2 + x_2^2, \tag{3.16}$$

together with the solution $(1,1,1)$ of this equation. A classical result in Diophantine approximation (see [25]) states that every solution $(a_0, a_1, a_2) \in \mathbb{Z}^3_{>0}$ of (3.16) can be obtained from the *minimal solution* $(1,1,1)$ by a finite number of the following transformations[5], (applied in some order):

$$\begin{aligned}(a_0, a_1, a_2) &\mapsto (3a_1a_2 - a_0, a_1, a_2);\\ (a_0, a_1, a_2) &\mapsto (a_0, 3a_0a_2 - a_1, a_2);\\ (a_0, a_1, a_2) &\mapsto (a_0, a_1, 3a_0a_1 - a_2).\end{aligned} \tag{3.17}$$

Thus (3.16) determines a connected graph, $\mathcal{G}$, whose vertices, $(a_0, a_1, a_2) \in \mathbb{Z}^3_{>0}$, are solutions to (3.16) and in which two vertices are joined by an edge if and only one is obtained from the other by a single arithmetic mutation (3.17). Since the Markov equation (3.16) is invariant under permuting the indices of x_0, x_1, x_2, the graph $\mathcal{G}$ admits an action of the symmetric group S_3 defined by: $\sigma \cdot (a_0, a_1, a_2) = (a_{\sigma(0)}, a_{\sigma(1)}, a_{\sigma(2)})$ for all $\sigma \in S_3$. The aim of this example is to show that the arithmetic mutation graph $\mathcal{G}/S_3$ is isomorphic to the rank 1 mutation graph $\Gamma_1(\mathbb{P}^2) := \Gamma_1(T)$. For notational convenience, we will not distinguish between a solution $(a_0, a_1, a_2) \in \mathrm{verts}(\mathcal{G})$ and its equivalence class in $\mathrm{verts}(\mathcal{G}/S_3)$.

Firstly, if a solution $(a_0, a_1, a_2) \in \mathbb{Z}^3_{>0}$ to (3.16) is well-formed ($\gcd\{a_i, a_j\} = 1$ whenever $i \neq j$) then any arithmetic mutation (3.17) of it is also well-formed,

[5]Note that each of these transformations is self-inverse.

because if $p \mid a_j$ and $p \mid 3a_ja_k - a_i$ then $p \mid \gcd\{a_i, a_j\} = 1$. It now follows, from Lemma 3.17 and the constructions of Lemma 3.14 and Remark 3.15, that there is a one-to-one correspondence between verts$(\mathcal{G}/S_3)$ and the following set of weighted projective planes (with well-formed weight vectors):

$$S := \{\mathbb{P}(a_0^2, a_1^2, a_2^2) \mid (a_0, a_1, a_2) \in \mathrm{verts}(\mathcal{G}/S_3)\}.$$

Here, we have used the fact that $P(\lambda_0, \lambda_1, \lambda_2)$ is isomorphic to $P(\lambda_{\sigma(0)}, \lambda_{\sigma(1)}, \lambda_{\sigma(2)})$ for any $\sigma \in S_3$. Notice that $\mathbb{P}^2 \in S$. Secondly, if two weighted projective planes in verts$(\Gamma_1(\mathbb{P}^2))$ are joined by an edge (i.e. obtained from each other by a one-step mutation), then by Remark 3.19, the corresponding solutions to (3.16) are related by an arithmetic mutation (3.17). Conversely, suppose that $(a_0, a_1, a_2), (3a_1a_2 - a_0, a_1, a_2) \in \mathbb{Z}^3_{>0}$ are two adjacent solutions to (3.16) in $\mathcal{G}/S_3$. These correspond to the weighted projective planes $\mathbb{P}(a_0^2, a_1^2, a_2^2)$ and $\mathbb{P}((3a_1a_2 - a_0)^2, a_1^2, a_2^2)$ in S. Now $(3a_1a_2 - a_0)^2$ divides $(a_1^2 + a_2^2)^2$. This can be seen by substituting (a_0, a_1, a_2) into (3.16), rearranging and then squaring to find:

$$a_0^2(3a_1a_2 - a_0)^2 = (a_1^2 + a_2^2)^2.$$

Since $\mathrm{mult}(T) = 1$, it follows from Remark 3.12 that there exists a one-step mutation from $\mathbb{P}((3a_1a_2 - a_0)^2, a_1^2, a_2^2)$ to $\mathbb{P}(a_0^2, a_1^2, a_2^2)$. Thus two solutions to (3.16) are connected by an arithmetic mutation if and only if their corresponding weighted projective planes in S are related by a one-step mutation.

Since $\mathbb{P}^2 \in S$, this establishes an edge preserving bijection between verts$(\Gamma_1(\mathbb{P}^2))$ and verts$(\mathcal{G}/S_3)$. Thus $\Gamma_1(\mathbb{P}^2)$ and $\mathcal{G}/S_3$ are isomorphic graphs, with vertices related by: $(a_0, a_1, a_2) \leftrightarrow \mathbb{P}(a_0^2, a_1^2, a_2^2)$.

Remark 3.21. In fact, $\Gamma_1(\mathbb{P}^2) = \Gamma(\mathbb{P}^2)$. In order to prove this, it suffices to show that $\Gamma(\mathbb{P}^2)$ contains no lattice polygons with ≥ 4 vertices. Demonstrating this requires the notion of *singularity content*, introduced in Section 3.3. In particular, see Example 3.33.

We conclude with a remark on the structure of $\Gamma_1(T)$, for T a Fano triangle.

Definition 3.22. The *height* of the weight vector $(\lambda_0, \lambda_1, \lambda_2)$ is given by the sum $h := \lambda_0 + \lambda_1 + \lambda_2 \in \mathbb{Z}_{>0}$. We call the weights *minimal* if for any sequence of one-step mutations $(\lambda_0, \lambda_1, \lambda_2) \mapsto \ldots \mapsto (\lambda_0', \lambda_1', \lambda_2')$ we have that $h \leq h'$.

Lemma 3.23 ([3, Lemma 3.16]). *Given a well-formed weight vector* $(\lambda_0, \lambda_1, \lambda_2)$ *at height* h *there exists at most one one-step mutation at height* h' *such that* $h' \leq h$. *Moreover, if* $h' = h$ *then the weights are the same.*

Proof. Without loss of generality suppose we have two one-step mutations

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0}\right) \qquad \text{and} \qquad \left(\lambda_0, \frac{(\lambda_0 + \lambda_2)^2}{\lambda_1}, \lambda_2\right)$$

with respective heights h' and h'' such that $h' \leq h$ and $h'' \leq h$. Since $h' \leq h$ we obtain $(\lambda_1 + \lambda_2)^2 \leq \lambda_0^2$, and so $\lambda_1^2 + \lambda_2^2 < \lambda_0^2$. Similarly, from $h'' \leq h$ we obtain $\lambda_0^2 + \lambda_2^2 < \lambda_1^2$. Combining these two inequalities in the λ_i gives a contradiction, hence there exists at most one one-step mutation such that $h' \leq h$. If we suppose that $h' = h$ then $(\lambda_1 + \lambda_2)^2/\lambda_0 = \lambda_0$ and equality of the weights is immediate. □

The height imposes a natural direction on the rank 1 mutation graph generated by the Fano triangle with weight vector $(\lambda_0, \lambda_1, \lambda_2)$.

3.3 Singularity Content

Let $P \subset N_{\mathbb{Q}}$ be a lattice polygon. As discussed on page 34, the combinatorial mutation construction naturally leads one to subdivide the edges of P into sub-line segments whose width equals their lattice height above the origin. In this section, we work exclusively with Fano polygons, where studying such subdivisions of edges is equivalent to studying subdivisions of cones (over the edges) into subcones whose width equals their local index. The upshot of this investigation will be a new invariant of Fano polygons under combinatorial mutation.

Preliminaries

We follow the notation of Sections 2.1 and 3.1. Throughout this section, $C \subset N_{\mathbb{Q}}$ will be a (strictly convex, rational polyhedral) cone of dimension two, and u, v will denote the primitive lattice vectors defined by the rays of C. For notational convenience, we set $w := \mathrm{width}(C)$ and $\ell := \ell_C$. This will not cause confusion since width vectors (Section 2.3) do not play an important role in this section. We set $\tau, \rho \in \mathbb{Z}_{>0}$ to be such that:

$$w = \tau\ell + \rho \quad ; \quad 0 \leq \rho < \ell. \tag{3.18}$$

Singularity Content of Cones

Given C, u and v as above, and an integer m such that $0 \leq m \leq \tau + 1$, choose a sequence of lattice points $v_0, v_1, \ldots, v_{\tau+1}$ on the line segment $\operatorname{conv}(u, v)$ as follows:

(1) $v_0 = u$ and $v_{\tau+1} = v$;

(2) $v_{i+1} - v_i$ is a non-negative scalar multiple of $v - u$, for $i \in \{0, 1, \ldots, \tau\}$;

(3) The line segment $\operatorname{conv}(v_i, v_{i+1})$ has width ℓ for $i \in \{0, \ldots, \widehat{m}, \ldots, \tau\}$;

(4) The line segment $\operatorname{conv}(v_m, v_{m+1})$ has width ρ.

The sequence $v_0, \ldots, v_{\tau+1}$ is uniquely determined by m and the choice of u. We consider the partition of C into subcones $C_i := \operatorname{cone}\{v_i, v_{i+1}\}$, $0 \leq i \leq \tau$.

Proposition 3.24 ([4, Proposition 2.3])**.** *Let $C \subset N_{\mathbb{Q}}$ be a two-dimensional cone of singularity type $\frac{1}{r}(1, a-1)$. Let u, v be the primitive lattice vectors defined by the rays of C, ordered such that u, v, and $\frac{a-1}{r}u + \frac{1}{r}v$ generate N. Let $v_0, \ldots, v_{\tau+1}$ be as above. Then:*

(1) The lattice points $v_0, \ldots, v_{\tau+1}$ are primitive;

(2) The subcones C_i, $0 \leq i < m$, are of singularity type $\frac{1}{\ell^2}(1, \frac{\ell a}{w} - 1)$;

(3) If $\rho \neq 0$ then the subcone C_m is of singularity type $\frac{1}{\rho\ell}(1, \frac{\rho a}{w} - 1)$;

(4) The subcones C_i, $m < i \leq \tau$, are of singularity type $\frac{1}{\ell^2}(1, \frac{\ell \bar{a}}{w} - 1)$.

Here $\bar{a}$ is any integer satisfying $(a-1)(\bar{a}-1) \equiv 1 \pmod{r}$, and so exchanging the roles of u and v exchanges a and $\bar{a}$ in the above formulae. In particular, the singularity type of C_m depends only on C and not on the choice of m, u.

Proof. Without loss of generality, we may choose a basis in which $u = (0, 1)$ and $v = (r, 1-a)$. Similarly, we may assume that $m \neq 0$. The primitive vector in the direction $v - u$ is $(\alpha, \beta) := (\ell, -a/w)$. Thus $v_1 = (\alpha^2, 1 + \alpha\beta)$, and so v_1 is primitive. There exists a change of basis sending v_1 to $(0, 1)$ and leaving (α, β) unchanged. This change of basis sends v_i to v_{i-1} for each $1 \leq i \leq m$. It follows that the lattice points v_i are primitive and that the cones C_i are isomorphic for $1 \leq i \leq m$. Since

$$\tfrac{1}{\alpha^2}(\alpha^2, 1 + \alpha\beta) - \tfrac{1+\alpha\beta}{\alpha^2}(0, 1) = (1, 0),$$

the cone C_1 has singularity type $\frac{1}{\alpha^2}(1, -1 - \alpha\beta) = \frac{1}{\ell^2}(1, \frac{\ell a}{w} - 1)$. This proves (2). Switching the roles of u and v proves (1) and (4). To prove (3), we may again assume that $u = (0, 1)$ and $v = (r, 1-a)$. After applying the above change of basis m times, C_m has primitive generators $(0, 1)$ and $(\rho\alpha, 1 + \rho\beta)$. Since

$$\tfrac{1}{\rho\alpha}(\rho\alpha, 1 + \rho\beta) - \tfrac{1+\rho\beta}{\rho\alpha}(0, 1) = (1, 0),$$

we see that C_m has singularity type $\frac{1}{\rho\alpha}(1, -1-\rho\beta) = \frac{1}{\rho\ell}(1, \frac{\rho a}{w} - 1)$. $\square$

Remark 3.25. Both a/w and $\bar{a}/w$ in Proposition 3.24 are integers by Lemma 3.1.

Definition 3.26. Let $C \subset N_{\mathbb{Q}}$ be a cone of singularity type $\frac{1}{r}(1, a-1)$. Let ℓ and w be as above, and write $w = \tau\ell + \rho$ with $0 \leq \rho < \ell$. The *residue* of C is given by

$$\operatorname{res}(C) := \begin{cases} \frac{1}{\rho\ell}\left(1, \frac{\rho a}{w} - 1\right) & \text{if } \rho \neq 0, \\ \varnothing & \text{if } \rho = 0. \end{cases}$$

The *singularity content* of C is the pair $\operatorname{SC}(C) := (\tau, \operatorname{res}(C))$ [4, Definition 2.4].

Remark 3.27. The singularity content of a cone C determines, and is determined uniquely by the singularity type of C. Indeed, suppose C has singularity type $\frac{1}{r}(1, a)$ and $\operatorname{SC}(C) = (\tau, \{\frac{1}{s}(1, b-1)\})$. Then Proposition 3.24 shows that

$$r = \frac{s \cdot \operatorname{width}(C)}{\rho} \quad \text{and} \quad a = \frac{b \cdot \operatorname{width}(C)}{\rho},$$

where $\operatorname{width}(C) \equiv \rho \bmod \ell_C$. The quantities $\operatorname{width}(C)$ and ρ can be computed using r and a by Lemma 3.1, or by using s and b since by construction $\operatorname{width}(C) = \tau\ell_C + \gcd\{s, b\}$ and $\ell_C = s/\gcd\{s, b\}$.

Example 3.28. Suppose that C has singularity type $\frac{1}{60}(1, 23)$. Then $w = 12$, $\ell = 5$, and $\rho = 2$. Setting $m = 1$ we obtain a decomposition of C into three subcones: C_0 of singularity type $\frac{1}{25}(1, 9)$, C_1 of singularity type $\frac{1}{10}(1, 3)$, and C_2 of singularity type $\frac{1}{25}(1, 4)$. In particular, $\operatorname{res}(C) = \frac{1}{10}(1, 3)$.

Singularities with Empty Residue

Define the *residue* of a cyclic quotient singularity σ to be $\operatorname{res}(C)$, where C is any cone of singularity type σ. A *T-singularity* is a cyclic quotient singularity of the form $\frac{1}{dn^2}(1, dnc - 1)$, where $\gcd\{n, c\} = 1$ [21, Proposition 3.10]. Corollary 3.29 shows that these are precisely the cyclic quotient singularities with empty residue.

Corollary 3.29 ([4, Corollary 2.6])**.** *Let $C \subset N_{\mathbb{Q}}$ be a cone and let w, ℓ be as above. The following statements are equivalent:*

(1) $\operatorname{res}(C) = \varnothing$*;*

(2) There exists an integer τ such that $w = \tau\ell$;

(3) There is a subdivision of C into τ cones of type $\frac{1}{\ell^2}(1, \ell c - 1)$, $\gcd\{\ell, c\} = 1$;

(4) C corresponds to a T-singularity of type $\frac{1}{\tau\ell^2}(1, \tau\ell c - 1)$, $\gcd\{\ell, c\} = 1$.

Proof. (1) and (2) are equivalent by definition. (3) follows from (2) by Proposition 3.24, and (1) follows from (4) by Lemma 3.1. Assume (3) and let the singularity type of C be $\frac{1}{R}(1, A-1)$. The width of C is τ times the width of a given subcone. Since $\gcd\{\ell, c\} = 1$, Lemma 3.1 implies

$$\gcd\{R, A\} = w = \tau \cdot \gcd\left\{\ell^2, \ell c\right\} = \tau\ell.$$

The local index of a given subcone coincides, by construction, with the local index of C. By Lemma 3.1 we see that

$$R = \ell \cdot \gcd\{R, A\} = \tau\ell^2.$$

Proposition 3.24 implies $\ell A/w = \ell c$, hence $A = \tau\ell c$, and so (3) implies (4). □

Invariance Under Combinatorial Mutations

Let $P \subset N_{\mathbb{Q}}$ be a Fano polygon, and let Σ be the spanning fan of P in $N_{\mathbb{Q}}$. Let the two-dimensional cones of Σ be $C_1, \ldots, C_m$, numbered cyclically, with $\mathrm{SC}(C_i) = (\tau_i, \mathrm{res}(C_i))$. The *singularity content* of P is

$$\mathrm{SC}(P) := (\tau, \mathcal{B}),$$

where $\tau := \sum_{i=0}^{m} \tau_i$ and $\mathcal{B}$ is the cyclically ordered list $\{\mathrm{res}(C_1), \ldots, \mathrm{res}(C_m)\}$, with the empty residues $\mathrm{res}(C_i) = \varnothing$ omitted. We call $\mathcal{B}$ the *residual basket* of P.

Proposition 3.30 ([4, Proposition 3.6])**.** *Let P be a Fano polygon and let $Q := \mathrm{mut}_h(P, F)$. Then $\mathrm{SC}(P) = \mathrm{SC}(Q)$. Equivalently, singularity content is an invariant of Fano polygons under combinatorial mutation.*

Proof. The dual polygon $P^\vee \subset M_{\mathbb{Q}}$ is an intersection of cones

$$P^\vee = \bigcap \left(C_L^\vee - v_L\right),$$

where the intersection ranges over all facets L of P. Here $C_L \subset N_{\mathbb{Q}}$ is the cone over the facet L and v_L is the vertex of $P^\vee$ corresponding to L.

If F is a point then $P \cong Q$ and we are done. Let F be a line segment and let $P_{\max}$ and $P_{\min}$ (resp. $Q_{\max}$ and $Q_{\min}$) denote the faces of P (resp. Q) at maximum and minimum height with respect to h. By assumption the mutation Q exists, hence $P_{\min}$ must be a facet, and so there exists a corresponding vertex $v_0 \in M$ of

$P^\vee$. $P_{\max}$ can be either facet or a vertex. The argument is similar in either case, so we will assume that $P_{\max}$ is a facet with corresponding vertex $v_1 \in M$ of $P^\vee$.

The inner normal fan of F, denoted Σ, defines a decomposition of $M_\mathbb{Q}$ into half-spaces Σ^+ and Σ^-. The vertices v_0 and v_1 of $P^\vee$ lie on the rays of Σ; any other vertex lies in exactly one of Σ^+ or Σ^-. Mutation acts as an automorphism in both half-spaces. Thus the contribution to $\mathrm{SC}(Q)$ from cones over all facets excluding $Q_{\max}$ and $Q_{\min}$ is equal to the contribution to $\mathrm{SC}(P)$ from cones over all facets excluding $P_{\max}$ and $P_{\min}$. Finally, mutation acts by exchanging T-singular subcones between the facets $P_{\max}$ and $P_{\min}$, leaving the residue unchanged. Hence the contribution to $\mathrm{SC}(Q)$ from $Q_{\max}$ and $Q_{\min}$ is equal to the contribution to $\mathrm{SC}(P)$ from $P_{\max}$ and $P_{\min}$. □

Example 3.31. If two Fano polytopes are related by combinatorial mutations then their toric surfaces have the same degree, by Corollary 2.21. The Fano polygons $P_1 \subset \mathbb{Q}^2$, with vertex set $\{(0,1),(5,4),(-7,-8)\}$, and $P_2 \subset \mathbb{Q}^2$, with vertex set $\{(0,1),(3,1),(-112,-79)\}$, correspond to $\mathbb{P}(5,7,12)$ and $\mathbb{P}(3,112,125)$ respectively. These both have degree $48/35$. However, P_1 and P_2 are not related by a sequence of combinatorial mutations because their singularity contents differ:

$$\mathrm{SC}(P_1) = \left(12, \left\{\tfrac{1}{5}(1,1), \tfrac{1}{7}(1,1)\right\}\right), \quad \mathrm{SC}(P_2) = \left(5, \left\{\tfrac{1}{14}(1,9), \tfrac{1}{125}(1,79)\right\}\right).$$

Proposition 3.34 gives an explicit formula relating degree and singularity content.

Lemma 3.32 ([4, Lemma 3.8])**.** *Let P be a Fano polygon with $\mathrm{SC}(P) = (\tau, \mathcal{B})$, and let ρ_X denote the Picard rank of the toric surface X, defined by the spanning fan of P. Then $\rho_X \le \tau + |\mathcal{B}| - 2$.*

Proof. The cone over any facet of P admits a subdivision (in the sense of Section 3.3) into at least one subcone. Therefore we must have that $|\mathrm{verts}(P)| \le \tau + |\mathcal{B}|$. Recalling that $\rho_X = |\mathrm{verts}(P)| - 2$, we obtain the result. □

Example 3.33. In Section 3.2 we studied rank 1 mutation graphs of Fano triangles, T (fake weighted projective planes). Lemma 3.32 shows that $\Gamma_1(T) = \Gamma(T)$ whenever the singularity content, $(\tau, \mathcal{B})$, of the spanning fan of T satisfies $\tau + |\mathcal{B}| = 3$. In particular, Example 3.20 shows that $\Gamma(\mathbb{P}^2)$ is isomorphic to the arithmetic mutation graph, $\mathcal{G}/S_3$, of solutions to the Markov equation $3xyz = x^2 + y^2 + z^2$. Similarly, Example 3.10 shows that the Fano triangle defining the weighted projective plane $\mathbb{P}(3,5,11)$ does not admit *any* non-trivial combinatorial mutations.

Degree and Singularity Content

Let $P \subset N_{\mathbb{Q}}$ be a Fano polygon and let X denote the complete toric surface defined by the spanning fan of P. The singularity content of X is $\mathrm{SC}(X) := \mathrm{SC}(P)$. The degree of X, K_X^2, and the singularity content, $\mathrm{SC}(P)$, are two invariants of P under the combinatorial mutation construction (Corollary 2.21 and Proposition 3.30). We now describe a precise relationship between these two invariants.

We recall some standard facts about toric surfaces; see for instance [12]. Let X be a toric surface with singularity $\frac{1}{r}(1, a-1)$. Let $[b_1, \ldots, b_k]$ denote the Hirzebruch–Jung continued fraction expansion of $r/(a-1)$, having length $k \in \mathbb{Z}_{>0}$. For $i \in \{1, \ldots, k\}$, define $\alpha_i, \beta_i \in \mathbb{Z}_{>0}$ as follows: Set $\alpha_1 = \beta_k = 1$ and set

$$\begin{aligned} \alpha_i/\alpha_{i-1} &:= [b_{i-1}, \ldots, b_1], \quad 2 \le i \le k, \\ \beta_i/\beta_{i+1} &:= [b_{i+1}, \ldots, b_k], \quad 1 \le i \le k-1. \end{aligned}$$

If $\pi : \widetilde{X} \to X$ is a minimal resolution then

$$K_{\widetilde{X}} = \pi^* K_X + \sum_{i=1}^{k} d_i E_i, \tag{3.19}$$

where $E_i^2 = -b_i$ and $d_i = -1 + (\alpha_i + \beta_i)/r$ is the discrepancy (see for instance [17]).

Proposition 3.34 ([4, Proposition 3.3])**.** *Let X be a complete toric surface with $\mathrm{SC}(X) = (\tau, \mathcal{B})$. Then*

$$K_X^2 = 12 - \tau - \sum_{\sigma \in \mathcal{B}} A(\sigma), \qquad \textit{where } A(\sigma) := k_\sigma + 1 - \sum_{i=1}^{k_\sigma} d_i^2 b_i + 2 \sum_{i=1}^{k_\sigma - 1} d_i d_{i+1}.$$

Proof. Let Σ in $N_{\mathbb{Q}}$ be the fan of X. If $C \in \Sigma$ is a two-dimensional cone whose rays are generated by the primitive lattice vectors u and v then, possibly by adding an extra ray through a primitive lattice vector on the line segment $\mathrm{conv}(u, v)$, we can partition C as $C = S \cup R_C$, where S is a (possibly smooth) T-singularity or $S = \varnothing$, and $R_C = \mathrm{res}(C)$. Repeating this construction for all two-dimensional cones of Σ gives a new fan $\widetilde{\Sigma}$ in $N_{\mathbb{Q}}$. If $\widetilde{X}$ is the toric variety corresponding to $\widetilde{\Sigma}$ then the natural morphism $\widetilde{X} \to X$ is crepant. In particular $K_{\widetilde{X}}^2 = K_X^2$. Notice that $\mathrm{SC}(X) = (\tau, \mathcal{B}) = \mathrm{SC}(\widetilde{X})$.

By resolving singularities on all the nonempty cones R_C, we obtain a morphism $Y \to \widetilde{X}$ where the toric surface Y (whose fan we denote Σ_Y) has only

T-singularities. Thus by Noether's formula [16, Proposition 2.6]:

$$K_Y^2 + \rho_Y + \sum_{\sigma \in \mathrm{Sing}(Y)} \mu_\sigma = 10, \tag{3.20}$$

where ρ_Y is the Picard rank of Y, and μ_σ denotes the Milnor number of σ. But $\rho_Y + 2$ is equal to the number of two-dimensional cones in Σ_Y, and the Milnor number of a T-singularity $\frac{1}{dn^2}(1, dnc - 1)$ equals $d - 1$ by [22, Proposition 13]. Hence,

$$\rho_Y + \sum_{\sigma \in \mathrm{Sing}(Y)} \mu_\sigma = -2 + \tau + \sum_{\sigma \in \mathcal{B}} (k_\sigma + 1), \tag{3.21}$$

where k_σ denotes the length of the Hirzebuch–Jung continued fraction expansion $[b_1, \ldots, b_{k_\sigma}]$ of $\sigma \in \mathcal{B}$. It follows from Equation (3.19) that

$$K_Y^2 = K_X^2 + \sum_{\sigma \in \mathcal{B}} \left(-\sum_{i=1}^{k_\sigma} d_i^2 b_i + 2 \sum_{i=1}^{k_\sigma - 1} d_i d_{i+1} \right). \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.20) gives the desired formula. $\square$

The m-th Dedekind sum, $m \in \mathbb{Z}_{\geq 0}$, of the cyclic quotient singularity $\frac{1}{r}(a, b)$ is

$$\delta_m := \frac{1}{r} \sum \frac{\varepsilon^m}{(1 - \varepsilon^a)(1 - \varepsilon^b)},$$

where the summation is taken over those $\varepsilon \in \mu_r$ which satisfy $\varepsilon^a \neq 1$ and $\varepsilon^b \neq 1$. In the spirit of Proposition 3.34 and [6], we note that there is strong experimental evidence for the following:

Conjecture 3.35. Let X be a complete toric surface defined by the spanning fan of a Fano polygon. If the singularity content of X is $(\tau, \mathcal{B})$, then the Hilbert series of X admits a decomposition:

$$\mathrm{Hilb}(X, -K_X) = \frac{1 + (K_X^2 - 2)t + t^2}{(1 - t)^3} + \sum_{\sigma \in \mathcal{B}} Q_\sigma(t),$$

where $Q_{\frac{1}{r}(a,b)} := \left(\sum_{i=0}^{r-1} (\delta_{(a+b)i} - \delta_0) t^i \right) (1 - t^r)^{-1}$.

Chapter 4

Further Directions

4.1 Laurent Polynomial Mirrors

In this section, we restrict our attention to two dimensions and to Fano polygons. As seen in Section 2.4, if one pulls back a Laurent polynomial f by an algebraic mutation φ, so that the resulting rational function $g = \varphi^* f$ is itself a Laurent polynomial, then there is an induced transformation which realizes $\mathrm{Newt}(g)$ as a combinatorial mutation of $\mathrm{Newt}(f)$.

However, if one now adopts the combinatorial viewpoint as the primary one, it is possible for there to exist a polytope P, a Laurent polynomial f supported on P and data $(w, F = \mathrm{Newt}(A))$ such that $\mathrm{mut}_w(P, F)$ exists but $\varphi_A^* f$ is not a Laurent polynomial.

Example 4.1. Consider $f(x, y) := x + y^{-1}(1 + 2x) + x^{-1}y^{-2}(1 + x)^2$, supported on the lattice polygon $P \subset \mathbb{Q}^2$ with vertex set $\{(0, 1), (1, -1), (1, -2), (-1, -2)\}$. Choose $w = (0, 1)^t \in (\mathbb{Z}^2)^\vee$ and let $A(x) = 1+x \in \mathbb{C}[x^{\pm 1}]$, so that $F = \mathrm{Newt}(A) = \mathrm{conv}(\mathbf{0}, (1, 0))$. A direct check using Lemma 3.2 shows that $\mathrm{mut}_w(P, F)$ exists — it is the lattice polygon with vertex set $\{(0, 1), (1, 1), (0, -1), (-1, -2)\}$. On the other hand, the rational function $\varphi_A^* f$ is not a Laurent polynomial because $1 + x$ does not divide $1 + 2x$ in $\mathbb{C}[x^{\pm 1}]$.

Given a Fano polygon P, it is therefore natural to ask whether one can decorate its vertices with coefficients in such a way as to obtain a Laurent polynomial f supported on P with the property that the set of Laurent polynomials related to f by a single algebraic mutation is in one-to-one correspondence with the set of one-step mutations of P. If such a Laurent polynomial exists, it is also natural to ask whether it holds any significance from the viewpoint of the mirror duality discussed in Conjecture 1.2.

In this section, we present some examples addressing the two questions posed above. The method used in these examples was discovered by the author, together with A. Kasprzyk and K. Tveiten. A formal definition for the class of Laurent polynomials determined by this construction appears as Definition 4 in [1]. We also refer the reader to [20], where the definition and properties of these *maximally mutable* Laurent polynomials are established in all dimensions. Statements concerning mirror duality in all our examples are taken from the works [8, 24].

Given a Fano polygon $P \subset N_{\mathbb{Q}}$, our method begins by imposing the conventions[1]:

(1) Decorate the origin, $\mathbf{0} \in \operatorname{int}(P)$, with coefficient 0; and

(2) Decorate the vertices of P with coefficient 1.

The remaining lattice points of P are decorated with unknown (complex) coefficients. In this way, we obtain a family of Laurent polynomials, with general member f, supported on P. Next, we impose conventions which allow us to construct a Laurent polynomial supported on any non-trivial factor of P. Any such factor must be a line segment, for dimension reasons. Our conventions are:

(3) Decorate the vertices of any primitive line segment with coefficient 1; and

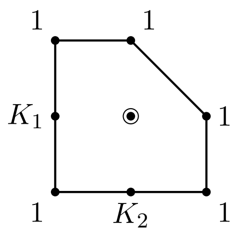
(4) If g_1, g_2 are Laurent polynomials supported on line segments $L_1, L_2 \subset N_{\mathbb{Q}}$, then their product $g_1 g_2$ is the Laurent polynomial supported on the Minkowski sum $L_1 + L_2$. Equivalently, the lattice point $(a, b) \in L_1 + L_2$ is decorated with the coefficient of $x^a y^b$ in $g_1 g_2$.

Now suppose $F \subset N_{\mathbb{Q}}$ is a non-trivial factor of P. Without loss of generality (Section 2.3), we may consider F up to translation. So after choosing a $\mathbb{Z}$-basis for N, we have $F = \operatorname{conv}(\mathbf{0}, (a, b))$, where $(a, b) \in \mathbb{Z}^2$. If (a, b) is primitive then, by Convention (3), we think of F as supporting the Laurent polynomial $A(x, y) = 1 + x^a y^b$. Otherwise $(a, b) = m \cdot (c, d)$, where $m > 0$ and $(c, d) \in \mathbb{Z}^2$ is primitive. In this case, F is the Minkowski sum of the line segments $\operatorname{conv}(\mathbf{0}, (m-1) \cdot (c, d))$ and $\operatorname{conv}(\mathbf{0}, (c, d))$. By induction on Convention (4), we think of F as supporting the Laurent polynomial $A(x, y) = (1 + x^c y^d)^m$. In this way, Conventions (3) and (4) give a unique Laurent polynomial A supported on any non-trivial factor F of P. Note that translating F by $(h, k) \in \mathbb{Z}^2$ amounts to multiplying A by $x^h y^k$. After a change of basis, we may write A as a polynomial in one variable only.

[1] We note that both conventions already appear in [8]. Furthermore, the author thanks A. Corti and T. Coates for pointing out that Convention (1) is imposed by Gromov-Witten theory via mirror duality: the linear term in the Taylor expansion of the regularized quantum period is always zero and Conjecture 1.2 implies that this Taylor expansion must coincide with the *period sequence* [7], $\sum \operatorname{coeff}_{\mathbb{1}}(f^k) t^k$, of a Laurent polynomial mirror f (Section 1.3). In particular, $\operatorname{coeff}_{\mathbb{1}}(f) = 0$ for any Laurent polynomial mirror to a Fano manifold.

The above setup allows us to associate an algebraic mutation to any combinatorial mutation of P; In the above notation, constructing $\mathrm{mut}_w(P,F)$ is interpreted as constructing the rational function $\varphi_A^* f$, where φ_A is the algebraic mutation associated to A (Section 1.3). Since $\mathrm{mut}_w(P,F)$ is a lattice polygon, it is natural to desire that $\varphi_A^* f$ is in fact a Laurent polynomial. This requirement imposes constraints on the coefficients of f. Thus, the content of our method can be summarized as follows: we wish to determine conditions on the unknown coefficients of f which ensure that f remains a Laurent polynomial under every algebraic mutation of f arising from a combinatorial mutation of P, as above. In practice, this amounts to repeated application of the following observation from Section 2.4: Let $f(x,y) = \sum C_h(x) y^h$ be a Laurent polynomial and choose $A(x) \in \mathbb{C}[x^{\pm 1}]$. Then the rational function $\varphi_A^* f = f \circ \varphi_A$ is a Laurent polynomial if and only if A^{-h} divides C_h in $\mathbb{C}[x^{\pm 1}]$ for all negative values of h.

Example 4.2. Let $P \subset \mathbb{Q}^2$ have vertex set $\{(0,1),(-1,1),(-1,-1),(1,-1),(1,0)\}$. By imposing Conventions (1) and (2), we are able to determine coefficients of all but two lattice points of P. Let K_1, K_2 denote the unknown coefficients, as illustrated in the figure below.



First choose $w = (1,0)^t \in (\mathbb{Z}^2)^\vee$. There is exactly one $P_{w,h}$ with $h < 0$, namely: $P_{w,-1} = \mathrm{conv}((-1,1),(-1,-1))$. We interpret this as the Laurent polynomial $x^{-1}y^{-1}(1 + K_1 y + y^2) = C_{-1}(y)x^{-1}$. Up to translation, there are two factors of P with respect to w, namely $F_1 := \mathrm{conv}(\mathbf{0},(0,1))$ which supports $A_1(y) := 1 + y$ and $F_2 := \mathrm{conv}(\mathbf{0},(0,2))$ which supports $A_2(y) := (1+y)^2$.

If we choose the factor F_1, then any Laurent polynomial f supported on P (with the given coefficients) will remain a Laurent polynomial after pullback by φ_{A_1} if and only if $A_1^{-(-1)} = A_1 = 1 + y$ divides C_{-1}. Thus K_1 must satisfy

$$(\alpha + \beta y)(1 + y) = 1 + K_1 y + y^2, \tag{4.1}$$

for some choice of $\alpha, \beta \in \mathbb{Z}$. Expanding and comparing coefficients in (4.1) shows that $\alpha = \beta = 1$ and K_1 must equal $\alpha + \beta = 2$. On the other hand, if we choose the factor F_2, then by the same reasoning, we would arrive at the condition:

$$\alpha(1+y)^2 = 1 + K_1 y + y^2, \tag{4.2}$$

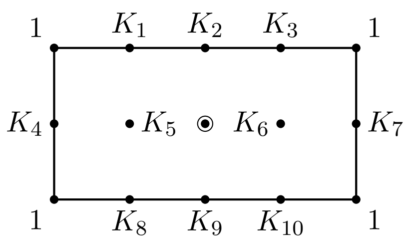
for some $\alpha \in \mathbb{Z}$. This again forces $K_1 = 2$, by comparing coefficients in (4.2).

Next choose $w = (0,1)^t \in (\mathbb{Z}^2)^\vee$. By replacing the roles of x and y above, we conclude that $K_2 = 2$ is the only choice which will ensure that a Laurent polynomial supported on P remains Laurent under pullback by the algebraic mutations φ_{A_3} and φ_{A_4}, where $A_3 = 1 + x$ and $A_4 = (1+x)^2$.

We conclude that $f(x,y) := x^{-1}y + y + 2x^{-1} + x + x^{-1}y^{-1} + 2y^{-1} + xy^{-1}$ is the unique Laurent polynomial supported on P which remains Laurent under every one-step mutation of P. By computing the first few terms of the period sequence of f (Section 1.3) we see from [8] that f is a Laurent polynomial mirror to the del Pezzo surface of degree 5.

Note that both factors F_1 and F_2 impose the same value on the unknown coefficient K_1 in Example 4.2. Generally speaking, the algebraic conditions imposed by the collection of all possible factors over all possible width vectors yield a system of linear equations in the unknown coefficients. Every such system of linear equations is shown to be consistent in [20]. Note also that in any example, it suffices to restrict attention to those width vectors for which there exists at least one non-trivial factor. This is a finite set, by Proposition 2.15 and Remark 2.16.

Example 4.3. Let $P \subset \mathbb{Q}^2$ have vertex set $\{(2,1),(-2,1),(-2,-1),(2,-1)\}$. After imposing Conventions (1) and (2) there remain 10 unknown coefficients, denoted $K_1, K_2, \ldots, K_{10}$ as shown in the following diagram:



First choose $w = (0,-1)^t \in (\mathbb{Z}^2)^\vee$. There is exactly one $P_{w,h}$ with $h < 0$ with respect to w. This is $P_{w,-1} := \mathrm{conv}((-2,1),(2,1))$, which supports $x^{-2}y^{-1}(1 + K_1 x + K_2 x^2 + K_3 x^3 + x^4) = x^{-2}y^{-1}C_{-1}(x)$. There are four factors of P with respect to w namely $F_s := \mathrm{conv}(\mathbf{0},(-s,0))$, supporting $A_s(x) := (1+x)^s$, for $s = 1,2,3,4$. The equations determined by these four factors are consistent, and so it suffices to restrict attention to the equations determined by F_4. A Laurent polynomial supported on P will remain a Laurent polynomial after pullback by φ_{A_4} if and only if A_4 divides C_{-1}. Thus K_1, K_2, K_3 must be chosen to satisfy:

$$k(1+x)^4 = 1 + K_1 x + K_2 x^2 + K_3 x^3 + x^4$$

for some $k \in \mathbb{Z}$. This forces $k = 1$ and determines $K_1 = 4, K_2 = 6$ and $K_3 = 4$.

For $w = (0,-1)^t$, similar reasoning determines $K_8 = 4, K_9 = 6$ and $K_{10} = 4$.

Now choose $w = (1,0)^t$. One must then consider equations arising from both $P_{w,-2} := \operatorname{conv}((-2,1),(-2,-1))$ and $P_{w,-1} := \operatorname{conv}((-1,1),(-1,-1))$. $P_{w,-2}$ supports[2] $x^{-1}y^{-2}(1 + K_4 x + x^2)$ and $P_{w,-1}$ supports $x^{-1}y^{-1}(K_1 + K_5 x + K_8 x^2) = x^{-1}y^{-1}(4 + K_5 x + 4x^2)$. There is one factor to consider: $F := \operatorname{conv}(\mathbf{0}, (0,-1))$, which supports $A(x) := 1 + x$. The divisibility conditions for $h = -1$ and $h = -2$ enforce the following equalities:

$$\begin{aligned} (\alpha + \beta x)(1+x) &= 4 + K_5 x + 4x^2, \text{ and} \\ k(1+x)^2 &= 1 + K_4 x + x^2, \end{aligned}$$

where $\alpha, \beta, k \in \mathbb{Z}$. Comparing coefficients determines $K_5 = 8$ and $K_4 = 2$.

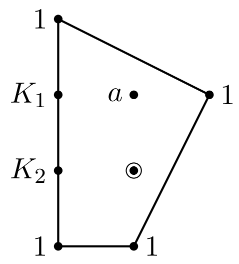
Finally, choosing $w = (-1,0)^t$ and applying similar reasoning as above determines $K_6 = 8$ and $K_7 = 2$.

We thus obtain the following Laurent polynomial supported on P: $f(x,y) := x^{-2}y(1+x)^4 + x^{-1}y^{-1}(1+x)^4 + 2x^{-2} + 8x^{-1} + 8x + 2x^2$. By construction, f has the property that it remains Laurent under every one-step mutation of P. By computing the first few terms of the period sequence of f (Section 1.3) and comparing with [8], we find that f is a Laurent polynomial mirror to the del Pezzo surface of degree 2.

Example 4.4. In both Examples 4.2 and 4.3, we obtain a single Laurent polynomial supported on the polytope P in question. In general however, a number of coefficients can remain undetermined so the above method yields not a single Laurent polynomial but a family of Laurent polynomials, parameterized by

[2]Note that the variables x, y for $w = (1,0)^t$ are not the same as the variables for $w = (0,-1)^t$, which we also denoted x, y. Both pairs of variables are related to each other under a change of variables $(x,y) \mapsto (x^a y^b, x^c y^d)$ with $(a,b),(c,d)$ the rows of some $M \in GL_2(\mathbb{Z})$. In practice however, this slight abuse of notation makes the calculations more transparent without affecting the final result. In short, we always choose y to be the variable corresponding to w and x to be a variable corresponding to a chosen unit vector orthogonal to w in $\mathbb{Z}^2$.

the unknown (complex) coefficients. Consider for instance the Fano polygon P shown in the following diagram: Here, the unknown coefficients K_1 and K_2 can



be determined by applying the method illustrated in Examples 4.2 and 4.3; the choice $w = (1,0)^t$ fixes $K_1 = 3$ and $K_2 = 3$. However, in order to determine the unknown coefficient a, it is necessary to consider the width vector $w = (-1,-2)^t$. If there exists a factor of P with respect to this choice of w, then in particular there must exist a factor F satisfying $\mathrm{width}(F) = 1$. But then, by Lemma 3.2, it would follow that:

$$1 = \mathrm{width}(\mathrm{conv}((1,0),(-1,-1))) \geq -(-2) \cdot \mathrm{width}(F) = 2,$$

which is a contradiction. Since no factor exists for this choice of w, we can not apply the method of the previous two examples. Thus, the coefficient a remains undetermined, because $(0,1)$ does not lie at negative height with respect to any other width vector of P contained in the set (3.1). We obtain a family of Laurent polynomials supported on P, which depends on the parameter $a \in \mathbb{C}$, namely:

$$f_a(x,y) := x^{-1}y^2 + 3x^{-1}y + ay + xy + 3x^{-1} + x^{-1}y^{-1} + y^{-1} \quad ; \quad a \in \mathbb{C}.$$

Each member of this family has the property that it remains a Laurent polynomial under pullback by the algebraic mutations induced by the one-step combinatorial mutations of P. In [24], this family is shown to correspond under mirror duality to the orbifold del Pezzo surface that is is the blow-up of the weighted projective plane $\mathbb{P}(1,1,3)$ in three general points.

4.2 Deformation Theory

The combinatorial mutations of a Fano polytope P are closely related to the deformation theory of the toric variety X_P defined by its spanning fan. This is seen in the following result, due to N. Ilten:

Theorem 4.5 ([18, Theorem 1.3])**.** *Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope and suppose $Q := \mathrm{mut}_w(P, F)$ exists. Then there exists a flat projective family $\pi : \mathcal{X} \to \mathbb{P}^1$ such that $\pi^{-1}(0) = X_P$ and $\pi^{-1}(\infty) = X_Q$.*

The study of deformation families arising from combinatorial mutations is currently a topic of active research.

The results of Section 3.3 can also be used to shed light on deformations of surface cyclic quotient singularities, in the spirit of Kollár–Shepherd-Barron. In [21], T-singularities are characterized as those surface singularities which admit a local $\mathbb{Q}$-Gorenstein one-parameter local smoothing. By Corollary 3.29, T-singularities are precisely the surface cyclic quotient singularities with empty residue. Combining these two viewpoints, we adapt an argument from [16] to obtain:

Proposition 4.6 ([4, Proposition 2.7])**.** *A surface cyclic quotient singularity σ admits a local $\mathbb{Q}$-Gorenstein smoothing if and only if* $\mathrm{res}(\sigma) = \varnothing$*. Otherwise there is a local $\mathbb{Q}$-Gorenstein deformation of σ whose general fibre is a cyclic quotient singularity of type* $\mathrm{res}(\sigma)$*.*

Proof. By the opening comments, σ admits a local $\mathbb{Q}$-Gorenstein smoothing if and only if it is a T-singularity. Thus the first statement follows from Corollary 3.29. Assume σ is not a T-singularity and let ω, ℓ, and ρ be as in Section 3.3. By Corollary 3.29 we must have $\rho > 0$. Now $\sigma = \frac{1}{r}(1, a-1)$ has index ℓ and canonical cover

$$\tfrac{1}{\omega}(1, -1) = (xy - z^{\omega}) \subset \mathbb{A}^3_{x,y,z}.$$

Taking the quotient by the cyclic group μ_ℓ, and noting that $\omega \equiv \rho \,(\mathrm{mod}\ \ell)$, we have:

$$\tfrac{1}{r}(1, a-1) = (xy - z^{\omega}) \subset \tfrac{1}{\ell}(1, \tfrac{\rho a}{\omega} - 1, \tfrac{a}{w}).$$

A local $\mathbb{Q}$-Gorenstein deformation is given by

$$(xy - z^{\omega} + tz^{\rho}) \subset \tfrac{1}{\ell}\left(1, \tfrac{\rho a}{\omega} - 1, \tfrac{a}{\omega}\right) \times \mathbb{A}^1_t,$$

and the general fibre of this family is the singularity $\frac{1}{\rho\ell}(1, \frac{\rho a}{\omega} - 1)$. □

By combining Proposition 4.6 above with [1, Lemma 6], which tells us that there are no local-to-global obstructions, we obtain:

Corollary 4.7. *Let H be a del Pezzo surface with cyclic quotient singularities. There exists a $\mathbb{Q}$-Gorenstein deformation of H to a surface H^{res} such that* $\mathrm{Sing}(H^{\mathrm{res}})$ *is equal to the multiset* $\{\mathrm{res}(\sigma) \mid \sigma \in \mathrm{Sing}(H),\ \mathrm{res}(\sigma) \neq \varnothing\}$*.*

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Appendix A

A.1 Singularity Type of Cones

We give a self-contained introduction to surface cyclic quotient singularities from the viewpoint of lattice theory. The material in this appendix is well-known; see [12, Secion 2.2]. Our own presentation intends to fix notation and emphasize those aspects of the theory which appear most prominently in the main text. We follow the conventions of Sections 2.1 and 3.1. In particular, $C \subset N_{\mathbb{Q}} \cong \mathbb{Q}^2$ always denotes a (strictly convex, rational polyhedral) cone of dimension two.

Definition A.1. Let $C \subset N_{\mathbb{Q}}$ be a cone, and let $L \subseteq N$ be the sublattice generated by the primitive lattice vectors along the rays of C. The *singularity type* of C is

$$\mathrm{ST}(C) := \{u \in N \mid u \notin L \text{ and } N = L + u\mathbb{Z}\}. \tag{A.1}$$

If C is a smooth cone then $L = N$, by definition, so the singularity type of C is the empty set.

Lemma A.2. *If $C \subset N_{\mathbb{Q}}$ is singular then it has nonempty singularity type.*

Proof. Let $L \subseteq N$ be the sublattice generated by the primitive lattice vectors along the rays of C, denoted $p, q \in C \cap N$. $L \neq N$, because C is not smooth, so the set $T := (\mathrm{conv}(\mathbf{0}, p, q) \cap N) \backslash \{\mathbf{0}, p, q\}$ is nonempty and finite. If $u \in T$ then $u \notin L$, because $\mathrm{conv}(\mathbf{0}, p, q) \cap N \cap L = \{\mathbf{0}, p, q\}$.

Choose inward pointing normals, $w_p, w_q \in M$, with $\langle w_i, i \rangle = 0$ for $i \in \{p, q\}$. Set S to be the (nonempty) intersection of T with the hyperplane $\{v \in N_{\mathbb{Q}} \mid \langle w_p, v \rangle = m_p\}$, where $m_p = \min\{\langle w_p, u \rangle \mid u \in T\}$. Take S' be the intersection of S with the hyperplane $\{v \in N_{\mathbb{Q}} \mid \langle w_p, v \rangle = m_{p,q}\}$, where $m_{p,q} = \min\{\langle w_q, u \rangle \mid u \in S\}$. Then S' contains a unique point, called x. By construction $x \in T$, so $x \notin L$. Furthermore, by minimality, the triangle $\mathrm{conv}(\mathbf{0}, x, p) \subset N_{\mathbb{Q}}$ contains no lattice points of N other than vertices, so $\{x, p\}$ is a $\mathbb{Z}$-basis for N which implies that $N = L + x\mathbb{Z}$. $\square$

Lemma A.3. *Let $C \subset N_{\mathbb{Q}}$ be a singular cone with primitive generators $p, q \in C \cap N$. For any $u \in \mathrm{ST}(C)$ there exists an expression:*

$$u = \frac{a}{r}p + \frac{b}{r}q \quad ; \quad r \in \mathbb{Z}_{>0},\ a, b \in \mathbb{Z}. \tag{A.2}$$

If we impose the additional condition that $\gcd\{r, a\} = 1$ *and* $\gcd\{r, b\} = 1$*, then* r, a, b *(and hence the expression* (A.2)*) are uniquely determined by* u*.*

Proof. Let the sublattice of N generated by p and q be denoted L. Take r to be the smallest positive integer such that $ru \in L$. Such an r exists, by well ordering. Indeed $0 < [N : L] < \infty$ and $[N : L]u \in L$, so the set $\{\alpha \in \mathbb{Z}_{>0} \mid \alpha u \in L\}$ is nonempty. The integer r is uniquely determined by u, and determines unique integers $a, b \in \mathbb{Z}$ such that:

$$ru = ap + bq. \tag{A.3}$$

Suppose that $d := \gcd\{r, a\} > 1$, and write $b = hd + k$, $0 \le k < d$. Now $\gcd\{b, d\} = \gcd\{b, \gcd\{r, a\}\} = \gcd\{r, a, b\}$, and this must equal 1. Otherwise it is possible to divide through (A.3) by $\gcd\{r, a, b\}$, and obtain a positive integer $r' < r$ such that $r'u \in L$, contradicting the minimality of r. So $k \neq 0$. Substituting the expression for b into (A.3) yields:

$$\frac{k}{d}q = \frac{r}{d}u - \frac{a}{d}p - hq. \tag{A.4}$$

This is a contradiction, because the right side of (A.4) is a lattice vector in N while the left side is not, because $0 < (k/d) < 1$ and q is primitive. Similar reasoning shows that $\gcd\{r, b\} = 1$. So an expression (A.2) satisfying the additional condition exists.

To prove uniqueness, we first claim that if $R \in \mathbb{Z}_{>0}$ satisfies $Ru \in L$ then r must divide R. Indeed, write $R = sr + t$, $0 \le t < r$. Then $Ru \in L$ implies that $sru + tu \in L$, and hence $tu \in L$ (since $sru \in L$). This contradicts minimality of r if $t \neq 0$. Now suppose there exists an expression:

$$Ru = Ap + Bq \quad ; \quad R, A, B \in \mathbb{Z}_{>0}, \tag{A.5}$$

with $\gcd\{R, A\} = 1$ and $\gcd\{R, B\} = 1$. By the claim, we can write $R = sr$ for some $s \in \mathbb{Z}_{>0}$, and substitute into (A.5). Combining this with (A.3) gives:

$$sap + sbq = sru = Ap + Bq,$$

which implies that $A = sa$ and $B = sb$, by linear independence of p, q in $N_{\mathbb{Q}}$. In particular, since $\gcd\{r, a\} = 1$, it follows that $1 = \gcd\{R, A\} = \gcd\{sr, sa\} = s$. This forces $R = r$, $A = a$, $B = b$, and establishes uniqueness. □

Corollary A.4. *In the notation of Lemma A.3, fix a choice of numbering on the primitive generators of C, say $p_1 = p$ and $p_2 = q$. Then every $u \in \mathrm{ST}(C)$ can be assigned a unique symbol $\frac{1}{r}(a, b)$, where $r \in \mathbb{Z}_{>0}$, $a, b \in Z$, $\gcd\{r, a\} = 1$ and $\gcd\{r, b\} = 1$. These symbols encode equations of the form* (A.2).

Example A.5. Let $C \subset \mathbb{Q}^2$ be the singular cone with primitive generators $(1, 0), (1, 4)$. Then $(1, 1), (1, 3), (-1, -3), (3, 3)$ all lie in $\mathrm{ST}(C)$ and are represented by the symbols $\frac{1}{4}(3, 1)$, $\frac{1}{4}(1, 3)$, $\frac{1}{4}(-1, -3)$, $\frac{1}{4}(9, 3)$, with respect to the numbering $p_1 := (1, 0), p_2 := (1, 4)$. On the other hand, given the same numbering of generators, there can not exist a $u \in \mathrm{ST}(C)$ which is represented by the symbol $\frac{1}{4}(3, 2)$ because $\gcd\{4, 2\} \neq 1$.

If $C \subset N_{\mathbb{Q}}$ is a singular cone, then $\mathrm{ST}(C)$ will always contain more than one element. After choosing a numbering of the primitive generators of C, each element of $\mathrm{ST}(C)$ is represented by a unique symbol $\frac{1}{r}(a, b)$ with $\gcd\{r, a\} = 1$ and $\gcd\{r, b\} = 1$. The next step is to determine how these symbols are related to one another.

Lemma A.6. *Let C be a singular cone and fix a numbering, p_1, p_2, on its primitive generators. Fix an element $u \in \mathrm{ST}(C)$ with symbol $\frac{1}{r}(a, b)$. If $u' \in \mathrm{ST}(C)$ has symbol $\frac{1}{R}(A, B)$ then $R = r$ and (A, B) is obtained from (a, b) by a finite sequence of the following transformations:*

(1) $(x, y) \mapsto (x + \alpha r, y)$, *for some* $\alpha \in \mathbb{Z}$;
(2) $(x, y) \mapsto (x, y + \beta r)$, *for some* $\beta \in \mathbb{Z}$;
(3) $(x, y) \mapsto (\gamma x, \gamma y)$, *for some* $\gamma \in \mathbb{Z}$ *with* $\gcd\{r, \gamma\} = 1$.

Conversely, any symbol $\frac{1}{R}(A, B)$ with $R = r$ and (A, B) obtained from (a, b) via a sequence of the above transformations defines an element of $\mathrm{ST}(C)$.

Proof. u, u' both lie in $\mathrm{ST}(C)$, so $p_1\mathbb{Z} + p_2\mathbb{Z} + u\mathbb{Z} = N = p_1\mathbb{Z} + p_2\mathbb{Z} + u'\mathbb{Z}$ and there exist $\alpha, \beta, \gamma \in \mathbb{Z}$ and $h, k, l \in \mathbb{Z}$ such that:

$$u' = \alpha p_1 + \beta p_2 + \gamma u \quad \text{and} \quad u = hp_1 + kp_2 + lu'. \tag{A.6}$$

To say that u has symbol $\frac{1}{r}(a, b)$ means that equation (A.2) holds, with $p = p_1$ and $q = p_2$. Multiply both equations (A.6) by r and the left hand equation by l.

Then substitute (A.2) to deduce that:

$$l(\alpha r + \gamma a)p_1 + l(\beta r + \gamma b)p_2 = (a - hr)p_1 + (b - kr)p_2.$$

The linear independence of p_1, p_2 in $N_{\mathbb{Q}}$ implies that $l(\alpha r + \gamma a) = a - hr$, and hence r divides $a(1 - l\gamma)$. This implies $l\gamma \equiv 1 \bmod r$, because $\gcd\{r, a\} = 1$. Therefore l, γ in (A.6) are both coprime to r. In particular, substituting equation (A.2) into the left hand equation (A.6) gives:

$$u' = \frac{(\alpha r + \gamma a)}{r} p_1 + \frac{(\beta r + \gamma b)}{r} p_2, \tag{A.7}$$

with $\gcd\{r, \gamma\} = 1$. If d divides both r and $\alpha r + \gamma a$ then it must divide γa and hence divides $\gcd\{r, \gamma a\} = 1$. So $\gcd\{r, \alpha r + \gamma a\} = 1$. Similar reasoning shows that $\gcd\{r, \beta r + \gamma b\} = 1$. We conclude that the unique symbol representing u' (with respect to the chosen numbering of the primitive generators of C) is $\frac{1}{r}(\alpha r + \gamma a, \beta r + \gamma b)$, for some $\alpha, \beta, \gamma \in \mathbb{Z}$ with $\gcd\{r, \gamma\} = 1$.

For the converse, let $L := p_1\mathbb{Z} + p_2\mathbb{Z}$. First consider the symbol $\frac{1}{r}(a + \alpha r, b)$ for a choice of $\alpha \in \mathbb{Z}$. This defines the following element of N:

$$u' = \frac{(a + \alpha r)}{r} p_1 + \frac{b}{r} p_2 = \alpha p_1 + u. \tag{A.8}$$

Equation (A.8) shows that $u' \notin L$ (because $u \notin L$ and $\alpha p_1 \in L$) and that $L + u'\mathbb{Z} = L + u\mathbb{Z} = N$. So $u' \in \mathrm{ST}(C)$. Similar reasoning shows that $\frac{1}{r}(a, b + \beta r)$ for a chosen $\beta \in \mathbb{Z}$ also defines an element of $\mathrm{ST}(C)$. Finally, consider the symbol $\frac{1}{r}(\gamma a, \gamma b)$ for a chosen $\gamma \in \mathbb{Z}$ coprime to r. This defines the following element of N:

$$u'' = \frac{\gamma a}{r} p_1 + \frac{\gamma b}{r} p_2 = \gamma u. \tag{A.9}$$

Now $u'' \notin L$ because the ratios $(\gamma a/r), (\gamma b/r)$ are not integers. Also $L + u''\mathbb{Z} \subseteq L + u\mathbb{Z}$ by (A.9). Furthermore, since $\gcd\{r, \gamma\} = 1$, there exists an equation $sr + t\gamma = 1$ with $s, t \in \mathbb{Z}$. Using (A.2) and (A.9), we have that:

$$u = 1 \cdot u = sru + t\gamma u = sap_1 + sbp_2 + tu'' \in L + u''\mathbb{Z}, \tag{A.10}$$

which establishes the reverse inclusion. So $u'' \in \mathrm{ST}(C)$. $\square$

Definition A.7. Let $C \subset N_{\mathbb{Q}}$ be a singular cone, and make a choice of numbering, p_1, p_2, on its primitive generators. We say that the symbol $\frac{1}{r}(a, b)$ represents the singularity type of C if the lattice vector u defined by (A.2) lies in $\mathrm{ST}(C)$. The singularity type of a smooth cone is represented by the symbol $\frac{1}{1}(1,1)$.

Remark A.8. By Lemma A.6, the singularity type of any cone C can be represented by a symbol of the form $\frac{1}{r}(1, a-1)$, with $\gcd\{r, a-1\} = 1$. This fact is used implicitly throughout the main text, most notably in Section 3.3.

Definition A.9. Two cones $C \subset N_Q \cong \mathbb{Q}^2$ and $C' \subset N'_{\mathbb{Q}} \cong \mathbb{Q}^2$ are said to have the same singularity type if there exists a lattice isomorphism $\varphi : N \cong N'$ which maps the primitive generators of C onto the primitive generators of C'.

Remark A.10. In the notation of Definition A.9, if C, C' have the same singularity type then $u \in \mathrm{ST}(C)$ has symbol $\frac{1}{r}(a,b)$ with respect to a numbering p_1, p_2 if an only if $\varphi(u) \in \mathrm{ST}(C')$ has symbol $\frac{1}{r}(a,b)$ with respect to the numbering $q_i := \varphi(p_i)$, $i \in \{1,2\}$. Note also that any automorphism of the ambient lattice preserves singularity type.

A.2 The Degree of a Complete Toric Surface

Let Σ be a complete fan in $N_{\mathbb{Q}} \cong \mathbb{Q}^2$, and let X denote the complete toric surface defined by Σ. Let $v_0, \ldots, v_n \in N \cong \mathbb{Z}^2$ denote the primitive lattice vectors defined by the rays of Σ, numbered in a clockwise manner. All indices in this section will be taken modulo $n+1$ so that, for instance, $v_{n+1} = v_0$ and $R^{n+2}_{n+1} = R^1_0$.

Lemma A.11. *For each $i \in \{0, \ldots, n\}$, the vectors v_{i-1}, v_i, v_{i+1} satisfy*

$$R_i^{i-1} v_{i-1} + (-R_i^i) v_i + R_i^{i+1} v_{i+1} = \mathbf{0}, \tag{A.11}$$

where $R_i^{i-1} := \det(\widehat{v_{i-1}}\ v_i\ v_{i+1})$, $R_i^i := \det(v_{i-1}\ \widehat{v_i}\ v_{i+1})$ *and* $R_i^{i+1} = \det(v_{i-1}\ v_i\ \widehat{v_{i+1}})$.

Proof. Since $\mathrm{cone}(v_{i-1}, v_i)$ is simplicial, the vectors v_{i-1} and $-v_i$ are linearly independent over $\mathbb{Q}$. Applying Cramer's rule to the equation:

$$\beta_1 v_{i-1} - \beta_2 v_i = -v_{i+1},$$

gives $\beta_1 = \det(v_i\ v_{i+1}) / \det(v_{i-1}\ v_i)$ and $\beta_2 = \det(v_{i-1}\ v_{i+1}) / \det(v_{i-1}\ v_i)$. □

Note that $R_i^{i-1} = R_{i+1}^{i+2}$ for all $i \in \{0, \ldots, n\}$.

Remark A.12. Let $L(i,j)$ denote the sublattice of N spanned by v_i and v_j. Since the v_i are all primitive, it follows that $[N : L(i, i+1)] = |\det(v_i\ v_{i+1})| = |R_i^{i-1}|$. Similarly, $|R_i^i| = [N : L(i-1, i+1)]$ and $|R_i^{i+1}| = [N : L(i-1, i)]$.

Proposition A.13. *In the above notation:*

$$K_X^2 = \sum_{i=0}^{n} \left(2 - \frac{R_i^i}{R_i^{i-1}}\right) \frac{1}{|R_i^{i+1}|}. \tag{A.12}$$

Proof. The ray in Σ spanned by v_i corresponds to a torus-invariant Weil divisor D_i in X. Since $-K_X = D_0 + \ldots + D_n$ [10, Theorem 8.2.3], we have the intersection product:

$$K_X^2 = \Lambda + \sum_{i=0}^{n} \left(2D_{i-1}D_i + D_i^2\right), \tag{A.13}$$

where Λ is a sum of terms of the form: D_iD_j, $j \notin \{i-1, i, i+1\}$. Since Σ is a simplicial fan, the intersection products on the right hand side of Equation (A.13) can be computed using [10, Proposition 6.4.4]. In the notation of the result just cited, the $n+1$ equations (A.11) are called wall relations and the quantity $[N : L(i,j)]$ (discussed in Remark A.12) is called the multiplicity of $\mathrm{cone}(v_i, v_j)$ in N. For each $i \in \{0, \ldots, n\}$, we thus obtain:

$$D_i^2 = \frac{-R_i^i}{R_i^{i-1}|R_i^{i+1}|} \quad ; \quad D_{i-1}D_i = \frac{1}{|R_i^{i+1}|} \quad ; \quad D_iD_j = 0 \text{ if } j \notin \{i-1, i, i+1\}.$$

Substituting these quantities into Equation (A.13) gives the desired formula. □

Remark A.14. Let L be the sublattice of N spanned by the vectors $v_0, \ldots, v_n$. The quantity $[N : L]$ is called the multiplicity of Σ (in N), and is denoted $\mathrm{mult}(\Sigma)$. If Σ is the spanning fan of a Fano polygon P, then the multiplicity of P is $\mathrm{mult}(P) := \mathrm{mult}(\Sigma)$. This definition agrees with the one given in Section 2.1 of the main text. There are inclusions:

$$L(i,j) \subseteq L \subseteq N,$$

for every $i, j \in \{0, \ldots, n\}$. This observation, together with Remark A.12, implies that every term R_i^j appearing in Equation (A.12) is divisible by $\mathrm{mult}(\Sigma)$. By writing these terms as $R_i^j = \mathrm{mult}(\Sigma) r_i^j$, Equation (A.12) can be rewritten as a formula explicitly involving $\mathrm{mult}(\Sigma)$:

$$\mathrm{mult}(\Sigma) \cdot K_X^2 = \sum_{i=0}^{n} \left(2 - \frac{r_i^i}{r_i^{i-1}}\right) \frac{1}{|r_i^{i+1}|}. \tag{A.14}$$

Example A.15. Let X be a fake weighted projective plane with weight vector $(\lambda_0, \lambda_1, \lambda_2)$ satisfying $\gcd\{\lambda_i, \lambda_j\} = 1$ if $i \neq j$. Choose any (necessarily complete) fan Σ in $N_{\mathbb{Q}} \cong \mathbb{Q}^2$ which defines X and let v_0, v_1, v_2 denote the primitive lattice vectors defined by the rays of Σ. By definition, the v_i must satisfy

$$\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 = \mathbf{0}. \tag{A.15}$$

Up to sign, this is the unique relation among the v_i in which the coefficients λ_i are pairwise coprime. Let L denote the sublattice of N spanned by v_0, v_1, v_2. Then Equation (A.15) shows that:

$$\lambda_0 = [L : L(1,2)] \quad ; \quad \lambda_1 = [L : L(0,2)] \quad ; \quad \lambda_2 = [L : L(0,1)].$$

Next, consider the equation:

$$r_0^2 v_2 - r_0^0 v_0 + r_0^1 v_1 = \mathbf{0}, \tag{A.16}$$

which is simply Equation (A.11), for $i = 0$, divided by $\mathrm{mult}(\Sigma)$. The definition of the r_i^j in Remark A.14 implies that $|r_0^2| = [L : L(0,1)] = \lambda_2$. Similarly, $|r_0^i| = \lambda_i$ for $i \in \{0,1,2\}$. It follows that $\gcd\{r_0^i, r_0^j\} = 1$ if $i \neq j$ and that $r_0^i = \pm\lambda_i$, where the signs are to be determined. The pairwise coprimeness of the r_0^i implies (by uniqueness) that Equations (A.15) and (A.16) are the same up to sign. This yields two possibilities, namely $(r_0^2, r_0^0, r_0^1) = (\lambda_2, -\lambda_0, \lambda_1)$ or $(r_0^2, r_0^0, r_0^1) = (-\lambda_2, \lambda_0, -\lambda_1)$. The choice is irrelevant, since in both cases we have:

$$\left(2 - \frac{r_0^0}{r_0^2}\right)\frac{1}{|r_0^1|} = \left(2 + \frac{\lambda_0}{\lambda_2}\right)\frac{1}{\lambda_1}. \tag{A.17}$$

By repeating this process for the remaining two equations of the form (A.16) and substituting the resulting expressions of the form (A.17) into (A.14), we recover the well-known formula for the degree of a fake weighted projective plane X:

$$\mathrm{mult}(\Sigma) \cdot K_X^2 = \frac{(\lambda_0 + \lambda_1 + \lambda_2)^2}{\lambda_0 \lambda_1 \lambda_2}.$$

Here, Σ is the complete fan which defines X and $(\lambda_0, \lambda_1, \lambda_2)$ is the weight vector of X satisfying $\gcd\{\lambda_i, \lambda_j\} = 1$ whenever $i \neq j$. As discussed in Section 3.2 of the main text, this formula plays an important role in the study of one-step mutations between Fano triangles.